J. OPERATOR THEORY
11(1984), 343–364

TOMITA-TAKESAKI THEORY FOR JORDAN ALGEBRAS

UFFE HAAGERUP and HARALD HANCHE-OLSEN

1. INTRODUCTION

The present paper is an attempt to generalize the Tomita-Takesaki Theory to JBW-algebras, the Jordan algebra analogues of W^*-algebras (cf. [19]). If $\mathscr{M}$ is a W^*-algebra and φ is a normal faithful state on $\mathscr{M}$, it is not possible to distinguish σ_t^φ and σ_{-t}^φ solely in terms of the Jordan product $x \circ y = \frac{1}{2}(xy + yx)$. Indeed if one computes the modular automorphism group of φ considered as a state on the opposite algebra $\mathscr{M}^{\mathrm{op}}$ (i.e. the vector space $\mathscr{M}$ equipped with the product $(x, y) \to yx$) one gets σ_{-t}^φ instead of σ_t^φ. However it turns out that the one parameter family

$$\rho_t^\varphi = \frac{1}{2}(\sigma_t^\varphi + \sigma_{-t}^\varphi)$$

can be generalized in a natural way to normal states on JBW-algebras (cf. Theorem 3.3 and Proposition 3.6). The construction of ρ_t^φ makes use of the full structure theory of JBW-algebras.

We reduce the general case to the following two cases:

(I) There exists a trace τ on the JBW-algebra M, such that $\varphi(x) = \tau(x \circ h)$ for some invertible $h \in M_+$.

(II) There exists a von Neumann algebra $\mathscr{M}$ and an involutive anti-automorphism Φ of $\mathscr{M}$, such that M is isomorphic to the self-adjoint part of $\mathscr{M}^\Phi = \{x \in M \mid \Phi(x) = x\}$.

In the case of W^*-algebras the K.M.S.-conditions give a very useful characterization of the modular automorphism group σ_t^φ. No analogue of the K.M.S.-conditions seems to be available for ρ_t^φ. However, it is possible to give a characterization of ρ_t^φ which does not involve structure theory. We prove that the one parameter family $\rho_t = \rho_t^\varphi$ is characterized by the following five conditions:

(i) The map $t \to \rho_t(x)$ is w*-continuous for all $x \in M$;
(ii) Each ρ_t is normal positive and preserves the unit;
(iii) $\rho_0 = \mathrm{id}_M$ and $\rho_s\rho_t = \frac{1}{2}(\rho_{s+t} + \rho_{s-t})$, $s, t \in \mathbf{R}$;
(iv) $\varphi(\rho_t(a) \circ b) = \varphi(a \circ \rho_t(b))$, $a, b \in M$;
(v) The bilinear form on M defined by

$$s(x, y) = \int_{-\infty}^{\infty} \varphi(\rho_t(a) \circ b) \cosh(\pi t)^{-1} \, \mathrm{d}t$$

is a self-polar form on M in the sense of Connes [6] and Woronowicz [26].

It follows from this result that for every state φ on a JB-algebra A, there is a unique self-polar form s_φ on A, such that $s(x, 1) = \varphi(x)$ (cf. Proposition 3.8). The corresponding result for C^*-algebras was proved by Woronowicz in [26].

Recently, Iochum has proved that there is a one-to-one correspondence between JBW-algebras and homogeneous self-dual cones in Hilbert spaces (cf. [12, Chapter VII]). The special case of JBW-algebras with a faithful tracial state was treated previously by Bellissard and Iochum in [4].

In Section 4 of this paper we show that the homogeneous, self-dual cone associated with a σ-finite JBW-algebra can be realized as the closure of M_+ in the real Hilbert space $H_\varphi^\natural$ obtained by completing M with respect to the norm coming from the inner product

$$(a, b)_\varphi^\natural = \int_{-\infty}^{\infty} \varphi(\rho_t(a) \circ b) \cosh(\pi t)^{-1} \, \mathrm{d}t.$$

Moreover, we show that the isomorphism between M and the set of self-adjoint derivations of the self-dual cone described in [5] and [12] can be expressed explicitly in terms of the one parameter family ρ_t^φ.

2. PRELIMINARIES ON JBW-ALGEBRAS

A *JB-algebra* A is a Jordan algebra over $\mathbf{R}$ with a complete norm, such that

$$\|a \circ b\| \leqslant \|a\| \, \|b\| \quad \text{and} \quad \|a^2 + b^2\| \geqslant \|a\|^2$$

for all $a, b \in A$ (cf. [1], [3], [19]). A *JBW-algebra* is a *JB-algebra* which is a dual Banach space. A JB-algebra is called a *JC-algebra* if it is isomorphic to a norm closed Jordan algebra of self-adjoint operators on a Hilbert space. Similarly a JBW-algebra is called a *JW-algebra* if it is isomorphic to a σ-weakly closed Jordan algebra of self-adjoint operators on a Hilbert space.

From [1] we know that a JB-algebra A is a JC-algebra if and only if no quotient of A is isomorphic to M_3^8, the exceptional algebra of self-adjoint 3×3 matrices over the Cayley numbers. By [19] every JBW-algebra M is of the form

$$M = M^{\text{sp}} \oplus M^{\text{ex}}$$

where M^{sp} is a W-algebra, and M^{ex} is isomorphic to $C(X, M_3^8)$ for some hyperstonean space X. Since M_3^8 is a finite dimensional Banach space, M^{ex} can also be written in the form $L^\infty(\Omega, \mu, M_3^8)$ for some Radon measure μ on a locally compact space Ω. Combining this result with the classical theory of JW-algebras due to Topping [24] we get that any JBW-algebra can be decomposed into a type I-part, a type II-part, and a type III-part,

$$M = M_{\text{I}} \oplus M_{\text{II}} \oplus M_{\text{III}}$$

and M_{I} can be decomposed further into its type I_n parts, where n can be any cardinal number. The exceptional part $M^{\text{ex}} = L^\infty(\Omega, \mu, M_3^8)$ should be considered as a part of M_{I_3}. Using Stacey's results from [20], [21] each of the type I_n-parts ($n < \infty$) of M can be decomposed into

$$M_{\text{I}_n} = \sum_\alpha L^\infty(\Omega_\alpha, \mu_\alpha, F_\alpha)$$

where the μ_α's are Radon measures on locally compact spaces Ω_α, and F_α are Jordan factors of type I_n. (A similar decomposition holds when n is an infinite cardinal number, but in this case one has to use w*-measurable functions from Ω_α to F_α, because the type I_n factors are no longer reflexive Banach spaces, when n is infinite.)

Besides the exceptional JBW-factor M_3^8 of type I_3, the only JBW-factors of type I_n, $n \geqslant 3$ are $M_n(\mathbf{R})_{\text{s.a.}}$, $M_n(\mathbf{C})_{\text{s.a.}}$ and $M_n(\mathbf{H})_{\text{s.a.}}$. The JBW-factors of type I_2 are the spin factors V_m, where m can be any finite or infinite cardinal number $\geqslant 2$.

Recall that a positive functional τ on a JB-algebra A is called a trace if

$$\tau(a \circ (b \circ c)) = \tau((a \circ b) \circ c), \quad a, b, c \in A$$

(cf. [4], [16]). Note that any JBW-factor of type I_n, $n < \infty$, has a unique tracial state τ_0.

The following three lemmas are all easy conseqences of known results.

LEMMA 2.1. *Let M be a* JBW-*algebra, which is a direct sum of type* I_n *algebras ($n < \infty$), and let φ be a normal, faithful state on M. Then there exists a tracial state τ on M, and a central decomposition*

$$M = \bigoplus_\alpha M_\alpha$$

of M, such that for each α the restriction of φ to M_α is of the form

$$\varphi(x) = \tau(h_\alpha \circ x), \quad x \in M_\alpha$$

for some invertible element $h_\alpha \in M_+$.

