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ON THE STRUCTURE OF THE NAIMARK DILATION

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One of the main ideas of classical work of I. Schur [9] is to give a concrete description of the structure of Fourier coefficients of a scalar contractive analytic function on the unit circle by means of an associated sequence of complex numbers $\{\gamma_n\}_{n=1}^{\infty}$ with the properties that $|\gamma_n| \leqslant 1$ for every $n \in \mathbf{N}$, and if $|\gamma_{n_0}| = 1$, then $\gamma_n = 0$ for $n > n_0$.

For the operator-valued case of this problem one needs a good generalization of such kind of sequence; this was done in [4] by introducing the notion of choice sequences (see Section 1 for the definition).

There is a large class of problems intimately connected with the previous Schur problem. Let us mention some extrapolation problems, the contractive intertwining dilations theory, the structure of positive Toeplitz forms, the Naimark dilation of semispectral measures, and so on. Choice sequences appear in all of these and seem to be a powerful tool in dealing with some specific problems of these topics.

The aim of this paper is to present the structure of the connection between the Naimark dilation of operator-valued semispectral measures on the unit circle and choice sequences. We use for this some improvements of the techniques developed in our previous paper [7] on the structure of positive Toeplitz forms. The matrix form (2.2) of the Naimark dilation can be thought as a generalization of the Schäffer form of Sz.-Nagy's unitary dilation of a contraction [11]. We must also mention that W_+ in (2.6) is in fact what is called in [5] an adequate isometry.

As an application we compute the prediction-error operator of a stationary process.

§ 1

We shall begin this section by reminding a series of facts and notation from [7] and some usual notation in the contractive intertwining dilations theory.

Thus, let $\mathscr{H}$ and $\mathscr{K}$ be two Hilbert spaces and let $T \in \mathscr{L}(\mathscr{H}, \mathscr{K})$ be a contraction (i.e. $\|T\| \leqslant 1$). We denote as usual $D_T = (I - T^*T)^{1/2}$ and $\mathscr{D}_T = \overline{D_T\mathscr{H}}$, the defect operator and the defect space of T.

We also consider the following well-known unitary operator:

$$(1.1)\qquad \begin{cases} J(T):\mathscr{H}\oplus\mathscr{D}_{T^*}\to\mathscr{H}\oplus\mathscr{D}_T \\ J(T)=\begin{pmatrix} T, & D_{T^*} \\ D_T, & -T^* \end{pmatrix}. \end{cases}$$

As a main labelling of the set of contractive intertwining dilations, the choice sequences was introduced in [4] and [1].

Further on, we shall call a choice sequence a sequence of contractions $\mathscr{G}=\{\Gamma_n\}_{n=1}^{\infty}$, $\Gamma_1\in\mathscr{L}(\mathscr{H})$ and $\Gamma_k\in\mathscr{L}(\mathscr{D}_{\Gamma_{k-1}},\mathscr{D}_{\Gamma^*_{k-1}})$, $k\geqslant 2$.

We fix a choice sequence $\mathscr{G}=\{\Gamma_n\}_{n=1}^{\infty}$ and we consider the following operators used in [5] and [1]: for $n\geqslant 1$,

$$(1.2)_n^1\qquad \begin{cases} J_n^1=J_n^1(\Gamma_1,\dots,\Gamma_n):\mathscr{H}\oplus\mathscr{D}_{\Gamma_1^*}\oplus\mathscr{D}_{\Gamma_2}\oplus\dots\oplus\mathscr{D}_{\Gamma_n}\to\mathscr{H}\oplus\mathscr{D}_{\Gamma_1}\oplus\dots\oplus\mathscr{D}_{\Gamma_n} \\ J_n^1=J(\Gamma_1)\oplus I_{n-1}; \end{cases}$$

for $n\geqslant 2$ and $2\leqslant k\leqslant n$,

$$(1.2)_n^k\qquad \begin{cases} J_n^k=J_n^k(\Gamma_1,\dots,\Gamma_n):\mathscr{H}\oplus\mathscr{D}_{\Gamma_1}\oplus\dots\oplus\mathscr{D}_{\Gamma_{k-1}}\oplus\mathscr{D}_{\Gamma_k^*}\oplus\dots\oplus\mathscr{D}_{\Gamma_n}\to \\ \qquad\to\mathscr{H}\oplus\mathscr{D}_{\Gamma_1}\oplus\dots\oplus\mathscr{D}_{\Gamma^*_{k-1}}\oplus\mathscr{D}_{\Gamma_k}\oplus\dots\oplus\mathscr{D}_{\Gamma_n} \\ J_n^k=I_{k-1}\oplus J(\Gamma_k)\oplus I_{n-k}. \end{cases}$$

Having these operators we shall continue by defining:

$$(1.3)_0\qquad V_0=I_{\mathscr{H}}$$

and for $n\geqslant 1$,

$$(1.3)_n\qquad \begin{cases} V_n=V_n(\Gamma_1,\dots,\Gamma_n):\mathscr{H}\oplus\mathscr{D}_{\Gamma_1}\oplus\dots\oplus\mathscr{D}_{\Gamma_{n-1}}\oplus\mathscr{D}_{\Gamma_n^*}\to\mathscr{H}\oplus\mathscr{D}_{\Gamma_1}\oplus\dots\oplus\mathscr{D}_{\Gamma_n} \\ V_n=J_n^1J_n^2\dots J_n^n \end{cases}$$

and

$$(1.4)_0\qquad U_0=I_{\mathscr{H}},$$

$$(1.4)_n\qquad \begin{cases} U_n=U_n(\Gamma_1,\dots,\Gamma_n):\mathscr{H}\oplus\mathscr{D}_{\Gamma_1^*}\oplus\dots\oplus\mathscr{D}_{\Gamma_n^*}\to\mathscr{H}\oplus\mathscr{D}_{\Gamma_1}\oplus\dots\oplus\mathscr{D}_{\Gamma_n} \\ U_n=V_n(U_{n-1}\oplus I_{\mathscr{D}_{\Gamma_n^*}}),\quad \text{for } n\geqslant 1. \end{cases}$$

