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PSEUDO-REGULAR SPECTRAL FUNCTIONS IN KREĬN SPACES

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1. INTRODUCTION

A pseudo-regular subspace is a subspace $\mathscr{M}$ of a Kreĭn space $\mathscr{K}$ such that $\mathscr{M} = \mathscr{M}^0[+]\mathscr{R}$, where $\mathscr{M}^0$ is the isotropic part of $\mathscr{M}$ and $\mathscr{R}$ is a regular subspace. If A is a definitizable operator then it is known [10] that A has an invariant maximal non-negative subspace. On the other hand, the question "when a definitizable operator has an invariant regular maximal non-negative subspace?" has a neat answer [12, Theorem 2].

This paper is concerned with the following question: "when a definitizable operator has an invariant pseudo-regular maximal non-negative subspace?". Thus, in Theorem 4.2 we prove that a sufficient condition for the existence of an invariant pseudo-regular maximal non-negative subspace of the definitizable operator A is a certain condition of pseudo-regularity on the spectral function of A in the neighbourhoods of the critical points of A. However, in Example 4.3 it is shown that in general this pseudo-regularity condition is not necessary in order that A has an invariant pseudo-regular maximal non-negative subspace. Corollary 4.4 gives a particular case in which the equivalence holds.

We note that concerning this pseudo-regularity condition on the spectral function, a slightly stronger condition appeared in [6], and also that the above considered problem can be related to [3].

The proof of the main result leans on Section 3 where the following problem is considered: "given a commutative family $\mathscr{P}$ of selfadjoint projections decide when $\bigvee_{P\in\mathscr{P}} P\mathscr{K}$ is a pseudo-regular subspace".

We have presented in Section 1 some terminology from Kreĭn space theory (slightly different from some known papers), according to what we need.

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2. REDUCTIONS BY NEUTRAL SUBSPACES

Let $\mathcal{K}$ be a Kreĭn space, i.e. the complex vector space $\mathcal{K}$ has an indefinite inner product $[\cdot,\cdot]$ and there exists an operator J on $\mathcal{K}$ such that $J^2 = I$ and the equality

$$(\xi|\eta)_J = [J\xi, \eta], \quad \xi, \eta \in \mathcal{K} \tag{2.1}$$

defines a positive definite inner product $(\cdot,\cdot)_J$ with respect to which $\mathcal{K}$ is a Hilbert space. The operator J is usually called *a fundamental symmetry* (f.s) of $\mathcal{K}$ and the quadratic norm $\xi \mapsto (\xi|\xi)_J^{1/2}$ is called *the J-norm.*

Since different fundamental symmetries yield equivalent norms [3, Theorem V. 1.1] the *norm topology* on $\mathcal{K}$ is any J-norm topology, for an arbitrary f.s. J of $\mathcal{K}$, and the usual topologies on $\mathcal{B}(\mathcal{K})$ ($\mathcal{B}(\mathcal{K})$ denotes the set of bounded linear operators on $\mathcal{K}$) are considered with respect to this norm topology.

If $\mathcal{L}$ is a subspace of $\mathcal{K}$ (i.e. $\mathcal{L}$ is a norm closed linear submanifold of $\mathcal{K}$) one usually denotes the *orthogonal subspace* of $\mathcal{L}$ by $\mathcal{L}^{[\perp]} = \{\xi \in \mathcal{K} \mid [\xi, \mathcal{L}] = \{0\}\}$ and the *isotropic part* of $\mathcal{L}$ by $\mathcal{L}^0 = \mathcal{L} \cap \mathcal{L}^{[\perp]}$. The subspace $\mathcal{L}$ is called *non--degenerate* if $\mathcal{L}^0 = \{0\}$ (equivalently $\mathcal{L} \vee \mathcal{L}^{[\perp]} = \mathcal{K}$) and it is called *regular* if $\mathcal{L} \dotplus \mathcal{L}^{[\perp]} = \mathcal{K}$.

