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ON SOME COMMUTATORS OF OPERATORS

J. A. ERDOS and S. GIOTOPOULOS

The questions considered in this paper are of the following general form: what conditions must operators X and Y satisfy so that their commutator $XY - YX$ belongs to a given set? Such questions are essentially concerned with derivations and have been extensively studied in relation to selfadjoint algebras. Non-selfadjoint theory has had less attention. However, a significant part of the recent progress in nest algebra theory consists of solutions to problems of this type (see [2, 4, 11, 13 and 14] and it is in this context that the investigation is continued here.

A set $\mathscr{E}$ of orthogonal projections on a Hilbert space H is called a *nest* if it is totally ordered. The set $\mathrm{Alg}\,\mathscr{E}$ of all (bounded, linear) operators leaving invariant the range of each member of $\mathscr{E}$ is called the *nest algebra* of $\mathscr{E}$. Nest algebras were introduced by Ringrose [18] and appear to be the most tractable class of non-selfadjoint operator algebras. We use the following notation: for any subalgebra $\mathscr{A}$ of $\mathscr{L}(H)$ and any subset $\mathscr{M}$ of $\mathscr{L}(H)$ which is a two-sided $\mathscr{A}$-module,

$$\mathrm{C}(\mathscr{A}, \mathscr{M}) = \{X \in \mathscr{L}(H) : AX - XA \in \mathscr{M} \text{ for all } A \in \mathscr{A}\}.$$

In other words, $\mathrm{C}(\mathscr{A}, \mathscr{M})$ denotes the commutant of $\mathscr{A}$ modulo $\mathscr{M}$. In this paper $\mathscr{A}$ will always be either a nest algebra $\mathrm{Alg}\,\mathscr{E}$, its diagonal $\mathscr{D} = \mathscr{E}'$ or its core $\mathscr{C} = \mathscr{E}''$ and $\mathscr{M}$ will always be some ideal of $\mathrm{Alg}\,\mathscr{E}$. It follows easily that $\mathrm{C}(\mathscr{A}, \mathscr{M}) \subseteq \mathrm{Alg}\,\mathscr{E}$ in these cases. In [7] it is asked whether in general

$$\mathrm{C}(\mathrm{Alg}\,\mathscr{E}, \mathscr{M}) \subseteq \mathscr{C} + \mathscr{M}.$$

This relation holds in all the cases considered here. Since every derivation of a nest algebra into $\mathscr{L}(H)$ is of the form $A \to AX - XA$ for some $X \in \mathscr{L}(H)$, [2], the determination of $\mathrm{C}(\mathrm{Alg}\,\mathscr{E}, \mathscr{M})$ is equivalent to the determination of all derivations of $\mathrm{Alg}\,\mathscr{E}$ into $\mathscr{M}$.

In Section 2 we use a method of Larson [16] to show that $\mathrm{C}(\mathscr{C}, \mathscr{I}) = \mathscr{D} + \mathscr{I}$ when $\mathscr{I}$ is any of the "diagonal" ideals defined in [18] and [8]. From this, we recover Larson's result [16] that $\mathrm{C}(\mathscr{C}, \mathscr{I}) = \mathscr{D} \oplus \mathscr{I}$ when $\mathscr{I}$ is the (Jacobson) radical of $\mathrm{Alg}\,\mathscr{E}$ and show that the same relation holds when $\mathscr{I}$ is any one of the ideals

$\mathscr{R}_0$, $\mathscr{R}_1$ or $\mathscr{R}_T$ which are related to the radical and which were introduced in [8] (definitions appear below). We also show that $\mathrm{C}(\mathscr{D}, \mathscr{I}) = \mathscr{C} + \mathscr{I}$ for all the ideals mentioned above. Section 3 is mainly devoted to a new proof of the fact that, if $\mathscr{R}$ is the radical of $\mathrm{Alg}\,\mathscr{E}$

$$\mathrm{C}(\mathrm{Alg}\,\mathscr{E}, \mathscr{R}) = \mathrm{C}^*(\mathscr{E}) \oplus \mathscr{R}.$$

($\mathrm{C}^*(\mathscr{U})$ denotes the smallest C^*-algebra containing each member of $\mathscr{U}$.) This result is due to Lance [15] and is also proved in a special case by Larson [16]. For nest subalgebras of von Neumann algebras the result is established by Gilfeather and Larson [13]. All these proofs use some form of functional representation and depend on the analysis of the resulting functions. The proof here is simpler and more direct and also establishes results for use in later sections. The remainder of the paper is concerned with determining $\mathrm{C}(\mathrm{Alg}\,\mathscr{E}, \mathscr{I})$ when $\mathscr{I}$ is $\mathscr{R}_0$, $\mathscr{R}_1$ or $\mathscr{R}_T$. The proofs for a continuous nest are very simple and are given in Section 4 and the general case is dealt with in Section 5.

The theory of nest algebras appears to be an appropriate setting for systems theory and elements of the radical appear there as the "strictly causal" operators, (see [12, 19]). The elements of $\mathscr{R}_0$ also have a systems theory interpretation [9]. However, for these applications the continuous nest case is by far the most important and this is further justification for giving separate proofs for this case.

1. PRELIMINARIES

The basic definitions concerning nest algebras may be found in [18, 6, 8]. In general we shall adopt the same conventions and notation as in [8]. Attention will be confined to *separable* Hilbert space.

We briefly review the definitions of the ideals which appear in this paper. If $E_A(\cdot)$ is the spectral measure of a positive invertible operator A then $\{E_A[0, a] : a \geqslant 0\}$ is called the *spectral nest* of A. Given a fixed nest $\mathscr{E}$, let $\mathscr{A}_0$ be the set of all positive invertible operators whose spectral nest has completion equal to $\mathscr{E}$. The subset $\mathscr{A}_1$ of $\mathscr{A}_0$ consists of operators whose spectral nest is the whole of $\mathscr{E}$. For each $A \in \mathscr{A}_0$, define

$$\mathscr{R}_A = \{X \in \mathscr{L}(H) : \|A^n X A^{-n}\| \to 0\}.$$

The set $\mathscr{R}_A$ is an ideal of the nest algebra $\mathrm{Alg}\,\mathscr{E}$. The ideals $\mathscr{R}_0$ and $\mathscr{R}_1$ are defined by

$$\mathscr{R}_0 = \bigcap \{\mathscr{R}_A : A \in \mathscr{A}_0\}$$
$$\mathscr{R}_1 = \bigcap \{\mathscr{R}_A : A \in \mathscr{A}_1\}.$$

The above definitions arise from a circle of ideas initiated by Deddens [5] and developed in [8].

Given $F, G \in \mathscr{E}$ with $F < G$, the seminorm $\Delta_{F,G}$ on Alg$\mathscr{E}$ is defined by

$$\Delta_{F,G}(X) = \|(G - F)X(G - F)\|.$$

The *diagonal ideals* $\mathscr{I}_E^+, \mathscr{I}_E^-, \mathscr{I}_E$ and $\mathscr{J}_E$ of Alg$\mathscr{E}$ are defined as follows. If $E = I$, $\mathscr{I}_E^+ = \text{Alg}\mathscr{E}$; otherwise

$$\mathscr{I}_E^+ = \{X \in \text{Alg}\mathscr{E} : \inf_{G>E} \Delta_{E,G}(X) = 0\}.$$

If $E = 0$, $\mathscr{I}_E^- = \text{Alg}\mathscr{E}$; otherwise

$$\mathscr{I}_E^- = \{X \in \text{Alg}\mathscr{E} : \inf_{F<E} \Delta_{F,E}(X) = 0\}.$$

If $E = 0$, $E = I$ or if $E \neq E^+$, $\mathscr{I}_E = \mathscr{I}_E^+ \cap \mathscr{I}_E^-$; otherwise

$$\mathscr{I}_E = \{X \in \text{Alg}\mathscr{E} : \inf_{F<E<G} \Delta_{F,G}(X) = 0\}.$$

Finally, if $E = 0$, $E = I$ or if E or E^- is an isolated point of $\mathscr{E}$, $\mathscr{J}_E = \mathscr{I}_E$; otherwise

$$\mathscr{J}_E = \{X \in \text{Alg}\mathscr{E} : \inf_{F<E^-, E<G} \Delta_{F,G}(X) = 0\}.$$

The ideals $\mathscr{I}_E^+$ and $\mathscr{I}_E^-$ were defined by Ringrose in [18] and $\mathscr{I}_E$ and $\mathscr{J}_E$ were introduced in [8].