1.1. Remark. For every $n\geqslant 0$, we have:

$$(1.5)_n\qquad U_n=\tilde{U}_n$$

where

$$(1.6)_n^1 \quad \begin{cases} \tilde{J}_n^1 = \tilde{J}_n^1(\Gamma_1,\ldots,\Gamma_n) : \mathscr{H} \oplus \mathscr{D}_{\Gamma_1^*} \oplus \mathscr{D}_{\Gamma_2^*} \oplus \ldots \oplus \mathscr{D}_{\Gamma_n^*} \to \mathscr{H} \oplus \mathscr{D}_{\Gamma_1} \oplus \mathscr{D}_{\Gamma_2^*} \oplus \ldots \oplus \mathscr{D}_{\Gamma_n^*} \\ \tilde{J}_n^1 = J(\Gamma_1) \oplus I_{n-1}, \quad \text{for } n \geqslant 1; \end{cases}$$

for $n \geqslant 2$, $2 \leqslant k \leqslant n$,

$$(1.7)_n^k \quad \begin{cases} \tilde{J}_n^k = \tilde{J}_n^k(\Gamma_1,\ldots,\Gamma_n) : \mathscr{H} \oplus \mathscr{D}_{\Gamma_1^*} \oplus \ldots \oplus \mathscr{D}_{\Gamma_{k-1}} \oplus \mathscr{D}_{\Gamma_1^*} \oplus \ldots \oplus \mathscr{D}_{\Gamma_n^*} \to \\ \to \mathscr{H} \oplus \mathscr{D}_{\Gamma_1^*} \oplus \ldots \oplus \mathscr{D}_{\Gamma_{k-1}^*} \oplus \mathscr{D}_{\Gamma_k} \oplus \ldots \oplus \mathscr{D}_{\Gamma_n^*} \\ \tilde{J}_n^k = I_{k-1} \oplus J(\Gamma_k) \oplus I_{n-k}, \end{cases}$$

$$(1.8)_0 \qquad \tilde{V}_0 = I_{\mathscr{H}}$$

and for $n \geqslant 1$,

$$(1.8)_n \quad \begin{cases} \tilde{V}_n = \tilde{V}_n(\Gamma_1,\ldots,\Gamma_n) : \mathscr{H} \oplus \mathscr{D}_{\Gamma_1^*} \oplus \ldots \oplus \mathscr{D}_{\Gamma_n^*} \to \mathscr{H} \oplus \mathscr{D}_{\Gamma_1^*} \oplus \ldots \oplus \mathscr{D}_{\Gamma_{n-1}^*} \oplus \mathscr{D}_{\Gamma_n} \\ \tilde{V}_n = \tilde{J}_n^n \cdot \tilde{J}_n^{n-1} \cdot \ldots \cdot \tilde{J}_n^1, \end{cases}$$

$$(1.9)_0 \qquad \tilde{U}_0 = I_{\mathscr{H}}$$

and for $n \geqslant 1$,

$$(1.9)_n \quad \begin{cases} \tilde{U}_n = \tilde{U}_n(\Gamma_1,\ldots,\Gamma_n) : \mathscr{H} \oplus \mathscr{D}_{\Gamma_1^*} \oplus \ldots \oplus \mathscr{D}_{\Gamma_n^*} \to \mathscr{H} \oplus \mathscr{D}_{\Gamma_1} \oplus \ldots \oplus \mathscr{D}_{\Gamma_n} \\ \tilde{U}_n = (\tilde{U}_{n-1} \oplus I_{\mathscr{D}_{\Gamma_n}})\tilde{V}_n . \end{cases}$$

The equalities $(1.5)_n$ are the reason for the difference between the notation in [7] and the one we shall use in this paper. ▨

Let us define the following spaces:

$$\mathscr{H}_n = \underbrace{\mathscr{H} \oplus \ldots \oplus \mathscr{H}}_{n}$$

$$\mathscr{K}_1 = \mathscr{H}$$

and for $n \geqslant 2$,

$$\mathscr{K}_n = \mathscr{H} \oplus \mathscr{D}_{\Gamma_1} \oplus \ldots \oplus \mathscr{D}_{\Gamma_{n-1}}$$

$$\mathscr{K}_+ = \mathscr{H} \oplus \bigoplus_{n=1}^{\infty} \mathscr{D}_{\Gamma_n}$$

and we denote by $P_n = P_{\mathscr{K}_n}^{\mathscr{K}_+}$ the orthogonal projection of $\mathscr{K}_+$ onto $\mathscr{K}_n$. (In order not to complicate the formulae, when necessary, $\mathscr{K}_n$ will be regarded as being embedded in $\mathscr{K}_+$.)

We continue this section with the study of the general form of a row-contraction. Thus, regarding the case of a finite row, the following result has been obtained in [6]:

1.2. LEMMA. (a) *The operator*

$$X_n = (T_1, T_2, \ldots, T_n): \mathscr{H}_n \to \mathscr{H}$$

is a contraction if and only if $T_1 = \Gamma_1: \mathscr{H} \to \mathscr{H}$ *is a contraction and, for* $2 \leqslant k \leqslant n$,

$$(1.10)_k \qquad T_k = D_{\Gamma_1^*} \ldots D_{\Gamma_{k-1}^*} \Gamma_k,$$

where $\Gamma_k: \mathscr{H} \to \mathscr{D}_{\Gamma_{k-1}^*}$ *are contractions.*

