J. OPERATOR THEORY
12(1984), 23-45

ON THE STRUCTURE OF (BCP)-OPERATORS AND RELATED ALGEBRAS. I

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1. INTRODUCTION

This paper is concerned with a class of (bounded, linear) operators on Hilbert space which we call (BCP)-operators, and which were first studied by S. Brown, B. Chevreau, and C. Pearcy in [7], where the existence of invariant subspaces for such operators was established. The class (BCP), together with various related classes of operators, has been further studied in a number of works including [1], [2], [3], [4], [8], and [13]. In particular, recent work of C. Foiaş, C. Pearcy, and B. Sz.-Nagy [10, 11] has given strong reason to hope that a sufficient understanding of (BCP)-operators would yield the existence of invariant subspaces for a large class of operators containing, for example, all Toeplitz operators and all hyponormal operators. For this reason, and others, results about the structure of (BCP)-operators are of considerable interest.

This paper is divided into six sections. In Section 2, we recall some preliminary material and establish our notational conventions. We also give here our definition of the class (BCP), which has the advantage that the class (BCP) is now both self-adjoint and more general than the class studied in [7]. We are able to work in this broader context by virtue of Lemma 3.4, Corollary 3.5, and Lemma 3.6 (B).

Operators of class (BCP) (together with some related classes of operators) were studied in [3] in the context of the Sz.-Nagy—Foiaş functional model for contraction operators. In Section 3 we obtain many of the key results of an early version of [3] for (our, more general) (BCP)-operators. Unlike [3], our proofs make no use of the Sz.-Nagy—Foiaş functional model. In addition to giving a more accessible approach to these results, our techniques admit certain extensions which will be taken up in the forthcoming Part II of this paper.

In Section 4 we use the results of Section 3 to obtain Theorem 4.2 which concerns the realization of certain operators as compressions of a (BCP)-operator to a semi-invariant subspace.

In Section 5 we obtain some density theorems which amplify the results of Sections 3 and 4.

Section 6 is devoted to showing that any (BCP)-operator can be represented, up to unitary equivalence, as a lower triangular, two-way infinite operator matrix in which the diagonal entries are all of class (BCP).

2. PRELIMINARIES

Throughout this paper, $\mathscr{H}$ will denote a separable, infinite-dimensional, complex Hilbert space, and $\mathscr{L}(\mathscr{H})$ will denote the algebra of all (bounded, linear) operators on $\mathscr{H}$. Denote by (τc) the Banach space of all trace-class operators on $\mathscr{H}$ under the trace norm $\|\cdot\|_\tau$. Recall from [9] that the bilinear form $\langle A, K\rangle = \mathrm{tr}(AK)$ on $\mathscr{L}(\mathscr{H}) \times (\tau c)$ allows the identification of $\mathscr{L}(\mathscr{H})$ as the dual of (τc), and that the weak* topology on $\mathscr{L}(\mathscr{H})$ under this identification coincides with the ultraweak operator topology. If $\mathscr{A}$ is any ultraweakly closed subspace of $\mathscr{L}(\mathscr{H})$, then we may consider its preannihilator ${}^\perp\mathscr{A}$ in (τc), which will be a closed subspace of (τc). Let $Q = (\tau c)/{}^\perp\mathscr{A}$; then the bilinear form $\langle A, [K]\rangle = \mathrm{tr}(AK)$ on $\mathscr{A} \times Q$ allows the identification of $\mathscr{A}$ with the dual of Q. For $x, y \in \mathscr{H}$ we let $x \otimes y$ denote, as usual, the operator $(x \otimes y)(z) = (z, y)x$. We have $x \otimes y \in (\tau c)$ and $\|x \otimes y\|_\tau = \|x \otimes y\| = \|x\|\,\|y\|$.

For an operator $T \in \mathscr{L}(\mathscr{H})$, the symbols $\sigma(T)$, $\sigma_e(T)$, $\sigma_{le}(T)$, and $\sigma_{re}(T)$ will denote, respectively, the spectrum, essential spectrum, left essential spectrum, and right essential spectrum, of T.

Let $\mathbf{D}$ denote the open unit disc in the complex plane, and let H^∞ denote the Banach algebra of all bounded, analytic functions on $\mathbf{D}$, under the supremum norm. We may identify H^∞ with a certain subspace of $L^\infty(\mathbf{T})$ (where the unit circle $\mathbf{T}$ is endowed with Haar measure), and moreover under this identification H^∞ is weak* closed in $L^\infty(\mathbf{T})$. For each $\lambda \in \mathbf{D}$, the map $f \mapsto f(\lambda)$ is weak* continuous on H^∞. For all of these facts we refer to [12].

A subset Λ of $\mathbf{D}$ is said to be *dominating* if for every $f \in H^\infty$, $\|f\| = \sup\limits_{\lambda\in\Lambda}|f(\lambda)|$. A theorem of L. Brown, A. Shields, and K. Zeller [6] asserts that $\Lambda \subset \mathbf{D}$ is dominating if and only if almost every point of $\mathbf{T}$ is a nontangential limit point of Λ.

Recall that given any completely nonunitary contraction T on $\mathscr{H}$, one has the Sz.-Nagy—Foiaş functional calculus $\Phi_T\colon H^\infty \to \mathscr{L}(\mathscr{H})$ for T, which is a contractive, unit-preserving algebra homomorphism such that $\Phi_T(f_1) = T$, where $f_1 \in H^\infty$ is the position function $f_1(\lambda) = \lambda$ (see [17], Chapter 3). If $\sigma(T) \cap \mathbf{D}$ is dominating, then Φ_T is, in addition, isometric. Moreover, in this case the range of Φ_T is precisely the ultraweakly closed subalgebra $\mathscr{A}_T$ of $\mathscr{L}(\mathscr{H})$ generated by T and $1_{\mathscr{H}}$, and Φ_T is a weak* homeomorphism of H^∞ onto $\mathscr{A}_T$ ([7], Theorem 3.2 and p. 129).

We denote by Q_T the quotient space $(\tau c)/{}^{\perp}\mathscr{A}_T$; then as noted earlier, we may identify $\mathscr{A}_T$ as the dual space Q_T^*. If $T \in \mathscr{L}(\mathscr{H})$ is a completely nonunitary contraction and $\sigma(T) \cap \mathbf{D}$ is dominating, then by what has already been said, for each $\lambda \in \mathbf{D}$, the map $f(T) \mapsto f(\lambda)$ is a (well-defined) weak* continuous linear functional on $\mathscr{A}_T$, and is hence an element of Q_T. We denote this element of Q_T by $[C_\lambda]$.

The main theorem of [7] asserts that a contraction T has a nontrivial invariant subspace whenever $\sigma(T) \cap \mathbf{D}$ is dominating. For the purpose of proving this result, one can easily reduce to the case where T is completely nonunitary, $\sigma(T) = \sigma_e(T) = \sigma_{le}(T) = \sigma_{re}(T)$, and $T^n \to 0$ strongly. After making these reductions, the authors of [7] assume that T is completely nonunitary, $\sigma_{le}(T) \cap \mathbf{D}$ is dominating, and $T^n \to 0$ strongly; they then prove the following structure theorem:

For any $[K] \in Q_T$, there exist vectors $x, y \in \mathscr{H}$ such that $[K] = [x \otimes y]$. In particular, this is true for $[K] = [C_0]$, and this yields easily the existance of invariant subspaces.

This result has been strengthened in several directions. C. Apostol, in [1], proved the above structure theorem under the assumptions that T is completely nonunitary and has no hyperinvariant subspaces, $T^n \to 0$ strongly, and (in lieu of requiring that $\sigma_{le}(T)$ be dominating) that the essential norm of the resolvent of T satisfies a certain growth condition. Then H. Bercovici, C. Foiaş and C. Pearcy, in [3], by means of the Sz.-Nagy—Foiaş functional model [17], obtained this structure theorem in a context which contained both of the above results, and without the assumption that $T^n \to 0$ strongly. Moreover, they obtained "$n \times n$ matrix versions" of this structure theorem (where n can be either a positive integer or $\aleph_0$) which in turn yield more detailed structural information about the operator T. For example, the matrix versions of this structure theorem allow one to prove, as Bercovici, Foiaş, J. Langsam, and Pearcy do in [2], that the operator T under study is reflexive. These results also make it possible to obtain operators of various prescribed classes as compressions of T to a semi-invariant subspace (up to some equivalence relation), as is done in [4] (and in Section 4 of the present paper).

Our aim in Section 3 below is to obtain "matrix versions" of the above structure theorem for the following class of operators.

DEFINITION. The class (BCP) consists of all completely nonunitary contractions T on $\mathscr{H}$ for which $\sigma_e(T) \cap \mathbf{D}$ is dominating.

Prior to the present work, these results had been obtained, in an early version of [3], for those completely nonunitary contractions T such that $\sigma_{le}(T) \cap \mathbf{D}$ is dominating. The class (BCP) as defined above has the dual advantages of greater generality and of being self-adjoint. Moreover, we are able to prove our results without recourse to the functional model.

We conclude this section with some remarks concerning our notation. When there is no ambiguity in context, we shall suppress the subscript on Φ, $\mathscr{A}$, and Q, and we shall usually write $f(T)$ rather than $\Phi_T(f)$. We write ker and ran for kernel and range, and the letter P will always designate an orthogonal projection. Sequences are indexed by the positive integers, unless otherwise specified.

3. THE STRUCTURE OF Q

Throughout this section, unless otherwise specified, let $T \in \mathscr{L}(\mathscr{H})$ be a fixed operator of class (BCP), and let $\mathscr{A} = \mathscr{A}_T$, $Q = Q_T$, as above. Let us choose, once and for all, a countable, dense subset $\Lambda = \{\lambda_k\}$ of $\sigma_e(T) \cap \mathbf{D}$, and a function $\varkappa : \mathbf{Z}^+ \to \{-1, 1\}$ such that $\varkappa(k) = -1$ [respectively, $\varkappa(k) = +1$] implies $\lambda_k \in \sigma_{le}(T)$ [$\lambda_k \in \sigma_{re}(T)$].

