J. OPERATOR THEORY
12(1984), 319–329

SIMILARITY OF SMOOTH TOEPLITZ OPERATORS

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1. INTRODUCTION

Let $\mathbf{T}$ denote the unit circle $|z| = 1$ in the complex plane $\mathbf{C}$, and $\mathrm{d}m(z) = (1/2\pi)\mathrm{d}t$ normalized Lebesgue measure on $\mathbf{T}$. Let L^2 be the Lebesgue space of (equivalence classes of) square integrable (with respect to $\mathrm{d}m(z)$) functions on $\mathbf{T}$, and H^2 the L^2-closure of polynomials. For a bounded $\mathrm{d}m(z)$-measurable function g, the associated Toeplitz operator $T_g : H^2 \to H^2$ is defined by $T_g h = \mathbf{P}(gh)$, where $\mathbf{P}$ is the orthogonal projection of L^2 onto H^2. The function g is called the symbol of T_g.

In this paper we shall be interested in the case when the symbol is smooth and its negative Fourier coefficients decay exponentially. More precisely, let $J^{(n)}$, $n \geqslant 1$, denote the set of all functions $g \in C^{(n)}(\mathbf{T})$ with Fourier series $g(\mathrm{e}^{\mathrm{i}t}) \sim \sum_{k=-\infty}^{\infty} a_k \mathrm{e}^{\mathrm{i}kt}$ satisfying $|a_{-k}| \leqslant cr^k$ for $k = 1, 2, 3, \ldots$, and positive constants c and r, $r < 1$. It is clear that $J^{(n)} \subset J^{(m)}$ if $n \geqslant m$. We obtain the following:

THEOREM 1. *If $F \in J^{(1)}$, F is one-to-one and F' never vanishes on $\mathbf{T}$, then T_F does not have an eigenvalue on the boundary of the spectrum $\sigma(T_F)$ of T_F. If in addition, $t \mapsto F(\mathrm{e}^{\mathrm{i}t})$ is orientation preserving, then T_F has no eigenvalues.*

THEOREM 2. *If $F \in J^{(4)}$, F is one-to-one, F' never vanishes on $\mathbf{T}$ and $t \mapsto F(\mathrm{e}^{\mathrm{i}t})$ is orientation preserving, then T_F is similar to an analytic Toeplitz operator T_τ, where τ is a Riemann mapping function of $|z| < 1$ onto the interior of the curve $F(\mathbf{T})$.*

Note that, with proper orientation, every function analytic and one-to-one in a neighborhood of $\mathbf{T}$ satisfies the hypotheses of the above two theorems. The origin of our similarity theorem (Theorem 2) dates bach to a paper of P. L. Duren [9]. There Duren obtained Corollary 1 of Section 4 below under the assumption that $F(z) = \beta z + \gamma/z$, with $|\beta| > |\gamma|$. In her dissertation [10] J. H. Morrel proved Theorem 2 for F a trigonometric polynomial. Subsequently, D. N. Clark and J. H. Morrel [5] obtained the same similarity theorem under the assumptions that F is rational with poles off $\mathbf{T}$ and is one-to-one on some closed annulus $0 < s \leqslant z \leqslant 1$. For a further (rational) generalization, see [3] and [4].

Call a function g, defined on a set $B \subset \mathbf{C}$, differentiable on B if for every $\omega \in B$ the limit quotient $[g(z) - g(\omega)]/(z - \omega)$ as z tends to ω, $z \in B$, exists. The limit function is called the derivative of g. We define the higher derivatives of g in a similar way. A function defined on B is said to be $C^{(n)}$ on B in case its n-th derivative is continuous on B. If B is a closed set, our definition of "$C^{(1)}$ on B" does not imply analyticity on B, as the latter means $C^{(1)}$ in some open set containing B. A function g defined on $\mathbf{T}$ is jn $J^{(n)}$ if and only if it can be extended to be $C^{(n)}$ on some closed annulus $0 < t \leqslant |z| \leqslant 1$. Furthermore, g is one-to-one and g' never vanishes on $\mathbf{T}$ if and only if its extension has non-vanishing derivative on $\mathbf{T}$ and is one-to-one on some annulus $0 < t < s \leqslant |z| \leqslant 1$. It is in this context that we have proved our theorem, patterned after the proof in [5].

Using the fact that every Cauchy kernel $C_\omega = 1/(1 - \bar{\omega}z)$, $|\omega| < 1$, is an eigenvector with eigenvalue $\bar{\tau}(\omega)$ for the adjoint of an analytic Toeplitz operator T_τ, it can be shown that the invertible operator $L: H^2 \to H^2$ implementing the similarity, (satisfying $LT_F = T_\tau L$), is of the form $(Lg)(\omega) = \langle g, h_{\bar{\tau}(\omega)} \rangle$, where $|\omega| < 1$, $g \in H^2$ and h_λ is an eigenvector for T_F^* with eigenvalue λ, and $\langle \cdot, \cdot \rangle$ is the inner product defined on H^2. In Section 2 we study eigenvectors of Toeplitz operators with smooth symbols. In particular we obtain an explicit formula for the eigenvectors. In Section 3 we introduce two special Toeplitz operators V_λ and S_λ. The introduction of these operators is the principal novel idea in this paper. They enable us to study $T_F - \lambda$ and $T_F^* - \lambda$ when λ is near or on the boundary of $\sigma(T_F)$ and $\sigma(T_F^*)$. Theorem 1 is proved here with the aid of V_λ. We also study the null vector k_λ of S_λ and obtain a suitable decomposition for h_λ, via a corresponding decomposition for k_λ, to pave the way for the proof of Theorem 2. In the earlier work [5], factorization of the rational function $F(z) - \lambda$ was used to obtain the corresponding results about h_λ. Finally, we prove Theorem 2 and state its consequences in Section 4.