(b) *There exists a unitary operator:*

$$(1.11)_n \qquad \begin{cases} \alpha_n: \mathscr{D}_{X_n} \to \mathscr{D}_{\Gamma_1} \oplus \mathscr{D}_{\Gamma_2} \oplus \ldots \oplus \mathscr{D}_{\Gamma_n} \\ \alpha_n D_{X_n} = \begin{pmatrix} D_{\Gamma_1}, & -\Gamma_1^*\Gamma_2, & -\Gamma_1^* D_{\Gamma_2^*}\Gamma_3, & \ldots, & -\Gamma_1^* D_{\Gamma_2^*} \ldots D_{\Gamma_{n-1}^*}\Gamma_n \\ 0, & D_{\Gamma_2}, & -\Gamma_2^*\Gamma_3, & \ldots, & -\Gamma_2^* D_{\Gamma_3^*} \ldots D_{\Gamma_{n-1}^*}\Gamma_n \\ 0, & 0, & D_{\Gamma_3}, & \ldots & \\ \vdots & & & & \\ 0, & 0, & \ldots & , & D_{\Gamma_n} \end{pmatrix} \end{cases}$$

and a unitary operator

$$(1.12)_n \qquad \begin{cases} \tilde{\alpha}_n: \mathscr{D}_{X_n^*} \to \mathscr{D}_{\Gamma_n^*} \\ \tilde{\alpha}_n D_{X_n^*} = D_{\Gamma_n^*} \ldots D_{\Gamma_1^*}. \end{cases}$$ ▨

By duality, a similar result is obtained for the contractions:

$$Y_n = (T_1, T_2, \ldots, T_n)^{\mathrm{t}}: \mathscr{H} \to \mathscr{H}_n$$

("t" standing for the matrix transpose).

For the fixed choice sequence $\mathscr{G} = \{\Gamma_n\}_{n=1}^{\infty}$, Lemma 1.2 enables us to define the following contractions:

$$(1.13)_1^1 \qquad X_1^1(\Gamma_1) = \Gamma_1,$$

and for $n \geqslant 2$,

$$(1.13)_n^1 \qquad \begin{cases} X_n^1 = X_n^1(\Gamma_1, \ldots, \Gamma_n): \mathscr{H}_n \to \mathscr{H} \\ X_n^1 = (\Gamma_1, D_{\Gamma_1^*}\Gamma_2, \ldots, D_{\Gamma_1^*} \ldots D_{\Gamma_{n-1}^*}\Gamma_n); \end{cases}$$

and for $k \geqslant 2$, $n \geqslant k$,

$$(1.13)_n^k \qquad \begin{cases} X_n^k = X_n^k(\Gamma_k, \ldots, \Gamma_n): \mathscr{D}_{\Gamma_{k-1}} \oplus \mathscr{D}_{\Gamma_k} \oplus \ldots \oplus \mathscr{D}_{\Gamma_{n-1}} \to \mathscr{H} \\ X_n^k = (\Gamma_k, D_{\Gamma_k^*}\Gamma_{k+1}, \ldots, D_{\Gamma_k^*} \ldots D_{\Gamma_{n-1}^*}\Gamma_n) . \end{cases}$$

Similarly, let us define the contractions:

$$(1.14)_1^1 \qquad Y_1^1(\Gamma_1) = \Gamma_1,$$

and for $n \geqslant 2$,

$$(1.14)_n^1 \qquad \begin{cases} Y_n^1 = Y_n^1(\Gamma_1, \ldots, \Gamma_n): \mathscr{H} \to \mathscr{H} \oplus \mathscr{D}_{\Gamma_1^*} \oplus \ldots \oplus \mathscr{D}_{\Gamma_{n-1}^*} \\ Y_n^1 = (\Gamma_1, \Gamma_2 D_{\Gamma_1}, \ldots, \Gamma_n D_{\Gamma_{n-1}} \ldots D_{\Gamma_1})^t, \end{cases}$$

and for $k \geqslant 2$, $n \geqslant k$,

$$(1.14)_n^k \qquad \begin{cases} Y_n^k = Y_n^k(\Gamma_k, \ldots, \Gamma_n): \mathscr{H} \to \mathscr{D}_{\Gamma_{k-1}^*} \oplus \ldots \oplus \mathscr{D}_{\Gamma_{n-1}^*} \\ Y_n^k = (\Gamma_k, \Gamma_{k+1} D_{\Gamma_k}, \ldots, \Gamma_n D_{\Gamma_{n-1}} \ldots D_{\Gamma_k})^t . \end{cases}$$

Now, we can pass to the case of an infinite row.

1.3. LEMMA. *For every $k \geqslant 1$ there exist the strong operatorial limits:*

$$\operatorname*{s-lim}_{n \to \infty} X_n^k P_n : \mathscr{K}_+ \to \mathscr{H}$$

and

$$\operatorname*{s-lim}_{n \to \infty} (X_n^k)^* : \mathscr{H} \to \mathscr{K}_+ .$$

Proof. For the sake of simplicity, we shall write the proof only for $k = 1$. The general case contains the same computations.

Let $f \in \bigcup_{n=1}^{\infty} \mathscr{K}_n$, then there exists $p \in \mathbf{N}$ so that $f \in \mathscr{K}_p$. For every $n \geqslant p$ we have $X_n^1 P_n f = X_p^1 f$, so the sequence $\{X_n^1 P_n f\}_{n=1}^{\infty}$ is a Cauchy sequence. But,

$$\|X_n^1 P_n\| \leqslant 1, \quad n \in \mathbf{N}$$

and

$$\overline{\bigcup_{n=1}^{\infty} \mathscr{K}_n} = \mathscr{K}_+$$

so, there results the existence of the strong limit

$$\operatorname*{s-lim}_{n \to \infty} X_n^1 P_n .$$

In order to show the existence of the second limit, we remark that $(X_n^1)^*$ being a contraction, for $h \in \mathscr{H}$

$$\sum_{m=1}^{n} \|\Gamma_m^* D_{\Gamma_{m-1}^*} \dots D_{\Gamma_1^*} h\|^2 \leqslant \|h\|^2 .$$

Thus, we obtain for $h \in \mathscr{H}$

$$\sum_{n=1}^{\infty} \|\Gamma_n^* D_{\Gamma_{n-1}^*} \dots D_{\Gamma_1^*} h\|^2 \leqslant \|h\|^2 ,$$

and, for $n, m \in \mathbf{N}$, $n \geqslant m$,

$$\|(X_n^1)^* h - (X_m^1)^* h\|^2 \leqslant \sum_{p=m+1}^{n} \|\Gamma_p^* D_{\Gamma_{p-1}^*} \dots D_{\Gamma_1^*} h\|^2 .$$

This concludes the proof of the existence of the second limit. ▨

We define $X_\infty^k = X_\infty^k(\mathscr{G}) = \text{s-}\lim\limits_{n\to\infty} X_n^k P_n : \mathscr{K}_+ \to \mathscr{H}$.