Proof. By the structure theory for type I algebras, we can assume that M is of the form

$$M = L^\infty(\Omega, \mu, F)$$

where μ is a Radon measure on a locally compact space Ω and F is a JBW-factor of type I_n, for some $n < \infty$. Let τ_0 be the normalized trace on F. Define a map T of M into $Z(M)$ (the center of M) by

$$(T(x))(\omega) = \tau_0(x(\omega))1, \quad x \in L^\infty(\Omega, \mu, F), \ \omega \in \Omega.$$

Clearly T is a w*-continuous linear map of M into $Z(M)$. In fact T is a center valued trace in the sense of [24], but we shall not need this fact. Define a tracial state τ on M by

$$\tau(x) = \varphi \circ T(x), \quad x \in M.$$

Let p be an abelian projection in M. Then $p(\omega)$ is 0 or a minimal projection in F for locally almost all $\omega \in \Omega$. Since $\tau_0(q) = 1/n$ for any minimal projection q in F, we get $T(p) = (1/n)c(p)$. (The central support $c(p)$ of p is 1 for those $\omega \in \Omega$ such that $p(\omega) \neq 0$, and 0 on the other ω's.) Hence we get

$$\varphi(p) \leqslant \varphi(c(p)) = \varphi(nT(p)) = n\tau(p).$$

Since any projection is the sum of abelian projections, the above inequality is true for all projections in M. Hence by spectral theory

$$\varphi(x) \leqslant n\tau(x)$$

for all $x \in M_+$. Sakai's linear Radon-Nikodym Theorem [18, Proposition 1.24.4] is true also for JBW-algebras (with the same proof as for W^*-algebras). Hence there exists a $h \in M_+$, $\|h\| \leqslant n$, such that

$$\varphi(x) = \tau(h \circ x), \quad x \in M.$$

Clearly $\varphi(1 - [h]) = 0$, where $[h]$ is the support projection of h. Hence $[h] = 1$. Let p_m be the spectral projection of h corresponding to the interval $[0, 1/m]$. Clearly

$$p_1 \geqslant p_2 \geqslant p_3 \geqslant \dots$$

and $\text{w}^*\text{-}\lim_{m\to\infty} p_m = 0$. Since $c(p_m) \leqslant n\,T(p_m)$ and

$$c(p_1) \geqslant c(p_2) \geqslant c(p_3) \geqslant \ldots$$

it follows that also

$$\text{w}^*\text{-}\lim_{m\to\infty} c(p_m) = 0.$$

Put now $q_m = c(p_m) - c(p_{m+1})$. Then (q_m) are orthogonal central projections with sum 1, and $h_m = q_m h$ is invertible in the Jordan algebra $q_m M$. This proves Lemma 2.1.

LEMMA 2.2. *Let M be a* JBW*-algebra without type* I_2 *and type* I_3 *parts. Then there exists a von Neumann algebra* $\mathcal{N}$, *and an involutive* $*$*-anti-automorphism* Φ *of* $\mathcal{N}$, *such that A is Jordan isomorphic to*

$$\{x \in \mathcal{N}_{\text{s.a.}} \mid \Phi(x) = x\}.$$

Proof. Since M has no type I_3 part it is a JW-algebra. Recall that a JW--algebra N is called *universally reversible*, if for any normal representation π of N on a Hilbert space H, and any finite set $y_1, \ldots, y_n$ of elements in $\pi(N)$, the operator $y_1 \cdot \ldots \cdot y_n + y_n \cdot \ldots \cdot y_1$ is also contained in $\pi(N)$. By [22, Theorem 6.6] any JW--algebra without a type I_2 part is universally reversible. Hence M is universally reversible. The existence of $(\mathcal{N}, \Phi)$ can now be proved as in the case of universally reversible JC-algebras (cf. [11, Proposition 2.3]).

LEMMA 2.3. a) *A unital* JB*-algebra generated by two elements (and* 1*) is a* JC*-algebra.*

b) *A* JBW*-algebra generated by two elements (and* 1*) is a* JW*-algebra.*

Proof. a) Because of the exceptional nature of M_3^8 [13, Chapter 1, Theorem 11] and the Shirshov-Cohn theorem [13, Chapter 1, Theorem 10] M_3^8 cannot be algebraically generated by two elements and the identity. Since M_3^8 is finite dimensional it also cannot be generated as a JB-algebra by two elements and 1. Hence, if A is any unital JB-algebra generated by two elements and the unit, it cannot have a quotient isomorphic to M_3^8, i.e. A is a JC-algebra.

b) Let M be a JBW-algebra generated by two elements x, y and the identity 1, and let A be the smallest norm closed Jordan algebra containing $x, y, 1$. The imbedding $i\colon A \to M$ can be extended to a normal Jordan homomorphism $\tilde{i}$ of A^{**} into M. Since A is dense in M, $\tilde{i}(A^{**}) = M$. Hence there exists a central projection $p \in A^{**}$ such that M is isomorphic to pA^{**} (cf. [7, Theorem 3.3]). From a) we know that A is a JC-algebra. Hence by [8, Theorem 1], A^{**} is a JW-algebra. This proves b).

3. THE ONE-PARAMETER FAMILY ρ_t^φ

In [6], [26], Connes and Woronowicz introduced the notion of self-polar forms on C^*-algebras. With obvious modifications Woronowicz' definition makes sense also in the case of JB-algebras.

DEFINITION 3.1. Let A be a unital JB-algebra. For any bilinear form $t: A \times A \to \mathbf{R}$ we let t^* denote the linear map from A to A^* defined by

$$\langle t^*(a), b\rangle = t(a, b), \quad a, b \in A.$$

A *self-polar form* on A is a positive, symmetric bilinear form s on A, for which

(i) $s(a, b) \geqslant 0, \quad a, b \in A_+$

(ii) $s^*([0, 1])$ is weak* dense in $[0, s^*(1)]$.

Here $[\xi, \eta]$ means $\{\zeta \mid \xi \leqslant \zeta \leqslant \eta\}$ in any ordered vector space.

REMARK 3.2. a) Using Woronowicz' result [26, Theorem 1.2] on the complexification of A, it follows that if s_1 and s_2 are two self-polar forms on a JB-algebra such that $s_1^*(1) = s_2^*(1)$, then $s_1 = s_2$.

b) If s is a self-polar form on a JBW-algebra, such that $s^*(1)$ is a normal functional on A, an elementary compactness argument shows that $s^*([0, 1]) = [0, s^*(1)]$.

We are now able to state the main result of this paper:

THEOREM 3.3. *Let M be a* JBW*-algebra with a normal, faithful state φ. Then there is a unique one-parameter family $(\rho_t)_{t\in\mathbf{R}}$ of operators on M, satisfying*

(i) *The map $t \to \rho_t(x)$ is* w**-continuous for all $x \in M$;*

(ii) *Each ρ_t is positive, normal and preserves the unit;*

(iii) $\rho_0 = \mathrm{id}_A, \ \rho_s\rho_t = \dfrac{1}{2}(\rho_{s+t} + \rho_{s-t}), \quad s, t \in \mathbf{R};$

(iv) $\varphi(\rho_t(a) \circ b) = \varphi(a \circ \rho_t(b)), \quad a, b \in M;$

(v) *The bilinear form on M defined by*

$$s(a, b) = \int_{-\infty}^{\infty} \varphi(\rho_t(a) \circ b) \cosh(\pi t)^{-1}\, dt, \quad a, b \in M,$$

is a self-polar form on M.

