OPERATOR THEORY
12(1984), 213-219

ARE COMMUTING SYSTEMS OF DECOMPOSABLE OPERATORS DECOMPOSABLE ?

JÖRG ESCHMEIER

The notion of decomposability has been introduced for a single continuous linear operator on a complex Banach space by C. Foiaş in [9]. Later it was generalized to the case of several commuting operators by Ş. Frunză in [11].

In the sense of Ş. Frunză a commuting system $a = (a_1, \ldots, a_N)$ of continuous linear operators on a complex Banach space X is called decomposable, if there is a spectral capacity E for a, i.e. a map $E: F(\mathbf{C}^N) \to \mathrm{Inv}(a)$ from the system of all closed subsets of $\mathbf{C}^N$ into the set of all closed subspaces of X invariant under $a_1, \ldots, a_N$, which satisfies

(i) $E(\emptyset) = \{0\},\ E(\mathbf{C}^N) = X$;

(ii) $E(\bigcap_{i\in I} F_i) = \bigcap_{i\in I} E(F_i)$ for each family $(F_i)_{i\in I}$ in $F(\mathbf{C}^N)$;

(iii) $X = E(\overline{U}_1) + \ldots + E(\overline{U}_n)$ for each open cover $\mathbf{C}^N = U_1 \cup \ldots \cup U_n$;

(iv) $\sigma(a, E(F)) \subset F$ for each $F \in F(\mathbf{C}^N)$.

Here $\sigma(a, E(F))$ denotes the Taylor spectrum (see [17]) of a restricted to $E(F)$. If $a = (a_1, \ldots, a_N)$ is decomposable, its spectral capacity is given by ([10], [11])

$$E(F) = X_a(F) = \{x \in X;\ \sigma_a(x) \subset F\}.$$

Here $\sigma_a(x)$ denotes the local spectrum of a relative to $x \in X$, i.e. the smallest closed subset of $\mathbf{C}^N$ such that on the complement x is locally of the form

$$x = (\lambda_1 - a_1)f_1(\lambda) + \ldots + (\lambda_N - a_N)f_N(\lambda)$$

with analytic X-valued functions $f_1, \ldots, f_N$ (see [1], [11], [6]).

If $a = (a_1, \ldots, a_N)$ is decomposable with spectral capacity E, then due to the projection property of the Taylor spectrum each a_i, $1 \leqslant i \leqslant N$, is decomposable with spectral capacity

$$E_i(F) = E(\pi_i^{-1}(F)), \quad F = \overline{F} \subset \mathbf{C},$$

where $\pi_i: \mathbf{C}^N \to \mathbf{C}$ denotes the projection of $\mathbf{C}^N$ onto its i-th component. One of the most natural questions concerns the converse of this statement. Is it true, that each commuting system $a = (a_1, \ldots, a_N)$ with decomposable components a_i, $1 \leqslant i \leqslant N$, is decomposable? This question was formulated by E. Albrecht, Ş. Frunză and F.-H. Vasilescu in [3].

We are going to prove that there is a Banach space X and a non-decomposable system $a = (a_1, a_2)$ consisting of two commuting decomposable operators a_1, a_2 on X. The construction is based on a result concerning the solvability of the $\bar{\partial}$-equation with uniform bounds on strictly pseudoconvex domains. For an intense study of this and related questions we refer the reader to Grauert—Lieb [12], Henkin [13], Lieb [14], Øvrelid [15] and Range—Siu [16].

PRELIMINARIES

If U is an open set in $\mathbf{C}^N$, we denote by $C(U)$ $(C_b(U))$ the space of all continuous (bounded continuous) complex valued functions on U, by $C^\infty(U)$ $(C_0^\infty(U))$ the space of all complex valued C^∞-functions on U (C^∞-functions with compact support in U) and by $\mathfrak{A}(U)$ $(\mathfrak{A}_b(U))$ the space of all complex valued analytic (bounded analytic) functions on U.

If B is one of these function spaces, $\Lambda^q(\mathrm{d}\bar{z}, B)$ stands for the space of all forms $u = \sum_{|I|=q} u_I \mathrm{d}\bar{z}_I$ of degree q in $\mathrm{d}\bar{z}_1, \ldots, \mathrm{d}\bar{z}_N$ having coefficients in B.