We end this section with the identification of the defect spaces of X_∞^k using the choice sequence. Again, for simplicity, we shall give the proofs only for $k = 1$.

1.4. Proposition. *There exists a unitary operator*

$$\alpha_+^k : \mathscr{D}_{X_\infty^k} \to \bigoplus_{n=k}^{\infty} \mathscr{D}_{\Gamma_n} .$$

Proof. We consider the operators

$$(1.15)_n \qquad \begin{cases} D_n^1 : \mathscr{K}_n \to \bigoplus\limits_{m=1}^{n} \mathscr{D}_{\Gamma_m} \\[2ex] D_n^1 = \begin{pmatrix} D_{\Gamma_1}, & -\Gamma_1^*\Gamma_2, \dots, & -\Gamma_1^* D_{\Gamma_2^*} \dots D_{\Gamma_{n-1}^*} \Gamma_n \\ 0, & D_{\Gamma_2}, \dots & \\ \vdots & & \\ 0, & \dots & D_{\Gamma_n} \end{pmatrix} \end{cases}$$

also mentioned in Lemma 1.2(b). Proceeding as in Lemma 1.3, it results that there exists

$$D_\infty^1 = \text{s-}\lim_{n\to\infty} D_n^1 P_n : \mathscr{K}_+ \to \bigoplus_{n=1}^{\infty} \mathscr{D}_{\Gamma_n} .$$

But, there also results that $\|D_n^1 P_n f\| \underset{n\to\infty}{\longrightarrow} \|D_\infty^1 f\|$ for $f \in \mathscr{K}_+$.

Having Lemma 1.3, there results that $\|D_{X_n^1}P_nf\| \underset{n\to\infty}{\longrightarrow} \|D_{X_\infty^1}f\|$ for $f \in \mathscr{K}_+$ and, from Lemma 1.2 there results the equality

$$\|D_n^1P_nf\| = \|D_{X_n^1}P_nf\|.$$

Consequently, the operator

$$(1.16)\qquad \begin{cases} \alpha_+^1 : \mathscr{D}_{X_\infty^1} \to \bigoplus_{n=1}^{\infty} \mathscr{D}_{\Gamma_n} \\ \alpha_+^1 D_{X_\infty^1} f = D_\infty^1 f, \quad f \in \mathscr{K}_+ \end{cases}$$

is an isometry.

Let us show now that the operator α_+^1 is a unitary one. For, let $d = (d_1, d_2, \ldots)^t \in \bigoplus_{n=1}^{\infty} \mathscr{D}_{\Gamma_n}$ so that $d \perp \operatorname{Ran} D_\infty^1$ ($\operatorname{Ran} D_\infty^1$ denotes the range of the operator D_∞^1). We choose first $(d_0', 0, 0, \ldots)^t \in \mathscr{K}_+$, $d_0' \in \mathscr{H}$, consequently, from the fact that $d \perp D_\infty^1(d_0', 0, 0, \ldots)^t$ it results $d_1 \perp \mathscr{D}_{\Gamma_1} d_0'$ and this means that $d_1 = 0$. Taking into account the upper triangular form of D_∞^1 we can continue to consider elements of $\mathscr{K}_+$ with only one nonzero component in order to obtain $d = 0$, so $\overline{\operatorname{Ran} D_\infty^1} = \bigoplus_{n=1}^{\infty} \mathscr{D}_{\Gamma_n}$ and the operator α_+^1 is a unitary one. ▨

In order to identify the spaces $\mathscr{D}_{(X_\infty^k)^*}$ we consider the operators:

$$(1.17)_n^k \qquad \begin{cases} G_n^k = G_n^k(\Gamma_k, \ldots, \Gamma_n) : \mathscr{H} \to \mathscr{H} \\ \quad G_n^k = D_{\Gamma_k^*} \ldots D_{\Gamma_1^*} \end{cases} \qquad \text{for } k \geqslant 1,\ n \geqslant k.$$

1.5. Lemma. *There exist the limits:*

$$\operatorname*{s-lim}_{n\to\infty} G_n^k (G_n^k)^*$$

for every $k \geqslant 1$.

Proof.

$$G_n^1(G_n^1)^* = G_{n-1}^1(G_{n-1}^1)^* - (D_{\Gamma_1^*} \ldots D_{\Gamma_{n-1}^*}\Gamma_n)(D_{\Gamma_1^*} \ldots D_{\Gamma_{n-1}^*}\Gamma_n)^*$$

consequently, $\{G_n^1(G_n^1)^*\}_{n=1}^{\infty}$ is a decreasing sequence of positive contractions; consequently, there exists the limit

$$\operatorname*{s-lim}_{n\to\infty} G_n^1 (G_n^1)^*.$$ ▨

We define

$$(1.18)_k \qquad \begin{cases} H_k : \mathscr{H} \to \mathscr{H} \\ H_k = \text{s-}\lim_{n\to\infty} G_n^k (G_n^k)^* . \end{cases}$$

These operators are positive contractions.

1.6. PROPOSITION. *There exists a unitary operator*

$$\tilde{\alpha}_+^k : \mathscr{D}_{(X_\infty^k)^*} \to \overline{\operatorname{Ran} H_k} .$$

Proof. For every $n \in \mathbf{N}$,

$$X_n^1 (X_n^1)^* + G_n^1 (G_n^1)^* = I$$

and for $n \to \infty$, we obtain

$$X_\infty^1 (X_\infty^1)^* + H_1 = I$$

consequently,

$$(1.19) \qquad \begin{cases} \tilde{\alpha}_+^1 : \mathscr{D}_{(X_\infty^1)^*} \to \overline{\operatorname{Ran} H_1} \\ \tilde{\alpha}_+^1 D_{(X_\infty^1)^*} h = H_1^{1/2} h , \quad h \in \mathscr{H} \end{cases}$$

is a unitary operator. ▨

We shall define $\mathscr{D}_* = \overline{\operatorname{Ran} H_1}$.