Proof of uniqueness of ρ_t. Let $H_\varphi^\#$ denote the completion of M with respect to the inner product

$$(a, b)_\varphi^\# = \varphi(a \circ b).$$

Assume that (ρ_t) satisfies the conditions of Theorem 3.3. If $x \in M$, $\|x\|_\varphi^\# = 1$, we find

$$(\|\rho_t(x)\|_\varphi^\#)^2 = \varphi(\rho_t(x) \circ \rho_t(x)) = \varphi(\rho_t^2(x) \circ x) =$$

$$= \frac{1}{2}\varphi((\rho_{2t}(x) + x) \circ x) = \frac{1}{2}((\rho_{2t}(x), x)_\varphi^\# + (\|x\|_\varphi^\#)^2) \leqslant$$

$$\leqslant \frac{1}{2}(\|\rho_{2t}(x)\|_\varphi^\# + 1).$$

By condition (ii) in Theorem 3.3, $\|\rho_t(x)\| \leqslant \|x\|$ for all t. Hence the constant

$$K = \sup_{t \in \mathbf{R}} \|\rho_t(x)\|_\varphi^\#$$

is finite. The above inequality shows that $K^2 \leqslant (1/2)(K + 1)$, which implies that $K \leqslant 1$. Hence $\|\rho_t(x)\|_\varphi^\# \leqslant 1$. Therefore each ρ_t can be extended to an operator u_t on $H_\varphi^\#$ with $\|u_t\| \leqslant 1$. It is easily seen that $(u_t)_{t \in \mathbf{R}}$ is a weakly continuous family of operators on the real Hilbert space $H_\varphi^\#$. By the conditions (iii) and (iv) of Theorem 3.3 we get

$$u_0 = 1, \quad u_s u_t = \frac{1}{2}(u_{s+t} + u_{s-t}) \quad \text{and} \quad u_s = u_s^*$$

for all $s, t \in \mathbf{R}$. By Kurepa's result [14, Theorem 2] one can derive fairly easily that there exists a (possibly unbounded) positive, self-adjoint operator D on $H_\varphi^\#$, such that $u_t = \cos(tD)$, $t \in \mathbf{R}$. In the appendix (Section 5) we give a direct proof of this fact. Since

$$\int_{-\infty}^{\infty} e^{itx} \cosh(\pi t)^{-1} dt = \cosh\left(\frac{x}{2}\right)^{-1}$$

for any $x \in \mathbf{R}$, we get

$$\int_{-\infty}^{\infty} u_t \cosh(\pi t)^{-1} dt = \cosh\left(\frac{D}{2}\right)^{-1} \qquad \text{(strongly)}.$$

Hence for $a, b \in M$ the self-polar form (v) in Theorem 3.3 can be expressed as

$$s(a, b) = \int_{-\infty}^{\infty} \varphi(\rho_t(a) \circ b) \cosh(\pi t)^{-1} dt = \left(\cosh\left(\frac{D}{2}\right)^{-1} \dot{a}, \dot{b}\right),$$

where $\dot{a}$ and $\dot{b}$ denote the natural imbeddings of a and b in $H_\varphi^\#$.

If $(\rho'_t)_{t\in\mathbf{R}}$ is another one-parameter family satisfying (i)—(v) in Theorem 3.3, we get in the same way operators u'_t and D' on H^*_φ, such that D' is positive and self-adjoint, $u'_t = \cos(tD')$

$$s'(a, b) = \left(\cosh\left(\frac{D'}{2}\right)^{-1} \dot{a}, \dot{b}\right), \quad a, b \in M$$

is a self-polar form on M. Clearly $s^*(1) = (s')^*(1) = \varphi$. Hence by the Woronowicz uniqueness Theorem (cf. Remark 3.2 a) $s = s'$, which implies that

$$\cosh\left(\frac{D}{2}\right)^{-1} = \cosh\left(\frac{D'}{2}\right)^{-1}$$

and since the function cosh is one-to-one on $[0, \infty[$, we conclude that $D = D'$. Therefore $u_t = u'_t$, $t \in \mathbf{R}$, which clearly implies that $\rho_t = \rho'_t$ for all $t \in \mathbf{R}$.

Proof of existence of ρ_t. Assume that $M = \sum_\alpha^\oplus M_\alpha$ and $\varphi = \sum^\oplus \lambda_\alpha \varphi_\alpha$, where φ_α are faithful normal states on M_α and λ_α are positive scalars with sum 1. If we can find for each α a one-parameter family $\rho_{\alpha,t}$ satisfying the conditions in Theorem 3.3 with respect to $(M_\alpha, \varphi_\alpha)$, then clearly

$$\rho_t = \sum^\oplus \rho_{\alpha,t}$$

satisfies the conditions with respect to (M, φ). Hence by Lemma 2.1 and Lemma 2.2 it is sufficient to treat the following two cases.

CASE I. There exists a faithful trace τ on M, and an invertible element h in M_+, such that $\varphi(x) = \tau(x \circ h)$ for all $x \in M$.

CASE II. There exists a von Neumann algebra $\mathcal{N}$ and an involutive $*$-automorphism Φ of $\mathcal{N}$, such that M is Jordan isomorphic to $\{x \in \mathcal{N}_{\text{s.a.}} \mid \Phi(x) = x\}$.

Proof of existence in Case I. We will show that

$$\rho_t(a) = \{h^{it} a h^{-it}\}, \quad a \in M$$

satisfies the conditions in Theorem 3.3. As usual $\{\cdots\}$ denotes the Jordan triple product

$$\{abc\} = a \circ (b \circ c) + c \circ (a \circ b) - b \circ (a \circ c)$$

and h^{it} is the element $\cos(t \log h) + \mathrm{i} \sin(t \log h)$ in the complex Jordan algebra $M^{\mathbf{C}} = M + \mathrm{i} M$. $M^{\mathbf{C}}$ has a natural norm, which makes it a "JB*-algebra" (cf. [27]), but we shall not need this fact. ρ_t maps M into itself because, for $a \in M$,

$$\rho_t(a) = \{\cos(t \log h)\, a \cos(t \log h)\} + \{\sin(t \log h)\, a \sin(t \log h)\}.$$

The conditions (i) and (ii) of Theorem 3.3 can easily be verified. Clearly $\rho_0 = \mathrm{id}_M$, so to verify (iii) it suffices to show that

$$\{h^{is}\{h^{it}ah^{-it}\}h^{-is}\} =$$

$$= \frac{1}{2}(\{h^{i(s+t)}ah^{-i(s+t)}\} + \{h^{i(s-t)}ah^{-i(s-t)}\})$$

for $a \in M$ and $s, t \in \mathbf{R}$. However, by Lemma 2.3, the JBW-algebra generated by a and h is a JW-algebra, and for two operators a, h on a Hilbert space the above equality is trivial.

Next we prove (iv). By use of Lemma 2.3 we find

$$\{h^{it}(a \circ h)h^{-it}\} = \{h^{it}ah^{-it}\} \circ h$$

for all $a \in M$. It follows from the trace property

$$\tau(a \circ (b \circ c)) = \tau((a \circ b) \circ c)$$

that for all $c, d, e, f \in M$,

$$\tau(\{ecf\} \circ d) = \tau(c \circ \{edf\})$$

and the formula is also true if $c, d, e, f \in M^{\mathbf{C}}$. Hence for $a, b \in M$:

$$\varphi(a \circ \rho_t(b)) = \tau(h \circ (a \circ \{h^{it}bh^{-it}\})) =$$

$$= \tau((h \circ a) \circ \{h^{it}bh^{-it}\}) = \tau(\{h^{it}(h \circ a)h^{-it}\} \circ b) =$$

$$= \tau((h \circ \{h^{it}ah^{-it}\}) \circ b) = \tau(h \circ (\{h^{it}ah^{-it}\} \circ b)) = \varphi(\rho_t(a) \circ b).$$

To prove (v), let $a, b \in M$, and put

$$a' = \int_{-\infty}^{\infty} \{h^{it}ah^{-it}\} \cosh(\pi t)^{-1}\,\mathrm{d}t.$$