As for notations, the bar in $\bar{g}, \bar{z}$, etc., denotes complex conjugation. The topological closure and interior of a set B other than a curve are denoted $\operatorname{cl} B$ and $\operatorname{int} B$, respectively. For convenience, we shall write $g(\mathbf{T})$ for the curve $t \mapsto g(e^{it})$. A point z is in the interior of $g(\mathbf{T})$ if the winding number of $g(\mathbf{T})$ about z is not zero. For an integrable function g, $\tilde{g}$ or $g^{\sim}$ is its (harmonic) conjugate function.

2. EIGENVECTORS OF SMOOTH TOEPLITZ OPERATORS

In this section we study eigenvectors of smooth Toeplitz operators. For the general theory of Toeplitz operators, the reader is referred to [8] and [12]. In particular, the following two facts will be used freely throughout this paper:

1) For a Toeplitz operator T_g, $g \not\equiv 0$, either $\operatorname{Ker} T_g = \{0\}$ or $\operatorname{Ker} T_g^* = \{0\}$.

2) For $g \in C(\mathbf{T})$, T_g is Fredholm if and only if g does not vanish on $\mathbf{T}$ and in this case $\operatorname{ind} T_g$ is equal to the negative of the winding number of the curve $t \mapsto g(e^{it})$ about 0.

Thus if $f \in C^{(n)}(\mathbf{T})$, $n \geqslant 1$, and $t \mapsto f(e^{it})$ is an orientation reversing simple closed curve, $T_{f-\lambda}$ is Fredholm of index 1 for every λ in the interior of $f(\mathbf{T})$. Hence every λ in the interior of $f(\mathbf{T})$ is a simple eigenvalue for T_f.

First we obtain an explicit formula for the eigenvector h_λ for T_f with eigenvalue λ satisfying $h_\lambda(0) = 1$. It turns out that h_λ is in $(H^\infty)^{-1}$ and the function $\lambda \mapsto h_\lambda$ is an analytic H^2-valued function. Here H^∞ is the algebra of bounded analytic function in the open unit disk $\mathbf{D}$ and $(H^\infty)^{-1}$ is the group of invertible elements in H^∞.

PROPOSITION 2.1. *Suppose $f \in C^{(n)}(\mathbf{T})$, $n \geqslant 1$, $0 \notin f(\mathbf{T})$ and T_f is Fredholm of index* 1. *Then every null vector is of the form*

$$h(e^{it}) = b/\left\{y(e^{it}) \exp \frac{1}{2} i[\psi(e^{it}) + i\tilde{\psi}(e^{it})]\right\},$$

where

$$y = \exp \frac{1}{2}[\log|f| + i(\log|f|)^\sim],$$

ψ is such that

$$f(e^{it}) = |f(e^{it})| \exp i[\psi(e^{it}) - t],$$

and b is a constant. Furthermore, $h \in C^{(n-1)}(\mathbf{T})$ and if h is not the zero vector $(h \not\equiv 0)$, then $h \in (H^\infty)^{-1}$.

Proof. Let $y = \exp \frac{1}{2}[\log|f| + i(\log|f|)^\sim]$. Clearly $y\bar{y} = |f|$ and $y \in (H^\infty)^{-1}$. Since $\log|f| \in C^{(n)}(\mathbf{T})$, $(\log|f|)^\sim \in C^{(n-1)}(\mathbf{T})$, [13, page 121]. We have $y \in C^{(n-1)}(\mathbf{T})$. Let $\psi(e^{it})$ be a $C^{(n)}$-determination of the argument of $e^{it}f(e^{it})$. Then

$$f = |f| e^{-it} e^{i\psi} = y\left[\exp \frac{1}{2} i(\psi + i\tilde{\psi})\right] e^{-it} \bar{y} \exp\left[\frac{1}{2} i(\psi - i\tilde{\psi})\right].$$

Since $\tilde{\psi} \in C^{(n-1)}(\mathbf{T})$, $\exp \frac{1}{2} i(\psi + i\tilde{\psi}) \in (H^\infty)^{-1}$ and $\exp \frac{1}{2} i(\psi - i\tilde{\psi}) \in \bar{H}^\infty = \{\bar{h} \mid h \in H^\infty\}$. Thus $f \cdot \left\{1/\left[y \exp \frac{1}{2} i(\psi + i\tilde{\psi})\right]\right\} \perp H^2$.

Since $\operatorname{ind} T_f = 1$, $\dim \ker T_f = 1$. Therefore if $h \in \ker T_f$, then $h = b/\left\{y \exp \frac{1}{2} i(\psi + i\tilde{\psi})\right\}$ for some constant b. The remaining conclusions now follow since both y and $\exp \frac{1}{2} i(\psi + i\tilde{\psi})$ are in $C^{(n-1)}(\mathbf{T})$, and in $(H^\infty)^{-1}$.

LEMMA 2.2. *Suppose $f \in C(\mathbf{T})$ and T_f is Fredholm of index* 1. *If $g \in \ker T_f$ and $g \not\equiv 0$, then $g(0) \neq 0$.*

Proof. Suppose $g(0) = 0$. Then $e^{-it}g \in H^2$ and $e^{-it}g \not\equiv 0$. Clearly, $T_{zf}(e^{-it}g) = 0$. But since the winding number of $t \mapsto e^{it}f(e^{it})$ about the origin is 0, $\dim \ker T_{zf} = 0$. This is a contradiction. Therefore $g(0) \neq 0$.