Note that the set Λ is obviously dominating. The following lemma is proved in the same way as [7, Lemma 4.7].

LEMMA 3.1. *The closed, absolutely convex hull* $\overline{\mathrm{aco}}\{[C_\lambda] : \lambda \in \Lambda\}$ *is the closed unit ball of* Q.

Recall [14, p. 10] that $\lambda \in \sigma_{le}(T)$ if and only if there exists an orthonormal sequence $\{x_n\}$ in $\mathscr{H}$ such that $\|(T - \lambda)x_n\| \to 0$. Obviously, then, $\lambda \in \sigma_{re}(T)$ if and only if there exists an orthonormal sequence $\{x_n\}$ in $\mathscr{H}$ such that $\|(T^* - \bar{\lambda})x_n\| \to 0$.

The following is [7, Lemma 4.4].

LEMMA 3.2. *Let* $\lambda \in \sigma_{le}(T) \cap \mathbf{D}$ *and let* $\{x_n\}$ *be an orthonormal sequence such that* $\|(T - \lambda)x_n\| \to 0$. *Then for any fixed* $y \in \mathscr{H}$, $\|[x_n \otimes y]\|_Q \to 0$.

Our next goal is Lemma 3.4, in preparation for which we establish the following result, which essentially appeared in [7]. The proof given here is perhaps somewhat more transparent. For Lemma 3.3 we shall assume only that T is a completely nonunitary contraction and that $\sigma(T) \cap \mathbf{D}$ [rather than $\sigma_e(T) \cap \mathbf{D}$] is dominating.

LEMMA 3.3. *Assume that* $T^n \to 0$ *strongly, and let* $\{x_n\} \subset \mathscr{H}$ *be a sequence such that* $x_n \to 0$ *weakly. Then for any* $y \in \mathscr{H}$, $\|[y \otimes x_n]\|_Q \to 0$.

Proof. We may assume, by [17, Chapter 2], that $T = V^*|\mathscr{H}$, where V is a unilateral shift of some multiplicity and $\mathscr{H}$ is an invariant subspace for V. We may also assume, by the uniform boundedness principle, that each $\|x_n\| \leqslant 1$.

For each n, we may choose, by the Hahn-Banach theorem (and since Φ_T is isometric), $f_n \in H^\infty$, $\|f_n\| = 1$, such that

$$\|[y \otimes x_n]\|_Q = \langle f_n(T), [y \otimes x_n]\rangle =$$

$$= \mathrm{tr}\{[f_n(T)](y \otimes x_n)\} = \mathrm{tr}\{[f_n(T)y] \otimes x_n\} = (f_n(T)y, x_n).$$

Let $\varepsilon > 0$. Choose $N > 0$ such that $\|P_{\mathrm{ran}(V^N)}y\| < \varepsilon/2$ and let $y_2 = P_{\mathrm{ran}(V^N)}y$, $y_1 = y - y_2 = P_{\ker(V^{*N})}y$. For each n, write $f_n(z) = p_n(z) + h_n(z)z^N$ where $h_n \in H^\infty$

and $p_n(z) = \sum_{k=0}^{N-1} \alpha_{nk} z^k$. Note that each $|\alpha_{nk}| \leqslant 1$. Since $x_n \to 0$ weakly, there exists $n_0 > 0$ such that $n \geqslant n_0$ implies that $|(V^{*k}y_1, x_n)| < \varepsilon/2N$ for $k = 0, \ldots, N-1$. Then for $n \geqslant n_0$,

$$\|[y \otimes x_n]\|_Q = |(f_n(T)y, x_n)| = |(f_n(V^*)y, x_n)| \leqslant |(f_n(V^*)y_1, x_n)| +$$

$$+ |(f_n(V^*)y_2, x_n)| \leqslant |(f_n(V^*)y_1, x_n)| + \varepsilon/2 \leqslant |(p_n(V^*)y_1, x_n)| +$$

$$+ |(h_n(V^*)V^{*N}y_1, x_n)| + \varepsilon/2 \leqslant \sum_{k=0}^{N-1} |\alpha_{nk}|\,|(V^{*k}y_1, x_n)| + 0 + \varepsilon/2 < \varepsilon.$$ ▨

We now revert to our standing convention that $T \in \mathscr{L}(\mathscr{H})$ is a fixed operator of class (BCP).

LEMMA 3.4. *Let $\lambda \in \sigma_{le}(T) \cap \mathbf{D}$ and let $\{x_n\} \subset \mathscr{H}$ be an orthonormal sequence such that $\|(T-\lambda)x_n\| \to 0$. Then for any $y \in \mathscr{H}$, $\|[y \otimes x_n]\|_Q \to 0$.*

Proof. Let $T_\lambda = (T-\lambda)(1-\bar{\lambda}T)^{-1}$. Since $\lambda \in \mathbf{D}$, $(1-\bar{\lambda}T)^{-1}$ is a power series in T, so $T_\lambda \in \mathscr{A}_T$ and hence $\mathscr{A}_{T_\lambda} \subset \mathscr{A}_T$. Moreover $(T_\lambda)_{-\lambda} = T$, so $\mathscr{A}_{T_\lambda} = \mathscr{A}_T$. Also, T_λ is a completely nonunitary contraction [17] and the obvious spectral mapping theorem holds, so that T_λ is of class (BCP). These comments show that we may assume that $\lambda = 0$.

Let V be the minimal isometric dilation of T^*. The space on which V acts may be decomposed as a direct sum $\ell^2(\mathscr{F}) \oplus \mathscr{R}$, where $\mathscr{F}$, $\mathscr{R}$ are Hilbert spaces, and $\ell^2(\mathscr{F})$ is the space of square-summable sequences (indexed by $\mathbf{Z}^+$) in $\mathscr{F}$. With respect to this decomposition, we have $V = V_1 \oplus U$ where V_1 is a unilateral shift and U is a unitary operator with spectral measure absolutely continuous with respect to Haar measure on $\mathbf{T}$. (See [17], Chapters 2 and 3.)

Write $x_n = x_n^1 \oplus x_n^2$, $y = y^1 \oplus y^2$ where $x_n^1, y^1 \in \ell^2(\mathscr{F})$ and $x_n^2, y^2 \in \mathscr{R}$. Now

$$\|[y \otimes x_n]\|_{Q_T} =$$

$$= \sup_{f \in H^\infty, \|f\|=1} |(f(T)y,\ x_n)| = \sup_{f \in H^\infty, \|f\|=1} |(f(V^*)y, x_n)| = \|[y \otimes x_n]\|_{Q_{V^*}},$$

so we need to show that the latter tends to zero. Since for any vectors y, v, w, we have

$$\|[y \otimes v]\|_{Q_{V^*}} \leqslant \|[y \otimes w]\|_{Q_{V^*}} + \|y\|\,\|v - w\|$$

and since

$$\|x_n - P_{\ker V^*}x_n\| = \|(1 - P_{\ker V^*})x_n\| = \|VV^*x_n\| = \|Tx_n\| \to 0,$$

we may assume that $\{x_n\} \subset \ker V^*$ and hence in particular that $x_n^2 = 0$ for all n. Thus, we have that

$$\|[y \otimes x_n]\|_{Q_{V^*}} = \sup_{\substack{f \in H^\infty \\ \|f\|=1}} |(f(V^*)y, x_n)| =$$

$$= \sup_{\substack{f \in H^\infty \\ \|f\|=1}} |([f(V_1^*) \oplus f(U^*)](y^1 \oplus y^2),\ x_n^1 \oplus x_n^2)| =$$

$$= \sup_{\substack{f \in H^\infty \\ \|f\|=1}} |(f(V_1^*)y^1, x_n^1)| = \|[y^1 \otimes x_n^1]\|_{Q_{V_1^*}}.$$

The last expression tends to zero by the previous lemma, since $V_1^{*n} \to 0$ strongly and $x_n^1 \to 0$ weakly. ▨

REMARK. In [7], where the goal was to show the existence of invariant subspaces for an operator T of class (BCP), the authors were able, as mentioned earlier, to reduce to the case where $T^n \to 0$ strongly, by virtue of [17, Chapter 2, Theorem 5.4]. They then showed that in this case $\|[y \otimes x_n]\|_Q \to 0$ for *any* orthonormal sequence $\{x_n\}$ (and any vector y). This reduction is unavailable for our present purposes, and the conclusion for an arbitrary sequence $\{x_n\}$ is false without the hypothesis on the powers of T (see the remarks at the end of Part II). The argument given above was suggested by H. Bercovici.

COROLLARY 3.5. *Let $\lambda \in \sigma_{re}(T) \cap \mathbf{D}$ and let $\{x_n\}$ be an orthonormal sequence such that $\|(T^* - \bar{\lambda})x_n\| \to 0$. Then for any $y \in \mathcal{H}$,*

(A) $$\|[x_n \otimes y]\|_Q \to 0$$

and

(B) $$\|[y \otimes x_n]\|_Q \to 0.$$

Proof. Note that for any $x, y \in \mathcal{H}$ we have

$$\|[x \otimes y]\|_{Q_T} = \sup_{\substack{A \in \mathcal{A}_T \\ \|A\|=1}} |(Ax, y)| =$$

$$= \sup_{\substack{A \in \mathcal{A}_T \\ \|A\|=1}} |(A^*y, x)| = \sup_{\substack{A \in \mathcal{A}_{T^*} \\ \|A\|=1}} |(Ay, x)| = \|[y \otimes x]\|_{Q_{T^*}}.$$

Since T^* is of class (BCP) if and only if T is, assertion (A) follows from Lemma 3.4 applied to T^*, and similarly, (B) follows from Lemma 3.2. ▨

From now on we shall refer to Lemmas 3.2, 3.4 and Corollary 3.5 as the "vanishing lemmas".