If $u \in \Lambda^q(\mathrm{d}\bar{z}, C(U))$ and $v \in \Lambda^{q+1}(\mathrm{d}\bar{z}, C(U))$, we write $\bar{\partial} u = v$, provided this equality holds in the distribution sense, that means

$$\sum_{\rho=1}^{q+1} (-1)^{\rho} \int_U u_{i_1 \ldots \hat{i}_\rho \ldots i_{q+1}} (\bar{\partial}_{i_\rho} \varphi)\, \mathrm{d}z = \int_U v_{i_1 \ldots i_{q+1}} \varphi \, \mathrm{d}z$$

for all increasing sequences $1 \leqslant i_1 < \ldots < i_{q+1} \leqslant N$ and all $\varphi \in C_0^\infty(U)$. Here the integral is the ordinary Lebesgue integral, $\bar{\partial}_i$ stands for $\partial/\partial\bar{z}_i$ and the circumflex in $i_1 \ldots \hat{i}_\rho \ldots i_{q+1}$ means that the index i_ρ has to be omitted.

For $u \in \Lambda^q(\mathrm{d}\bar{z}, C_b(U))$ we define

$$\|u\|_U = \max_{|I|=q} \sup_{z \in U} |u_I(z)|.$$

By $B^q(U)$ we denote the space of all forms $u \in \Lambda^q(\mathrm{d}\bar{z}, C_b(U))$ such that $\bar{\partial} u \in \Lambda^{q+1}(\mathrm{d}\bar{z}, C_b(U))$ in the sense explained above. The dominated convergence theorem shows that $B^q(U)$ equipped with the norm $\|u\| = \|u\|_U + \|\bar{\partial} u\|_U$ is a Banach space. For $q = N$ we get the appropriate definition, if we set $\bar{\partial} u = 0$ for each $u \in \Lambda^N(\mathrm{d}\bar{z}, C(U))$. Without proof we state the following important result.

THEOREM 1. *If U is a bounded strictly pseudoconvex domain in $\mathbf{C}^N$ with C^∞-boundary, then*

$$0 \to \mathfrak{A}_b(U) \xrightarrow{i} B^0(U) \xrightarrow{\bar{\partial}} B^1(U) \xrightarrow{\bar{\partial}} \ldots \xrightarrow{\bar{\partial}} B^N(U) \to 0$$

is an exact sequence of continuous linear operators between Banach spaces.

Here i simply denotes the embedding of $\mathfrak{A}_b(U)$ into $B^0(U)$.

It is well-known that $\mathfrak{A}_b(U) = \{u \in B^0(U);\ \bar{\partial} u = 0\}$. In [15] Øvrelid constructs an integral operator

$$T_q\colon B^q(U) \to \Lambda^{q-1}(\mathrm{d}\bar{z}, C_b(U))$$

for $q \geqslant 1$ such that with some constant C_q

$$u = C_q[\bar{\partial} T_q u - T_{q+1}\bar{\partial} u]$$

holds for all $u \in B^q(U)$ (see Proposition 6.2 and 6.4 of [15]).

For another proof of Theorem 1 see Range—Siu, Theorem 3.9 of [16].

If u and $\bar{\partial} u$ are forms on U having continuous functions as coefficients and θ is a form of degree r on U with C^∞-functions as coefficients, then a routine computation shows that

$$\bar{\partial}(\theta \wedge u) = (\bar{\partial}\theta) \wedge u + (-1)^r \theta \wedge (\bar{\partial} u)$$

holds in the distribution sense. Hence $\theta \wedge u$ and $\bar{\partial}(\theta \wedge u)$ both have continuous functions as coefficients.

THE EXAMPLE

From now on let $U \subset \mathbf{C}^2$ be a fixed bounded strictly pseudoconvex domain with C^∞-boundary such that $\bar{D}_1(0) = \{(z_1, z_2);\ \max(|z_1|, |z_2|) \leqslant 1\} \subset U$. We define a commuting system $b = (b_1, b_2)$ of continuous linear operators on the Banach space $B^1(U)$ by

$$u = u_1 \mathrm{d}\bar{z}_1 + u_2 \mathrm{d}\bar{z}_2 \to \pi_i u = (\pi_i u_1)\mathrm{d}\bar{z}_1 + (\pi_i u_2)\mathrm{d}\bar{z}_2, \quad i = 1, 2.$$