We claim that $a' \circ h = \{h^{1/2}ah^{1/2}\}$. By Lemma 2.3 it is enough to check that the formula is true when a and h are operators on a Hilbert space. But in this case the formula is due to Van Daele and Pedersen (see [25] and [15, Proposition 3]). Therefore

$$s(a, b) = \int_{-\infty}^{\infty} \varphi(\rho_t(a) \circ b) \cosh(\pi t)^{-1}\,\mathrm{d}t =$$

$$= \tau(h \circ (a' \circ b)) = \tau((h \circ a') \circ b) = \tau(\{h^{1/2}ah^{1/2}\} \circ b) =$$

$$= \tau(\{h^{1/4}\{h^{1/4}ah^{1/4}\}h^{1/4}\} \circ b) = \tau(\{h^{1/4}ah^{1/4}\} \circ \{h^{1/4}bh^{1/4}\}).$$

This formula shows that s is a positive symmetric form on M. Moreover $s(a, b) \geqslant 0$ for $a, b \in M_+$ (cf. [16, Theorem (iii)]). From the the formula

$$s(a, b) = \tau(\{h^{1/2}ah^{1/2}\} \circ b)$$

we have $s^*(1) = \varphi$. Let now $\psi \in [0, \varphi]$. By the generalisation of Sakai's linear Radon-Nikodym Theorem to JBW-algebras, there exists $k \in M_+$ such that

$$\psi(x) = \tau(k \circ x), \quad x \in M_+.$$

Let $(k - h)_+$ denote the positive part of $k - h$. Since

$$0 \geqslant (\psi - \varphi)((k - h)_+) = \tau((k - h)(k - h)_+) = \tau((k - h)_+^2)$$

it follows that $k \leqslant h$. Put $a = \{h^{-1/2}kh^{-1/2}\}$. Then $0 \leqslant a \leqslant 1$, and

$$s(a, x) = \tau(\{h^{1/2}ah^{1/2}\} \circ x) = \tau(k \circ x) = \varphi(x)$$

for all $x \in M$, i.e. $s^*(a) = \psi$. Hence $s^*([0, 1]) = [0, s^*(1)]$ and we conclude that s is a self-polar form.

For the existence proof in case II we shall need the following lemma:

LEMMA 3.4. *Let $\mathcal{N}$ be a von Neumann algebra, and let φ be a normal faithful state on $\mathcal{N}$. Then*

$$s(a, b) = \int_{-\infty}^{\infty} \varphi(\sigma_t^\varphi(a) \circ b) \cosh(\pi t)^{-1}\, dt$$

is a self-polar form on $\mathcal{N}_{s.a.} = \{x \in \mathcal{N} \mid x = x^*\}$, and $s^*([0, 1]) = [0, \varphi]$.

Proof. We may assume that $\mathcal{N}$ acts on a Hilbert space H with a cyclic and separating vector ξ_0, such that

$$\varphi(x) = (x\xi_0, \xi_0), \quad x \in \mathcal{N}.$$

Let S, F, J, Δ be the usual operators from Tomita-Takesaki theory coming from $(\mathcal{N}, \xi_0)$. From [6] we know that the self-polar form s on $\mathcal{N}_{s.a.}$, for which $s^*(1) = \varphi$ is given by

$$s(a, b) = (\Delta^{1/2}a\xi_0, b\xi_0), \quad a, b \in \mathcal{N}_{s.a.}.$$

(Strictly speaking, Connes defines s as a sesquilinear form on $\mathcal{N}$, but it is clear that when we restrict to $\mathcal{N}_{s.a.}$, we get a self-polar form in the sense of Definition 3.1.)

Let $a, b \in \mathcal{N}_{s.a.}$ and put

$$a' = \int_{-\infty}^{\infty} \sigma_t^\varphi(a) \cosh(\pi t)^{-1}\, dt.$$

Then

$$a'\xi_0 = \int_{-\infty}^{\infty} \Delta^{it} a\xi_0 \cosh(\pi t)^{-1} \mathrm{d}t = 2(\Delta^{1/2} + \Delta^{-1/2})^{-1} a\xi_0$$

because

$$\int_{-\infty}^{\infty} \mathrm{e}^{its} \cosh(\pi t)^{-1} \mathrm{d}t = \cosh\left(\frac{s}{2}\right)^{-1}, \quad s \in \mathbf{R}$$

and $\cosh\left(\frac{1}{2}\log\Delta\right) = \frac{1}{2}(\Delta^{1/2} + \Delta^{-1/2})$. Therefore $a'\xi_0 \in D(\Delta^{1/2}) \cap D(\Delta^{-1/2})$ and

$$\frac{1}{2}(\Delta^{1/2} + \Delta^{-1/2}) a'\xi_0 = a\xi_0 .$$

Hence

$$(\Delta^{1/2} a\xi_0, b\xi_0) = (a\xi_0, \Delta^{1/2} b\xi_0) =$$

$$= \frac{1}{2}((\Delta^{1/2} + \Delta^{-1/2}) a'\xi_0, \Delta^{1/2} b\xi_0) = \frac{1}{2}((\Delta^{1/2} a'\xi_0, \Delta^{1/2} b\xi_0) + (a'\xi_0, b\xi_0)) =$$

$$= \frac{1}{2}((JSa'\xi_0, JSb\xi_0) + (a'\xi_0, b\xi_0)) = \frac{1}{2}((b\xi_0, a'\xi_0) + (a'\xi_0, b\xi_0)) =$$

$$= \varphi(a' \circ b) = \int_{-\infty}^{\infty} \varphi(\sigma_t^{\varphi}(a) \circ b) \cosh(\pi t)^{-1} \mathrm{d}t.$$

This proves Lemma 3.4.

Proof of existence in Case II. In this case we can assume that

$$M = \{a \in \mathcal{N} \mid a = a^* = \Phi(a)\}$$

for some von Neumann algebra $\mathcal{N}$, and some involutive antiautomorphism Φ of $\mathcal{N}$. Let $\overline{\varphi}$ be the state on $\mathcal{N}$ for which

$$\overline{\varphi}(a) = \frac{1}{2}\varphi(a + \Phi(a)), \quad a \in \mathcal{N}_{\text{s.a.}}.$$

Clearly $\overline{\varphi}$ is a normal faithful Φ-invariant extension of φ. Using that Φ is an antiisomorphism one checks easily that $\overline{\varphi}$ satisfies the K.M.S.-conditions with respect to the one parameter automorphism group $t \to \Phi \circ \sigma_{-t}^{\overline{\varphi}} \circ \Phi$. Hence

$$\sigma_t^{\overline{\varphi}} = \Phi \circ \sigma_{-t}^{\overline{\varphi}} \circ \Phi, \quad t \in \mathbf{R}$$

or equivalently

$$\Phi \circ \sigma_t^{\bar\varphi} = \sigma_{-t}^{\bar\varphi} \circ \Phi, \quad t \in \mathbf{R}.$$

From this it follows that $\frac{1}{2}(\sigma_t^{\bar\varphi} + \sigma_{-t}^{\bar\varphi})$ maps M into itself. Let ρ_t be the restriction of $\frac{1}{2}(\sigma_t^{\bar\varphi} + \sigma_{-t}^{\bar\varphi})$ to M. We will show that ρ_t satisfies the conditions of Theorem 3.3. The conditions (i)—(iv) are easily verified. Let us prove (v). By Lemma 3.4

$$\bar s(a, b) = \int_{-\infty}^{\infty} \bar\varphi(\sigma_t^{\bar\varphi}(a) \circ b) \cosh(\pi t)^{-1} \, dt$$

is a self-polar form on $\mathcal{N}_{\text{s.a.}}$. The form on M given by

$$s(a, b) = \int_{-\infty}^{\infty} \varphi(\rho_t(a) \circ b) \cosh(\pi t)^{-1} \, dt$$

is simply the restriction of $\bar s$ to M. Hence s is a positive symmetric form on M and $s(a, b) \geqslant 0$ for $a, b \in M_+$. Moreover $s^*(1) = \varphi$. Let $\omega \in M^*$, $0 \leqslant \omega \leqslant \varphi$. Let $\bar\omega$ be the state on $\mathcal{N}$, for which

$$\bar\omega(a) = \frac{1}{2}\omega(a + \Phi(a)), \quad a \in \mathcal{N}_{\text{s.a.}}.$$