If $f \in C(\mathbf{T})$ and $T_{f-\lambda}$ is Fredholm of index 1 for every λ in the interior of $f(\mathbf{T})$, then $\dim\ker T_{f-\lambda} = 1$. By the above lemma, there is a unique $h_\lambda \in \ker T_{f-\lambda}$ such that $h_\lambda(0) = 1$.

PROPOSITION 2.3. *With T_f and h_λ as above, the function $\lambda \mapsto h_\lambda$ is an analytic H^2-valued function for λ in the interior of $f(\mathbf{T})$.*

Proof. For λ in the interior of $f(\mathbf{T})$, $T_{z(f-\lambda)}$ is Fredholm of index 0. Hence $T_{z(f-\lambda)}$ is invertible. If $g_\lambda = T_{z(f-\lambda)}^{-1}1$, then $g_\lambda \in \ker T_{f-\lambda}$ and $g_\lambda \not\equiv 0$. Since $\lambda \mapsto T_{z(f-\lambda)}^{-1}$ is an analytic operator-valued function, $\lambda \mapsto g_\lambda$ is an analytic H^2-valued function. Thus $\lambda \mapsto h_\lambda = g_\lambda/g_\lambda(0)$ is analytic since $g_\lambda(0) \neq 0$ by Lemma 2.2.

As the referee has pointed out, Proposition 2.3 also follows from Proposition 1.11 of [6].

3. THE OPERATORS V_λ AND S_λ

If $F \in J^{(n)}$, F is one-to-one and F' never vanishes on $\mathbf{T}$, then there is a closed annulus $N = \{z \mid 0 < s \leqslant |z| \leqslant 1\}$ such that F extends to be $C^{(n)}$ and one-to-one on N. We shall denote the extension of $F(e^{it})$ by $F(z)$, $z \in N$. Let $1/D\colon F(N) \to N$ be the continuous inverse of $F(z)$. For $\lambda \in F(N_1)$, where $N_1 = \{z \mid s < s_1 \leqslant |z| \leqslant 1\}$, we let V_λ be the Toeplitz operator with symbol $Q(z, \lambda)$, defined to be $[F(z) - \lambda]/[1 - D(\lambda)z]$ if $z \neq 1/D(\lambda)$, and $Q(z, \lambda) = F'(1/D(\lambda))$ if $z = 1/D(\lambda)$.

With F as above, let $f(z) = \bar{F}(1/\bar{z})$. Then $f(z)$ is $C^{(n)}$ and one-to-one on the closed annulus $M = \{z \mid 1 \leqslant |z| \leqslant 1/s\}$. For $\lambda \in f(M_1)$, where $M_1 = \{z \mid 1 \leqslant |z| \leqslant 1/s_1\}$, we let S_λ be the Toeplitz operator with symbol $q(z, \lambda)$, defined to be $(f(z) - \lambda)/(1 - d(\lambda)z)$ if $z \neq 1/d(\lambda)$, and $q(z, \lambda) = f'(1/d(\lambda))$ if $z = 1/d(\lambda)$, where $1/d\colon f(M) \to M$ is the inverse of $f(z)$.

The introduction of V_λ and S_λ enable us to study $T_{F-\lambda}$ ($T_{f-\lambda}$) even for $\lambda \in F(\mathbf{T})$ ($\lambda \in f(\mathbf{T})$, respectively). The operators V_λ and S_λ turn out to be Fredholm operators. With the aid of V_λ, Theorem 1 is proved in this section. As for the operator S_λ, we are interested in the case when $t \mapsto F(e^{it})$ is orientation preserving (and hence $t \mapsto f(e^{it})$ is orientation reversing). In this case, the index of S_λ is 1. We shall derive a suitable decomposition for the null vector k_λ of S_λ which in turn will enable us to obtain a desired decomposition of h_λ, the unique eigenvector of $T_F^* = T_f$ satisfying $h_\lambda(0) = 1$, to pave the way for the proof of Theorem 2.

Throughout this section, we shall denote the following four annuli $0 < s \leqslant |z| \leqslant 1$, $s < s_1 \leqslant |z| \leqslant 1$, $1 \leqslant |z| \leqslant 1/s$ and $1 \leqslant |z| \leqslant 1/s_1$ by N, N_1, M and M_1, respectively.

LEMMA 3.1. *Suppose the function g is $C^{(n)}$, $n \geqslant 1$, on N. Let the function $G(z, \omega)$ on $N \times N$ be defined to be $[g(z) - g(\omega)]/(z - \omega)$ if $z \neq \omega$, and $g'(\omega)$ if $z = \omega$. Then $\partial^m G/\partial z^m$ is continuous on $N_1 \times N_1$ for $0 \leqslant m \leqslant n - 1$.*

Proof. The continuity of $\partial^m G/\partial z^m$ at every $(z_0, \omega_0) \in N_1 \times N_1$, $z_0 \neq \omega_0$ is clear. For each $\omega_0 \in N_1$, there is a convex set $U \subset N$ which is open in the relative topology on N such that $\omega_0 \in U$. The convexity of U implies $G(z, \omega) = \int_0^1 g'(t\omega + (1-t)z)\,dt$ for $(z, \omega) \in U \times U$. Since $(\partial^m/\partial z^m)G(z, \omega) = \int_0^1 g^{(m+1)}(t\omega + (1-t)z)(1-t)^m dt$ and since $g^{(m+1)}$ is continuous on N, $\partial^m G/\partial z^m$ is continuous at every $(\omega_0, \omega_0) \in N_1 \times N_1$.