For two subspaces $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{K}$ we use the notation $\mathcal{A}[\dotplus]\mathcal{B}$ whenever $[\mathcal{A}, \mathcal{B}] = \{0\}$ and the algebraic sum $\mathcal{A} \dotplus \mathcal{B}$ is direct and closed. Then for a regular subspace $\mathcal{L}$ of $\mathcal{K}$ we can write $\mathcal{L}[\dotplus]\mathcal{L}^{[\perp]} = \mathcal{K}$ and note that $\mathcal{L}$ is also a Kreĭn space with respect to the induced indefinite inner product (see [1], [4], [8]).

A subspace $\mathcal{L}$ of $\mathcal{K}$ is called *pseudo-regular* if there exists a regular subspace $\mathcal{R}$ such that $\mathcal{L} = \mathcal{L}^0[\dotplus]\mathcal{R}$, equivalently if $\mathcal{L} + \mathcal{L}^{[\perp]} = \mathcal{L} \vee \mathcal{L}^{[\perp]}$ (see [3], [5], [6]).

Let $\mathcal{N}$ be a *neutral* subspace of $\mathcal{K}$, i.e. $\mathcal{N} \subset \mathcal{N}^{[\perp]}$. If J is a f.s. of $\mathcal{K}$ let us denote $\mathcal{L}_1 = \mathcal{N}$ and $\mathcal{L}_3 = J\mathcal{N}$. Clearly, $\mathcal{L}_1$ and $\mathcal{L}_3$ are orthogonal with respect to the J-inner product. Let $\mathcal{L}_2 = \mathcal{K} \ominus (\mathcal{L}_1 \oplus \mathcal{L}_3)$ (the symbols $\ominus$ and $\oplus$ are used with respect to the J-inner product). The decomposition

$$\mathcal{K} = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3,$$

which was used in [9], will be called the J-reduction of $\mathcal{K}$ by the neutral subspace $\mathcal{N}$ (cf. [14]).

Finally we recall that on $\mathcal{B}(\mathcal{K})$ one can consider the involution $\#$, i.e. for every $T \in \mathcal{B}(\mathcal{K})$ the equality

$$[T\xi, \eta] = [\xi, T^{\#}\eta], \quad \xi, \eta \in \mathcal{K},$$

defines an operator $T^{\#} \in \mathcal{B}(\mathcal{K})$. *Selfadjoint* operators will be considered with respect to this involution.

3. COMMUTATIVE FAMILIES OF SELFADJOINT PROJECTIONS

Let P be a selfadjoint projection on the Kreĭn space $\mathscr{K}$. Then $I - P$ is also a selfadjoint projection. A subspace $\mathscr{L}$ of $\mathscr{K}$ is the range of a selfadjoint projection P iff $\mathscr{L}$ is a regular subspace and in this case $\mathscr{L}^{[\perp]} = (I - P)\mathscr{K}$.

A projection $P \in \mathscr{B}(\mathscr{K})$ is called *positive* if $[P\xi, \xi] \geq 0$, $\xi \in \mathscr{K}$, and *negative* if $[P\xi, \xi] \leq 0$, $\xi \in \mathscr{K}$. Clearly positive (negative) projections are selfadjoint. A projection is called *definite* if it is either positive or negative.

A subspace $\mathscr{L}$ of $\mathscr{K}$ is the range of a positive projection iff $\mathscr{L}$ is regular and *non-negative* (i.e. $[\xi, \xi] \geq 0$, $\xi \in \mathscr{L}$), correspondingly a subspace $\mathscr{L}$ is the range of a negative projection iff $\mathscr{L}$ is regular and *non-positive* (i.e. $[\xi, \xi] \leq 0$, $\xi \in \mathscr{L}$).