Then

$$\Phi\colon C^\infty(\mathbf{C}^2) \to L(B^1(U)), \quad \Phi(\theta)u = \theta u$$

defines a $C^\infty(\mathbf{C}^2)$-functional calculus for b in the sense of [2]. In particular, b is decomposable with spectral capacity

$$E(F) = \bigcap(\operatorname{Ker}\Phi(\theta);\ \operatorname{supp}(\theta) \cap F = \emptyset) \quad \text{(Theorem 4 of [2]).}$$

Furthermore, b has the single valued extension property (Theorem 3.1 of [11]). Since on the other hand $E(F) = \{u \in B^1(U);\ \sigma_b(u) \subset F\}$, it is easy to deduce that

$$\sigma_b(u) = \operatorname{supp}(u)$$

for $u \in B^1(U)$. Let us give a more direct proof for the non-trivial inclusion $\sigma_b(u) \subset \operatorname{supp}(u)$. For $\lambda^0 \notin \operatorname{supp}(u)$ we choose functions $\theta_1, \theta_2 \in C^\infty(\mathbf{C}^2)$ and $r > 0$ such that $\theta_1 + \theta_2 = 1$ on an open neighbourhood of $\operatorname{supp}(u)$ and

$$\operatorname{supp}(\theta_i) \subset \{z \in \mathbf{C}^2\ ;\ |z_i - \lambda_i^0| > r\}$$

for $i = 1, 2$. It follows that

$$u = (\lambda_1 - b_1)\Phi\left(\frac{\theta_1}{\lambda_1 - \pi_1}\right)u + (\lambda_2 - b_2)\Phi\left(\frac{\theta_2}{\lambda_2 - \pi_2}\right)u$$

for $\lambda \in D_r(\lambda^0) = D_r(\lambda_1^0) \times D_r(\lambda_2^0)$. One way to prove that

$$\lambda \to \Phi\left(\frac{\theta_i}{\lambda_i - \pi_i}\right) \quad (i = 1, 2)$$

is analytic near λ^0 is to develop this map into a power series with center λ^0 converging in $L(B^1(U))$:

$$\Phi\left(\frac{\theta_i}{\lambda_i - \pi_i}\right) = \Phi\left(\frac{\theta_i}{\lambda_i^0 - \pi_i}\right) \sum_{n=0}^{\infty} \Phi\left(\frac{x_i}{\lambda_i^0 - \pi_i}\right)^n (\lambda_i^0 - \lambda_i)^n.$$

Here x_i is an arbitrary element of $C^\infty(\mathbf{C}^2)$ satisfying $x_i = 1$ on $\operatorname{supp}(\theta_i)$ and $\operatorname{supp}(x_i) \subset \{z \in \mathbf{C}^2 ; |z_i - \lambda_i^0| > r\}$.

Due to Theorem 1 the Banach spaces

$$X = B^0(U)/\mathfrak{A}_b(U) \quad \text{and} \quad Y = \{u \in B^1(U) \; ; \; \bar{\partial} u = 0\}$$

are topologically isomorphic. A topological isomorphism is given by $[u] \to \bar{\partial} u$, where $[u]$ denotes the equivalence class represented by $u \in B^0(U)$. The commuting system $a = (a_1, a_2)$ defined by

$$a_i : X \to X, \quad a_i[u] = [\pi_i u], \quad i = 1, 2,$$

is similar to the restriction of b to Y.

LEMMA 1. *The commuting system $a = (a_1, a_2)$ has the single valued extension property.*

Proof. For the definition of the single valued extension property in case of N-tuples we refer the reader to [11], [19], [7]. Since a is similar to $b \mid Y$, it is sufficient to prove that the single valued extension property is inherited from b to $b \mid Y$. This easily follows from the observation that for $i = 1, 2$ and $\lambda \in \mathbf{C}$ each element $u \in B^1(U)$ satisfying $(\lambda - b_i)u \in Y$ belongs to Y.