We denote the (Jacobson) radical of Alg$\mathscr{E}$ by $\mathscr{R}$ (for the definition of the radical see, for example [1]). In [18], Ringrose determined the radical in terms of diagonal ideals and it is shown in [8] that the ideals $\mathscr{R}_1$ and $\mathscr{R}_0$ may be described in a similar way. These characterizations are:

$$\mathscr{R} = \bigcap\{\mathscr{I}_E^+ \cap \mathscr{I}_E^- : E \in \mathscr{E}\}$$

$$\mathscr{R}_1 = \bigcap\{\mathscr{I}_E : E \in \mathscr{E}\}$$

$$\mathscr{R}_0 = \bigcap\{\mathscr{J}_E : E \in \mathscr{E}\}.$$

Recall that $\mathscr{E}$ is a compact topological space when equipped with its order topology and that the order topology coincides with the strong topology on $\mathscr{E}$.

2. COMMUTANTS OF THE DIAGONAL AND CORE

Let $\mathscr{C}$ be an abelian $\mathscr{W}^*$-algebra and denote its commutant $\mathscr{C}'$ by $\mathscr{D}$. We briefly review a standard construction of a projection of $\mathscr{L}(H)$ onto $\mathscr{D}$. Take any invariant mean $M(\cdot)$ on the unitary group of $\mathscr{C}$. Write $\mathscr{L}_*$ for the set of trace class operators and use the relation $(X,f) = \text{tr}(Xf)$ $(X \in \mathscr{L}(H), f \in \mathscr{L}_*)$ to identify $\mathscr{L}(H)$ with $(\mathscr{L}_*)^*$. The mean of the function $U \mapsto (U^*XU, f)$ will be written as

$M_U(U^*XU,f)$. For each $X \in \mathscr{L}(H)$, $\psi(X)$ is defined by

$$(\psi(X),f) = M_U(U^*XU,f) \quad (f \in \mathscr{L}_*).$$

In view of our identification, $\psi(X) \in \mathscr{L}(H)$ and it can be shown that ψ is a projection of $\mathscr{L}(H)$ onto $\mathscr{D}$. Also, for all $X \in \mathscr{L}(H)$, $D \in \mathscr{D}$, $\psi(XD) = \psi(X) \cdot D$, $\psi(DX) = D \cdot \psi(X)$. The map ψ depends on the choice of M. Such a ψ is called a *diagonal projection*. In this paper $\mathscr{C}$ and $\mathscr{D}$ will be the core and diagonal of the nest algebra under consideration. That is, $\mathscr{C} = \mathscr{E}''$ and $\mathscr{D} = \mathscr{E}' = (\mathrm{Alg}\,\mathscr{E})^* \cap \mathrm{Alg}\,\mathscr{E}$ for some nest $\mathscr{E}$.

The following lemma is essentially contained in the proof of Theorem 2.1 in Johnson and Parrott's paper [14].

LEMMA 1. *Let (P_n) be a sequence of mutually orthogonal projections of an abelian $\mathscr{W}^*$-algebra $\mathscr{C}$ and let ψ be a diagonal projection onto $\mathscr{D} = \mathscr{C}'$. If $T \in \mathscr{L}(H)$ satisfies $\psi(T) = 0$ and, for each $A \in \mathscr{C}$,*

$$\lim_{n\to\infty} \|P_n(AT - TA)P_n\| = 0$$

then

$$\lim_{n\to\infty} \|P_nTP_n\| = 0.$$

Proof. Since $\psi(T) = 0$, $\psi(P_nTP_n) = 0$ and so for any $f \in \mathscr{L}_*$

$$(P_nTP_n,f) = M_U([P_nTP_n - U^*P_nTP_nU],f)$$

Thus $\|P_nTP_n\| \leqslant \sup_U \|P_nTP_n - U^*P_nTP_nU\|$. Since $P_n(UT - TU)P_n = U(P_nTP_n - U^*P_nTP_nU)$, it follows that, for each n, there exists a unitary operator U_n of $\mathscr{C}$ such that

$$\|P_n(U_nT - TU_n)P_n\| \geqslant \frac{1}{2} \|P_nTP_n\|.$$

As P_n are mutually orthogonal projections and $\|U_n\| = 1$, the series $\sum U_nP_n$ converges strongly to some element A of $\mathscr{C}$. Then, for each n,

$$P_n(AT - TA)P_n = P_n(U_nT - TU_n)P_n$$

and hence

$$\|P_nTP_n\| \leqslant 2\|P_n(AT - TA)P_n\|.$$

Thus, since $\lim_{n\to\infty} \|P_n(AT - TA)P_n\| = 0$ we have that

$$\lim_{n\to\infty} \|P_nTP_n\| = 0.$$

In view of Lemma 1, it is useful to have a description of the diagonal ideals in terms of sequences of mutually orthogonal projections.

LEMMA 2. *Let $\mathscr{E}$ be a complete nest of subspaces of a separable Hilbert space. If $E \in \mathscr{E}$ let $\mathscr{F}(E)$ denote the set of all strictly increasing sequences of elements of $\mathscr{E}$ converging to E and let $\mathscr{G}(E)$ denote the set of all strictly decreasing sequences of elements of $\mathscr{E}$ converging to E.*

(i) *$X \in \mathscr{I}_E^-$ if and only if $X \in \mathrm{Alg}\,\mathscr{E}$, $(E - E^-)X(E - E^-) = 0$ and, for any (F_n) in $\mathscr{F}(E)$*

$$\lim_{n\to\infty} \|(F_n - F_{n-1})X(F_n - F_{n-1})\| = 0.$$

(ii) *$X \in \mathscr{I}_E^+$ if and only if $X \in \mathrm{Alg}\,\mathscr{E}$, $(E^+ - E)X(E^+ - E) = 0$ and, for any (G_n) in $\mathscr{G}(E)$,*

$$\lim_{n\to\infty} \|(G_n - G_{n+1})X(G_n - G_{n+1})\| = 0.$$

(iii) *$X \in \mathscr{I}_E$ if and only if $X \in \mathscr{I}_E^+ \cap \mathscr{I}_E^-$ and for any (F_n) in $\mathscr{F}(E)$ and (G_n) in $\mathscr{G}(E)$,*

$$\lim_{n\to\infty} \|(E - E^-)X(G_n - G_{n+1})\| = 0$$

$$\lim_{n\to\infty} \|(F_n - F_{n-1})X(G_n - G_{n+1})\| = 0.$$

(iv) *$X \in \mathscr{J}_E$ if and only if $X \in \mathscr{I}_E$ and, for any (F_n) in $\mathscr{F}(E^-)$ and (G_n) in $\mathscr{G}(E)$,*

$$\lim_{n\to\infty} \|(F_n - F_{n-1})X(G_n - G_{n+1})\| = 0.$$

Proof. The only if implication is obvious in each case. Suppose now that $X \notin \mathscr{I}_E^-$. If $E = 0$ or $E \neq E^-$ then $\mathscr{F}(E)$ is empty and the result (i) is trivial. If $E = E^-$ and $E \neq 0$ then, for some $\delta > 0$, $\|(E - F)X(E - F)\| > \delta$ for all $F < E$. Let $F_0 < E$ be arbitrary. Since, in the strong operator topology

$$\lim_{F \uparrow E} (F - F_0)X(F - F_0) = (E - F_0)X(E - F_0)$$

and since the norm is strongly lower semicontinuous, there exists $F_1 > F_0$ such that $\|(F_1 - F_0)X(F_1 - F_0)\| > \delta$. By an obvious induction we choose $(F_n) \in \mathscr{F}(E)$ such that $\|(F_n - F_{n-1})X(F_n - F_{n-1})\| > \delta$. Thus (i) follows.

The other parts are proved in a similar way. One merely needs to check that the cases when the sequential conditions are vacuous corresponds to the definitions in the right way. For example in (iv), if $\mathscr{F}(E^-) = \emptyset$ then either E^- is isolated or $E = 0$ or $E = E^- \neq (E^-)^-$ and in all these cases $\mathscr{I}_E = \mathscr{J}_E$ by definition.