LEMMA 3.6. (A) *Let $\lambda \in \sigma_{le}(T) \cap \mathbf{D}$ and let $\{x_n\}$ be an orthonormal sequence such that $\|(T-\lambda)x_n\| \to 0$. Then $\|[x_n \otimes x_n] - [C_\lambda]\|_Q \to 0$.*

(B) *Let $\lambda \in \sigma_{re}(T) \cap \mathbf{D}$ and let $\{x_n\}$ be an orthonormal sequence such that $\|(T^* - \bar{\lambda})x_n\| \to 0$. Then $\|[x_n \otimes x_n] - [C_\lambda]\|_Q \to 0$.*

Proof. Assertion (A) is [7, Lemma 4.3] and is proven in a similar manner as (B).

To prove (B), choose, via the Hahn-Banach theorem, $f_n \in H^\infty$ such that

$$\|f_n(T)\| = \|f_n\| = 1$$

and

$$\|[x_n \otimes x_n] - [C_\lambda]\|_Q = \langle f_n(T), [x_n \otimes x_n] - [C_\lambda]\rangle.$$

We may write $f_n(\zeta) = f_n(\lambda) + (\zeta - \lambda)g_n(\zeta)$ where $g_n \in H^\infty$ and $\|g_n\| \leqslant 2(1-|\lambda|)^{-1}$. We then have

$$\|[x_n \otimes x_n] - [C_\lambda]\|_Q =$$

$$= \langle f_n(\lambda) + (T-\lambda)g_n(T),\ [x_n \otimes x_n] - [C_\lambda]\rangle =$$

$$= \langle f_n(\lambda),\ [x_n \otimes x_n]\rangle - \langle f_n(\lambda), [C_\lambda]\rangle +$$

$$+ \langle (T-\lambda)g_n(T),\ [x_n \otimes x_n]\rangle - \langle (T-\lambda)g_n(T),\ [C_\lambda]\rangle =$$

$$= f_n(\lambda)(x_n, x_n) - f_n(\lambda) + \langle (T-\lambda)g_n(T), [x_n \otimes x_n]\rangle - 0 =$$

$$= ((T-\lambda)g_n(T)x_n, x_n) \leqslant \|g_n\|\, \|(T^* - \bar{\lambda})x_n\| \to 0.$$ ▨

LEMMA 3.7. *Let $\lambda \in \sigma_{le}(T)$ (respectively, $\lambda \in \sigma_{re}(T)$), and let $\mathscr{F}$ be any finite-dimensional subspace of $\mathscr{H}$. Then there exists an orthonormal sequence $\{x_n\}$ in $\mathscr{F}^\perp$ such that $\|(T-\lambda)x_n\| \to 0$ ($\|(T^* - \bar{\lambda})x_n\| \to 0$).*

Proof. One constructs $\{x_n\}$ inductively, using the fact that if $\lambda \in \sigma_{le}(T)$ ($\lambda \in \sigma_{re}(T)$) then $(T-\lambda)$ ($(T^* - \bar{\lambda})$) cannot be bounded below on any subspace of finite co-dimension in $\mathscr{H}$. ▨

LEMMA 3.8. *There exists an orthonormal family $\{e_n^k : k,\ n > 0\}$ such that*

(A) $$\|(T-\lambda_k)e_n^k\| \to 0 \quad \text{as } n \to \infty \text{ whenever } \varkappa(k) = -1,$$

(B) $$\|(T^* - \bar{\lambda}_k)e_n^k\| \to 0 \quad \text{as } n \to \infty \text{ whenever } \varkappa(k) = +1,$$

(C) $$\|[e_n^k \otimes e_n^l]\|_Q \to 0 \quad \text{as } n \to \infty \text{ whenever } k \neq l.$$

Proof. Let $i \to (k(i), n(i))$ be the enumeration of $\mathbf{Z}^+ \times \mathbf{Z}^+$ suggested by the matrix below:

$$\begin{matrix} 1 & 2 & 4 & 7 & . \\ 3 & 5 & 8 & . & . \\ 6 & 9 & . & . & . \\ 10 & . & . & . & . \\ . & . & . & . & . & . \end{matrix}$$

We proceed by induction on i. For $i = 1$, we choose a unit vector e_1^1 in $\mathscr{H}$ such that either $\|(T - \lambda_1)e_1^1\| < 1$ or $\|(T^* - \bar{\lambda}_1)e_1^1\| < 1$, depending on whether $\varkappa(1) = -1$ or $+1$.

Suppose that mutually orthogonal unit vectors $e_{n(i)}^{k(i)}$ have been chosen for $1 \leqslant i \leqslant N$, such that

$$\|(T - \lambda_{k(i)})e_{n(i)}^{k(i)}\| < i^{-1} \quad \text{whenever} \quad \varkappa(k(i)) = -1,$$

$$\|(T^* - \bar{\lambda}_{k(i)})e_{n(i)}^{k(i)}\| < i^{-1} \quad \text{whenever} \quad \varkappa(k(i)) = +1,$$

and

$$\|[e_{n(i)}^{k(i)} \otimes e_{n(j)}^{k(j)}]\|_Q < [\max\{i,j\}]^{-1} \quad \text{whenever} \quad i \neq j.$$

Let $\mathscr{F} = \bigvee \{e_{n(i)}^{k(i)} : 1 \leqslant i \leqslant N\}$. Applying Lemma 3.7, there exists an orthonormal sequence $\{x_m\}$ in $\mathscr{F}^\perp$ such that either

$$\|(T - \lambda_{k(N+1)})x_m\| \to 0$$

or

$$\|(T^* - \bar{\lambda}_{k(N+1)})x_m\| \to 0 \quad \text{as } m \to \infty$$

depending on whether $\varkappa(k(N+1)) = -1$ or $+1$. Applying the appropriate vanishing lemmas, we have that for any sufficiently large index M and for each $1 \leqslant i \leqslant N$,

$$\|[e_{n(i)}^{k(i)} \otimes x_M]\|_Q < (N+1)^{-1},$$

and

$$\|[x_M \otimes e_{n(i)}^{k(i)}]\|_Q < (N+1)^{-1}.$$

In addition, we may also ensure, by taking M sufficiently large, that either $\|(T - \lambda_{k(N+1)})x_M\| < (N+1)^{-1}$ or $\|(T^* - \bar{\lambda}_{k(N+1)})x_M\| < (N+1)^{-1}$, depending on whether $\varkappa(k(N+1)) = -1$ or $+1$. We now set $e_{n(N+1)}^{k(N+1)} = x_M$, where M is large enough that the aforementioned inequalities hold.

This constructs by induction an orthonormal family $\{e_n^k\}$ which clearly has the desired properties. ▨

We now partition the family $\{e_n^k\}$ of Lemma 3.8 into a family of mutually orthogonal "drawers" $\mathscr{D}_{ij}$, indexed by $\mathbf{Z}^+ \times \mathbf{Z}^+$, so that each drawer consists of an orthonormal family of vectors which has the same properties as the family $\{e_n^k\}$. More precisely, each drawer

$$\mathscr{D}_{ij} = \{e_n^{k,i,j} : k, n > 0\}$$

satisfies

(D1) $$\|(T - \lambda_k)e_n^{k,i,j}\| \to 0 \quad \text{as } n \to \infty, \text{ whenever } \varkappa(k) = -1;$$

(D2) $$\|(T^* - \bar{\lambda}_k)e_n^{k,i,j}\| \to 0 \quad \text{as } n \to \infty, \text{ whenever } \varkappa(k) = +1;$$

and

(D3) $$\|[e_n^{k,i,j} \otimes e_n^{l,i,j}]\|_Q \to 0 \text{ as } n \to \infty, \text{ whenever } k \neq l.$$

Set $\mathscr{M}_i = \bigvee_j \mathscr{D}_{ij}$ and $\mathscr{M}^j = \bigvee_i \mathscr{D}_{ij}$.

LEMMA 3.9. *Let $N > 0$ and suppose $u_i, v_j \in \mathscr{H}$ and $[L_{ij}] \in Q$ ($1 \leqslant i, j \leqslant N$). Assume that*

$$\|[u_i \otimes v_j] - [L_{ij}]\|_Q < \varepsilon_{ij}.$$

Let $1 \leqslant i_0, j_0 \leqslant N$ and let $0 < \delta < \varepsilon_{i_0 j_0}$. Then there exist $u'_{i_0}, v'_{j_0} \in \mathscr{H}$ such that

(A) $$\|[u'_{i_0} \otimes v'_{j_0}] - [L_{i_0 j_0}]\|_Q < \delta,$$

(B) $$\|[u'_{i_0} \otimes v_j] - [L_{i_0 j}]\|_Q < \varepsilon_{i_0 j} \quad \textit{for all } j,$$

(C) $$\|[u_i \otimes v'_{j_0}] - [L_{i j_0}]\|_Q < \varepsilon_{i j_0} \quad \textit{for all } i,$$

(D) $$\|u'_{i_0} - u_{i_0}\| < \varepsilon_{i_0 j_0}^{1/2},$$

and

(E) $$\|v'_{j_0} - v_{j_0}\| < \varepsilon_{i_0 j_0}^{1/2}.$$

Moreover, we can arrange matters so that $u'_{i_0} - u_{i_0} \in \mathscr{M}_{i_0}$ and $v'_{j_0} - v_{j_0} \in \mathscr{M}^{j_0}$.

Proof. Set $d = \|[u_{i_0} \otimes v_{j_0}] - [L_{i_0 j_0}]\|_Q$. We may assume that $d > 0$ since otherwise we can simply take $u'_{i_0} = u_{i_0}$, $v'_{j_0} = v_{j_0}$.