Then clearly $0 \leqslant \bar\omega \leqslant \bar\varphi$. By Lemma 3.4, there exists $a \in \mathcal{N}_{\text{s.a.}}$, $0 \leqslant a \leqslant 1$, such that

$$\bar\omega(b) = \bar s(a, b) \quad \text{for} \quad b \in \mathcal{N}_{\text{s.a.}}.$$

But since $\bar\omega$ is Φ-invariant, we have also

$$\bar\omega(b) = \bar s(a, \Phi(b)).$$

Using $\bar\varphi \circ \Phi = \bar\varphi$ and $\sigma_{-t}^{\bar\varphi} \circ \Phi = \Phi \circ \sigma_t^{\bar\varphi}$ we get that $\bar s(a, \Phi(b)) = \bar s(\Phi(a), b)$. Hence

$$\bar\omega(b) = \frac{1}{2}(\bar s(a, b) + \bar s(\Phi(a), b)) = \bar s(a', b)$$

where $a' = \frac{1}{2}(a + \Phi(a)) \in M$, and $0 \leqslant a' \leqslant 1$. Hence s is a self-polar form on M. This completes the proof of Theorem 3.3.

DEFINITION 3.5. The unique one-parameter family ρ_t satisfying the conditions of Theorem 3.3 we call *the modular cosine family associated with* φ, and it will be denoted $(\rho_t^\varphi)_{t\in\mathbf{R}}$.

The following proposition can be extracted from the proof of Theorem 3.3.

PROPOSITION 3.6. *Let M be a* JBW-*algebra and φ a faithful normal state on M.*

a) *If there exists a trace τ on M and an invertible element $h \in M_+$, such that $\varphi(a) = \tau(h \circ a)$, $a \in M$, then*

$$\rho_t^\varphi(a) = \{h^{it} a h^{-it}\} =$$

$$= \{\cos(t \log h)\, a \cos(t \log h)\} + \{\sin(t \log h)\, a \sin(t \log h)\}.$$

b) *If M is the self-adjoint part of a von Neumann algebra $\mathcal{M}$, then*

$$\rho_t^\varphi(a) = \frac{1}{2}(\sigma_t^\varphi(a) + \sigma_{-t}^\varphi(a)), \quad a \in M.$$

c) *If M is of the form $\{a \in \mathcal{N}_{\text{s.a.}} \mid \Phi(a) = a\}$, where $\mathcal{N}$ is a von Neumann algebra, and Φ is an involutive antiisomorphism on $\mathcal{N}$, then*

$$\rho_t^\varphi(a) = \frac{1}{2}(\sigma_t^{\overline{\varphi}}(a) + \sigma_{-t}^{\overline{\varphi}}(a)), \quad a \in M$$

where $\overline{\varphi}(a) = \dfrac{1}{2}\varphi(a + \Phi(a))$, $a \in \mathcal{N}_{\text{s.a.}}$.

REMARK 3.7. If φ is a tracial state, then by Proposition 3.6 a), $\rho_t^\varphi = \mathrm{id}_M$ for all $t \in \mathbf{R}$. The converse is also true. Indeed, if $\rho_t^\varphi = \mathrm{id}_M$, then $s(a, b) = \varphi(a \circ b)$ is a self-polar form on M. In particular $\varphi(a \circ b) \geqslant 0$ for all $a, b \in M_+$. But this implies that φ is a trace (cf. [16]).

The following proposition was proved by Woronowicz in the case where A is the self-adjoint part of a unital C^*-algebra (cf. [26, Section 2]).

PROPOSITION 3.8. *For any state φ on a unital* JB-*algebra, there exists one and only one self-polar form s on A such that $s^*(1) = \varphi$.*

Proof. The uniqueness is due to Woronowicz (cf. Remark 3.2 (a)). Theorem 3.3 shows the existence of s in the case where A is a JBW-algebra, and φ is a faithful normal state. The condition of faithfulness can easily be removed by passing to the reduced Jordan algebra $\{pAp\}$ where p is the support projection of φ.

Let now φ be an arbitrary state on a unital JB-algebra A. Then A^{**} is a JBW-algebra (cf. [19], [10]) and φ has a unique extension to a normal state $\tilde{\varphi}$ on A^{**}. Let $\tilde{s}$ be the self-polar form on A^{**} associated with $\tilde{\varphi}$, and let s be the restriction of $\tilde{s}$ to $A \times A$. To show that s is a self-polar form on A, it suffices to verify condition (iv) in Definition 3.1. Note first, that the range of the map $\tilde{s}^{*}: A^{**} \to A^{***}$ is contained in A^{*}. This is true, because for every $a \in (A^{**})_{+}$, the positive functional $\tilde{s}^{*}(a)$ is dominated by a multiple of the normal functional $\tilde{\varphi}$, which implies that $\tilde{s}^{*}(a)$ is also normal. Let $\psi \in A^{*}$, $0 \leqslant \psi \leqslant \varphi$, and let $\tilde{\psi}$ be the normal extension of ψ to A^{**}. By Remark 3.2 (b), there exists a $b \in A^{**}$, $0 \leqslant b \leqslant 1$, such that $\tilde{\psi}(a) = \tilde{s}(a, b)$ for all $a \in A^{**}$. Choose now a net (b_α) in A, such that $0 \leqslant b_\alpha \leqslant 1$ and $b_\alpha \to b$ in the weak* topology. Then for all $a \in A$

$$\lim_\alpha \langle s^{*}(b_\alpha), a \rangle = \lim_\alpha \langle s^{*}(a), b_\alpha \rangle = \langle \tilde{s}^{*}(a), b \rangle = \psi(a).$$

Hence $s^{*}([0, 1])$ is weak* dense in $[0, \varphi]$.

4. THE SELF-DUAL CONE $P_\varphi^\natural$

Let P be a cone in a real or complex Hilbert space H. P is called *self-dual* if P coincides with the cone $P^0 = \{\xi \in H \mid (\xi, \eta) \geqslant 0, \eta \in P\}$. Following the notation of Connes [6], an operator $D \in B(H)$ is called a *derivation* of P if

$$\exp(tD)P = P, \quad t \in \mathbf{R}.$$

Moreover, a self-dual cone P is called *homogeneous* if for any face F in P, $e_F - e_{F^\perp}$ is a derivation of P. Here e_F and $e_{F^\perp}$ are the projections on the closed linear spans of F and $F^\perp$. ($F^\perp = \{\xi \in P \mid (\xi, \eta) = 0, \eta \in F\}$.)

In [6, Section 4] Connes associated to any W^{*}-algebra $\mathscr{M}$ acting standardly on a Hilbert space $\mathscr{H}$ a homogeneous self-dual cone $\mathscr{P}^\natural \subseteq \mathscr{H}$, such that $\mathscr{M}_{\mathrm{s.a.}}$ is isometrically isomorphic to the set of self-adjoint derivations of $\mathscr{P}^\natural$ (see also [2], [9]). Moreover, he proved that there is a one-to-one correspondence between W^{*}--algebras and those homogeneous self-dual cones which satisfy a certain condition of orientability (cf. [6, Section 5]).