§ 2

In this section we shall obtain a concrete realization of the Naimark dilation of a semispectral measure F on $\mathbf{T}$, the unit circle. A semispectral measure on $\mathbf{T}$ a linear means positive map:

$$F : C(\mathbf{T}) \to \mathscr{L}(\mathscr{H})$$

where $C(\mathbf{T})$ denotes the set of continuous functions on $\mathbf{T}$.

Let $\{S_n\}_{n\in\mathbf{Z}}$ be the Fourier coefficients of F,

$$S_n = F(\chi_n), \quad n \in \mathbf{Z} ,$$

where $\chi_n(e^{it}) = e^{int}$. As the function

$$\mathbf{Z} \ni n \to S_n \in \mathscr{L}(\mathscr{H})$$

is a positive-definite one, there results that we can use Theorem 1.2 from [7]. Thus, there exists a one-to-one correspondence between the set of semispectral measures F on $\mathbf{T}$ with $F(1) = I$ and the set of choice sequences $\mathscr{G} = \{\Gamma_n\}_{n=1}^{\infty}$, given by the formulae:

$$(*) \qquad \begin{cases} S_1 = \Gamma_1 \\ S_n = X_{n-1}^1 U_{n-2} Y_{n-1}^1 + D_{\Gamma_1^*} \ldots D_{\Gamma_{n-1}^*} \Gamma_n D_{\Gamma_{n-1}} \ldots D_{\Gamma_1}, \quad n \geqslant 2, \end{cases}$$

where $X_n^1 = X_n^1(\Gamma_1, \ldots, \Gamma_n)$, $Y_n^1 = Y_n^1(\Gamma_1, \ldots, \Gamma_n)$, $U_n = U_n(\Gamma_1, \ldots, \Gamma_n)$ are given by $(1.13)_n^1$, $(1.14)_n^1$ and $(1.4)_n$.

Further on, we fix a semispectral measure F on $\mathbf{T}$ with $F(1) = I$ and let $\mathscr{G} = \{\Gamma_n\}_{n=1}^{\infty}$ be the associated choice sequence. Consider also $X_{\infty}^1 = X_{\infty}^1(\mathscr{G})$. Now, we can pass on to the construction of the Naimark dilation of F. First, let us consider the unitary operator

$$\begin{pmatrix} D_{(X_{\infty}^1)^*}, & X_{\infty}^1 \\ -(X_{\infty}^1)^*, & D_{X_{\infty}^1} \end{pmatrix} : \begin{matrix} \mathscr{D}_{(X_{\infty}^1)^*} \\ \oplus \\ \mathscr{K}_+ \end{matrix} \to \begin{matrix} \mathscr{H} \\ \oplus \\ \mathscr{D}_{X_{\infty}^1} \end{matrix}$$

and then we define the unitary operator

$$(2.1) \qquad \begin{cases} W_{\text{red}} = W_{\text{red}}(F) : \mathscr{D}_* \oplus \mathscr{K}_+ \to \mathscr{H} \oplus \bigoplus_{n=1}^{\infty} \mathscr{D}_{\Gamma_n} \\[2ex] W_{\text{red}} = \begin{pmatrix} I & 0 \\ 0 & \alpha_+^1 \end{pmatrix} \begin{pmatrix} D_{(X_{\infty}^1)^*}, & X_{\infty}^1 \\ -(X_{\infty}^1)^*, & D_{X_{\infty}^1} \end{pmatrix} \begin{pmatrix} (\tilde{\alpha}_+^1)^* & 0 \\ 0 & I \end{pmatrix} \end{cases}$$

where α_+^1 and $\tilde{\alpha}_+^1$ are given by (1.16) and (1.19).

The last step is to define

$$\mathscr{K} = \ldots \oplus \mathscr{D}_* \oplus \mathscr{D}_* \oplus \boxed{\mathscr{H}} \oplus \bigoplus_{n=1}^{\infty} \mathscr{D}_{\Gamma_n}$$

and the unitary operator

$$(2.2) \qquad \begin{cases} W = W(F) : \mathscr{K} \to \mathscr{K} \\ W = I \oplus W_{\text{red}} \end{cases}$$

where W is written with respect to the decompositions

$$\mathscr{K} = (\ldots \oplus \mathscr{D}_*) \oplus (\mathscr{D}_* \oplus \mathscr{K}_+) \quad \text{and} \quad \mathscr{K} = (\ldots \oplus \mathscr{D}_*) \oplus \left(\mathscr{H} \oplus \bigoplus_{n=1}^{\infty} \mathscr{D}_{\Gamma_n}\right).$$

We can write down a matricial form of W:

$$W = \left(\begin{array}{ccc|c} \ddots & \vdots & \vdots & \\ I & 0 & 0 & 0 \\ \dots 0 & I & 0 & \\ \hline \dots\;\dots & 0 & D_{*1} & \\ \dots\;\dots & 0 & -Z_1 & W|\mathscr{K}_+ \\ \dots\;\dots & 0 & -Z_2 & \\ & \vdots & \vdots & \end{array}\right)$$

where

$$D_{*1} = H_1^{1/2} \tag{2.3}$$

and

$$(2.4)_k \qquad \begin{cases} Z_k : \mathscr{D}_* \to \mathscr{D}_{\Gamma_k} \\ Z_k = P^{\mathscr{K}_+}_{\mathscr{D}_{\Gamma_k}} \alpha^1_+ (X^1_\infty)^* (\tilde{\alpha}^1_+)^*, \quad k \geqslant 1. \end{cases}$$