The result below, for the cases $\mathscr{I} = \mathscr{I}_E^+$ and $\mathscr{I} = \mathscr{I}_E^-$ is essentially contained in the proof of Theorem 2.4 of [16]. Recall that, by assumption, our Hilbert space is separable.

THEOREM 3. *Let $\mathscr{C}$ be the core and $\mathscr{D}$ be the diagonal of the nest algebra $\mathrm{Alg}\,\mathscr{E}$ and let $\mathscr{I}$ be an ideal of any one of the following types: $\mathscr{I}_E^+$, $\mathscr{I}_E^-$, $\mathscr{I}_E$ or $\mathscr{J}_E$.*

Then

$$C(\mathscr{C}, \mathscr{I}) = \mathscr{D} + \mathscr{I}.$$

Proof. The inclusion $\mathscr{D} + \mathscr{I} \subseteq C(\mathscr{C}, \mathscr{I})$ is clear. Suppose now that $X \in C(\mathscr{C}, \mathscr{I}_E^-)$. If $E \neq E^-$ it follows that $(E - E^-)X(E - E^-) \in \mathscr{C}' = \mathscr{D}$. Since $X - (E - E^-) \cdot \cdot X(E - E^-) \in \mathscr{I}_E^-$ the result is proved for this case. If $E = E^-$, let ψ be a diagonal projection onto $\mathscr{D}$. Put $T = X - \psi(X)$ so that $\psi(T) = 0$. Since, for any $A \in \mathscr{C}$, $AT - TA \in \mathscr{I}_E^-$, we have from Lemma 2 that

$$\lim_{n\to\infty} \|(F_n - F_{n-1})(AT - TA)(F_n - F_{n-1})\| = 0$$

where (F_n) is any member of $\mathscr{F}(E)$ (notation as in Lemma 2). Hence, from Lemma 1,

$$\lim_{n\to\infty} \|(F_n - F_{n-1})T(F_n - F_{n-1})\| = 0$$

and so, from Lemma 2, $T \in \mathscr{I}_E^-$. Thus the proof is complete for this case.

The proofs for the other cases are similar. We outline one further instance, that of $\mathscr{I} = \mathscr{I}_E = \mathscr{J}_E$ when $E^- = E = E^+$ and leave the remainder to the reader. Suppose $X \in C(\mathscr{C}, \mathscr{I}_E)$ and put $T = X - \psi(X)$ as above. From the previous case, since $\mathscr{I}_E^- \cap \mathscr{I}_E^+ \supseteq \mathscr{I}_E$, it follows that $T \in \mathscr{I}_E^- \cap \mathscr{I}_E^+$. Now let (F_n) and (G_n) be arbitrary members of $\mathscr{F}(E)$ and $\mathscr{G}(E)$ and put $P_n = (G_{n+1} - G_n) + (F_n - F_{n-1})$. Since $(I - E)P_n = G_{n+1} - G_n$ and $P_n E = F_n - F_{n-1}$ and since for any $A \in \mathscr{C}$

$$E(AT - TA)(I - E) = AET(I - E) - ET(I - E)A \in \mathscr{I}_E,$$

Lemmas 1 and 2 show that

$$\lim_{n\to\infty} \|P_n ET(I - E)P_n\| = \lim_{n\to\infty} \|(F_n - F_{n-1})T(G_{n+1} - G_n)\| = 0.$$

Thus, from Lemma 2, $T \in \mathscr{I}_E$ and so $X = \psi(X) + T \in \mathscr{D} + \mathscr{I}_E$.

COROLLARY 4. *Let $\mathscr{S}$ be any intersection of diagonal ideals. Then*

$$C(\mathscr{C}, \mathscr{S}) = \mathscr{D} + \mathscr{S}.$$

In particular, the result holds when $\mathscr{S}$ is any one of the ideals $\mathscr{R}, \mathscr{R}_0, \mathscr{R}_1$ or $\mathscr{R}_A$ as defined in Section 1 and, in these cases the sum is a direct sum of vector spaces.

Proof. If $X \in C(\mathscr{C}, \mathscr{S})$, $X - \psi(X) \in \mathscr{S}$ as in the proof of the theorem. All of the ideals $\mathscr{R}, \mathscr{R}_0, \mathscr{R}_1$ and $\mathscr{R}_A$ are of the required type (for $\mathscr{R}, \mathscr{R}_0$ and $\mathscr{R}_1$ see Section 1, for $\mathscr{R}_A$ see Theorem 14 of [8]). The fact that the sum $\mathscr{D} \oplus \mathscr{R}$ is direct, suppose $D \in \mathscr{D} \cap \mathscr{R}$. Then $D^* \in \mathscr{D}$ and, since $\mathscr{R}$ is a quasinilpotent ideal, D^*D is quasinilpotent. Hence $D = 0$. Since $\mathscr{R}_0, \mathscr{R}_1$ and $\mathscr{R}_A$ are all subsets of $\mathscr{R}$, all the sums are direct.

For the case of the radical, the above, in effect, reproduces Larson's proof of the same result [16].

We are indebted to the referee for pointing out that a result of Christensen [3] may be used to give a simpler proof of the theorem below. We have retained our original version to keep the presentation self-contained. A sketch of the alternative argument is given at the end of the proof.

THEOREM 5. *Let $\mathscr{C}$ be the core and $\mathscr{D}$ be the diagonal of the nest algebra* Alg$\mathscr{E}$ *and let $\mathscr{I}$ be an ideal of any one of the following types: $\mathscr{I}_E^+$, $\mathscr{I}_E^-$, $\mathscr{I}_E$ or $\mathscr{J}_E$. Then*

$$C(\mathscr{D}, \mathscr{I}) = \mathscr{C} + \mathscr{I}.$$

Proof. Clearly $\mathscr{C} + \mathscr{I} \subseteq C(\mathscr{D}, \mathscr{I})$. If $X \in C(\mathscr{D}, \mathscr{I})$ then $X \in C(\mathscr{C}, \mathscr{I})$ and, if ψ is a diagonal projection, we have from Theorem 3 that $X - \psi(X) \in \mathscr{I}$. We use standard von Neumann algebra theory (see e.g., [20]). Let z be a separating vector for $\mathscr{C}$ and let Q be the cyclic projection generated by $\mathscr{C}$ and z. Then $Q \in \mathscr{D}$ and, since $\mathscr{C} \mid Q$ admits a cyclic vector, $\mathscr{C} \mid Q$ is maximal abelian ([20], III.1.3) and so $\mathscr{C} \mid Q = \mathscr{C}' \mid Q = \mathscr{D} \mid Q$. Since $\psi(X) \in \mathscr{D}$, there exists $Y \in \mathscr{C}$ such that $Q[\psi(X) - Y]Q = 0$. Put $Z = \psi(X) - Y$.

We now show that, given any sequence (P_n) of mutually orthogonal projections of $\mathscr{C}$, if for each $D \in \mathscr{D}$,

$$\lim_{n\to\infty} \|P_n(DZ - ZD)P_n\| = 0$$

then

$$\lim_{n\to\infty} \|P_n Z P_n\| = 0.$$

Indeed, if $\|P_n Z P_n\| = \|ZP_n\| \nrightarrow 0$, choose unit vectors x_n in the range of P_n such that $\|Zx_n\| \nrightarrow 0$. Let F_n be the cyclic projection generated by $\mathscr{C}$ and x_n. Since z is cyclic for $\mathscr{D}$, the central support of Q is I and (e.g., by V.1.10 of [20]) $F_n \precsim Q$. Since $P_n \in \mathscr{C} = \mathscr{D}'$, $F_n = F_n P_n < QP_n$. Thus there exist partial isometries U_n in $\mathscr{D}$ such that $U_n U_n^* = F_n$ and $U_n^* U_n \leqslant QP_n$. As both (F_n) and (QP_n) are mutually orthogonal sequences of projections, the series $\sum U_n$ converges in the strong operator topology to an element U of $\mathscr{D}$. Since $QZQ = 0$ and $U_n = U_n Q$ we have that $UZU_n^* = 0$. Thus

$$P_n(ZU - UZ)P_n U_n^* x_n = P_n Z F_n x_n = Zx_n$$

and this shows that $\|P_n(UZ - ZU)P_n\| \nrightarrow 0$.