LEMMA 3.2. *Suppose F is $C^{(n)}$, $n \geqslant 1$, and one-to-one on N. Let $1/D\colon F(N) \to N$ be the inverse of F. Let the function $Q(z, \lambda)$ be as above. Then $\partial^m Q/\partial z^m$ is continuous on $N_1 \times F(N_1)$ for $0 \leqslant m \leqslant n-1$.*

Proof. Since

$$[F(z) - \lambda]/(1 - D(\lambda)z) =$$

$$= (-1/D(\lambda))[F(z) - F(1/D(\lambda))]/(z - 1/D(\lambda)),$$

the conclusion follows from Lemma 3.1 and the continuity of $1/D$.

PROPOSITION 3.3. *With notation and hypothesis as in the above lemma, suppose, in addition, F' never vanishes on $\mathbf{T}$. Let V_λ be the Toeplitz operator with symbol $Q(z, \lambda)$, $\lambda \in F(N_1)$. Then V_λ is Fredholm. Furthermore, the index of V_λ is equal to 0 if $t \mapsto F(e^{it})$ is orientation preserving, and 1 if $t \mapsto F(e^{it})$ is orientation reversing.*

Proof. Since $Q(z, \lambda)$ is continuous on $N_1 \times F(N_1)$ and F' never vanishes on $\mathbf{T}$, $Q(\cdot, \lambda)$ is continuous and never vanishes on $\mathbf{T}$ for each $\lambda \in F(N_1)$. That V_λ is Fredholm for $\lambda \in F(N_1)$ now follows. If $t \mapsto F(e^{it})$ is orientation preserving and if $\lambda \in F(N_1)\setminus F(\mathbf{T})$, then the winding number of $t \to Q(e^{it}, \lambda)$ about 0 is the winding number of $F(e^{it}) - F(1/D(\lambda))$ minus that of $e^{it} - 1/D(\lambda)$, and so is equal to 0. Therefore ind $V_\lambda = 0$ for $\lambda \in F(N_1)\setminus F(\mathbf{T})$. If $\lambda \in F(\mathbf{T})$, we may pick $\lambda_n \in F(N_1)\setminus F(\mathbf{T})$, $n = 1,2,3,\ldots$, such that λ_n tends to λ. The uniform continuity of $Q(z, \lambda)$ on the compact set $\mathbf{T} \times F(N_1)$ implies $Q(\cdot, \lambda_n)$ tends uniformly to $Q(\cdot, \lambda)$ on $\mathbf{T}$. Hence V_{λ_n} tends to V_λ in norm and we have ind $V_\lambda = 0$. If $t \mapsto F(e^{it})$ is orientation reversing, then the winding number of $t \mapsto Q(e^{it}, \lambda)$ about 0 is equal to -1 for $\lambda \in F(N_1)\setminus F(\mathbf{T})$. Therefore ind $V_\lambda = 1$ for every $\lambda \in F(N_1)\setminus F(\mathbf{T})$. Arguing as above shows ind $V_\lambda = 1$ for $\lambda \in F(\mathbf{T})$.

Now we can give the

Proof of Theorem 1. We first note that the hypotheses on F here are equivalent to those of the previous proposition. Also note that the boundary of $\sigma(T_F)$ in this case is $F(\mathbf{T})$. Suppose $\lambda \in F(\mathbf{T})$ is an eigenvalue for T_F and suppose g_λ is a nonzero eigenvector with eigenvalue λ for T_F. Then $(1 - D(\lambda)z)g_\lambda$ is clearly a nonzero null vector for V_λ.

If $t \mapsto F(e^{it})$ is orientation preserving, then ind $V_\lambda = 0$, by the previous proposition, and we have a contradiction. On the other hand if $t \mapsto F(e^{it})$ is orientation reversing, then ind $V_\lambda = 1$ and the null vector $(1 - D(\lambda)z)g_\lambda$ has to be in $(H^\infty)^{-1}$ and $C(\mathbf{T})$ by Proposition 2.1. Since $(1 - D(\lambda)z)g_\lambda = 0$ at $z = 1/D(\lambda)$, we have a contradiction again. Therefore T_F has no boundary eigenvalue. The last statement of the theorem is now immediate.

It is interesting to note that the degree of smoothness of the symbol of T_F plays an important role in our proof of the nonexistence of boundary eigenvalues for T_F. K.F. Clancey [2] has given an example of a continuous F such that T_F does have boundary eigenvalues.

The remaining part of this section concerns the operator S_λ and its null vector k_λ.

By using a construction similar to that of Lemma 3.1, the following lemma can be proved in a manner similar to Lemma 3.2; we shall, therefore, omit its proof.

LEMMA 3.4. *Suppose f is $C^{(n)}$, $n \geqslant 1$, and one-to-one on M. Let $1/d: f(M) \to M$ be the inverse of f. Let the function $q(z, \lambda)$ on $M \times f(M)$ be defined as above. Then $\partial^m q/\partial z^m$ is continuous on $M_1 \times f(M_1)$ for $0 \leqslant m \leqslant n-1$.*

PROPOSITION 3.5. *With notation and hypothesis as in the above lemma, suppose in addition, f' never vanishes on $\mathbf{T}$ and $t \mapsto f(e^{it})$ is orientation reversing. Let S_λ be the Toeplitz operator with symbol $q(z, \lambda)$, $\lambda \in f(M_1)$. Then S_λ is Fredholm of index 1.*

Proof. The proof is similar to that of Proposition 3.3 and hence is omitted. Observe that if $t \mapsto f(e^{it})$ is orientation reversing, then the winding number of $t \mapsto q(e^{it}, \lambda)$ about 0 is equal to -1 for $\lambda \in f(M_1) \setminus f(\mathbf{T})$.