LEMMA 2. *For each $u \in B^0(U)$ we have*

$$\sigma_a([u]) = \operatorname{supp}(\bar{\partial}_1 u) \cup \operatorname{supp}(\bar{\partial}_2 u).$$

Proof. Due to similarity, it is sufficient to show that

$$\sigma_{b|Y}(u) = \sigma_b(u)$$

for all $u \in Y$. Since the inclusion $\sigma_b(u) \subset \sigma_{b|Y}(u)$ is obvious and since $\sigma_b(u) = \operatorname{supp}(u)$, it suffices to show that

$$\sigma_{b|Y}(u) \subset \operatorname{supp}(u)$$

for $u \in Y$. Now fix $u \in Y$ and let $\lambda^0 \notin \text{supp}(u)$. We choose $\theta_1, \theta_2, x_1, x_2$ as above and define an analytic function g near λ^0 with values in $B^2(U)$ by

$$g(\lambda) = \frac{u \wedge \bar{\partial}\theta_1}{(\lambda_1 - \pi_1)(\lambda_2 - \pi_2)} = \frac{u_1\bar{\partial}_2\theta_1 - u_2\bar{\partial}_1\theta_1}{(\lambda_1 - \pi_1)(\lambda_2 - \pi_2)} \, d\bar{z}_1 \wedge d\bar{z}_2 .$$

If we identify $B^2(U)$ with the Banach algebra $C_b(U)$, then g can be represented as a power series converging in $C_b(U)$ near λ^0

$$g(\lambda) = g(\lambda^0) \sum_{n \in \mathbf{N}^2} \left(\frac{x_1}{\lambda_1 - \pi_1}\right)^{n_1} \left(\frac{x_2}{\lambda_2 - \pi_2}\right)^{n_2} (\lambda^0 - \lambda)^n .$$

Using Theorem 1 and Theorem 2.2 of [17] we get an analytic function f with values in $B^1(U)$ defined in a neighbourhood W of λ^0 and satisfying $g(\lambda) = \bar{\partial} f(\lambda)$ for $\lambda \in W$. For $\{i, j\} = \{1, 2\}$

$$h_i : W \to B^1(U), \quad h_i(\lambda) = \Phi\left(\frac{\theta_i}{\lambda_i - \pi_i}\right) u + (-1)^{i-1}(\lambda_j - b_j) f(\lambda)$$

is an analytic function such that $u = (\lambda_1 - b_1)h_1(\lambda) + (\lambda_2 - b_2)h_2(\lambda)$ on W. Since u belongs to Y, we have for $\lambda \in W$ and $\{i, j\} = \{1, 2\}$

$$\bar{\partial} h_i(\lambda) = \bar{\partial}\left(\frac{\theta_i}{\lambda_i - \pi_i} u\right) + (-1)^{i-1}(\lambda_j - \pi_j) g(\lambda) = \frac{\bar{\partial}\theta_i \wedge u}{\lambda_i - \pi_i} + \frac{u \wedge \bar{\partial}\theta_i}{\lambda_i - \pi_i} = 0.$$

Thus we have shown that $\lambda^0 \notin \sigma_{b|Y}(u)$.

As easy consequences of Lemma 2 we obtain.

COROLLARY 1. *All spectral subspaces of a, i.e. all spaces*

$$X_a(F) = \{x \in X \, ; \, \sigma_a(x) \subset F\}, \quad F = \bar{F} \subset \mathbf{C}^2,$$

are closed.

COROLLARY 2. *Both components a_i of $a = (a_1, a_2)$ are decomposable.*

Proof. Theorem 2.1 of [7] shows that for $F = \bar{F} \subset \mathbf{C}$

$$X_{a_i}(F) = X_a(\pi_i^{-1}(F)) = \{[u] \, ; \, u \in B^0(U) \ \text{ with } \ \pi_i(\text{supp}(\bar{\partial}u)) \subset F\}.$$

Since a_i has the single valued extension property and $B^0(U)$ admits partitions of unity, it easily follows that $E_i(F) = X_{a_i}(F)$ defines a spectral capacity for a_i (see also [4], Chapter I, Proposition 3.8).

THEOREM 2. *The commuting system $a = (a_1, a_2)$ is not decomposable.*

Proof. Choose $u \in B^0(U)$ with $u=0$ on $D_{1/2}(0) = \{(z_1, z_2) \in \mathbf{C}^2; \max(|z_1|, |z_2|) < 1/2\}$, $u = 1$ on $U \setminus D_1(0)$ and define $F = \bar{D}_1(0) \setminus D_{1/2}(0)$. Then $[u] \in X_a(F)$. Let us suppose that a is decomposable.