Let $[K] = [L_{i_0 j_0}] - [u_{i_0} \otimes v_{j_0}]$. By Lemma 3.1 we may choose $m \in \mathbf{Z}^+, \lambda_1, \ldots, \lambda_m \in \Lambda$, $\alpha_1, \ldots, \alpha_m \in \mathbf{C}$ such that

$$\sum_{\nu=1}^{m} |\alpha_\nu| \leqslant 1,$$

and

$$\left\| d^{-1}[K] - \sum_{\nu=1}^{m} \alpha_\nu [C_{\lambda_\nu}] \right\|_Q < \delta/(2d).$$

Choose $\gamma_1, \ldots, \gamma_m \in \mathbf{C}$ such that $\gamma_\nu^2 = \alpha_\nu d$, and select sequences $\{e_n^{1, i_0, j_0}\}, \ldots$ $\ldots, \{e_n^{m, i_0, j_0}\}$ from drawer $\mathscr{D}_{i_0 j_0}$. For simplicity of notation, we will write $\{e_n^1\}, \ldots$ $\ldots, \{e_n^m\}$ for these, respectively.

Set $s_n = \sum_{\nu=1}^{m} \gamma_\nu e_n^\nu$ and $t_n = \sum_{\nu=1}^{m} \bar{\gamma}_\nu e_n^\nu$. We claim that we may take

$$u'_{i_0} = u_{i_0} + s_n, \quad v'_{j_0} = v_{j_0} + t_n. \tag{3.1}$$

First, note that obviously for any such choice, we will have $u'_{i_0} - u_{i_0} \in \mathscr{M}_{i_0}$ and $v'_{j_0} - v_{j_0} \in \mathscr{M}^{j_0}$.

Next, observe that

$$\|(u_{i_0} + s_n) - u_{i_0}\|^2 = \sum_{\nu=1}^{m} |\gamma_\nu|^2 = \sum_{\nu=1}^{m} |\alpha_\nu|\, d \leqslant d < \varepsilon_{i_0 j_0},$$

and likewise

$$\|(v_{j_0} + t_n) - v_{j_0}\|^2 < \varepsilon_{i_0 j_0}.$$

Hence any choice of u'_{i_0} and v'_{j_0} given by (3.1) for some n will satisfy conditions (D) and (E).

Next we show that we may satisfy condition (A). For any n, we have

$$\|[(u_{i_0} + s_n) \otimes (v_{j_0} + t_n)] - [L_{i_0 j_0}]\|_Q \leqslant$$

$$\leqslant \|[u_{i_0} \otimes v_{j_0}] - [L_{i_0 j_0}] + [s_n \otimes t_n]\|_Q + \|[s_n \otimes v_{j_0}]\|_Q + \|[u_{i_0} \otimes t_n]\|_Q =$$

$$= \|[s_n \otimes t_n] - [K]\|_Q + \|[s_n \otimes v_{j_0}]\|_Q + \|[u_{i_0} \otimes t_n]\|_Q \leqslant$$

$$\leqslant \left\| \sum_{\nu=1}^{m} \gamma_\nu^2 [C_\lambda] - [K] \right\|_Q + \left\| [s_n \otimes t_n] - \sum_{\nu=1}^{m} \gamma_\nu^2 [C_{\lambda_\nu}] \right\|_Q +$$

$$+ \|[s_n \otimes v_{j_0}]\|_Q + \|[u_{i_0} \otimes t_n]\|_Q <$$

$$< \frac{\delta}{2} + \left\| \left[\left(\sum_{\nu=1}^{m} \gamma_\nu e_n^\nu \right) \otimes \left(\sum_{\nu=1}^{m} \bar{\gamma}_\nu e_n^\nu \right) \right] - \sum_{\nu=1}^{m} \gamma^2 [C_{\lambda_\nu}] \right\|_Q +$$

$$+ \sum_{\nu=1}^{m} |\gamma_\nu| \, \|[e_n^\nu \otimes v_{j_0}]\|_Q + \sum_{\nu=1}^{m} |\gamma_\nu| \, \|[u_{j_0} \otimes e_n^\nu]\|_Q <$$

$$< \frac{\delta}{2} + \sum_{\nu=1}^{m} |\gamma_\nu|^2 \|[e_n^\nu \otimes e_n^\nu] - [C_{\lambda_\nu}]\|_Q + \sum_{\substack{\nu_1,\nu_2=1\\ \nu_1\neq\nu_2}}^{m} |\gamma_{\nu_1}\gamma_{\nu_2}| \, \|[e_n^{\nu_1} \otimes e_n^{\nu_2}]\|_Q +$$

$$+ \sum_{\nu=1}^{m} |\gamma_\nu| \, \|[e_n^\nu \otimes v_{j_0}]\|_Q + \sum_{\nu=1}^{m} |\gamma_\nu| \, \|[u_{i_0} \otimes e_n^\nu\|_Q .$$

As $n \to \infty$, each summand in the second term approaches 0 by Lemma 3.6; each summand in the third term approaches 0 by condition (D3); and each summand in the last two terms approaches 0 by the appropriate vanishing lemmas. Hence we may satisfy condition (A) by a choice of u'_{i_0}, v'_{j_0} as in (3.1) for any sufficiently large n.

Finally note that

$$\|[(u_{i_0} + s_n) \otimes v_j] - [L_{i_0 j}]\|_Q \leqslant$$

$$\leqslant \|[u_{i_0} \otimes v_j] - [L_{i_0 j}]\|_Q + \sum_{\nu=1}^{m} |\gamma_\nu| \, \|[e_n^\nu \otimes v_j]\|_Q,$$

and

$$\|[u_i \otimes (v_{j_0} + t_n)] - [L_{ij_0}]\|_Q \leqslant$$

$$\leqslant \|[u_i \otimes v_{j_0}] - [L_{ij_0}]\|_Q + \sum_{\nu=1}^{m} |\gamma_\nu| \, \|[u_i \otimes e_n^\nu]\|_Q .$$

Hence, by the appropriate vanishing lemmas, we may satisfy conditions (B) and (C) by choosing n sufficiently large in (3.1). ▨

By N^2 successive applications of Lemma 3.9 we immediately obtain the following result.

LEMMA 3.10. *Let* $N > 0$, u_i, $v_j \in \mathscr{H}$ *and* $[L_{ij}] \in Q$ $(i, j = 1, \ldots, N)$. *Assume that*

$$\|[u_i \otimes v_j] - [L_{ij}]\|_Q < \varepsilon_{ij},$$

and let $0 < \delta_{ij} \leqslant \varepsilon_{ij}$. *Then there exist* u'_i, $v'_j \in \mathscr{H}$ *such that*

(A) $$\|[u'_i \otimes v'_j] - [L_{ij}]\|_Q < \delta_{ij} \quad \textit{for all } i, j,$$

(B) $$\|u'_i - u_i\| < \sum_{j=1}^{N} \varepsilon_{ij}^{1/2} \quad \textit{for all } i,$$

and

(C) $$\|v'_j - v_j\| < \sum_{i=1}^{N} \varepsilon_{ij}^{1/2} \quad \textit{for all } j.$$

Moreover we can arrange it so that $u'_i - u_i \in \mathscr{M}_i$ *and* $v'_j - v_j \in \mathscr{M}^j$.

THEOREM 3.11. *Let* $N > 0$, $u_i, v_j \in \mathscr{H}$, *and* $[L_{ij}] \in Q$ $(i, j = 1, \ldots, N)$. *Let* $\varepsilon > 0$ *and let* $d_{ij} = \|[u_i \otimes v_j] - [L_{ij}]\|_Q$. *Then there exist* $u_i', v_j' \in \mathscr{H}$ *such that*

$$\text{(A)} \qquad [u_i' \otimes v_j'] = [L_{ij}] \quad \textit{for all } i, j,$$

$$\text{(B)} \qquad \|u_i' - u_i\| < \sum_{j=1}^{N} d_{ij}^{1/2} + \varepsilon \quad \textit{for all } i,$$

and

$$\text{(C)} \qquad \|v_j' - v_j\| < \sum_{i=1}^{N} d_{ij}^{1/2} + \varepsilon \quad \textit{for all } j.$$

Moreover we can arrange it so that $u_i' - u_i \in \mathscr{M}_i$ *and* $v_j' - v_j \in \mathscr{M}^j$.

Proof. Since $\varepsilon > 0$ is arbitrary, it clearly suffices to prove the theorem with (B) and (C) replaced by

$$\text{B}') \qquad \|u_i' - u_i\| < \sum_{j=1}^{N} \varepsilon_{ij}^{1/2} + \varepsilon$$

$$\text{C}') \qquad \|v_j' - v_j\| < \sum_{i=1}^{N} \varepsilon_{ij}^{1/2} + \varepsilon$$

where $\varepsilon_{ij} > d_{ij}$ are arbitrary.

Let $u_i^{(0)} = u_i$ and $v_j^{(0)} = v_j$. By Lemma 3.10 there exist $u_i^{(1)}, v_j^{(1)}$ in $\mathscr{H}$ such that

$$\|[u_i^{(1)} \otimes v_j^{(1)}] - [L_{ij}]\|_Q < \frac{\varepsilon^2}{2^{2\cdot 1}N^2}, \tag{3.2}$$

$$\|u_i^{(1)} - u_i^{(0)}\| < \sum_{j=1}^{N} \varepsilon_{ij}^{1/2}, \tag{3.3}$$

and

$$\|v_j^{(1)} - v_j^{(0)}\| < \sum_{i=1}^{N} \varepsilon_{ij}^{1/2}. \tag{3.4}$$

Moreover we can have $u_i^{(1)} - u_i^{(0)} \in \mathscr{M}_i$ and $v_j^{(1)} - v_j^{(0)} \in \mathscr{M}^j$.

By induction and Lemma 3.10 we can construct a sequence of families $\{u_i^{(0)}\}$, $\{v_j^{(0)}\}$, $\{u_i^{(1)}\}$, $\{v_j^{(1)}\}$, $\ldots$, $\{u_i^{(n)}\}$, $\{v_j^{(n)}\}$, $\ldots$ such that

$$\|[u_i^{(n)} \otimes v_j^{(n)}] - [L_{ij}]\|_Q < \frac{\varepsilon^2}{2^{2n}N^2} \quad \text{for all } n \geqslant 1, \tag{3.5}$$

$$\|u_i^{(n)} - u_i^{(n-1)}\| < \frac{\varepsilon}{2^{n-1}} \quad \text{for all } n > 1 \text{ and all } i, \tag{3.6}$$

and

$$\|v_j^{(n)} - v_j^{(n-1)}\| < \frac{\varepsilon}{2^{n-1}} \quad \text{for all } n > 1 \text{ and all } j. \tag{3.7}$$

Moreover we can have $u_i^{(n)} - u_i^{(n-1)} \in \mathcal{M}_i$ and $v_j^{(n)} - v_j^{(n-1)} \in \mathcal{M}^j$.