2.1. Remark. It is fairly easy to compute the operators Z_k, $k \geqslant 1$. For $h \in \mathscr{H}$, we have

$$Z_k D_{*1} h = P^{\mathscr{K}_+}_{\mathscr{D}_{\Gamma_k}} \alpha^1_+ (X^1_\infty)^* (\tilde{\alpha}^1_+)^* D_{*1} h = P^{\mathscr{K}_+}_{\mathscr{D}_{\Gamma_k}} \alpha^1_+ (X^1_\infty)^* D_{(X^1_\infty)^*} h =$$

$$= P^{\mathscr{K}_+}_{\mathscr{D}_{\Gamma_k}} \alpha^1_+ D_{X^1_\infty} (X^1_\infty)^* h = P^{\mathscr{K}_+}_{\mathscr{D}_{\Gamma_k}} D^1_\infty (X^1_\infty)^* h =$$

$$= \Gamma_k^* (I - X^{k+1}_\infty (X^{k+1}_\infty)^*) D_{\Gamma_k^*} \dots D_{\Gamma_1^*} h = \Gamma_k^* H_{k+1} D_{\Gamma_k^*} \dots D_{\Gamma_1^*} h.$$

But, on the other hand,

$$D_{\Gamma_k^*} G^{k+1}_n (G^{k+1}_n)^* D_{\Gamma_k^*} = G^k_n (G^k_n)^*, \quad n, k \in \mathbf{N}$$

consequently, if $n \to \infty$, it results:

$$D_{\Gamma_k^*} H_{k+1} D_{\Gamma_k^*} = H_k$$

or

$$(H^{1/2}_{k+1} D_{\Gamma_k^*})^* (H^{1/2}_{k+1} D_{\Gamma_k^*}) = H_k^{1/2} H_k^{1/2}.$$

Then there exists a partial isometry C_k such that

$$C_k H_k^{1/2} = H^{1/2}_{k+1} D_{\Gamma_k^*}.$$

It results

$$H_{k+1}^{1/2}D_{\Gamma_k^*}\dots D_{\Gamma_1^*}=C_kH_k^{1/2}D_{\Gamma_{k-1}^*}\dots D_{\Gamma_1^*}=\dots=C_kC_{k-1}\dots C_1D_{*1}$$

so

$$Z_kD_{*1}h=\Gamma_k^*H_{k+1}^{1/2}H_{k+1}^{1/2}D_{\Gamma_k^*}\dots D_{\Gamma_1^*}=\Gamma_k^*H_{k+1}^{1/2}C_k\dots C_1D_{*1}h.$$

If we define

$$D_{*k+1}=H_{k+1}^{1/2}C_k\dots C_1$$

we obtain

$$Z_k=\Gamma_k^*D_{*k+1},\quad k\geqslant 1.$$

In order to continue the study of the operator W, we introduce the following operators. (They have also appeared in [7] in a particular case.)

$$(2.5)_1\qquad\begin{cases}W_1=W_1(\Gamma_1):\mathscr{H}\to\mathscr{H}\\ W_1=\Gamma_1\end{cases}$$

and, for $n\geqslant 2$,

$$(2.5)_n\qquad\begin{cases}W_n=W_n(\Gamma_1,\dots,\Gamma_n):\mathscr{K}_n\to\mathscr{K}_n\\ W_n=V_{n-1}(I\oplus\Gamma_n)\end{cases}$$

where $V_{n-1}=V_{n-1}(\Gamma_1,\dots,\Gamma_{n-1})$ is defined by $(1.3)_{n-1}$.

2.2. LEMMA. *There exists the operator*

$$(2.6)\qquad\begin{cases}W_+=W_+(\mathscr{G}):\mathscr{K}_+\to\mathscr{K}_+\\ W_+=\operatorname{s-lim}_{n\to\infty}W_nP_n.\end{cases}$$

Proof. In [6] the following identity was proved:

$$V_n=\begin{pmatrix}X_n^1, & G_n^1\\ D_n^1, & K_n\end{pmatrix},\quad n\geqslant 1,$$

where D_n^1 and G_n^1 are given by $(1.15)_n$ and $(1.17)_n^1$, and

$$K_n=(-\Gamma_1^*D_{\Gamma_2^*}\dots D_{\Gamma_n^*},-\Gamma_2^*D_{\Gamma_3^*}\dots D_{\Gamma_n^*},\dots,-\Gamma_{n-1}^*D_{\Gamma_n^*},-\Gamma_n^*)^t.$$

Consequently,

$$W_n = \begin{pmatrix} X_{n-1}^1, & G_{n-1}^1 \\ D_{n-1}^1, & K_{n-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \Gamma_n \end{pmatrix} =$$

$$= \begin{pmatrix} X_{n-2}^1, & D_{\Gamma_1^*} \dots D_{\Gamma_{n-2}^*}\Gamma_{n-1}, & D_{\Gamma_1^*} \dots D_{\Gamma_{n-1}^*}\Gamma_n \\ D_{n-2}^1, & K_{n-2}\Gamma_{n-1}, & \\ 0, & D_{\Gamma_{n-1}}, & K_{n-1}\Gamma_n \end{pmatrix} =$$

$$= \begin{pmatrix} W_{n-1}, & D_{\Gamma_1^*} \dots D_{\Gamma_{n-1}^*}\Gamma_n \\ 0, \dots, D_{\Gamma_{n-1}}, & K_{n-1}\Gamma_n \end{pmatrix}.$$

Let $f \in \bigcup_{n=1}^{\infty} \mathscr{K}_n$, then there exists $p \in \mathbf{N}$ so that $f \in \mathscr{K}_p$, $f = (h_0, h_1, \dots, h_{p-1}, 0, \dots)^t$ and, for $n > p$,

$$W_n P_n f = (W_p f,\ D_{\Gamma_p} h_{p-1}, 0, \dots)^t$$

then $\{W_n P_n f\}_{n=1}^{\infty}$ is a Cauchy sequence and as $\|W_n P_n\| \leqslant 1$, there exists $\text{s-}\lim_{n\to\infty} W_n P_n$. ▨

Before proving the main property of W_+ we need a technical result.