Let $\mathscr{P}$ be a commutative family of selfadjoint projections on $\mathscr{K}$. We say that $\bigvee_{P \in \mathscr{P}} P$ exists if the subspace $\bigvee_{P \in \mathscr{P}} P\mathscr{K}$ is a regular subspace and in this case $\bigvee_{P \in \mathscr{P}} P$ is by definition the associated selfadjoint projection. If, in addition, $\mathscr{P}$ is finite then $\bigvee_{P \in \mathscr{P}} P$ always exists. Moreover, if $\mathscr{P}$ is an arbitrary commutative family of positive projections then the subspace $\bigvee_{P \in \mathscr{P}} P\mathscr{K}$ is non-negative [10, Lemma 2.1].

If S is an arbitrary set then we denote by $\mathscr{F}(S)$ the class of finite subsets of S, partially ordered by inclusion.

3.1. LEMMA. *Let $\mathscr{P}$ be a commutative family of positive projections and consider the associated net of mutually commuting positive projections $(P_K)_{K \in \mathscr{F}(\mathscr{P})}$,*

$$P_K := \bigvee_{P \in K} P, \quad K \in \mathscr{F}(\mathscr{P}).$$

The following conditions are equivalent:

(i) *There exists* $\bigvee_{P \in \mathscr{P}} P$.

(ii) $(P_K)_{K \in \mathscr{F}(\mathscr{P})}$ *is bounded.*

(iii) $(P_K)_{K \in \mathscr{F}(\mathscr{P})}$ *is* so-*convergent.*

Moreover, in this case $\bigvee_{P \in \mathscr{P}} P = \text{so-}\lim_{K \in \mathscr{F}(\mathscr{P})} P_K$.

The proof is an easy consequence of the Banach-Steinhaus principle. Clearly the above lemma can be formulated with the word negative instead of the word positive. However, the implication (i) $\Rightarrow$ (ii) does not extend even in a well-behaved situation, namely when the projections considered are all definite, as the following example proves.

3.2. EXAMPLE. Let $\mathscr{H}$ be a separable complex Hilbert space and $\{e_j\}_{j=1}^{\infty}$ an orthonormal basis of $\mathscr{H}$. One can organize the space $\mathscr{K} = \mathscr{H} \oplus \mathscr{H}$ as a Kreĭn space by

$$[\xi_1 \oplus \eta_1, \xi_2 \oplus \eta_2] = (\xi_1 \mid \xi_2) - (\eta_1 \mid \eta_2), \quad \xi_i, \eta_i \in \mathscr{H}.$$

Let $T \in \mathscr{B}(\mathscr{H})$ be defined by

$$Te_j = \frac{j}{j+1} e_j, \quad j \in \mathbf{N}.$$

It follows that $\mathscr{M} = \{\xi \oplus T\xi \mid \xi \in \mathscr{H}\} \subset \mathscr{K}$ is a maximal non-negative subspace of $\mathscr{K}$, non-degenerate but also non-regular (see [4, Theorem 6.3]).

Consider now, for every $n \in \mathbf{N}$, the subspace

$$\mathscr{H}_n = \operatorname{lin}\{e_j \mid j = \overline{1, n}\}.$$

Then $\dim \mathscr{H}_n = n$ and

$$\mathscr{H}_1 \subset \mathscr{H}_2 \subset \ldots \subset \mathscr{H}_n \subset \mathscr{H}_{n+1} \subset \ldots \subset \mathscr{H}.$$

If we put $\mathscr{L}_n = \{\xi \oplus T\xi \mid \xi \in \mathscr{H}_n\}$ and $\mathscr{L}'_n = \{T\xi \oplus \xi \mid \xi \in \mathscr{H}_n\}$ it follows that

$$\mathscr{L}_1 \subset \mathscr{L}_2 \subset \ldots \subset \mathscr{L}_n \subset \mathscr{L}_{n+1} \subset \ldots \subset \mathscr{M},$$

$$\mathscr{L}'_1 \subset \mathscr{L}'_2 \subset \ldots \subset \mathscr{L}'_n \subset \mathscr{L}'_{n+1} \subset \ldots \subset \mathscr{M}^{[\perp]},$$

and

$$\bigvee_{n \in \mathbf{N}} \mathscr{L}_n = \mathscr{M}, \quad \bigvee_{n \in \mathbf{N}} \mathscr{L}'_n = \mathscr{M}^{[\perp]}.$$