Using the result of the above paragraph and choosing the sequence P_n appropriate to the diagonal ideal $\mathscr{I}$, it follows from Lemma 2 that $Z \in \mathscr{I}$. Hence $X - Y = X - \psi(X) + \psi(X) - Y \in \mathscr{I}$ and, since $Y \in \mathscr{C}$ the theorem is proved.

The following alternative argument is due to the referee. Theorem 2.3 of [3] states that, if $\mathscr{D}_X(Z) = ZX - XZ$ and $\mathscr{D}_X$ maps $\mathscr{C}'$ into $\mathscr{C}$, then subject to certain conditions on $\mathscr{C}$ (which hold in the present case),

$$\inf_{Y \in \mathscr{C}'} \|Y - X\| \leqslant \|\mathscr{D}_X\|.$$

Thus, using the notation of the proof above, there exist $Y_n \in \mathscr{C}' = \mathscr{D}$ with $P_n Y_n P_n = Y_n$ such that

$$\|P_n \psi(X) P_n - Y_n\| \leqslant 2\|\mathscr{D}_{P_n(X)P_n}\|.$$

Since $X - \psi(X) \in \mathscr{I}$, $D\psi(X) - \psi(X)D \in \mathscr{I}$ and so

$$\|DP_n \psi(X) P_n - P_n \psi(X) P_n D\| \to 0.$$

An easy argument shows that $\|\mathscr{D}_{P_n \psi(X) P_n}\| \to 0$. Writing $Y = \sum Y_n$ we have that $\psi(X) - Y \in \mathscr{I}$ and the theorem follows.

COROLLARY 6. *Let $\mathscr{S}$ be any intersection of diagonal ideals. Then*

$$\mathrm{C}(\mathscr{D}, \mathscr{S}) = \mathscr{C} + \mathscr{S}.$$

In particular, the result holds when $\mathscr{S}$ is any one of the ideals $\mathscr{R}, \mathscr{R}_0, \mathscr{R}_1$ or $\mathscr{R}_A$ as defined in Section 1 and, in these cases the sum is a direct sum of vector spaces.

Proof. This follows from the proof of the theorem in the same way as Corollary 4 followed from Theorem 3. One needs to note that the element Y of the core was chosen independently of the sequence (P_n) of projections. In the cases when the sum is direct the corollary may also be easily deduced from Corollary 4.

COROLLARY 7. *Let $\mathscr{S}$ be any intersection of diagonal ideals. Then*

$$\mathrm{C}(\mathrm{Alg}\,\mathscr{E}, \mathscr{S}) \subseteq \mathscr{C} + \mathscr{S}.$$

Proof. Obvious, since $\mathscr{D} \subseteq \mathrm{Alg}\,\mathscr{E}$.

The corollary above settles for intersections of diagonal ideals the question mentioned at the beginning of the paper. It also implies that, when considering commutants of $\mathrm{Alg}\,\mathscr{E}$ modulo such ideals we may confine our attention to the core. Since the core is an abelian *-algebra, this means that only normal operators need be considered. In fact, if we write $\mathscr{C}_{\mathscr{S}}$ for $\mathrm{C}(\mathrm{Alg}\,\mathscr{E}, \mathscr{S}) \cap \mathscr{C}$ we shall see later (Section 5) that $\mathscr{C}_{\mathscr{S}}$ is a *-subalgebra of $\mathscr{C}$.

3. COMMUTANTS MODULO THE RADICAL

We first consider commutants of the nest algebra $\mathrm{Alg}\,\mathscr{E}$ modulo some of the diagonal ideals. The presence of atoms introduce certain complications in the case of $\mathscr{I}_E$ and $\mathscr{J}_E$ and we postpone these to Section 5.

THEOREM 8. *Let $\mathscr{E}$ be a nest of projections on a separable Hilbert space. Then for each $E \in \mathscr{E}$,*

$$\mathrm{C}(\mathrm{Alg}\,\mathscr{E}, \mathscr{I}_E^-) = \mathbf{C}I + \mathscr{I}_E^-$$

$$\mathrm{C}(\mathrm{Alg}\,\mathscr{E}, \mathscr{I}_E^+) = \mathbf{C}I + \mathscr{I}_E^+$$

and, if $E^- = E = E^+$,

$$\mathrm{C}(\mathrm{Alg}\,\mathscr{E}, \mathscr{I}_E) = \mathbf{C}I + \mathscr{I}_E.$$

Proof. Let $Z \in \mathrm{C}(\mathrm{Alg}\,\mathscr{E}, \mathscr{I}_E^-)$. Then from Corollary 7, $Z = C + Y$ with $C \in \mathscr{C}$ and $Y \in \mathscr{I}_E^-$. If $E \neq E^-$, we have that $C(E - E^-) = \lambda(E - E^-)$ for some scalar λ and so $C - \lambda I \in \mathscr{I}_E^-$.

Now suppose $E = E^-$ and let $E_n \in \mathscr{F}(E)$ (notation as in Lemma 2). We use spectral theory and show that, for some scalar λ,

$$(*) \qquad \bigcap_{n=1}^{\infty} \sigma(C|(E - E_n)) = \{\lambda\}.$$

Suppose the intersection (which is non-empty by compactness) contains two distinct points a and b with $|b - a| > 2\varepsilon > 0$. Let α and β be open discs in $\mathbf{C}$ of radii ε and centres a and b respectively. Denote the spectral measure of C by $G(\cdot)$. For each n, $G(\alpha)(E - E_n) \neq 0$ and so there exists $m > n$ such that $G(\alpha)(E_m - E_n) \neq 0$ (otherwise $G(\alpha)(E - E_n) = \text{strong}\lim_{n\to\infty} G(\alpha)(E_m - E_n) = 0$). The similar fact is true for $G(\beta)$. Thus, choosing a subsequence inductively if need be, we may assume that for each n,

$$G(\alpha)(E_{2n+1} - E_{2n}) \neq 0, \quad G(\beta)(E_{2n+2} - E_{2n+1}) \neq 0.$$

Let x_n and y_n be arbitrary unit vectors in the ranges of $(E_{2n+2} - E_{2n+1})$ and $(E_{2n+1} - E_{2n})$ respectively. Then $x_n \otimes y_n \in \mathrm{Alg}\,\mathscr{E}$ (recall that $(x \otimes y)t = \langle t, x\rangle y$) and so $\sum_{n=1}^{\infty} x_n \otimes y_n$ converges to some element X of $\mathrm{Alg}\,\mathscr{E}$ (in the strong operator topology). From spectral theory, we have that

$$\|Cx_n - ax_n\| < \varepsilon, \quad \|Cy_n - by_n\| < \varepsilon.$$

Thus,

$$\|(E_{2n+2} - E_{2n})\,[CX - XC]\,(E_{2n+2} - E_{2n})\| = \|C(x_n \otimes y_n) - (x_n \otimes y_n)C\| =$$

$$= \|x_n \otimes (C - b)y_n - (C^* - \bar{a})x_n \otimes y_n + (b - a)(x_n \otimes y_n)\| > |b - a| - 2\varepsilon.$$

Thus, using Lemma 2 we see that $CX - XC \notin \mathscr{I}_E^-$ and this establishes $(*)$. Since $\|(E - E_n)(C - \lambda I)(E - E_n)\|$ equals the spectral radius of $(C - \lambda I)|(E - E_n)$, it follows that $C - \lambda I \in \mathscr{I}_E^-$. The proof for $\mathscr{I}_E^+$ is similar.