In this case we would have $\sigma(a, X_a(F)) \subset F$. In particular, we could choose $v \in \mathfrak{A}_b(U)$, $[u_1]$, $[u_2] \in X_a(F)$ with

$$(*) \qquad u - v = \pi_1 u_1 + \pi_2 u_2 .$$

Let us denote by $\tilde{u}_1$, $\tilde{u}_2$ the analytic extensions of $u_1 \mid U \setminus \bar{D}_1(0)$, $u_2 \mid U \setminus \bar{D}_1(0)$ onto U. On the one hand $1 - v = \pi_1 \tilde{u}_1 + \pi_2 \tilde{u}_2$ implies $v(0) = 1$, on the other hand $(*)$ implies $v(0) = 0$.

Hence a cannot be decomposable.

The proof of Theorem 2 is based on the fact that unlike the 1-dimensional case it may happen for N-tuples that the spectrum of a restricted to one of its spectral subspaces $X_a(F)$ is not contained in F, even if $X_a(F)$ is closed and a has the single valued extension property.

REFERENCES

1. ALBRECHT, E., Funktionalkalküle in mehreren Veränderlichen für stetige lineare Operatoren auf Banachräumen, *Manuscripta Math.*, **14**(1974), 1–40.
2. ALBRECHT, E.; FRUNZĂ, Ş., Non-analytic functional calculi in several variables, *Manuscripta Math.*, **18**(1976), 327–336.
3. ALBRECHT, E.; FRUNZĂ, Ş.; VASILESCU, F.-H., Some problems in the theory of decomposable operators, in *Topics in modern operator theory*, pp. 329–330, Birkhäuser Verlag, 1981.
4. COLOJOARĂ, I.; FOIAŞ, C., *Theory of generalized spectral operators*, New York, Gordon and Breach, 1968.
5. ESCHMEIER, J., Operator decomposability and weakly continuous representations of locally compact abelian groups, *J. Operator theory*, **7**(1982), 201–208.
6. ESCHMEIER, J., On two notions of the local spectrum for several commuting operators, submitted for publication.
7. ESCHMEIER, J., Local properties of Taylor's analytic functional calculus, *Invent. Math.*, to appear.
8. ESCHMEIER, J., Equivalence of decomposability and 2-decomposability for several commuting operators, submitted for publication.
9. FOIAŞ, C., Spectral maximal spaces and decomposable operators in Banach spaces, *Arch. Math.*, **14**(1963), 341–349.
10. FOIAŞ, C., Spectral capacities and decomposable operators, *Rev. Roumaine Math. Pures Appl.*, **13**(1968), 1539–1545.
11. FRUNZĂ, Ş., The Taylor spectrum and spectral decompositions, *J. Functional Analysis*, **19**(1975), 390–421.
12. GRAUERT, H.; LIEB, I., Das Ramirezsche Integral und die Lösung der Gleichung $\bar{\partial} f = \alpha$ im Bereich der beschränkten Formen, *Rice Univ. Studies*, **56**(1970), 26–50.
13. HENKIN, G. M., Integral representations in strictly pseudoconvex domains and applications to the $\bar{\partial}$-problem (Russian), *Mat. Sb.*, **82**(1970), 300–308; English translation, *Math. USSR Sb.*, **11**(1970), 273–281.
14. LIEB, I., Die Cauchy-Riemannschen Differentialgleichungen auf streng pseudokonvexen Gebieten. I, *Math. Ann.*, **190**(1970), 6–44; II, *Math. Ann.*, **199**(1972), 2–256.

15. ØVRELID, N., Integral representation formulas and L^p-estimates for the $\bar{\partial}$-equation, *Math. Scand.*, **29**(1971), 137–160.
16. RANGE, R. M.; SIU, Y. T., Uniform estimates for the $\bar{\partial}$-equation on domains with piecewise smooth strictly pseudoconvex boundaries, *Math. Ann.*, **206**(1973), 325–354.
17. TAYLOR, J. L., A joint spectrum for several commuting operators, *J. Functional Analysis*, **6**(1970), 172–191.
18. TAYLOR, J. L., The analytic functional calculus for several commuting operators, *Acta Math.*, **125**(1970), 1–38.
19. VASILESCU, F.-H., *Calcul funcţional analitic multidimensional*, Bucureşti, Editura Academiei, 1979.

JÖRG ESCHMEIER
Mathematisches Institut,
Universität Münster,
Einsteinstraße 64,
4400 Münster,
West Germany.

Received October 8, 1982.