2.3. LEMMA. *For every $n \geqslant 1$ we have the equality*

(2.7) $$P_1 W_+^n P_1 = P_1 W_n^n P_1 .$$

Proof. From the way W_+ was constructed there results

$$P_n W_+ P_n = W_n, \quad n \geqslant 1$$

and using the recursive formula for W_n obtained in Lemma 2.2, we have

$$P_{n+1} W_+ P_n = W_+ P_n, \quad n \geqslant 1.$$

Having these two relations it is easy to check the following too

$$P_{n+k} W_+^k P_n = W_+^k P_n, \quad n, k \geqslant 1.$$

Now, we can prove the desired formula:

$$P_1 W_n^n P_1 = P_1 \underbrace{(P_n W_+ P_n) \dots (P_n W_+ P_n)}_{n} P_1 =$$

$$= P_1 \underbrace{(P_n W_+) \dots (P_n W_+)}_{n-1} P_n W_+ P_1 =$$

$$= P_1 \underbrace{(P_n W_+) \dots (P_n W_+)}_{n-2} P_n W_+^2 P_1 = \dots = P_1 P_n W_+ W_+^{n-1} P_1 =$$

$$= P_1 W_+^n P_1 .$$ ▨

The main property of W_+ is an improvement of Proposition 3.4 from [7]; it connects the Fourier coefficients of F with the operator W_+.

2.4. LEMMA. *For every $n \geq 1$,*

$$S_n = P_1 W_+^n P_1 . \tag{2.8}$$

Proof. From (2.7) we have to prove that

$$S_n = P_1 W_n^n P_1, \quad n \geq 1.$$

Using the identity

$$W_n = \begin{pmatrix} X_{n-1}^1, & D_{\Gamma_1^*} \dots D_{\Gamma_{n-1}^*} \Gamma_n \\ D_{n-1}^1, & K_{n-1}\Gamma_n \end{pmatrix}$$

it results $(I, 0_{n-1}) W_n = X_n^1$.

Then we shall prove by induction the following equality:

$$W_n^{n-1}\begin{pmatrix} I \\ 0_{n-1} \end{pmatrix} = \begin{pmatrix} U_{n-2} Y_{n-1}^1 \\ D_{\Gamma_{n-1}} \dots D_{\Gamma_1} \end{pmatrix}, \quad n \geq 2.$$

For $n = 2$, the equality is immediately verified. Then,

$$W_{n+1}^n \begin{pmatrix} I \\ 0_n \end{pmatrix} = \begin{pmatrix} W_n, & * \\ 0, \dots, D_{\Gamma_n}, & * \end{pmatrix}^n \begin{pmatrix} I \\ 0_n \end{pmatrix} =$$

$$= \begin{pmatrix} W_n, & * \\ 0, \dots, D_{\Gamma_n}, & * \end{pmatrix} \begin{pmatrix} W_n^{n-1}\begin{pmatrix} I \\ 0_{n-1} \end{pmatrix} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} W_n \begin{pmatrix} U_{n-2} Y_{n-1}^1 \\ D_{\Gamma_{n-1}} \dots D_{\Gamma_1} \end{pmatrix} \\ D_{\Gamma_n} \dots D_{\Gamma_1} \end{pmatrix}.$$

But,

$$W_n \begin{pmatrix} U_{n-2} Y_{n-1}^1 \\ D_{\Gamma_{n-1}} \dots D_{\Gamma} \end{pmatrix} = V_{n-1} \begin{pmatrix} I_{n-1} & 0 \\ 0 & \Gamma_n \end{pmatrix} \begin{pmatrix} U_{n-2} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} Y_{n-1}^1 \\ D_{\Gamma_{n-1}} \dots D_{\Gamma} \end{pmatrix} =$$

$$= V_{n-1} \begin{pmatrix} U_{n-2} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I_{n-1} & 0 \\ 0 & \Gamma_n \end{pmatrix} \begin{pmatrix} Y_{n-1}^1 \\ D_{\Gamma_{n-1}} \dots D_{\Gamma_1} \end{pmatrix} = U_{n-1} Y_n^1 .$$

Now, in order to conclude the proof, we have:

$$P_1 W_n^n P_1 = P_1 \begin{pmatrix} I & 0 \\ 0 & 0_{n-1} \end{pmatrix} W_n W_n^{n-1} \begin{pmatrix} I & 0 \\ 0 & 0_{n-1} \end{pmatrix} P_1 =$$

$$= X_{n-1}^1 U_{n-2} Y_{n-1}^1 + D_{\Gamma_1^*} \dots D_{\Gamma_{n-1}^*} \Gamma_n D_{\Gamma_{n-1}} \dots D_{\Gamma_1} = S_n .$$

▨

Let E be the spectral measure of W. We can now state the main result of this section.

2.5. THEOREM. *E is the Naimark dilation of F.*

Proof. It is easy to notice that $W_+ = W|\mathscr{K}_+$, consequently, from Lemma 2.4 it results that E is a dilation of F. Thus, we have only to prove the minimality of E.

First, let us consider $f = (h_0, h_1, \ldots)^t \in \mathscr{K}_+$, $f \perp W^n\mathscr{H}$, $n \in \mathbf{N}$; but,

$$W_+^n h = (\ldots, *, \ldots, *, \underset{\underset{n+1}{\uparrow}}{D_{\Gamma_n}} \ldots D_{\Gamma_1} h, 0, \ldots, 0, \ldots)^t, \quad h \in \mathscr{H}$$

then, we obtain $h_n = 0$, $n \in \mathbf{N}$ so

$$\mathscr{K}_+ = \bigvee_{n=0}^{\infty} W_+^n \mathscr{H}.$$

Further on, for $f \in \mathscr{K}, f \perp W^n\mathscr{H}$, $n \in \mathbf{Z}$, if $f = (\ldots h_{-1}, h_0, h_1 \ldots)^t$, we obtain $h_n = 0$, $n \geqslant 0$. As

$$W^{*n}h = (0, \overset{\overset{n}{\downarrow}}{D_{*1}h}, *, *, \ldots)^t$$

it also results $h_n = 0, n < 0$ so $\mathscr{K} = \bigvee_{n=-\infty}^{\infty} W^n\mathscr{H}$ and this means that E is the minimal spectral dilation of F. ▨