REMARK 3.6. If f is as in the above proposition, then a similar argument as in Theorem 1 shows T_f has no boundary eigenvalues.

REMARK 3.7. Following the comment after Lemma 2.2, we shall let k_λ be the unique null vector for S_λ, $\lambda \in f(M_1)$, satisfying $k_\lambda(0) = 1$. By Proposition 2.1, $k_\lambda = b_\lambda \Big/ \left\{ y_\lambda \exp \frac{1}{2} i[\psi_\lambda + i\tilde{\psi}_\lambda] \right\}$, where $y_\lambda = \exp \frac{1}{2} [\log |q| + i(\log |q|)^\sim]$, ψ_λ satisfies $q(e^{it}, \lambda) = |q(e^{it}, \lambda)| \exp i[\psi_\lambda(e^{it}) - t]$ and b_λ is a constant.

LEMMA 3.8. *Suppose B is a compact set and the function g defined on $\mathbf{T} \times B$ is such that $(\partial^2/\partial t^2)\, g(e^{it}, \lambda)$ is continuous on $\mathbf{T} \times B$. Then $g(e^{it}, \lambda)^\sim$ is continuous on $\mathbf{T} \times B$. Here the harmonic conjugate is taken with respect to the variable e^{it}.*

Proof. For each $\lambda \in B$, let $g(e^{it}, \lambda) \sim \sum_{n=-\infty}^{\infty} a_n(\lambda) e^{int}$ be the Fourier series for g. Then $g(e^{it}, \lambda)^\sim \sim i \sum_{n=-\infty}^{-1} a_n(\lambda) e^{int} - i \sum_{n=1}^{\infty} a_n(\lambda) e^{int}$. Since

$$a_n(\lambda) = \frac{1}{2\pi} \int_0^{2\pi} g(e^{it}, \lambda) e^{-int}\, dt,$$

integrating by parts twice we have

$$a_n(\lambda) = \frac{-1}{n^2}\,\frac{1}{2\pi}\int_0^{2\pi}\left[\frac{\partial^2}{\partial t^2}g(e^{it},\lambda)\right]e^{-int}dt.$$

The continuity of $(\partial^2/\partial t^2)g(e^{it},\lambda)$ implies $a_n(\lambda)$ is continuous in λ, for every n, and the series $\sum_{n=-\infty}^{\infty}|a_n(\lambda)|$ is uniformly convergent for $\lambda\in B$. Thus $g(e^{it},\lambda)^{\sim}$ is continuous on $\mathbf{T}\times B$.

PROPOSITION 3.9. *Let f and S_λ be as in Proposition 3.5 with $n = 4$, and let k_λ be as in Remark 3.7. The functions*

(1) $$(z,\lambda)\mapsto k_\lambda(z),$$
and
(2) $$(z,\lambda)\mapsto k_\lambda'(z)$$

are continuous on $\mathrm{cl}\,\mathbf{D}\times f(M_1)$.

Proof. (1) By Proposition 2.1, it is easily seen that $k_\lambda(z)$ is the Poisson extension of $k_\lambda(e^{it})$. Since f is $C^{(4)}$ on M, $q(e^{it},\lambda)$ never vanishes, and $(\partial^2/\partial t^2)q$ is continuous on $\mathbf{T}\times f(M_1)$ by Lemma 3.4, we have $\log|q|$, ψ_λ and $(\partial^2/\partial t^2)\psi_\lambda$ are continuous on $\mathbf{T}\times f(M_1)$. Lemma 3.8 now implies $(\log|q|)^{\sim}$ and $\tilde{\psi}_\lambda$ are continuous on $\mathbf{T}\times f(M_1)$. Thus $y_\lambda\exp\frac{1}{2}\,i[\psi_\lambda+i\tilde{\psi}_\lambda]$ is continuous on $\mathbf{T}\times f(M_1)$. A simple application of Poisson integral would then show that $y_\lambda\exp\frac{1}{2}\,i[\psi_\lambda+i\tilde{\psi}_\lambda]$ is continuous on $\mathrm{cl}\,\mathbf{D}\times f(M_1)$. Finally, $k_\lambda(0)=1$ implies b_λ is continuous in $\lambda\in f(M_1)$. We therefore have that the function (1) is continuous on $\mathrm{cl}\,\mathbf{D}\times f(M_1)$.

(2) Because $k_\lambda(e^{it})\in C^{(3)}(\mathbf{T})$ by Proposition 2.1, it is not difficult to show $k_\lambda'(z)\in A$, the disk algebra. In this case, we have $\frac{d}{dt}k_\lambda(e^{it}) = ie^{it}k_\lambda'(e^{it})$. It is therefore sufficient to show that $\frac{d}{dt}y_\lambda(e^{it})$, $\frac{d}{dt}\psi_\lambda(e^{it})$ and $\frac{d}{dt}\tilde{\psi}_\lambda(e^{it})$ are continuous on $\mathbf{T}\times f(M_1)$.

The continuity of $\frac{d}{dt}\psi_\lambda$ follows from that of $\frac{\partial}{\partial t}q$. Since $\frac{\partial}{\partial t}(\log|q|)^{\sim} = \left[\frac{\partial}{\partial t}\log|q|\right]^{\sim}$ and $\frac{d}{dt}\tilde{\psi}_\lambda = \left[\frac{d}{dt}\psi_\lambda\right]^{\sim}$ for each fixed λ, and $(\partial^3/\partial t^3)\log|q|$ and $(d^3/dt^3)\psi_\lambda$ are continuous on $\mathbf{T}\times f(M_1)$, $\frac{d}{dt}y_\lambda = y_\lambda\left\{\frac{1}{2}\left[\frac{\partial}{\partial t}\log|q| + i\frac{\partial}{\partial t}(\log|q|)^{\sim}\right]\right\}$ and $\frac{d}{dt}\tilde{\psi}_\lambda$ are continuous on $\mathbf{T}\times f(M_1)$.