In [4], [5] Bellissard and Iochum studied the connection between JBW-algebras and homogeneous self-dual cones (without orientation) and recently Iochum established a one-to-one correspondence between JBW-algebras and homogeneous self-dual cones (cf. [12, Chapter VII]).

The aim of the present section is to prove that the self-dual cone $P^\natural$, which Iochum associates to a JBW-algebra M, and the isometry δ of M onto the set of self-adjoint derivations of $P^\natural$ can be expressed in terms of the modular cosine families ρ_t^φ.

THEOREM 4.1. *Let M be a JBW-algebra with a faithful normal state φ, and let s_φ be the self-polar form on M associated with φ. Let $H_\varphi^\natural$ denote the completion of M with respect to the inner product*

$$(a, b)_\varphi^\natural = s_\varphi(a, b)$$

and let $P_\varphi^\natural$ be the closure of M_+ in $H_\varphi^\natural$. Then

a) *$P_\varphi^\natural$ is a homogeneous, self-dual cone in $H_\varphi^\natural$;*

b) *For any $a \in M$, there is a unique operator $\delta_\varphi(a) \in B(H_\varphi^\natural)$, such that*

$$(*) \qquad (\delta_\varphi(a)x, y)_\varphi^\natural = \frac{1}{2}((\tilde{a} \circ x, y)_\varphi^\natural + (x, \tilde{a} \circ y)_\varphi^\natural), \quad x, y \in M$$

where

$$\tilde{a} = 2 \int_{-\infty}^{\infty} \frac{\rho_t^\varphi(a)}{\cosh(2\pi t)} \, dt.$$

Moreover, $\delta_\varphi : M \to B(H_\varphi^\natural)$ is an isometry of M onto the set of self-adjoint derivations of $P^\natural$.

Proof. We divide the proof in two cases as in the proof of Theorem 3.3.

Case I. Assume that there exists a tracial state τ and a positive invertible operator $h \in M_+$, such that

$$\varphi(a) = \tau(h \circ a), \quad a \in M.$$

By Remark 3.7 we have $\rho_t^\tau = \mathrm{id}_M$, $t \in \mathbf{R}$ and $s_\tau(a, b) = \tau(a \circ b)$. Thus $H_\tau^\natural$ is the completion of M with respect to the norm $\tau(a^2)^{1/2}$. Moreover the formula defining $\delta_\tau(a)$ reduces to

$$(\delta_\tau(a)x, y)_\tau^\natural = \frac{1}{2}(\tau((a \circ x) \circ y) + \tau(x \circ (a \circ y))) = (a \circ x, y)_\tau^\natural$$

i.e. $\delta_\tau(a)x = a \circ x$, $x \in M_+$. Hence by [4], $P_\tau^\natural$ is a homogeneous self-dual cone in H, and δ_τ is an isometry of M onto the self-adjoint derivations of $P_\tau^\natural$.

By the proof of Theorem 3.3 (Case I) we have

$$s_\varphi(a, b) = \tau(\{h^{1/4} a h^{1/4}\} \circ \{h^{1/4} b h^{1/4}\}), \quad a, b \in M.$$

Hence the map $u_0(a) = \{h^{1/4} a h^{1/4}\}$ extends to a unitary map of $H_\varphi^\natural$ onto $H_\tau^\natural$. Since $u_0(M_+) = M_+$ we have $u(P_\varphi^\natural) = P_\tau^\natural$. Therefore $P_\varphi^\natural$ is also a homogeneous, self-dual cone and the map $\delta_\varphi\colon M_+ \to B(H_\varphi^\natural)$ given by $\delta_\varphi(a) = u^*\delta_\tau(a)u$, $a \in M_+$ is an isometry of M_+ onto the self-adjoint derivations of $P_\varphi^\natural$. For $x, y \in M$

$$(\delta_\varphi(a)x, y)_\varphi^\natural = (\delta_\tau(a)ux, uy)_\tau^\natural =$$

$$= \tau((a \circ \{h^{1/4}xh^{1/4}\}) \circ \{h^{1/4}yh^{1/4}\}) = \tau(a \circ (\{h^{1/4}xh^{1/4}\} \circ \{h^{1/4}yh^{1/4}\})).$$

However

$$\{h^{1/4}xh^{1/4}\} \circ \{h^{1/4}yh^{1/4}\} = \{h^{1/4}\{xh^{1/2}y\}h^{1/4}\}.$$

Indeed, this formula is trivial for $x = y$ by Lemma 2.3, and since both sides are symmetric in (x, y) the general case follows by polarization. Thus

$$(\delta_\varphi(a)x, y)_\varphi^\natural = \tau(a \circ \{h^{1/4}\{xh^{1/2}y\}h^{1/4}\}) =$$

$$= \tau(\{h^{1/4}ah^{1/4}\} \circ \{xh^{1/2}y\}).$$

As in the proof of Theorem 3.3 (case I) we have

$$\{h^{1/4}ah^{1/4}\} = \tilde{a} \circ h^{1/2}$$

where

$$\tilde{a} = \int_{-\infty}^{\infty} \{h^{it/2}ah^{-it/2}\} \cosh(\pi t)^{-1}\, dt =$$

$$= 2\int_{-\infty}^{\infty} \{h^{it}ah^{-it}\} \cosh(2\pi t)^{-1}\, dt =$$

$$= 2\int_{-\infty}^{\infty} \rho_t^\varphi(a) \cosh(2\pi t)^{-1}\, dt.$$

Therefore

$$(\delta_\varphi(a)x, y)_\varphi^\natural = \tau((\tilde{a} \circ h^{1/2}) \circ \{xh^{1/2}y\}) =$$

$$= \tau(\tilde{a} \circ (h^{1/2} \circ \{xh^{1/2}y\})).$$

By Lemma 2.3 and polarization, we get

$$h^{1/2} \circ \{xh^{1/2}y\} = \frac{1}{2}(\{h^{1/2}xh^{1/2}\} \circ y + \{h^{1/2}yh^{1/2}\} \circ x),$$

Hence

$$(\delta_\varphi(a)x, y)_\varphi^{h} = \frac{1}{2}\tau((\tilde{a} \circ y) \circ \{h^{1/2}xh^{1/2}\}) + \frac{1}{2}\tau((\tilde{a} \circ x) \circ \{h^{1/2}yh^{1/2}\}) =$$

$$= \frac{1}{2}((x, \tilde{a} \circ y)_\varphi^{h} + (\tilde{a} \circ x, y)_\varphi^{h}).$$

This completes the proof in Case I.

For the proof in Case II we need the following lemma.

LEMMA 4.2 (Van Daele, Pedersen). *Let $\mathscr{H}$ be a complex Hilbert space, let $a \in B(\mathscr{H})$, and let h be a non singular, positive self-adjoint (possibly unbounded) operator on $\mathscr{H}$. If we put*

$$a' = \int_{-\infty}^{\infty} h^{it} a h^{-it} \cosh(\pi t)^{-1} dt$$

then for $\xi, \eta \in D(h)$:

$$\frac{1}{2}((a'\xi, h\eta) + (a'h\xi, \eta)) = (ah^{1/2}\xi, h^{1/2}\eta).$$

In particular, if h is bounded, then

$$a' \circ h = h^{1/2} a h^{1/2}.$$

Proof. As already mentioned in Section 3, the case h bounded was treated in [15] (see also [25, Section 4]). Assume now that h is unbounded, and let p_n be the spectral projection of h corresponding to the interval $[0, n]$. Let $\xi, \eta \in \mathscr{H}$. Since h is bounded on $p_n(\mathscr{H})$, we have

$$\frac{1}{2}((a'\xi_n, h\eta_n) + (a'h\xi_n, \eta_n)) = (ah^{1/2}\xi_n, h^{1/2}\eta_n)$$

where $\xi_n = p_n\xi$, $\eta_n = p_n\eta$. By spectral theory

$$\lim_{n\to\infty} \|h^\alpha(\xi_n - \xi)\| = \lim_{n\to\infty} \|h^\alpha(\eta_n - \eta)\| = 0$$

for $0 \leqslant \alpha \leqslant 1$. Hence in the limit we get the stated equality.