By (3.3) and (3.6) we have that

$$\sum_{n=1}^{\infty} \|u_i^{(n)} - u_i^{(n-1)}\| < \sum_{j=1}^{N} \varepsilon_{ij}^{1/2} + \varepsilon.$$

Hence for each i the sequence $\{u_i^{(n)}\}$ converges in norm to some u_i' satisfying (B′) and such that $u_i' - u_i \in \mathcal{M}_i$. Similarly for each j the sequence $\{v_j^{(n)}\}$ converges in norm to some v_j' satisfying (C′) and such that $v_j' - v_j \in \mathcal{M}^j$. Finally, assertion (A) follows in an obvious way from (3.5). ▨

REMARK. Theorem 3.11 is the crucial ingredient in the proof [2] that every operator of class (BCP) is reflexive.

THEOREM 3.12. *Let $[L_{ij}] \in Q$ $(i, j \geqslant 1)$ and assume that*

$$\sum_{j=1}^{\infty} \|[L_{ij}]\|_Q^{1/2} < \infty \quad \textit{for each } i,$$

and

$$\sum_{i=1}^{\infty} \|[L_{ij}]\|_Q^{1/2} < \infty \quad \textit{for each } j.$$

Let $\varepsilon > 0$. Then there exist $u_i \in \mathcal{M}_i$ and $v_j \in \mathcal{M}^j$, such that

(A) $$[u_i \otimes v_j] = [L_{ij}] \quad \textit{for all } i \textit{ and } j,$$

(B) $$\|u_i\| < \sum_{j=1}^{\infty} \|[L_{ij}]\|_Q^{1/2} + \varepsilon/2^{i-1} \quad \textit{for each } i,$$

and

(C) $$\|v_j\| < \sum_{i=1}^{\infty} \|[L_{ij}]\|_Q^{1/2} + \varepsilon/2^{j-1} \quad \textit{for each } j.$$

Proof. Let $u_i^{(0)} = v_j^{(0)} = 0$ for all $i, j > 0$. By Theorem 3.11 (with $N = 1$) there exist $u_1^{(1)} \in \mathcal{M}_1$ and $v_1^{(1)} \in \mathcal{M}^1$ such that $[u_1^{(1)} \otimes v_1^{(1)}] = [L_{11}]$,

$$\|u_1^{(1)}\| = \|u_1^{(1)} - u_1^{(0)}\| < \|[L_{11}]\|_Q^{1/2} + \varepsilon/2,$$

and

$$\|v_1^{(1)}\| < \|[L_{11}]\|_Q^{1/2} + \varepsilon/2.$$

For $i > 1$ and $j > 1$ set $u_i^{(1)} = v_j^{(1)} = 0$.

Assume that $n \geqslant 1$ and that sequences $\{u_i^{(k)}\}$, $\{v_j^{(k)}\}$ $(0 \leqslant k \leqslant n)$ have been chosen so that

$$u_i^{(k)} \in \mathcal{M}_i \text{ and } v_j^{(k)} \in \mathcal{M}^j \quad \text{for all } i, j, \text{ and } k, \tag{3.8}$$

$$[u_i^{(k)} \otimes v_j^{(k)}] = [L_{ij}] \tag{3.9}$$

whenever $1 \leqslant i,\ j \leqslant k$,

$$u_i^{(k)} = v_j^{(k)} = 0 \quad \text{for } i > k \text{ and } j > k,$$

$$\|u_i^{(k)} - u_i^{(k-1)}\| < \|[L_{ik}]\|_Q^{1/2} + \varepsilon/2^k \quad \text{for } 1 \leqslant i < k, \tag{3.11}$$

$$\|u_k^{(k)} - u_k^{(k-1)}\| < \sum_{j=1}^{k} \|[L_{kj}]\|_Q^{1/2} + \varepsilon/2^k,$$

and

$$\|v_j^{(k)} - v_j^{(k-1)}\| < \|[L_{kj}]\|_Q^{1/2} + \varepsilon/2^k \quad \text{for } 1 \leqslant j < k, \tag{3.12}$$

$$\|v_k^{(k)} - v_k^{(k-1)}\| < \sum_{i=1}^{k} \|[L_{ik}]\|_Q^{1/2} + \varepsilon/2^k.$$

By Theorem 3.11 (with $N = n + 1$), we obtain $u_i^{(n+1)} \in \mathcal{M}_i$, $v_j^{(n+1)} \in \mathcal{M}^j (1 \leqslant i, j \leqslant n + 1)$ such that

$$[u_i^{(n+1)} \otimes v_j^{(n+1)}] = [L_{ij}] \quad \text{for } 1 \leqslant i, j \leqslant n + 1, \tag{3.13}$$

$$\|u_i^{(n+1)} - u_i^{(n)}\| < \|[L_{i\,n+1}]\|_Q^{1/2} + \varepsilon/2^{n+1} \quad \text{for } 1 \leqslant i < n + 1, \tag{3.14}$$

$$\|u_{n+1}^{(n+1)} - u_{n+1}^{(n)}\| < \sum_{j=1}^{n+1} \|[L_{n+1\,j}]\|_Q^{1/2} + \varepsilon/2^{n+1},$$

$$\|v_j^{(n+1)} - v_j^{(n)}\| < \|[L_{n+1\,j}]\|_Q^{1/2} + \varepsilon/2^{n+1} \quad \text{for } 1 \leqslant j < n + 1, \tag{3.15}$$

and

$$\|v_{n+1}^{(n+1)} - v_{n+1}^{(n)}\| < \sum_{i=1}^{n+1} \|[L_{i\,n+1}]\|_Q^{1/2} + \varepsilon/2^{n+1}.$$

Now set $u_i^{(n+1)} = v_j^{(n+1)} = 0$ for $i > n + 1$ and $j > n + 1$, and observe that (3.8) — — (3.12) hold for all $1 \leqslant k \leqslant n + 1$. Thus by induction, we have constructed sequences $\{u_i^{(k)}\}$, $\{v_j^{(k)}\}$ such that (3.8) — (3.12) hold for every $k \geqslant 1$.

Now consider any fixed $i \geqslant 1$. Then

$$\sum_{k=1}^{\infty} \|u_i^{(k)} - u_i^{(k-1)}\| = \sum_{k=1}^{i-1} \|u_i^{(k)} - u_i^{(k-1)}\| +$$

$$+ \|u_i^{(i)} - u_i^{(i-1)}\| + \sum_{k=i+1}^{\infty} \|u_i^{(k)} - u_i^{(k-1)}\| <$$

$$< 0 + \sum_{j=1}^{i} \|[L_{ij}]\|_Q^{1/2} + \varepsilon/2^i + \sum_{k=i+1}^{\infty} (\|[L_{ik}]\|_Q^{1/2} + \varepsilon/2^k) =$$

$$= \sum_{j=1}^{\infty} \|[L_{ij}]\|_Q^{1/2} + \varepsilon/2^{i-1}.$$

It follows (recall $u_i^{(0)} = 0$) that the sequence $\{u_i^{(k)}\}$ converges in norm to some $u_i \in \mathcal{M}_i$ with

$$\|u_i\| < \sum_{j=1}^{\infty} \|[L_{ij}]\|_Q^{1/2} + \varepsilon/2^{i-1}.$$

Similarly, for each $j \geqslant 1$ the sequence $\{v_j^{(k)}\}$ converges in norm to some $v_j \in \mathcal{M}^j$ which satisfies (C).

Finally, since for each $i, j \geqslant 1$ we have that $[u_i^{(k)} \otimes v_j^{(k)}] = [L_{ij}]$ (provided that $k \geqslant i$ and $k \geqslant j$) it readily follows that $[u_i \otimes v_j] = [L_{ij}]$. ▨

LEMMA 3.13. *Let $a_{ij} \geqslant 0$ $(i, j \geqslant 1)$. Then there exist $b_i > 0$, $c_j > 0$ such that $a_{ij}/b_i c_j < (ij)^{-4}$.*

Proof. Choose b_1, c_1 so that $a_{11}/b_1 c_1 < 1$. Having chosen $b_1, \ldots, b_n, c_1, \ldots$ $\ldots, c_n$, we may clearly choose b_{n+1}, c_{n+1} so that

$$\frac{a_{n+1,j}}{b_{n+1} c_j} < [(n+1)j]^{-4} \quad (j = 1, \ldots, n+1)$$

and

$$\frac{a_{i,n+1}}{b_i c_{n+1}} < [i(n+1)]^{-4} \quad (i = 1, \ldots, n+1).$$

The lemma follows by induction. ▨

THEOREM 3.14. *Let $[L_{ij}] \in Q$ $(i, j \geqslant 1)$. Then there exist orthogonal sequences $\{x_i\}$ and $\{y_j\}$ in $\mathcal{H}$ such that*

$$[x_i \otimes y_j] = [L_{ij}] \quad \textit{for all } i \textit{ and } j.$$

Proof. Let $a_{ij} = \|[L_{ij}]\|_Q$ and choose b_i, c_j as in the preceding lemma. By Theorem 3.12 there are $u_i \in \mathscr{M}_i$, $v_j \in \mathscr{M}^j$ such that

$$[u_i \otimes v_j] = (b_i c_j)^{-1} [L_{ij}].$$

Set $x_i = b_i u_i$ and $y_j = c_j v_j$. Then clearly we have that $[x_i \otimes y_j] = [L_{ij}]$. Since $\mathscr{M}_i \perp \mathscr{M}_{i'}$ unless $i = i'$, the sequence $\{x_i\}$ is orthogonal and, likewise, so is $\{y_j\}$. ▨

4. COMPRESSIONS OF (BCP)-OPERATORS

Theorem 3.14 may be used to obtain results about the structure of operators of class (BCP). Theorem 4.2 below is one such result. For its proof we shall require Proposition 4.1, which is a generalization of a familiar fact from linear algebra. (We claim no originality for the proposition, but we include the proof for lack of a suitable reference.)