LEMMA 3.10. *For* $(z, \lambda) \in \mathrm{cl}\,\mathbf{D} \times f(M_1)$, *let* $B_\lambda(z)$ *be defined to be* $[k_\lambda(z) - k_\lambda(\bar{d}(\lambda))]/[z - \bar{d}(\lambda)]$ *if* $z \neq \bar{d}(\lambda)$, *and* $k'_\lambda(\bar{d}(\lambda))$ *if* $z = \bar{d}(\lambda)$. *Then,*

(1) $B_\lambda(z)$ *is bounded on* $\mathrm{cl}\,\mathbf{D} \times f(M_1)$ *and hence* $B_\lambda \in H^\infty$ *for every* $\lambda \in f(M_1)$.

(2) $k_\lambda(z) = k_\lambda(\bar{d}(\lambda)) + [z - \bar{d}(\lambda)]B_\lambda(z)$.

Proof. (1) By Proposition 3.9, $B_\lambda(z)$ is continuous on the compact set $\mathrm{cl}\,\mathbf{D} \times f(M_1)$ and hence bounded. The definition of B_λ clearly implies $B_\lambda(z)$ is analytic in z for each fixed λ. Thus $B_\lambda \in H^\infty$ for each $\lambda \in f(M_1)$.

(2) The decomposition follows from the definition of $B_\lambda(z)$.

PROPOSITION 3.11. *For* $(z, \lambda) \in \mathrm{cl}\,\mathbf{D} \times f(M_1)$, *let* $Q_\lambda(z) = [(z - \bar{d}(\lambda))/(1 - d(\lambda)z)]B_\lambda(z)$, *where* B_λ *is as in Lemma* 3.10. *Then,*

(1) $Q_\lambda(z)$ *is bounded on* $\mathrm{cl}\,\mathbf{D} \times f(M_1)$ *and hence* $Q_\lambda \in H^\infty$ *for every* $\lambda \in f(M_1)$.

(2) *The function* $\lambda \mapsto Q_\lambda$ *is a continuous* H^2*-valued function for* $\lambda \in f(M_1)$.

(3) $k_\lambda(z)/[1 - d(\lambda)z] = k_\lambda(\bar{d}(\lambda))/[1 - d(\lambda)z] + Q_\lambda(z)$, *for* $\lambda \in f(M_1) \setminus f(\mathbf{T})$.

Proof. (1) It suffices to show that $[z - \bar{d}(\lambda)z]/[1 - d(\lambda)z]$ is bounded on $\mathrm{cl}\,\mathbf{D} \times f(M_1)$. If $\lambda \in f(\mathbf{T})$, then $|d(\lambda)| = 1$. In this case $d(\lambda) = 1/\bar{d}(\lambda)$. We have $[z - \bar{d}(\lambda)]/[1 - d(\lambda)z] = -\bar{d}(\lambda)$. Since $d(\lambda)$ is continuous on $f(M_1)$, $[z - \bar{d}(\lambda)]/[1 - d(\lambda)z]$ is bounded on $\mathrm{cl}\,\mathbf{D} \times f(M_1)$. On the other hand if $\lambda \in f(M_1) \setminus f(\mathbf{T})$, then $|d(\lambda)| < 1$ and $z \mapsto [z - \bar{d}(\lambda)]/[1 - d(\lambda)z]$ is a Möbius transformation sending $\mathrm{cl}\,\mathbf{D}$ onto $\mathrm{cl}\,\mathbf{D}$. Hence $[z - \bar{d}(\lambda)]/[1 - d(\lambda)z]$ is bounded on $\mathrm{cl}\,\mathbf{D} \times f(M_1)$.

(2) To show $\lambda \mapsto Q_\lambda$ is continuous, we fix $\lambda_0 \in f(M_1)$ and pick a sequence $\{\lambda_n\}$ in $f(M_1)$ such that $\lambda_n \to \lambda_0$. For each fixed $z \in \mathbf{T}$, $z \neq \bar{d}(\lambda_n)$, $n = 0, 1, 2, \ldots$, $Q_{\lambda_n}(z)$ tends to $Q_{\lambda_0}(z)$. Since $\{Q_{\lambda_n}\}$, $\lambda = 0, 1, 2, \ldots$, is uniformly bounded on $\mathbf{T}$ by (1), Lebesgue's dominated convergence theorem implies $\|Q_{\lambda_n} - Q_{\lambda_0}\|_2 \to 0$. Hence $\lambda \mapsto Q_\lambda$ is continuous.

(3) The decomposition follows from the corresponding decomposition for k_λ in the previous lemma.

For each $\lambda \in f(M_1) \setminus f(\mathbf{T})$, let h_λ be the uniques eigenvector with eigenvalue λ for T_f satisfying $h_\lambda(0) = 1$. By definition of k_λ (Remark 3.7), $h_\lambda(z) = k_\lambda(z)/[1 - d(\lambda)z]$ for $\lambda \in f(M_1) \setminus f(\mathbf{T})$, where k_λ is the unique null vector of S_λ satisfying $k_\lambda(0) = 1$. Thus Proposition 3.11 (3) gives a decomposition for h_λ when $\lambda \in f(M_1) \setminus f(\mathbf{T})$. This decomposition will be used in the proof of Theorem 2.

4. PROOF OF THE SIMILARITY THEOREM

We are now ready to prove Theorem 2. All pertinent notations introduced in the previous sections will be retained.