Proof of Theorem 4.1 (continued).

Case II. Assume next that M is of the form

$$M = \{x \in \mathscr{N}_{\mathrm{s.a.}} \mid \Phi(x) = x\},$$

where $\mathcal{N}$ is a W^*-algebra and Φ is an involutive antiisomorphism. The state φ can be extended to a state $\overline{\varphi}$ on $\mathcal{N}$, given by

$$\overline{\varphi}(a) = \frac{1}{2}\varphi(a + \Phi(a)), \quad a \in \mathcal{N}_{s.a.}.$$

We may assume that $\mathcal{N}$ acts on a Hilbert space $\mathcal{H}$ with a cyclic and separating vector ξ_0. The self-polar form on $\mathcal{N}_{s.a.}$ associated with $\overline{\varphi}$ is

$$\overline{s}(a, b) = (\Delta^{1/2}a\xi_0, b\xi_0), \quad a, b \in \mathcal{N}_{s.a.}$$

(cf. Proof of Lemma 3.4) and by the proof of Theorem 3.3 (Case II) the self-polar form s_φ associated with φ is simply the restriction of $\overline{s}$ to M. Hence the completion $H_\varphi^\natural$ of M with respect to the norm

$$\|a\|_\varphi^\natural = s_\varphi(a, b)$$

can be identified with the closure of $\Delta^{1/4}M\xi_0$ in $\mathcal{H}$, and $P_\varphi^\natural$ can be identified with the closure of $\Delta^{1/4}M_+\xi_0$. Thus $P_\varphi^\natural$ coincides with the cone $P_{M,\varphi}^\natural$ considered by Iochum in [12, Chapter VII, Section 2]. Iochum proves that this cone is self-dual and homogeneous. Moreover he shows that the map

$$a \to \frac{1}{2}(a + JaJ) \quad (\text{restricted to } H_\varphi^\natural)$$

is an isometry of M onto the set of self-adjoint derivations of $P_\varphi^\natural$.

Let $\delta_\varphi(a)$ denote the restriction of $\frac{1}{2}(a + JaJ)$ to $H_\varphi^\natural$. It remains to be proved that $\delta_\varphi(a)$ is given by the formula (*) stated in Theorem 4.1. From Proposition 3.6(c) we have

$$\rho_t^\varphi(a) = \frac{1}{2}(\sigma_t^{\overline{\varphi}}(a) + \sigma_{-t}^{\overline{\varphi}}(a)), \quad a \in M.$$

Let $x, y \in M$. Since $x, y \in \mathcal{N}_{s.a.}$, $J\Delta^{1/4}x\xi_0 = \Delta^{1/4}x\xi_0$ and $J\Delta^{1/4}y\xi_0 = \Delta^{1/4}y\xi_0$. Hence

$$(\delta_\varphi(a)x, y)_\varphi^\natural = \frac{1}{2}((a + JaJ)\Delta^{1/4}x\xi_0, \Delta^{1/4}y\xi_0) =$$

$$= \mathrm{Re}(a\Delta^{1/4}x\xi_0, \Delta^{1/4}y\xi_0).$$

Put

$$\tilde{a} = 2 \int_{-\infty}^{\infty} \rho_t^{\varphi}(a) \cosh(2\pi t)^{-1}\, dt =$$

$$= 2 \int_{-\infty}^{\infty} \sigma_t^{\varphi}(a) \cosh(2\pi t)^{-1}\, dt =$$

$$= \int_{-\infty}^{\infty} \Delta^{it/2} a \Delta^{-it/2} \cosh(\pi t)^{-1}\, dt.$$

By Lemma 4.2,

$$(a\Delta^{1/4}x\xi_0, \Delta^{1/4}y\xi_0) = \frac{1}{2}((\tilde{a}x\xi_0, \Delta^{1/2}y\xi_0) + (\Delta^{1/2}x\xi_0, \tilde{a}y\xi_0)) =$$

$$= \frac{1}{2}(\Delta^{1/4}\tilde{a}x\xi_0, \Delta^{1/4}y\xi_0) + (\Delta^{1/4}x\xi_0, \Delta^{1/4}\tilde{a}y\xi_0)).$$

Since $(\Delta^{1/4}c\xi_0, \Delta^{1/4}d\xi_0)$ is real for $c, d \in \mathcal{N}_{\text{s.a.}}$, we get by splitting $\tilde{a}x$ and $\tilde{a}y$ in their hermitean and skew-hermitean parts, that

$$\operatorname{Re}(a\Delta^{1/4}x\xi_0, \Delta^{1/4}y\xi_0) = \frac{1}{2}(\Delta^{1/4}(\tilde{a} \circ x)\xi_0, y\xi_0) + (x\xi_0, \Delta^{1/4}(\tilde{a} \circ y)\xi_0)).$$

This proves that

$$(\delta_\varphi(a)x, y)_\varphi^\natural = (\tilde{a} \circ x, y)_\varphi^\natural + (x, \tilde{a} \circ y)_\varphi^\natural.$$

General case. It is easily seen that if $(P_i)_{i \in I}$ is a family of homogeneous self--dual cones in real Hilbert spaces $(H_i)_{i \in I}$, thne the cone

$$P = \{\xi \in \bigoplus_{i \in I} H_i \mid \xi = (\xi_i),\ \xi_i \in P_i \text{ for all } i \in I\}$$

is a homogeneous self-dual cone in $H = \bigoplus_{i \in I} H_i$. Moreover $d \in B(H)$ is a derivation of P if and only if $d = \bigoplus_{i \in I} d_i$, where $d_i \in B(H_i)$ are derivations of P_i. Hence by Lemma 2.1 and Lemma 2.2 the general case can be reduced to Case I and Case II by a central decomposition of the algebra.

5. APPENDIX

We will give a short proof of the following result due to Kurepa (cf. [14, Theorem 2]).

PROPOSITION 5.1. *Let $(u_t)_{t\in\mathbf{R}}$ be a weakly continuous one-parameter family of bounded self-adjoint operators on a real or complex Hilbert space H, such that*

(i) $u_0 = 1$;

(ii) $\|u_t\| \leqslant 1, \quad t \in \mathbf{R}$;

(iii) $u_s u_t = \frac{1}{2}(u_{s+t} + u_{s-t}), \quad s, t \in \mathbf{R}$.

Then there exists a positive self-adjoint operator D on H, such that

$$u_t = \cos(tD), \quad t \in \mathbf{R}.$$

Proof. It is enough to treat the case where H is a complex Hilbert space. (If H is a real Hilbert space we can pass to $H^{\mathbf{C}} = H + \mathrm{i}H$.)