Recall that an operator S is said to be *algebraic* if $p(S) = 0$ for some nonzero polynomial p. In this case we can speak of the minimal polynomial of S.

PROPOSITION 4.1. *Let S be an algebraic operator on a Hilbert space $\mathscr{H}$ and let $p(z)$ be its minimal polynomial. Assume that $p(z)$ has distinct roots, say $\alpha_1, \ldots, \alpha_N$. Then S is similar to $\alpha_1 \oplus \ldots \oplus \alpha_N$ where each α acts on some nonzero Hilbert space.*

Proof. By the spectral mapping theorem, $\sigma(S) \subset \{\alpha_1, \ldots, \alpha_N\}$. (In fact, equality holds, as we shall see.) By relabelling if necessary, we may assume that $\sigma(S) = \{\alpha_1, \ldots, \alpha_M\}$ for some $M \leqslant N$.

By the Riesz decomposition theorem [15, Theorem 2.10], by Chapter 13, Problem C of [5], and by induction, there exist invariant subspaces $\mathscr{H}_1, \ldots, \mathscr{H}_M$ for S such that $\mathscr{H} = \mathscr{H}_1 \dotplus \ldots \dotplus \mathscr{H}_M$, $\sigma(S|\mathscr{H}_i) = \{\alpha_i\}$ for each i, and for which there exists an invertible linear map of $\mathscr{H}$ onto the (external) direct sum $\mathscr{H}_1 \oplus \ldots \ldots \oplus \mathscr{H}_M$ which is the identity on each $\mathscr{H}_i$. To complete the proof we will show that $S|\mathscr{H}_i = \alpha_i$ for each i.

Let i be fixed and let $x \in \mathscr{H}_i$. The operator $S|\mathscr{H}_i$ is algebraic, and hence the cyclic subspace $\mathscr{M}_x = \bigvee \{S^n x : n \geqslant 0\}$ generated by x is finite-dimensional. Hence $\sigma(S|\mathscr{M}_x)$ consists of eigenvalues, so that $\sigma(S|\mathscr{M}_x) \subset \sigma(S|\mathscr{H}_i) = \{\alpha_i\}$. Moreover the minimal polynomial of $S|\mathscr{M}_x$ has distinct roots since it divides the minimal polynomial of S. Since $\sigma(S|\mathscr{M}_x) = \{\alpha_i\}$ this minimal polynomial must be $(z - \alpha_i)$, which shows that $S|\mathscr{M}_x = \alpha_i$. Since $x \in \mathscr{H}_i$ was arbitrary we obtain that $S|\mathscr{H}_i = \alpha_i$.

The fact that each α acts on a nonzero space, and thus also the parenthetical remark at the beginning of the proof, now follows from the minimality of $p(z)$. ▨

Let T be an operator on $\mathscr{H}$. Recall that a *semi-invariant subspace* for T is a subspace of the form $\mathscr{M} \ominus \mathscr{N}$ where $\mathscr{M} \supset \mathscr{N}$ are invariant subspaces for T.

Recall also that for any subspace $\mathscr{S}$ of $\mathscr{H}$, the *compression* $T_{\mathscr{S}}$ is the operator on $\mathscr{S}$ defined by $T_{\mathscr{S}}x = P_{\mathscr{S}}Tx$ $(x \in \mathscr{S})$. A result of D. Sarason [16] asserts that $\mathscr{S}$ is semi-invariant for T if and only if $p(T_{\mathscr{S}}) = p(T)_{\mathscr{S}}$ for every polynomial $p(z)$.

THEOREM 4.2. *Let $T \in \mathscr{L}(\mathscr{H})$ be of class* (BCP) *and let $\alpha_1, \ldots, \alpha_N$ be (distinct) elements of* **D**. *Then there exist invariant subspaces $\mathscr{M}, \mathscr{N}$ for T with $\mathscr{M} \supset \mathscr{N}$ and such that $T_{\mathscr{M}\ominus\mathscr{N}}$ is similar to $\alpha_1 \oplus \ldots \oplus \alpha_N$ where each α acts on an infinite-dimensional space.*

REMARK. Thus with respect to the decomposition $\mathscr{H} = \mathscr{N} \oplus (\mathscr{M} \ominus \mathscr{N}) \oplus \mathscr{M}^{\perp}$ we obtain that

$$T = \begin{pmatrix} * & * & * \\ 0 & T_{\mathscr{M}\ominus\mathscr{N}} & * \\ 0 & 0 & * \end{pmatrix}$$

where $T_{\mathscr{M}\ominus\mathscr{N}}$ is, up to similarity, $\alpha_1 \oplus \ldots \oplus \alpha_N$.

Proof of Theorem 4.2. Let $\{\lambda_i\}$ be a sequence from $\{\alpha_1, \ldots, \alpha_N\}$ in which each α occurs infinitely often. By Theorem 3.14 there exist sequences $\{x_i\}$ and $\{y_j\}$ such that $[x_i \otimes y_j] = \delta_{ij}[C_{\lambda_i}]$, where δ_{ij} is the Kronecker delta.

Let

$$\mathscr{M} = \bigvee \{T^n x_i : i > 0,\ n \geqslant 0\},$$

$$\mathscr{M}_* = \bigvee \{T^{*m} y_j : j > 0,\ m \geqslant 0\},$$

and let $\mathscr{N} = \mathscr{M} \ominus \mathscr{M}_*$. Clearly $\mathscr{M}$ and $\mathscr{N}$ are invariant for T and $\mathscr{M} \supset \mathscr{N}$.

We claim that $\mathscr{M} \ominus \mathscr{N}$ is infinite-dimensional. Let $z_i = P_{\mathscr{M}\ominus\mathscr{N}} x_i$. We will show that $\{z_i\}$ is linearly independent. Suppose that $\sum_{i=1}^{k} c_i z_i = 0$. Then for each i_0,

$$0 = \sum_{i=1}^{k} c_i(z_i, y_{i_0}) = \sum_{i=1}^{k} c_i(P_{\mathscr{M}\ominus\mathscr{N}} x_i, y_{i_0}) = \sum_{i=1}^{k} c_i(x_i, P_{\mathscr{M}\ominus\mathscr{N}} y_{i_0}) =$$

$$= \sum_{i=1}^{k} c_i(x_i, y_{i_0}) = \qquad \text{(because } y_{i_0} \in \mathscr{M}_*\text{)}$$

$$= c_{i_0} \qquad \text{(because } [x_i \otimes y_{i_0}] = \delta_{ii_0}[C_{\lambda_i}]\text{)}.$$

Now let $p(z) = \prod_{i=1}^{N} (z - \alpha_i)$. We claim that $p(T_{\mathscr{M}\ominus\mathscr{N}}) = 0$. Since $\mathscr{M} \ominus \mathscr{N}$ is semi-invariant for T, we have that $p(T_{\mathscr{M}\ominus\mathscr{N}}) = p(T)_{\mathscr{M}\ominus\mathscr{N}}$. Since $\mathscr{M}$ is spanned by vectors of the form $T^n x_i$ and $\mathscr{M}_*$ is spanned by vectors of the form $T^{*m} y_j$, to show that $p(T)_{\mathcal{M}\ominus\mathcal{N}} = 0$ it suffices to show that $(p(T)\, T^n x_i, T^{*m} y_j) = 0$ for each i, j, m and n. Let i, j, m, and n be fixed and let $q(z) = p(z) z^{m+n}$. If $i \neq j$ it is obvious that

$$(p(T)T^n x_i, T^{*m} y_j) = (p(T)T^{m+n} x_i, y_j) = 0$$

because $[x_i \otimes y_j] = \delta_{ij}[C_{\lambda_i}] = 0$. If $i = j$ we have that

$$(p(T)T^n x_i) = (q(T)x_i, y_i) = q(\lambda_i) = 0$$

since $[x_i \otimes y_i] = [C_{\lambda_i}]$. Now, for each $i > 0$, $\lambda_i = \alpha_{k(i)}$ for some $1 \leqslant k(i) \leqslant N$. A simple calculation of the sort carried out above shows that $T_{\mathcal{M}\ominus\mathcal{N}} z_i = \alpha_{k(i)} z_i$ where $z_i = P_{\mathcal{M}\ominus\mathcal{N}} x_i$. Hence for each k, $\ker(T_{\mathcal{M}\ominus\mathcal{N}} - \alpha_k)$ is infinite-dimensional. The theorem now follows easily from Proposition 4.1. ▧

As a corollary we obtain the following result which was first obtained in an early version of [3].

COROLLARY 4.3. *If T is of class* (BCP) *then there exists an invariant subspace $\mathcal{M}$ for T such that $\mathcal{M} \ominus [T\mathcal{M}]^-$ is infinite-dimensional.*

Proof. Take $N = 1$ and $\alpha_1 = 0$ in the previous result. ▨

REMARK. Theorem 4.2 may be expressed by saying that for any (BCP)-operator T and any operator A of the form $\alpha_1 \oplus \ldots \oplus \alpha_N$ ($\alpha_k \in \mathbf{D}$), A can be obtained, up to similarity, as the compression of T to a semi-invariant subspace. Theorem 3.14 can be used to obtain other results of this sort. For example, the authors of [4] showed that if T is of class (BCP) and if A is any contraction of class C_0 [17], then A can be obtained, up to *quasi-similarity*, as the compression of T to a semi-invariant subspace. Moreover, they showed that if T is of class (BCP) and if $\|A\| < 1$, then A can be obtained, up to unitary equivalence, as a compression of T to a semi-invariant subspace. We have included the preceding, much weaker, result as an illustration of the power of Theorem 3.14, and, more importantly, so that we may refer to its proof in Part II.