Proof of Theorem 2. Note that under the assumption on $F(z)$, the mapping function τ extends to be continuous and one-to-one on $\mathrm{cl}\,\mathbf{D}$ and sends $\mathbf{T}$ onto $F(\mathbf{T})$;

see, for example [11]. Let $f(z) = \bar{F}(1/\bar{z})$. Then $T_F^* = T_f$ and f satisfies the hypothesis of Proposition 3.5. Thus f is $C^{(4)}$ and one-to-one on M_1, f' never vanishes on $\mathbf{T}$ and $t \mapsto f(e^{it})$ is orientation reversing. If $|\omega| < 1$, then $\tau(\omega) \in \operatorname{int} \sigma(T_F)$, the interior of $F(\mathbf{T})$, and $\bar{\tau}(\omega) \in \operatorname{int} \sigma(T_f)$, the interior of $f(\mathbf{T})$.

Our goal is to find an operator $L: H^2 \to H^2$ which is invertible and satisfies the intertwining relation $LT_F = T_\tau L$. We shall pattern the proof after that of [5], breaking it into steps.

Step 1. We first define L from $\mathbf{P}^2$ into H^2. Here $\mathbf{P}^2$ is the dense linear manifold of H^2 consisting of polynomials.

For $|\omega| < 1$ and $g \in \mathbf{P}^2$, define $Lg(\omega) = \langle g, h_{\bar{\tau}(\omega)} \rangle$, where h_λ is the eigenvector for T_f corresponding to $\lambda \in \operatorname{int} \sigma(T_f)$ satisfying $h_\lambda(0) = 1$. Clearly L is linear. Since h_λ depends analytically in λ by Proposition 2.3, $Lg(\omega)$ is analytic in $|\omega| < 1$. The function $\bar{\tau}(\omega)$ extends to be one-to-one and continuous from $|\omega| \leqslant 1$ onto $\sigma(T_f)$ and maps $\mathbf{T}$ onto $f(\mathbf{T})$. Hence there is a positive constant $0 < c < 1$ such that $\bar{\tau}$ maps the annulus $c \leqslant |\omega| \leqslant 1$ one-to-one onto $f(M_1)$. By Proposition 3.11 and the remark following it, we have

$$h_{\bar{\tau}(\omega)}(z) = k_{\bar{\tau}(\omega)}(z)/[1 - d(\bar{\tau}(\omega))z] =$$

$$= k_{\bar{\tau}(\omega)}(\bar{d}(\bar{\tau}(\omega)))/[1 - d(\bar{\tau}(\omega))z] + Q_{\bar{\tau}(\omega)}(z),$$

for $c < |\omega| < 1$. Thus

$$Lg(\omega) = \langle g, k_{\bar{\tau}(\omega)}(\bar{d}(\bar{\tau}(\omega)))/[1 - d(\bar{\tau}(\omega))z] \rangle + \langle g, Q_{\bar{\tau}(\omega)} \rangle =$$

$$= \bar{k}_{\bar{\tau}(\omega)}(d(\bar{\tau}(\omega))g(\bar{d}(\bar{\tau}(\omega)))) + \langle g, Q_{\bar{\tau}(\omega)} \rangle,$$

for $c < |\omega| < 1$.

Since $\lambda \mapsto k_\lambda(\bar{d}(\lambda))$ and $\lambda \mapsto Q_\lambda$ are continuous on $f(M_1)$ by Propositions 3.9 and 3.11, $Lg(\omega)$ is readily seen to be continuously extendable to $c < |\omega| \leqslant 1$. Therefore L maps $\mathbf{P}^2$ into H^2.

Step 2. L is bounded on $\mathbf{P}^2$, so L extends by continuity to a bounded operator, again denoted L, acting on H^2. Furthermore, this extended L intertwines T_F and T_τ, and has closed range and finite dimensional kernel.

Define L_s and L_c on $\mathbf{P}^2$ by $L_s g(\omega) = \bar{k}_{\bar{\tau}(\omega)}(d(\bar{\tau}(\omega)))g(\bar{d}(\bar{\tau}(\omega)))$, and $L_c g(\omega) = \langle g, Q_{\bar{\tau}(\omega)} \rangle$, $|\omega| = 1$ and $g \in \mathbf{P}^2$. Clearly L_s and L_c are linear and map $\mathbf{P}^2$ into $C(\mathbf{T})$. We have only that $L_s g$ and $L_c g$ are in L^2, but we know from Step 1, $Lg = L_s g + L_c g$ is in H^2.

Introducting the change of variable $\omega = \tau^{-1}(F(z))$, for $|z| = 1$, we have

$$\|L_s g\|_2^2 = \int_{|\omega|=1} |\bar{k}_{\bar{\tau}(\omega)}(d(\bar{\tau}(\omega)))|^2 |g(\bar{d}(\bar{\tau}(\omega)))|^2 \, dm(\omega) =$$

$$= \int_{|z|=1} |\bar{k}_{f(z)}(d(f(z)))|^2 |g(z)|^2 \, dm(\tau^{-1}(F(z))).$$

Since, by Proposition 2.1 and 3.9, k_λ never vanishes on $\mathbf{T}$ for $\lambda \in f(M_1)$, $|\overline{k}_{f(z)}(\overline{d}(f(z)))|$ is bounded and bounded away from 0 on $|z| = 1$. By Lemma 3.2 of [5], the measures $\mathrm{d}m(z)$ and $|\overline{k}_{f(z)}(\overline{d}(f(z)))|^2 \mathrm{d}m(\tau^{-1}(F(z)))$ are mutually boundedly absolutely continuous. Thus $c_1 \|g\|_2 \leqslant \|L_s g\|_2 \leqslant c_2 \|g\|_2$ for some positive constants c_1 and c_2. Clearly then L_s extends to be bounded and bounded below (in L^2 norm) on H^2. This extension, again denoted L_s, is thus semi-Fredholm with 0 kernel (as an operator from H^2 to L^2).