For the case of $\mathscr{I}_E$ when $E^- = E = E^+$ we use the same methods. As before, we need only consider an element C of $\mathscr{C} \cap \mathrm{C}(\mathrm{Alg}\,\mathscr{E}, \mathscr{I}_E)$. Let $(F_n) \in \mathscr{F}(E)$, $(G_n) \in \mathscr{G}(E)$. Since $\mathscr{I}_E^+ \cap \mathscr{I}_E^- \supseteq \mathscr{I}_E$, it follows from above that $\lim_{n\to\infty} \|(C - \lambda^- I)(E - F_n)\| = \lim_{n\to\infty} \|(C - \lambda^+ I)(G_n - E)\| = 0$ for some scalars λ^-, λ^+. Choose unit vectors x_n and y_n in the ranges of $G_n - G_{n+1}$ and $F_{n+1} - F_n$ respectively. Then

$X = \sum_{n=1}^{\infty} x_n \otimes y_n \in \mathrm{Alg}\,\mathscr{E}$ and

$$\|(G_n - F_n)(CX - XC)(G_n - F_n)\| \geqslant \|(F_{n+1} - F_n)(CX - XC)(G_n - G_{n+1})\| =$$

$$= \|C(x_n \otimes y_n) - (x_n \otimes y_n)C\| =$$

$$= \|x_n \otimes (C - \lambda^-)y_n - (C^* - \bar{\lambda}^+)x_n \otimes y_n + (\lambda^+ - \lambda^-)(x_n \otimes y_n)\|.$$

Since $CX - XC \in \mathscr{J}_E$ and the right hand side converges to $|\lambda^+ - \lambda^-|$, it follows that $\lambda^+ = \lambda^- = \lambda$ and $C - \lambda I \in \mathscr{J}_E$ completing the proof of the theorem.

The determination of $\mathrm{C}(\mathrm{Alg}\,\mathscr{E}, \mathscr{R})$ is now completed by a simple compactness argument below. This result is originally due to Lance [15]. The proof by our route appears to be simpler than the proofs in any of [13, 15, 16]. Recall that, if $\mathscr{U}$ is any set of operators, $C^*(\mathscr{U})$ denotes the C^*-algebra generated by $\mathscr{U}$.

THEOREM 9. *Let $\mathscr{E}$ be a nest of projections on a separable Hilbert space and let $\mathscr{R}$ be the Jacobson radical of* $\mathrm{Alg}\,\mathscr{E}$. *Then*

$$\mathrm{C}(\mathrm{Alg}\,\mathscr{E}, \mathscr{R}) = C^*(\mathscr{E}) \oplus \mathscr{R}.$$

Proof. Since $\mathrm{C}(\mathrm{Alg}\,\mathscr{E}, \mathscr{R}) \subseteq \mathrm{C}(\mathscr{D}, \mathscr{R})$ for the inclusion $\mathrm{C}(\mathrm{Alg}\,\mathscr{E}, \mathscr{R}) \subseteq C^*(\mathscr{E}) \oplus \mathscr{R}$ it suffices to prove that if $C \in \mathscr{C} \cap \mathrm{C}(\mathrm{Alg}\,\mathscr{E}, \mathscr{R})$ then $C \in C^*(\mathscr{E})$. Let $\varepsilon > 0$ be given. From Ringrose's characterization of $\mathscr{R}$ ([18], see Section 1) and Theorem 8 if E is not 0 or I, there exist scalars λ_E^+ and λ_E^- such that

$$C - \lambda_E^+ I \in \mathscr{J}_E^+, \quad C - \lambda_E^- I \in \mathscr{J}_E^-.$$

Since $E \in \mathscr{J}_E^+$, $C - \lambda_E^- E - \lambda_E^+(I - E) = C - \lambda_E^+ I + (\lambda_E^+ - \lambda_E^-)E \in \mathscr{J}_E^+$. Also, as $(I - E) \in \mathscr{J}_E^-$, $C - \lambda_E^- E + \lambda_E^+(I - E) = C - \lambda_E^- I - (\lambda_E^+ - \lambda_E^-)(I - E) \in \mathscr{J}_E^-$. Thus

$$C - \lambda_E^- E - \lambda_E^+(I - E) \in \mathscr{J}_E^+ \cap \mathscr{J}_E^-.$$

Hence, as $C \in \mathscr{C}$, there exists $P_E, Q_E \in \mathscr{E}$ with $P_E < E < Q_E$ such that for all P, Q satisfying $P_E \leqslant P < Q \leqslant Q_E$ we have

$$(*) \qquad \|(Q - P)[C - \lambda_E^- E - \lambda_E^+(I - E)](Q - P)\| < \varepsilon.$$

The set $\{(P_E, Q_E) : E \in \mathscr{E} \setminus \{0, I\}\}$ together with the analogous intervals $[0, Q_0)$ and $(P_I, I]$ for 0, I forms an open cover of $\mathscr{E}$. Since $\mathscr{E}$ is compact (in the order topology) there is a finite subcover

$$[0, Q_0), \{(P_{E_i}, Q_{E_i}) : 1 \leqslant i \leqslant n - 1\}, (P_I, I].$$

Write $E_0 = 0$, $E_n = I$. It is easy to see that by relabelling and, if need be, shrinking the intervals, one can arrange that $\{P_{E_i}\}$, $\{Q_{E_i}\}$, and $\{E_i\}$ all increase with i. Choose, for $1 \leqslant i \leqslant n-1$, F_i such that $P_{E_i} \leqslant F_i \leqslant Q_{E_{i-1}}$. Define C_ε by

$$(\dagger) \qquad C_\varepsilon = \sum_{i=1}^{n} \lambda_{E_i}^{-}(E_i - F_i) + \sum_{i=0}^{n-1} \lambda_{E_i}^{+}(F_{i+1} - E_i).$$

Then $C_\varepsilon \in C^*(\mathscr{E})$ and, using $(*)$ we have that

$$\|C - C_\varepsilon\| = \max_i \|(C - C_\varepsilon)(F_i - F_{i-1})\| < \varepsilon.$$

Thus $C \in C^*(\mathscr{E})$.

For the reverse inclusion, note that if $E \in \mathscr{E}$, and $X \in \mathrm{Alg}\mathscr{E}$ then $EX - XE = EX(I - E) \in \mathscr{R}$ and so, taking linear combinations and norm limits, $CX - XC \in \mathscr{R}$ for any $C \in C^*(\mathscr{E})$.

4. COMMUTANTS MODULO $\mathscr{R}_0$ AND $\mathscr{R}_A$: THE CONTINUOUS CASE

For the formulation of the next result, recall that the nest $\mathscr{E}$ has a projection valued measure $E(\cdot)$ associated with it [6]. The measure $E(\cdot)$ satisfies $E[0, E] = E$. If $f\colon \mathscr{E} \to \mathbf{C}$ is a bounded Borel function, the operator T_f given by

$$T_f = \int_{\mathscr{E}} f(E)\,\mathrm{d}E$$

is a member of the core (in fact every member of the core is of this form). An operator of the form T_f where f is continuous on $\mathscr{E}$ will be called an *$\mathscr{E}$-continuous* operator.

The ideals $\mathscr{R}_0$, $\mathscr{R}_1$ and $\mathscr{R}_A$ (see Section 1) were introduced in [8]. Note that $\mathscr{R}_0 = \mathscr{R}_1$ when $\mathscr{E}$ is continuous (i.e., when $E = E^-$ for each $E \in \mathscr{E}$).

THEOREM 10. *Let $\mathscr{E}$ be a continuous nest. Then*

$$\mathrm{C}(\mathrm{Alg}\mathscr{E}, \mathscr{R}_0) = \mathscr{C}_0 \oplus \mathscr{R}_0$$

where $\mathscr{C}_0$ is the algebra of $\mathscr{E}$-continuous operators.

Proof. Since $\|T_f\| = \|f\|_\infty$, if $F < E_0 < G$ it is easy to see that

$$\|[T_f - f(E_0)](G - F)\| = \operatorname*{ess\,sup}_{F<E<G} |f(E) - f(E_0)|.$$

Thus, if f is continuous, $T_f - f(E)I \in \mathscr{I}_E$. Therefore $T_f \in \mathrm{C}(\mathrm{Alg}\mathscr{E}, \mathscr{I}_E)$ for each E and so $T_f \in \mathrm{C}(\mathrm{Alg}\mathscr{E}, \mathscr{R}_0)$. This shows that

$$\mathscr{C}_0 \oplus \mathscr{R}_0 \subseteq \mathrm{C}(\mathrm{Alg}\mathscr{E}, \mathscr{R}_0).$$

For the converse it is sufficient (by Corollary 7) to prove that if $C \in \mathscr{C} \cap \mathrm{C}(\mathrm{Alg}\mathscr{E}, \mathscr{R}_0)$ then $C \in \mathscr{C}_0$. Using Theorem 8 and the characterization of $\mathscr{R}_0$ from

[7] mentioned in Section 1, for each $E \in \mathscr{E}$,

$$C - \lambda(E)I \in \mathscr{I}_E$$

for some scalar $\lambda(E)$. As in Theorem 9, given $\varepsilon > 0$, we find P_E, Q_E such that, for $P_E \leqslant P < Q \leqslant Q_E$,

$$\|(Q - P)(C - \lambda(E)I)(Q - P)\| < \varepsilon.$$

Now, if $F \in (P_E, Q_E)$ and $(P, Q) \subseteq (P_E, Q_E) \cap (P_F, Q_F)$ we also have that

$$\|(Q - P)(C - \lambda(F)I)(Q - P)\| < \varepsilon,$$

which shows that $|\lambda(E) - \lambda(F)| < 2\varepsilon$. Thus $\lambda(E)$ is a continuous function of E.