Note that by (iii)

$$u_t = u_{-t}, \quad t \in \mathbf{R}.$$

The weak continuity of $(u_t)_{t\in\mathbf{R}}$ actually implies strong continuity, because

$$\|(u_s - u_t)\xi\|^2 = ((u_s^2 + u_t^2 - u_s u_t - u_t u_s)\xi, \xi) =$$
$$= \frac{1}{2}((u_{2s} + u_{2t} + 2 - 2u_{s+t} - 2u_{s-t})\xi, \xi)$$

for all $\xi \in H$. For $f \in L^1(\mathbf{R})$ we put

$$u(f) = \int_{-\infty}^{\infty} f(t)u_t \, \mathrm{d}t \quad \text{(strongly)}.$$

A simple computation shows that for $f, g \in L^1(\mathbf{R})$:

$$u(f)u(g) = \frac{1}{2}(u(f*g) + u(f*\check{g}))$$

where $\check{g}(t) = g(-t)$. Hence if f and g are *even* functions, then $u(f)u(g) = u(f*g)$. The functions $(f_\lambda)_{\lambda>0}$ given by

$$f_\lambda(t) = \frac{1}{\pi}\,\frac{\lambda}{\lambda^2 + t^2}, \quad \lambda > 0$$

form a continuous convolution semigroup of even functions in $L^1(\mathbf{R})$. Note that $f_\lambda \geqslant 0$, $\|f_\lambda\|_1 = 1$ and that $(f_\lambda)_{\lambda>0}$ is an approximating unit for $\lambda \to 0$. Therefore $(u(f_\lambda))_{\lambda>0}$ is a strongly continuous semigroup of self-adjoint contractions and $u(f_\lambda) \to 1$

strongly for $\lambda \to 0$. Let $(-D)$ be the generator of this semigroup. Then D is positive self-adjoint and

$$u(f_\lambda) = \exp(-\lambda D), \quad \lambda > 0.$$

The semigroup (f_λ) can be extended to a holomorphic semigroup $(f_\lambda)_{\mathrm{Re}\,\lambda>0}$, where f_λ is given by the same formula as for λ real. Since both $u(f_\lambda)$ and $\exp(-\lambda D)$ are strongly holomorphic in the half plane $\mathrm{Re}\,\lambda > 0$, we have

$$u(f_\lambda) = \exp(-\lambda D), \quad \mathrm{Re}\,\lambda > 0.$$

For $t \in \mathbf{R}$ and $\sigma > 0$

$$\cos(tD)\,\mathrm{e}^{-\sigma D} = \frac{1}{2}(u(f_{\sigma+\mathrm{i}t}) + u(f_{\sigma-\mathrm{i}t})).$$

A simple computation shows that

$$f_{\sigma+\mathrm{i}t}(s) + f_{\sigma-\mathrm{i}t}(s) = f_\sigma(s-t) + f_\sigma(s+t), \quad s \in \mathbf{R}.$$

Therefore

$$\cos(tD)\mathrm{e}^{-\sigma D} = \frac{1}{2}\int_{-\infty}^{\infty} (f_\sigma(s-t) + f_\sigma(s+t))u_s\,\mathrm{d}s$$

and since $(f_\sigma)_{\sigma>0}$ is an approximating unit, we conclude that

$$\cos(tD) = \operatorname*{strong-lim}_{\sigma\to 0}(\cos(tD)\mathrm{e}^{-\sigma D}) = $$

$$= \frac{1}{2}(u_t + u_{-t}).$$

REMARK 5.2. Kurepa's setting-up is more general than stated in Proposition 5.1. He considers weakly continuous one-parameter families of bounded *normal* operators satisfying (i) and (iii) and proves the existence of a normal operator D, such that $D - D^*$ is bounded and $u_t = \cos(tD)$, $t \in \mathbf{R}$. However, it is clear that the extra conditions $u_t = u_t^*$ and $\|u_t\| \leqslant 1$ force D to be self-adjoint, and by exchanging D with $|D|$ one can get D positive. Kurepa considers only separable Hilbert spaces, but this condition is not essential, because the Hilbert space can always be written as a direct sum of separable u_t-invariant subspaces.

REFERENCES

1. ALFSEN, E. M.; SHULTZ, F. W.; STØRMER, E., A Gelfand-Neumark theorem for Jordan algebras, *Advances in Math.*, **28**(1978), 11–56.
2. ARAKI, H., Some properties of modular conjugation operator of von Neumann algebras and a non commutative Radon-Nikodym theorem with a chain rule, *Pacific J. Math.*, **50**(1974), 309–354.

3. BEHNKE, H., Hermitian Jordan Banach algebras, *J. London Math. Soc. (2)*, **20**(1979), 327–333.
4. BELLISSARD, J.; IOCHUM, B., Homogeneous self-dual cones versus Jordan algebras. The theory revisited, *Ann. Inst. Fourier (Grenoble)*, **28**(1978), 27–67.
5. BELLISSARD, J.; IOCHUM, B., L'algebre de Jordan d'un cône autopolaire facialement homogène. *C.R. Acad. Sci. Paris*, **288**(1979), 229–232.
6. CONNES, A., Caractérisation des espaces vectoriels ordonnés sous-jacent aux algèbres de von Neumann, *Ann. Inst. Fourier (Grenoble)*, **24**(1974), 121–155.
7. EDWARDS, C. M., Ideal theory in JB-algebras, *J. London Math. Soc. (2)*, **16**(1977), 507–513.
8. EFFROS, E. G.; STØRMER, E., Jordan algebras of self-adjoint operators, *Trans. Amer. Math. Soc.*, **127**(1967), 313–316.
9. HAAGERUP, U., The standard form of von Neumann algebras, *Math. Scand.*, **37**(1975), 271–283.
10. HANCHE-OLSEN, H., A note on the bidual of a JB-algebra, *Math. Z.*, **175**(1980), 29–31.
11. HANCHE-OLSEN, H., On the structure and tensor products of JC-algebras, *Canad. J. Math.*, 1984, to appear.
12. IOCHUM, B., *Cones autopolaires et algèbres de Jordan*, Springer Lecture Notes in Math., **1049**(1984).
13. JACOBSON, N., Structure and representations of Jordan algebras, *Amer. Math. Soc. Colloq. Publ.*, **39**, Providence, R.I., 1968.
14. KUREPA, S., A cosine functional equation in Hilbert space, *Canad. J. Math.*, **12**(1960), 45–50.
15. PEDERSEN, G. K., On the operator equation $HT + TH = 2K$, *Indiana Univ. Math. J.*, **25**(1976), 1029–1033.
16. PEDERSEN, G.K.; STØRMER, E., Traces on Jordan algebras, *Canad. J. Math.*, **34**(1982), 370–373.
17. PEDERSEN, G.K.; TAKESAKI, M., The operator equation $THT = K$, *Proc. Amer. Math. Soc.*, **36**(1972), 311–312.
18. SAKAI, S., *C^*-algebras and W^*-algebras*, Springer-Verlag, Berlin — Heidelberg — New York, 1971.
19. SHULTZ, F. W., Jordan algebras which are Banach dual spaces, *J. Functional Analysis*, **31**(1979), 360–376.
20. STACEY, P. J., The structure of type I JBW-algebras, *Math. Proc. Cambridge Philos. Soc.*, **90**(1981), 477–482.
21. STACEY, P. J., Type I_2 JBW-algebras, *Quart. J. Math. Oxford*, **33**(1982), 115–127.
22. STØRMER, E., Jordan algebras of type I, *Acta Math.*, **115**(1966), 165–184.
23. TAKESAKI, M., *Tomita's theory of modular Hilbert algebras and its applications*, Springer Lecture Notes in Math., **128**(1970).
24. TOPPING, D. M., Jordan algebras of self-adjoint operators, *Mem. Amer. Math. Soc.*, **53**(1965).
25. VAN DAELE, A., A new approach to the Tomita-Takesaki theory of generalized Hilbert algebras, *J. Functional Analysis*, **15**(1974), 378–393.
26. WORONOWICZ, S. L., Selfpolar forms and their applications to the C^*-algebra theory, *Rep. Math. Phys.*, **6**(1974), 487–495.
27. WRIGHT, S., Jordan C^*-algebras, *Michigan Math. J.*, **24**(1977), 291–302.

UFFE HAAGERUP
Matematisk Institut,
Odense Universitet,
DK-5230, Odense M,
Denmark.

HARALD HANCHE-OLSEN
Matematisk Institutt,
Universitetet i Oslo,
Blindern, Oslo 3,
Norway.

Received February 23, 1983.