For L_c, we have

$$L_c g(\mathrm{e}^{it}) = \langle g, Q_{\bar{\tau}(\mathrm{e}^{it})} \rangle = \int_{|z|=1} g(z) \bar{Q}_{\bar{\tau}(\mathrm{e}^{it})}(z)\, \mathrm{d}m(z).$$

L_c is seen to be an integral operator whose kernel $\bar{Q}_{\tau(\omega)}(z)$ is bounded on $\mathbf{T} \times \mathbf{T}$, by Proposition 3.11. Hence L_c extends to act on H^2 and is a compact operator. Thus $L = L_s + L_c$ extends to be bounded as an operator acting on H^2.

Being a compact perturbation of a semi-Fredholm operator with 0 kernel, L must have closed range in L^2 (and hence in H^2), and finite dimensional kernel. The extended L still satisfies $Lg(\omega) = \langle g, h_{\bar{\tau}(\omega)} \rangle$ for $g \in H^2$ and $|\omega| < 1$. The intertwining property follows, since

$$LT_F g(\omega) = \langle T_F g, h_{\bar{\tau}(\omega)} \rangle = \langle g, T_f h_{\bar{\tau}(\omega)} \rangle = \tau(\omega) \langle g, h_{\bar{\tau}(\omega)} \rangle =$$

$$= T_\tau L g(\omega), \quad |\omega| < 1.$$

Step 3. (The extended) L is onto.

By Step 2, it is sufficient to prove that L has dense range. Since $L(1)(\omega) = \overline{h}_{\bar{\tau}(\omega)}(0) = 1$, $|\omega| < 1$, we have $L1 = 1$, and thus $L(T_F^n 1) = \tau^n$, $n = 0, 1, 2, \ldots$. The range of L therefore contains all polynomials in τ. By Mergelyan's Theorem, there is a sequence of polynomials p_n tending uniformly to τ^{-1} on $\sigma(T_F)$. Hence any polynomial p is the uniform limit of the sequence $p(p_n(\tau))$ of polynomials in τ. The range of L is thus dense.

Step 4. (The extended) L is one-to-one and hence invertible by the open mapping theorem.

Since $LT_F g = T_\tau L g$ for $g \in H^2$, the kernel of L, which is finite dimensional by Step 2, must be invariant under T_F. The operator T_F then must have an eigenvalue. This contradicts Theorem 1, unless the kernel of L is $\{0\}$. The proof of Theorem 2 is now complete.

In conclusion, we state some consequences of our similarity theorem.

COROLLARY 1. *The invariant subspace lattice of T_F is isomorphic to the lattice of inner functions.*

Proof. The lattice of invariant subspaces of T_τ and hence of T_F is the lattice of invariant subspaces of T_z. See [9].

COROLLARY 2. *The commutant* $\{T_F\}'$ *of* T_F *satisfies* $\{T_F\}' = L^{-1}\{T_g \mid g \in H^\infty\}L$, *here L is the operator implementing the similarity between* T_F *and* T_τ.

Proof. This follows from the fact that τ is univalent and hence $\{T_\tau\}' = \{T_z\}' = \{T_g \mid g \in H^\infty\}$. See [1] and [7].

COROLLARY 3. *The closed linear span of the eigenvectors of* T_F^* *is* H^2.

Proof. This follows since T_τ^* has the stated property and T_F^* is similar to T_τ^*.

The paper is based in part on the author's Ph. D. thesis written under the direction of Professor Douglas N. Clark at the University of Georgia. The author whishes to express his deepest gratitude to Professor Clark for his guidance.

REFERENCES

1. BROWN, A.; HALMOS, P. R., Algebraic properties of Toeplitz operators, *J. Reine Angew. Math.*, **213**(1964), 89–102.
2. CLANCEY, K. F., Toeplitz models for operators with one dimensional self-commutators, preprint.
3. CLARK, D. N., On a similarity theory for rational Toeplitz operators, *J. Reine Angew. Math.*, **320**(1980), 6–31.
4. CLARK, D. N., On the structure of rational Toeplitz operators, in *Contributions to Analysis and Geometry*, Amer. J. Math. supplement (1981), 63–72.
5. CLARK, D. N.; MOREL, J. H., On Toeplitz operators and similarity, *Amer. J. Math.*, **100**(1978), 973–986.
6. COWEN, M. J.; DOUGLAS, R. G., Complex geometry and operator theory, *Acta Math.*, **141**(1978), 187–261.
7. DEDDENS, J. A.; WONG, T. K., The commutant of analytic Toeplitz operators, *Trans. Amer. Math. Soc.*, **184**(1973), 261–273.
8. DOUGLAS, R. G., *Banach algebra techniques in operator theory*, Academic Press, New York, 1972.
9. DUREN, P. L., Extension of a result of Beurling on invariant subspaces, *Trans. Amer. Math. Soc.*, **99**(1961), 320–324.
10. MORREL, J. H., *A similarity theorem for polynomial Toeplitz operators*, Dissertation, University of Georgia, 1976.
11. RUDIN, W., *Real and complex analysis*, 2nd ed., McGraw-Hill, New York, 1974.
12. SARASON, D. E., *Lecture notes on Toeplitz operators*, University of Kentucky, 1970.
13. ZYGMUND, A., *Trigonometric series*, 2nd ed., Cambridge University Press, 1959.

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Received June 14, 1983.