We now proceed just as in Theorem 9, but here $\lambda_E^+ = \lambda_E^- = \lambda(E)$. The expresion (†) for C_ε becomes

$$C_\varepsilon = \sum_{i=0}^{n} \lambda(E_i)(F_{i+i} - F_i)$$

where we put $F_0 = 0$ and $F_{n+1} = I$. This is clearly an approximating sum for the integral of $\lambda(E)$ and, by choice of the cover we have that

$$\left\| \int_{\mathscr{E}} \lambda(E) \mathrm{d}E - C_\varepsilon \right\| < 2\varepsilon.$$

As before, $\|C - C_\varepsilon\| < \varepsilon$ and, since ε is arbitrary $C \in \mathscr{C}_0$ as required.

We now turn to the determination of $\mathscr{R}_A$ when the spectral nest of A is continuous. Recall that for a continuous nest, $\mathscr{A}_0 = \mathscr{A}_1$ (see Section 4 of [8]).

THEOREM 11. *Let A be an invertible positive operator whose spectral nest $\mathscr{E}$ is continuous. Then*

$$\mathrm{C}(\mathrm{Alg}\,\mathscr{E}, \mathscr{R}_A) = C^*(A) \oplus \mathscr{R}_A.$$

Proof. Theorem 20 of [8] states that, for any $X \in \mathrm{Alg}\,\mathscr{E}$, $AX - XA \in \mathscr{R}_A$. Hence $CX - XC \in \mathscr{R}_A$ for all $C \in C^*(A)$ and it follows that

$$C^*(A) \oplus \mathscr{R}_A \subseteq C(\mathrm{Alg}\,\mathscr{E}, \mathscr{R}_A).$$

For the opposite inclusion it is sufficient (by Corollary 7) to show that, if $C \in \mathscr{C} \cap \mathrm{C}(\mathrm{Alg}\mathscr{E}, \mathscr{R}_A)$ then $C \in C^*(A)$. Let $\sigma \subseteq \mathbf{R}^+$ be the spectrum of A and write E_s for the spectral projections of A (that is, $E_s = E_A[0, s]$). Then $\mathscr{E} = \{E_s : s \geqslant 0\}$. Recall [8] that $E \in \mathscr{E}$ is called a *jump* of A if $\{s : E_s = E\}$ contains an open interval of $\mathbf{R}$. Theorem 14 of [8] states that

$$\mathscr{R}_A = \mathscr{R} \cap \bigcap \{\mathscr{I}_E : E \notin J(A)\}$$

where $J(A)$ denotes the jumps of A. Now let t be any point of σ and let $t_- = \inf\{s : E_s = E_t\}$ and $t_+ = \sup\{s : E_s = E_t\}$. If $t_- = t_+ = t$ then $E_t \notin J(A)$ and so $\mathscr{R}_A \subseteq \mathscr{I}_E$. Define $\lambda(t)$ as the scalar such that

$$C - \lambda(t)I \in \mathscr{I}_E.$$

Such $\lambda(t)$ exists by Theorem 8 and is clearly unique. If $t_- < t_+$ then $E_t \in J(A)$ and, since $(t_-, t_+) \subseteq \rho(A)$ either $t = t_-$ or $t = t_+$. (Note: if $E_t = I$, $t_+ = \infty$ and $t = t_-$; similarly if $E_t = 0$, $t = t_+$.) Define $\lambda(t)$ by

$$C - \lambda(t)I \in \mathscr{I}_E^+ \quad \text{or} \quad C - \lambda(t)I \in \mathscr{I}_E^-$$

according as $t = t_+$ or $t = t_-$. Since $\mathscr{R}_A \subseteq \mathscr{R} \subseteq \mathscr{I}_E^\pm$, Theorem 8 shows that $\lambda(t)$ is well-defined.

Since the map $s \mapsto E_s$ is order preserving and onto $\mathscr{E}$, for any $t \in \sigma$, given $\varepsilon > 0$ there exists $a < t < b$ such that

$$\|(E_b - E_a)(C - \lambda(t)I)(E_b - E_a)\| < \varepsilon.$$

It follows easily (cf. the proof of Theorem 10) that $\lambda(t)$ is continuous on σ and that

$$C = \int_\sigma \lambda(t)\, \mathrm{d}E_t = \lambda(A).$$

Thus $C \in C^*(A)$ and

$$\mathrm{C}(\mathrm{Alg}\,\mathscr{E}, \mathscr{R}_A) = C^*(A) \oplus \mathscr{R}_A.$$

Theorems 10 and 11 clearly overlap. For example it can be shown that if $A_0 \in \mathscr{A}_0$ has no jumps then $C^*(A_0)$ consists of the $\mathscr{E}$-continuous operators and Theorem 11 implies that

$$\mathscr{C} \cap \mathrm{C}(\mathrm{Alg}\,\mathscr{E}, \mathscr{R}_0) = \bigcap\{C^*(A) : A \in \mathscr{A}_0\} = C^*(A_0).$$

However, it seems worthwhile to present the simple proof of Theorem 10 which is independent of Theorems 14 and 20 of [8].

5. COMMUTANTS MODULO $\mathscr{R}_0$, $\mathscr{R}_1$ AND $\mathscr{R}_A$: THE GENERAL CASE

There are two facts which complicate the theory when the nest $\mathscr{E}$ has atoms (i.e., elements E with $E \neq E^-$). Firstly $\mathscr{A}_0 \neq \mathscr{A}_1$ so $\mathscr{R}_0 \neq \mathscr{R}_1$ and secondly the commutants of Alg $\mathscr{E}$ modulo diagonal ideals are not always trivial. Once appropriate adjustments have been made to compensate for these facts, natural extensions of the proofs above hold in the general case. In view of this some details will be omitted.

First we deal with $\mathrm{C}(\mathrm{Alg}\,\mathscr{E}, \mathscr{I})$ when $\mathscr{I}$ is a diagonal ideal. Reference to the definitions shows that we need only consider $\mathscr{I}_E$ when $E^- \neq E = E^+$ and $\mathscr{J}_E$ when $(E^-)^- = E^- \neq E = E^+$ or when $E^- = E \neq E^+$ since all other cases are covered by intersections of ideals dealt with in Theorem 8. The last case may be written as $\mathscr{J}_{E^-}$ when $(E^-)^- = E^- \neq E$.

LEMMA 12. *Let $\mathscr{J}$ be a diagonal ideal of* Alg$\mathscr{E}$ *of one of the following forms:* $\mathscr{J}_E$ *with* $E^- \neq E = E^+$, $\mathscr{J}_E$ *with* $(E^-)^- = E^- \neq E = E^+$ *or* $\mathscr{J}_{E^-}$ with $(E^-)^- = E^- \neq E$. *Then, if* $E - E^-$ *has infinite-dimensional range,*

$$\mathrm{C}(\mathrm{Alg}\,\mathscr{E}, \mathscr{J}) = \mathbf{C}I + \mathscr{J}$$

and, if $E - E^-$ *has finite-dimensional range,*

$$\mathrm{C}(\mathrm{Alg}\mathscr{E}, \mathscr{J}) = C^*(I, E - E^-) + \mathscr{J}.$$

Proof. First consider $\mathscr{J}_E$ when $(E^-)^- = E^- \neq E = E^+$. Let $(F_n) \in \mathscr{F}(E^-)$, $(G_n) \in \mathscr{G}(E)$ (notation as in Lemma 2). If $C \in \mathscr{C} \cap \mathrm{C}(\mathrm{Alg}\mathscr{E}, \mathscr{J}_E)$ since $\mathscr{J}_E \subseteq \mathscr{J}_E^+ \cap \mathscr{J}_{E^-}^-$, it follows from Theorem 8 that, for some scalars λ^+, λ^-,

$$\lim_{n\to\infty} \|(C - \lambda^- I)(E^- - F_n)\| = 0 = \lim_{n\to\infty} \|(C - \lambda^+ I)(G_n - E)\|.$$