5. DENSITY THEOREMS

Throughout this section, let $T \in \mathcal{L}(\mathcal{H})$ be a fixed operator of class (BCP), and let $Q = Q_T$. Let $c_0(\mathcal{H})$ denote the Banach space of all null sequences $\overline{x} = (x_n)$ in $\mathcal{H}$, under the norm $\|\overline{x}\| = \sup\|x_n\|$.

THEOREM 5.1. *Let $[L_{ij}] \in Q$ $(i, j \geqslant 1)$ and suppose that*

$$\sum_{i=1}^{\infty} \|[L_{ij}]\|_Q^{1/2} < \infty \quad \textit{for each } j.$$

Then the set of all $\bar{x} \in c_0(\mathscr{H})$ *for which there exists an orthogonal sequence* $\{y_j\} \subset \mathscr{H}$ *such that* $[x_i \otimes y_j] = [L_{ij}]$ *for all* i *and* j, *is dense in* $c_0(\mathscr{H})$.

Proof. Let $\bar{x} \in c_0(\mathscr{H})$ and let $\delta > 0$. We will show that there exist $\bar{x}' \in c_0(\mathscr{H})$ and an orthogonal sequence $\{y_j\} \subset \mathscr{H}$ such that $[x_i' \otimes y_j] = [L_{ij}]$ for all i and j and $\|\bar{x}' - \bar{x}\| \leqslant \delta$. It is clear that we may assume that for some $N_0 \geqslant 1$, $x_i = 0$ for $i > N_0$. Moreover, we may replace each "column" $\{[L_{ij}] : i \geqslant 1\}$ by $\{c_j[L_{ij}] : : i \geqslant 1\}$ where the $c_j > 0$ are arbitrary (since $[x_i \otimes y_j] = c_j[L_{ij}]$ implies that $[x_i' \otimes c_j^{-1}y_j] = [L_{ij}]$). Therefore, we may assume that $\|[L_{ij}]\|_Q < \dfrac{\delta^2}{2^{2(j+1)}}$ for all i and j.

We now proceed much as in the proof of Theorem 3.12. We begin by setting

$$u_i^{(0)} = x_i \quad \text{for } 1 \leqslant i \leqslant N_0$$

$$u_i^{(0)} = 0 \quad \text{for } i > N_0$$

and

$$v_j^{(0)} = 0 \quad \text{for all } j.$$

By Theorem 3.11 (with $N = N_0$ and $\varepsilon = \delta/2^{2\cdot 1}$) there exist $u_i^{(1)}$, $v_j^{(1)}$ $(1 \leqslant i, j \leqslant N_0)$ such that

$$[u_i^{(1)} \otimes v_j^{(1)}] = [L_{ij}] \quad \text{for } 1 \leqslant i, j \leqslant N_0,$$

$$\|u_i^{(1)} - u_i^{(0)}\| < \sum_{j=1}^{N_0} \left(\frac{\delta^2}{2^{2(j+1)}} \right)^{1/2} + \frac{\delta}{2^{2\cdot 1}},$$

$$\|v_j^{(1)} - v_j^{(0)}\| < \sum_{i=1}^{N_0} \|[L_{ij}]\|_Q^{1/2} + \frac{\delta}{2^{2\cdot 1}},$$

and $v_j^{(1)} \in \mathscr{M}^j$ for each j (where $\mathscr{M}^j$ is as in Section 3). We let $u_i^{(1)} = v_j^{(1)} = 0$ for $i > N_0$ and $j > N_0$.

Just as in the proof of Theorem 3.12, we construct, by induction and Theorem 3.11, a sequence of pairs $\{u_i^{(k)}\}$, $\{v_j^{(k)}\}$ such that, in addition to the conditions above, for each $k \geqslant 2$,

$$[u_i^{(k)} \otimes v_j^{(k)}] = [L_{ij}] \quad \text{for } 1 \leqslant i, j \leqslant N_0 + k - 1,$$

$$\|u_i^{(k)} - u_i^{(k-1)}\| < \frac{\delta}{2^{N_0+k}} + \frac{\delta}{2^{2k}} \quad \text{for } i < k,$$

$$\|u_k^{(k)} - u_k^{(k-1)}\| < \sum_{j=1}^{k} \frac{\delta}{2^{j+1}} + \frac{\delta}{2^{2k}},$$

$$\|v_j^{(k)} - v_j^{(k-1)}\| < \|[L_{N_0+k-1\, j}]\|_Q^{1/2} + \frac{\delta}{2^{2k}} \quad \text{for } j < k,$$

$$\|v_k^{(k)} - v_k^{(k-1)}\| < \sum_{i=1}^{N_0+k-1} \|[L_{i\, N_0+k-1}]\|_Q^{1/2} + \frac{\delta}{2^{2k}},$$

and

$$v_j^{(k)} \in \mathcal{M}^j \quad \text{for all } j.$$

It follows, as in the proof of Theorem 3.12, that for each i the sequence $\{u_i^{(k)}\}$ converges in norm to some vector x_i' such that $\|x_i' - x_i\| \leqslant \delta$, that for each j the sequence $\{v_j^{(k)}\}$ converges to some vector $y_j \in \mathcal{M}^j$, and that $[x_i' \otimes y_j] = [L_{ij}]$ for all i and j. ▨

As obvious corollaries, we obtain the following result, part (A) of which was obtained in an early version of [3].

COROLLARY 5.2. (A) *$\{x \in \mathcal{H}$: there exists $y \in \mathcal{H}$, $y \neq 0$ such that $[x \otimes y] = 0\}$ is dense in $\mathcal{H}$.*

(B) *For any $[K] \in Q$, $\{x \in \mathcal{H}$: there exists $y \in \mathcal{H}$ such that $[x \otimes y] = [K]\}$ is dense in $\mathcal{H}$.* ▨

For $x \in \mathcal{H}$, let $\mathcal{M}_x = \bigvee \{T^n x : n \geqslant 0\}$ be the cyclic subspace for T generated by x. Recall that x is said to be *noncyclic* (for T) in case $\mathcal{M}_x \neq \mathcal{H}$. It is easy to see that Corollary 5.2 (A) is equivalent to saying that the set of vectors which are noncyclic for T is dense in $\mathcal{H}$. We can sharpen this result.

COROLLARY 5.3. *$\{x \in \mathcal{H} : \mathcal{H} \ominus \mathcal{M}_x$ is infinite-dimensional$\}$ is dense in $\mathcal{H}$.*

Proof. Let $\varepsilon > 0$ and let $x \in \mathcal{H}$. Let $\bar{x} = (x, 0, 0, 0, \ldots) \in c_0(\mathcal{H})$, let $[K]$ be any nonzero element of Q, and let

$$[L_{ij}] = 0 \quad \text{for } i \neq 2 \text{ and all } j,$$

$$[L_{2j}] = [K] \quad \text{for all } j.$$

By Theorem 5.1 there exist $\bar{x}' = (x_i) \in c_0(\mathcal{H})$ and an orthogonal sequence $\{y_j\} \subset \mathcal{H}$ such that

$$\|\bar{x}' - \bar{x}\| < \varepsilon \quad \text{and} \quad [x_i \otimes y_j] = [L_{ij}] \quad \text{for all } i \text{ and } j.$$

Since $[x_i' \otimes y_j] = 0$ for all j, we have that each $y_j \in \mathcal{H} \ominus \mathcal{M}_{x_i}$. Since $\{y_j\}$ is orthogonal and each $y_j \neq 0$ (because $[x_2' \otimes y_j] = [K] \neq 0$), $\mathcal{H} \ominus \mathcal{M}_{x_i'}$ is infinite-dimensional. Since $\|x_1' - x_1\| \leqslant \|\bar{x}' - \bar{x}\| < \varepsilon$, the proof is complete. ▨

Let $\mathcal{H}^\omega$ be the space of all sequences $\bar{x} = (x_i)$ in $\mathcal{H}$, under the locally convex topology determined by the seminorms $\rho_i(\bar{x}) = \|x_i\|$. Since the finitely nonzero sequences are dense in $\mathcal{H}^\omega$, the proof of Theorem 5.1 also establishes the following.

THEOREM 5.4. *Let $[L_{ij}] \in Q$ $(i, j \geqslant 1)$ and assume that $\sum_{i=1}^{\infty} \|[L_{ij}]\|_Q^{1/2} < \infty$ for each j. Then the set of all $\bar{x} \in \mathcal{H}^\omega$ for which there exists an orthogonal sequence $\{y_j\} \subset \mathcal{H}$ such that $[x_i \otimes y_j] = [L_{ij}]$ is dense in $\mathcal{H}^\omega$.*

In view of the method of proof of Theorem 4.2, this theorem has various consequences, of which we are content to mention the following. For $\bar{x} = (x_i) \in \mathcal{H}^\omega$ write

$$\mathcal{M}_{\bar{x}} = \bigvee \{T^n x_i : n \geqslant 0,\ i \geqslant 1\}.$$

COROLLARY 5.5. *The set of* $\bar{x} \in \mathscr{H}^{\omega}$ *for which* $\mathscr{M}_{\bar{x}} \ominus [T\mathscr{M}_{\bar{x}}]^{-}$ *is infinite-dimensional, is dense in* $\mathscr{H}^{\omega}$. ▨

REMARK. Each of the above density results of course has a counterpart in which the roles of x and y are interchanged, the proofs of which are, *mutatis mutandis*, the same.

6. A DECOMPOSITION THEOREM

The aim of this section is to prove Theorem 6.3, which may be interpreted as saying that any operator of class (BCP) can be represented, up to unitary equivalence, as a lower triangular, two-way infinite operator matrix, in which the diagonal entries are all of class (BCP). We begin with the following result, which is interesting in itself, in view of the fact that the restriction of a (BCP)-operator to an (infinite-dimensional) invariant subspace need not be of class (BCP).