2.6. COROLLARY. *If $\mathscr{K}_+ = M_+(\mathscr{L}_+) \oplus \mathscr{R}_+$ is the Wold decomposition of W_+ then $\mathscr{L}_+ = W(\ldots 0 \oplus \mathscr{D}_* \oplus \underset{\sim}{0} \oplus 0 \ldots)$.* ▨

2.7. REMARK. We must also notice that W is the unitary extension of the isometry W_+, in the sense that $\mathscr{K} = M(\mathscr{L}_+) \oplus \mathscr{R}_+$. ▨

§ 3

In this section we indicate a consequence of Theorem 2.5, namely, we compute the prediction-error operator of a stationary process. In order to obtain this, we briefly remind the elements involved by a stationary process (the papers [12], [13] are basic for this subject; for the extension to the operatorial case, see [10]).

One shall take into account a semispectral measure on the Hilbert space $\mathscr{H}$ (the parameter space),

$$F : C(\mathbf{T}) \to \mathscr{L}(\mathscr{H})$$

uniquely determined by the process, the minimal spectral dilation E of F on the Hilbert space $\mathscr{K}$ (the measuring space) and W the unitary operator associated to the spectral measure E.

Having these elements, if $F(1) = I$, the process can be represented as a sequence of operators, $\mathcal{V} = \{\mathcal{V}_n\}_{n=-\infty}^{\infty}, \mathcal{V}_n \in \mathcal{L}(\mathcal{H}, \mathcal{K})$

$$\mathcal{V}_n h = W^{*n} h, \quad h \in \mathcal{H}. \tag{3.1}$$

If we define

$$\mathcal{K}_+ = \bigvee_{n=0}^{\infty} W^n \mathcal{H}$$

then, the prediction-error operator of the process $\mathcal{V} = \{\mathcal{V}_n\}_{n=-\infty}^{\infty}$ is defined as follows:

$$\Delta(\mathcal{V}) = P_{\mathcal{H}}^{\mathcal{K}} W(I - P_{\mathcal{K}_+}^{\mathcal{K}}) W^* |\mathcal{H} \tag{3.2}$$

(we do not forget the unique ness of the Naimark dilation).

Let us consider $\mathcal{G} = \{\Gamma_n\}_{n=1}^{\infty}$ the choice sequence associated to the semi-spectral measure F of the process $\mathcal{V}$. Then, we can compute the prediction-error operator in a simple manner.

3.1. PROPOSITION. *We have the equality:*

$$\Delta(\mathcal{V}) = H_1 \tag{3.3}$$

where H_1 *is given by* $(1.18)_1$.

Proof. In fact, using Theorem 2.5, the proof consists in an obvious computation:

$$\Delta(\mathcal{V})h = (P_{\mathcal{H}}^{\mathcal{K}} W(I - P_{\mathcal{K}_+}^{\mathcal{K}}) W^* |\mathcal{H})h = P_{\mathcal{H}}^{\mathcal{K}} W(I - P_{\mathcal{K}_+}^{\mathcal{K}})(0, D_{*1}h, (X_\infty^1)^* h)^t =$$

$$= P_{\mathcal{H}}^{\mathcal{K}} W(0, \ldots, D_{*1}h, \boxed{0}, 0, \ldots)^t = P_{\mathcal{H}}^{\mathcal{K}}(\ldots 0, \boxed{D_{*1}^2 h}, *)^t = D_{*1}^2 h = H_1 h.$$ ▨

Let us connect this fact with some classical results. In [12], the matricial form (the case when $\mathcal{H}$ has finite dimension) of the Szegö formula is obtained:

$\Delta(\mathcal{V})$ is invertible if and only if $\log\det \frac{dF}{dt} \in L^1(\mathbf{T})$ and $\det \Delta(\mathcal{V}) = G(F)$,

where $G(F) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log\det \frac{dF}{dt}(t)\,dt\right)$ is the geometrical mean of F.

But, as $H_1 = \operatorname{s-lim}_{n\to\infty} G_n^1 (G_n^1)^*$ and, in matricial case,

$$\det G_n^1 (G_n^1)^* = \prod_{k=1}^{n} \det D_{\Gamma_n^*}^2$$

it results that

$$\det H_1 = \prod_{n=1}^{\infty} \det D^2_{\Gamma_n^*} .$$

So, from Proposition 3.1, we obtain the following equality:

$$\prod_{n=1}^{\infty} \det D^2_{\Gamma_n^*} = \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \log\det \frac{dF}{dt}(t)\, dt\right). \tag{3.4}$$

There is also a connection between choice sequences and orthogonal polynomials. In the scalar case, let $\{\varphi_n\}_{n=1}^{\infty}$ denote the orthogonal polynomials associated with the measure F,

$$\varphi_n(z) = k_n \cdot z^n + \ldots$$

and define $\Phi_n(z) = \dfrac{\varphi_n(z)}{k_n}$, $c_n = -\overline{\Phi_n(0)}$.

Then, the following Verblunsky's formula holds (see [8]):

$$\exp\left(\frac{1}{2\pi} \int_0^{2\pi} \log \frac{dF}{dt}(t)\, dt\right) = \prod_{n=1}^{\infty} (1 - |c_n|^2).$$

On the other hand, in [7] we considered the polynomials $p_n(z) = \det(z - W_n^*)$. A lengthly computation shows that $p_n = \Phi_n$, $n \in \mathbf{N}$, so, $\gamma_n = c_n$, $n \in \mathbf{N}$, where $\{\gamma_n\}_{n=1}^{\infty}$ is the choice sequence associated with F. Thus, the scalar case of (3.4) is a variant of classical Verblunsky's formula.

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