Also, since $\mathscr{J}_E \subseteq \mathscr{J}_E^-$, for some scalar μ, $(C - \mu I)(E - E^-) = 0$. Now let x_n and y_n be unit vectors in the ranges of $(G_n - G_{n+1})$ and $(F_{n+1} - F_n)$ respectively and let $X = \sum x_n \otimes y_n$. Exactly as in Theorem 8 the fact that $XC - CX \in \mathscr{J}_E$ implies that $\lambda^+ = \lambda^-$. This shows that

$$\mathrm{C}(\mathrm{Alg}\mathscr{E}, \mathscr{J}_E) \subseteq C^*(I, E - E^-) + \mathscr{J}_E.$$

If $E - E^-$ has infinite-dimensional range, taking y_n (instead of as above) to be an orthogonal sequence in the range of $E - E^-$, the same calculation proves that $\lambda^+ = \mu$. Hence in this case

$$\mathrm{C}(\mathrm{Alg}\mathscr{E}, \mathscr{J}_E) = \mathbf{C}I + \mathscr{J}_E.$$

If $E - E^-$ is finite-dimensional then for any $X \in \mathrm{Alg}\mathscr{E}$, since $(F_n - G_n) \to E - E^-$ strongly and $X(E - E^-)$ is compact,

$$\lim_{n\to\infty} \|(F_n - G_n)[X(E - E^-) - (E - E^-)X](F_n - G_n)\| = 0.$$

Thus $E - E^- \in \mathrm{C}(\mathrm{Alg}\mathscr{E}, \mathscr{J}_E)$ and so the lemma is proved for this case.

The results for the other cases may be proved by identical arguments.

COROLLARY 13. *Let* $\mathscr{S}$ *be any intersection of diagonal ideals and let* $\mathscr{C}_{\mathscr{S}} = \mathrm{C}(\mathrm{Alg}\mathscr{E}, \mathscr{S}) \cap \mathscr{C}$. *Then* $\mathscr{C}_{\mathscr{S}}$ *is a* C^**-algebra.*

Proof. For any diagonal ideal $\mathscr{J}$, since $\mathscr{C} \cap \mathscr{J}$ is clearly self-adjoint, it follows from Theorem 8 and the lemma that $\mathrm{C}(\mathrm{Alg}\mathscr{E}, \mathscr{J}) \cap \mathscr{C}$ is self-adjoint. As $\mathscr{C}_{\mathscr{S}}$ is an intersection of such algebras, the result is proved.

For the proofs which follow it is convenient to re-write the characterizations of $\mathscr{R}_0$ and $\mathscr{R}_1$ to take account of some automatic inclusions between certain diagonal ideals.

LEMMA 14. (i) $\mathscr{R}_0 = \bigcap\{\mathscr{I}_E : E \textit{ not isolated}\} \cap \bigcap\{\mathscr{I}_E^- : E \textit{ isolated}\}$.
(ii) $\mathscr{R}_1 = \bigcap\{\mathscr{I}_E : E = E^+\} \cap \bigcap\{\mathscr{I}_E^- : E \neq E^+\}$.

Proof. These follow from the characterizations mentioned in Section 1 and trivial observations such as: if $E \neq E^+$, $\mathscr{I}_E^+ = \mathscr{I}_{E^+}^-$ and if E^+ is not isolated, $\mathscr{I}_{E^+}^- \subseteq \mathscr{I}_{E^+}$.

Note that the notation $\mathscr{C}_0$ as defined below does not conflict with its use in Theorem 10 since the core of a continuous nest algebra contains no non-zero compact operators.

THEOREM 15. *Let $\mathscr{E}$ be a nest of projections on a separable Hilbert space. Let $\mathscr{C}_1$ be the algebra generated by the compact operators of the core and the $\mathscr{E}$-continuous operators. Let $\mathscr{C}_0$ be the algebra generated by the compact operators of the core and the $\mathscr{E}$-continuous operators corresponding to functions f such that $f(E^-) = f(E) = f(E^+)$ whenever neither E nor E^- is isolated. Then*

(i) $C(\operatorname{Alg}\mathscr{E}, \mathscr{R}_0) = \mathscr{C}_0 \oplus \mathscr{R}_0$

(ii) $C(\operatorname{Alg}\mathscr{E}, \mathscr{R}_1) = \mathscr{C}_1 \oplus \mathscr{R}_1$.

Proof. This is modelled on the proof of Theorem 10. We first consider (ii) and note that from Corollary 7, $C(\operatorname{Alg}\mathscr{E}, \mathscr{R}_1) \subseteq \mathscr{C} + \mathscr{R}_1$. Let $C \in \mathscr{C} \cap C(\operatorname{Alg}\mathscr{E}, \mathscr{R}_1)$. Using Theorem 8 and Lemmas 12 and 14, we obtain scalars $\lambda(E)$, $\mu(E)$ such that

$$C - \lambda(E)I - \mu(E)(E - E^-) \in \mathscr{I}_E \quad E = E^+$$

$$C - \lambda(E)I \in \mathscr{I}_E^- \quad E \neq E^+$$

where $\mu(E) = 0$ unless $(E - E^-)$ has finite, non-zero dimension. Note that $\mu(E)$ does not appear for $\mathscr{I}_E^-$ since $E - E^- = I \pmod{\mathscr{I}_E^-}$ when $E^- \neq E \neq E^+$. The fact that $\lambda(E)$ is continuous is proved as in Theorem 10, ($\mu(E)$ does not affect this since continuity from below is automatic when $E \neq E^-$). We now find a finite cover as in Theorem 9 and obtain the same approximation to C but with the addition of a finite linear combination of finite-dimensional projections of the form $E - E^-$. Specifically, we have, for a given $\varepsilon > 0$,

$$\|C - C_\varepsilon - K_\varepsilon\| < \varepsilon$$

where K_ε is a compact operator of the core and C_ε is as in Theorem 10 and satisfies

$$\left\| \int_{\mathscr{E}} \lambda(E)\, dE - C_\varepsilon \right\| < 2\varepsilon .$$

This shows that $C \in \mathscr{C}_1$ and therefore $C(\operatorname{Alg}\mathscr{E}, \mathscr{R}_1) \subseteq \mathscr{C}_1 \oplus \mathscr{R}_1$.

The fact that $C(\operatorname{Alg}\mathscr{E}, \mathscr{R}_1)$ contains the $\mathscr{E}$-continuous operators follows exactly as in Theorem 10. If K is a compact operator of the core it is normal and its (finite-dimensional) spectral projections are of the form $E - E^-$. Since $E - E^-$ is in every diagonal ideal except those contained in $\mathscr{I}_E^-$, it follows from Lemma 12 that $K \in C(\operatorname{Alg}\mathscr{E}, \mathscr{R}_1)$. This completes the proof of (ii).

The proof of (i) is virtually identical to the proof above. The equality $\lambda(E^-) = \lambda(E) = \lambda(E^+)$ when, for example $(E^-)^- = E^- \neq E = E^+$, arises from the fact that in this case $\mathscr{I}_{E^-} = \mathscr{I}_E$ and the other proofs of the equality come from similar observations. The details of the proof are left to the reader.