PROPOSITION 6.1. *Let* $T \in \mathscr{L}(\mathscr{H})$ *be of class* (BCP), *and let* $y \in \mathscr{H}$, $\varepsilon > 0$. *Then there exist an invariant subspace* $\mathscr{M}$ *for* T *and a vector* $y_0' \in \mathscr{H}$ *such that* $y_0' \perp \mathscr{M}$, $\|y_0' - y\| < \varepsilon$, *and such that the restriction* $T|\mathscr{M}$ *and the compression* $T_{\mathscr{H} \ominus \mathscr{M}}$ *are both of class* (BCP).

REMARK. Hence, both $\mathscr{M}$ and $\mathscr{H} \ominus \mathscr{M}$ are infinite-dimensional.

Proof. Let Λ, $\{e_n^k\}$, and $i \mapsto (k(i), n(i))$ be as in the proof of Lemma 3.8. We consider the sequence $\{e_{n(i)}^{k(i)} : i \geqslant 1\}$ and for convenience, we give this sequence two names, $\{x_i : i \geqslant 1\}$ and $\{y_j : j \geqslant 1\}$. We set $y_0 = y$. Let $\{\varepsilon_\nu : \nu > 1\}$ be any sequence of positive numbers which decreases to zero, and let $\varepsilon_0 = \varepsilon$.

Using the vanishing lemmas and Theorem 3.11, we construct by induction increasing sequences $\{i_\nu : \nu \geqslant 1\}$ and $\{j_\nu : \nu \geqslant 0\}$ of positive integers, with $j_0 = 0$, and a pair of orthogonal sequences $\{x_\nu^{(N)} : \nu \geqslant 1\}$, $\{y_\nu^{(N)} : \nu \geqslant 0\}$ for each $N \geqslant 1$, such that for each N we have

$$\|x_N^{(N)} - x_{i_N}\| < \frac{\varepsilon_N}{2},$$

$$\|x_\nu^{(N)} - x_\nu^{(N-1)}\| < \frac{\varepsilon_\nu}{2^N} \quad \text{for } 1 \leqslant \nu < N,$$

$$\|y_N^{(N)} - y_{j_N}\| < \frac{\varepsilon_N}{2},$$

$$\|y_0^{(1)} - y_{j_0}\| < \frac{\varepsilon_0}{2},$$

$$\|y_\nu^{(N)} - y_\nu^{(N-1)}\| < \frac{\varepsilon_\nu}{2^N} \quad \text{for } 0 \leqslant \nu < N,$$

$$x_\nu^{(N)} = y_\nu^{(N)} = 0 \quad \text{for } \nu > N,$$

and

$$[x_{\nu_1}^{(N)} \otimes y_{\nu_2}^{(N)}] = 0 \quad \text{for all } \nu_1 \geqslant 1,\ \nu_2 \geqslant 0.$$

We will say that at the N-th stage we pay our respects to an element $\lambda \in \Lambda$ in case $\lambda_{k(i_N)} = \lambda_{k(j_N)} = \lambda$ (where $k\colon \mathbf{Z}^+ \to \mathbf{Z}^+$ is as in the first paragraph). In the inductive construction above, we may and we do, ensure that we pay our respects to each element of Λ infinitely often.

It follows, as in previous arguments, that for each ν the sequence $\{x_\nu^{(N)}\}$ converges to some x_ν' such that $\|x_\nu' - x_i'\| < \varepsilon_\nu$.

Likewise each sequence $\{y_\nu^{(N)}\}$ converges to some y_ν' with $\|y_\nu' - y_{j_\nu}\| < \varepsilon_\nu$. Moreover we have $[x_{\nu_1}' \otimes y_{\nu_2}'] = 0$ for all $\nu_1 \geqslant 1$ and $\nu_2 \geqslant 0$.

Let $M = \bigvee\{T^m x_\nu : m \geqslant 0,\ \nu > 0\}$. Clearly $\mathcal{M}$ is invariant for T. Also, $y_0' \perp \mathcal{M}$ and $\|y_0' - y\| < \varepsilon$. Since we paid our respects to each element of Λ infinitely often, it follows easily that $\Lambda \subset \sigma_e(T|\mathcal{M}) \cap \sigma_e(T_{\mathcal{H}\ominus\mathcal{M}})$. Hence $T|\mathcal{M}$ and $T_{\mathcal{H}\ominus\mathcal{M}}$ satisfy the spectral condition in the definition of the class (BCP). Since compressions (hence restrictions) of completely nonunitary contractions are again such, the proof is complete. ▨

PROPOSITION 6.2. *Let $T \in \mathcal{L}(\mathcal{H})$ be of class* (BCP). *Then there exists a chain of invariant subspaces $\mathcal{M}_0 \supset \mathcal{M}_1 \supset \ldots \supset \mathcal{M}_n \supset \ldots$ for T such that $T_{\mathcal{H}\ominus\mathcal{M}_0}$ and each $T_{\mathcal{M}_n\ominus\mathcal{M}_{n+1}}$ are of class* (BCP), *and such that $\bigcap\{\mathcal{M}_n : n \geqslant 0\} = \{0\}$.*

Proof. Let $\{y_n\} \subset \mathcal{H}$ be a dense sequence in $\mathcal{H}$ in which each term is repeated infinitely often. Let $\{\varepsilon_n\}$ be a sequence of positive numbers which decreases to zero. By inducton and Proposition 6.1 we can construct a sequence of invariant subspaces $\mathcal{M}_0 \supset \mathcal{M}_1 \supset \ldots \supset \mathcal{M}_n \supset \ldots$ and a sequence $\{y_n'\} \subset \mathcal{H}$ such that the operators $T_{\mathcal{H}\ominus\mathcal{M}_0}$, $T|\mathcal{M}_n$, and $T_{\mathcal{M}_n\ominus\mathcal{M}_{n+1}}$ are all of class (BCP), each $y_n' \in \mathcal{M}_{n-1} \ominus \mathcal{M}_n$ (we put $\mathcal{M}_{-1} = \mathcal{H}$) and $\|y_n' - P_{\mathcal{M}_{n-1}} y_n\| < \varepsilon_n$. It follows easily that $\bigcap\{\mathcal{M}_n : n \geqslant 0\} = \{0\}$. ▨

THEOREM 6.3. *Let $T \in \mathcal{L}(\mathcal{H})$ be of class* (BCP). *Then there exist invariant subspaces $\{\mathcal{M}_n : n \in \mathbf{Z}\}$ for T such that $\mathcal{M}_n \supset \mathcal{M}_{n+1}$ and $T_{\mathcal{M}_n\ominus\mathcal{M}_{n+1}}$ is of class* (BCP) *for each n, $\bigcap\{\mathcal{M}_n : n \in \mathbf{Z}\} = (0)$ and $\bigvee\{M_n : n \in \mathbf{Z}\} = \mathcal{H}$.*

Proof. We obtain the $\mathcal{M}_n$ for $n \geqslant 0$ by the previous result, and then obtain the $\mathcal{M}_n$ for $n < 0$ by applying the previous result to $T^*|\mathcal{H} \ominus \mathcal{M}_0$. ▨

REMARK. Corollary 5.3 also follows from this result.

Acknowledgements. This paper and its sequel form my doctoral dissertation. I want to thank my advisor, Professor Carl Pearcy, for his advice and encouragement; Hari Bercovici and Joe Langsam for several valuable conversations; and the members of the Department of Mathematics of The University of Michigan who provided a congenial environment during my graduate years. I would also like to thank the referee for providing several corrections.

BIBLIOGRAPHY

1. APOSTOL, C., Ultraweakly closed operator algebras, *J. Operator Theory*, **2**(1979), 49–61.
2. BERCOVICI, H.; FOIAŞ, C.; LANGSAM, J.; PEARCY, C., (BCP)-operators are reflexive, *Michigan Math. J.*, **29**(1982), 371–379.
3. BERCOVICI, H.; FOIAŞ, C.; PEARCY, C., A matricial factorization theorem and the structure of (BCP)-operators, preprint.
4. BERCOVICI, H.; FOIAŞ, C.; PEARCY, C., Dilation theory and systems of simultaneous equations in the predual of an operator algebra. I, *Michigan Math. J.*, to appear.
5. BROWN, A.; PEARCY, C., *Introduction to operator theory. I: Elements of functional analysis*, Springer-Verlag, New York, 1977.
6. BROWN, L.; SHIELDS, A.; ZELLER, K., On absolutely convergent exponential sums, *Trans. Amer. Math. Soc.*, **96**(1960), 162 – 183.
7. BROWN, S.; CHEVREAU, B.; PEARCY, C., Contractions with rich spectrum have invariant subspaces, *J. Operator Theory*, **1**(1979), 123–136.
8. CHEVREAU, B.; PEARCY, C.; SHIELDS, A., Finitely connected domains G, representations of $H^{\infty}(G)$, and invariant subspaces, *J. Operator Theory*, **6**(1981), 375–405.
9. DIXMIER, J., *Von Neumann algebras*, North-Holland, Amsterdam, 1981.
10. FOIAŞ, C.; PEARCY, C., (BCP)-operators and enrichment of invariant subspace lattices, *J. Operator Theory*, **9**(1983), 187–202.
11. FOIAŞ, C.; PEARCY, C.; SZ.-NAGY, B., Contractions with spectral radius one and invariant subspaces, *Acta. Sci. Math. (Szeged)*, **43**(1981), 273–280.
12. HOFFMAN, K., *Banach spaces of analytic functions*, Prentice-Hall, Englewood Cliffs, 1962.
13. LANGSAM, J., Ph. D. Thesis, University of Michigan, 1982.
14. PEARCY, C., *Some recent developments in operator theory*, CBMS Regional Conference Series, No. 36, Amer. Math. Soc., Providence, 1978.
15. RADJAVI, H.; ROSENTHAL, P., *Invariant subspaces*, Springer-Verlag, Berlin, 1973.
16. SARASON, D., On spectral sets having connected complement, *Acta. Sci. Math. (Szeged)*, **26**(1965), 289–299.
17. SZ.-NAGY, B.; FOIAŞ, C., *Harmonic analysis of operators on Hilbert space*, North-Holland, Amsterdam, 1970.

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Received November 15, 1982; revised September 28, 1983.