The special case when $\mathscr{E}$ is a maximal nest is of interest. It is true in general that any compact operator K of $\mathrm{Alg}\,\mathscr{E}$ can be written as $K = K_0 + K_1$ where $K_0 \in \mathscr{D}$ and $K_1 \in \mathscr{R}$ (for a maximal nest this is essentially in [17] and the general case is Theorem 4.4 of [10]). [This fact may also be easily proved using Corollary 4 as follows. Let $C \in \mathscr{C}$ and let (C_n) be a net of elements of $C^*(\mathscr{E})$ converging strongly to C. Since K is compact $C_nK - KC_n$ converges in norm to $CK - KC$ and so Theorem 9 shows that $CK - KC \in \mathscr{R}$. Thus $K \in \mathscr{D} \oplus \mathscr{R}$ by Corollary 4.] Denote the compact operators of $\mathrm{Alg}\,\mathscr{E}$ by $\mathscr{K}$. If $\mathscr{E}$ is maximal then $E - E^-$ has dimension 1 whenever $E \neq E^-$ and it follows easily that $\mathscr{K} \cap \mathscr{D} = \mathscr{K} \cap \mathscr{C}$. Since $\mathscr{K} \cap \mathscr{R} = \mathscr{K} \cap \mathscr{R}_0 = \mathscr{K} \cap \mathscr{R}_1$ ([8], Theorem 10) it follows from above that $\mathscr{K} + \mathscr{R}_i = \mathscr{K} \cap \mathscr{C} \oplus \mathscr{R}_i$ $(i = 0,1)$. Thus for a maximal nest, the compact perturbation can be absorbed into the ideal and Theorem 15 (ii) may be re-written as

$$\mathrm{C}(\mathrm{Alg}\,\mathscr{E}, \mathscr{R}_1) = \mathscr{C}_c + (\mathscr{R}_1 + \mathscr{K})$$

where $\mathscr{C}_c$ is the algebra $\mathscr{E}$-continuous operators. Now for all $X, A \in \mathrm{Alg}\,\mathscr{E}$, $(E - E^-)(AX - XA)(E_1 - E^-) = 0$ and so it follows easily that $\mathrm{C}(\mathrm{Alg}\,\mathscr{E}, \mathscr{R}_1 + \mathscr{K}) = \mathrm{C}(\mathrm{Alg}\,\mathscr{E}, \mathscr{R}_1)$. Thus

$$\mathrm{C}(\mathrm{Alg}\,\mathscr{E}, \mathscr{R}_1 + \mathscr{K}) = \mathscr{C}_c + (\mathscr{R}_1 + \mathscr{K}).$$

A similar reformulation of (i) holds with $\mathscr{C}_c$ replaced by the appopriate smaller algebra.

We now consider the ideals $\mathscr{R}_A$. Let $A \in \mathscr{A}_0$ have spectral projections $\{E_s : s \geqslant 0\}$. For each $t \in \mathbf{R}$, define

$$\mathscr{I}_t(A) = \{X \in \mathrm{Alg}\,\mathscr{E} : \inf_{x<t<y} \|(E_y - E_x)\, X(E_y - E_x)\| = 0\}.$$

Clearly each $\mathscr{I}_t(A)$ is a diagonal ideal of $\mathrm{Alg}\,\mathscr{E}$. Also $\mathscr{I}_t \neq \mathrm{Alg}\,\mathscr{E}$ if and only if $t \in \sigma(A)$.

LEMMA 16. *Let $A \in \mathscr{A}_0$. Then*

$$\mathscr{R}_A = \bigcap\{\mathscr{I}_t(A) : t \in \sigma(A)\}.$$

Proof. A trival check shows that this is a paraphrase of Theorem 14 of [8].

THEOREM 17. *Let $\mathscr{E}$ be a complete nest of projections on a separable Hilbert space and let $A \in \mathscr{A}_0$. Then*

$$\mathrm{C}(\mathrm{Alg}\,\mathscr{E}, \mathscr{R}_A) = \mathscr{C}_A \oplus \mathscr{R}_A$$

where $\mathscr{C}_A$ is the algebra generated by A and the compact operators of the core.

Proof. If $C \in \mathrm{C}(\mathrm{Alg}\,\mathscr{E}, \mathscr{R}_A) \cap \mathscr{C}$ then, from the results above, if each $t \in \sigma(A)$ there exists scalars $\lambda(t), \mu(t)$ such that

$$C - \lambda(t)I - \mu(t)(E_t - E_t^-) \in \mathscr{I}_t(A)$$

where $\mu(t) = 0$ unless $E_t - E_t^-$ is finite-dimensional. The methods of Theorems 11 and 15 are now used to show that λ is continuous on $\sigma(A)$ and that $C - \lambda(A)$ is a compact operator of $\mathscr{C}$. Thus $\mathrm{C}(\mathrm{Alg}\,\mathscr{E}, \mathscr{R}_A) \subseteq \mathscr{C}_A \oplus \mathscr{R}_A$. For the reverse inclusion note that Theorem 20 of [8] states that $A \in \mathrm{C}(\mathrm{Alg}\,\mathscr{E}, \mathscr{R}_A)$. The fact that $\mathrm{C}(\mathrm{Alg}\,\mathscr{E}, \mathscr{R}_A)$ contains all compact operators of the core is proved as in Theorem 15. Thus $\mathrm{C}(\mathrm{Alg}\,\mathscr{E}, \mathscr{R}_A) \supseteq \mathscr{C}_A$ and the theorem follows.

For the case of a maximal nest $\mathscr{E}$, Theorem 17 may be re-written as

$$\mathrm{C}(\mathrm{Alg}\,\mathscr{E}, \mathscr{R}_A) = C^*(A) + (\mathscr{R}_A + \mathscr{K}) = \mathrm{C}(\mathrm{Alg}\,\mathscr{E}, \mathscr{R}_A + \mathscr{K})$$

where $\mathscr{K}$ denotes the compact operators of $\mathrm{Alg}\,\mathscr{E}$. This is merely an observation based on the comment following Theorem 15.

The results of Section 2 show that, if $\mathscr{S}$ is any intersection of diagonal ideals, $\mathrm{C}(\mathrm{Alg}\,\mathscr{E}, \mathscr{S}) = \mathscr{C}_{\mathscr{S}} + \mathscr{S}$ where $\mathscr{C}_{\mathscr{S}}$ is a subalgebra of the core. Theorem 8 and Lemma 12 effectively determine the elements of $\mathscr{C}_{\mathscr{S}}$ in terms of "local" properties; that is, giving a condition for each projection E of $\mathscr{E}$. These may be translated into local properties of functions in some functional representation of the core. Such ideas appear in [15, 16, 13]. Alternatively these properties may be formulated in abstract spectral terms as follows. For each $C \in \mathscr{C}$ and $E \in \mathscr{E}$, define

$$\sigma_+(C, E) = \bigcap\{\sigma_e(C \mid G - E) : G > E\}$$

$$\sigma_-(C, E) = \bigcap\{\sigma_e(C \mid E - F) : F < E\}$$

$$\sigma_1(C, E) = \bigcap\{\sigma_e(C \mid G - F) : F < E < G\}$$

$$\sigma_0(C, E) = \bigcap\sigma_e(C \mid G - F) : F < E^-,\ E < G\},$$

where σ_e denotes the essential spectrum. Let $\delta_+(A, E)$ be the diameter of the set $\sigma_+(A, E)$ and define $\delta_-(A, E)$, $\delta_1(A, E)$ and $\delta_0(A, E)$ in the analogous way. Suppose the ideal $\mathscr{S}$ is the intersection of the four sets $\bigcap\{\mathscr{I}_E^+ : E \in \mathscr{E}_+\}$, $\cap\, \{\mathscr{I}_E^- : E \in \mathscr{E}_-\}$, $\bigcap\{\mathscr{I}_E : E \in \mathscr{E}_1\}$ and $\bigcap\{\mathscr{I}_E : E \in \mathscr{E}_0\}$ where $\mathscr{E}_i$ $(i = +, -, 1, 0)$ are subsets of $\mathscr{E}$ and where it is arranged that $E = E^+$ for all $E \in \mathscr{E}_1$ (otherwise delete E from $\mathscr{E}_1$ and adjoin E to both $\mathscr{E}_+$ and $\mathscr{E}_-$) and similarly neither E nor E^- is isolated for $E \in \mathscr{E}_0$. Then it is easy to prove from the foregoing that $\mathscr{C}_{\mathscr{S}}$ consists of all elements of $\mathscr{C}$ such that $\delta_i(A, E) = 0$ when $E \in \mathscr{E}_i$ $(i = +, -, 1, 0)$. However this really says little more than Theorem 8 and Lemma 12. The point of the results above when $\mathscr{S}$ is $\mathscr{R}$, $\mathscr{R}_0$, $\mathscr{R}_1$ or $\mathscr{R}_A$ is that the elements of $\mathscr{C}_{\mathscr{S}}$ are described in a global way.

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J. A. ERDOS
Department of Mathematics,
King's College, Strand,
London, WC2R2LS,
Great Britain.

S. GIOTOPOULOS
Department of Mathematics
University of Athens,
Athens,
Greece.

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