Let P_n and P'_n be the selfadjoint projections on $\mathscr{L}_n$ and respectively $\mathscr{L}'_n$ and $\mathscr{P} = \{P_1, P'_1, P_2, P'_2, \ldots\}$. Then $\mathscr{P}$ is a commutative family of definite selfadjoint projections; since $\mathscr{M}$ is non-degenerate we have $\bigvee_{P \in \mathscr{P}} P = I$ but, for every f.s. J on $\mathscr{K}$ $\sup_{n \in \mathbf{N}} \|P_n\|_J = \sup_{n \in \mathbf{N}} \|P'_n\|_J = +\infty$, (otherwise $\mathscr{M}$ would be regular, by Lemma 3.1).

We remark that other pathologies of this kind are exhibited in [11].

The implication (ii) $\Rightarrow$ (i) from Lemma 3.1 can easily be extended to the case when all the projections are definite (by applying the lemma to the subfamilies of positive and negative projections), namely we have the following:

3.3. Proposition. *Let $\mathscr{P}$ be a commutative family of definite projections and consider the net $(P_K)_{K \in \mathscr{F}(\mathscr{P})}$ of mutually commuting selfadjoint projections*

$$P_K = \bigvee_{P \in K} P, \quad K \in \mathscr{F}(\mathscr{P}).$$

If the net $(P_K)_{K \in \mathscr{F}(\mathscr{P})}$ is bounded then $\bigvee_{P \in \mathscr{P}} P$ does exist.

In order to investigate the problem of pseudo-regularity of the subspace generated by a family of regular subspaces we first record some facts.

Let $\mathscr{N}$ be a neutral subspace in $\mathscr{K}$, J a f.s. on $\mathscr{K}$ and $\mathscr{K} = \mathscr{L}_1 \oplus \mathscr{L}_2 \oplus \mathscr{L}_3$ the J-reduction of $\mathscr{K}$ by $\mathscr{N}$. If P is a selfadjoint projection and $\mathscr{N}$ is P-invariant then clearly

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ 0 & P_{22} & P_{23} \\ 0 & 0 & P_{33} \end{bmatrix}, \quad \text{w.r.t. } \mathscr{K} = \mathscr{L}_1 \oplus \mathscr{L}_2 \oplus \mathscr{L}_3 . \tag{3.1}$$

A trivial computation now proves that P_{22} itself is a selfadjoint projection in the Kreĭn space $\mathscr{L}_2$. Sometimes we will consider P_{22} as a selfadjoint projection in $\mathscr{K}$ such that $P_{22} \mid \mathscr{L}_2^{[\perp]} = 0$.

Suppose now that $\mathscr{M}$ is a subspace of $\mathscr{K}$ and that P is a selfadjoint projection. Then $\mathscr{M}$ is P-invariant if and only if there exist two subspaces $\mathscr{M}_1$ and $\mathscr{M}_2$ of $\mathscr{M}$ such that $\mathscr{M}_1 \subset P\mathscr{K}$, $\mathscr{M}_2 \subset (I-P)\mathscr{K}$ and $\mathscr{M} = \mathscr{M}_1[+]\mathscr{M}_2$. It follows that if P is definite and $\mathscr{M}$ is semi-definite of opposite sign then

$$P\mathscr{M} \subset \mathscr{M} \quad \text{iff} \quad P\mathscr{M} = \{0\}. \tag{3.2}$$

3.4. Lemma. *Let $\mathscr{P}$ be a commutative family of positive projections in $\mathscr{K}$, J a* f.s. *on $\mathscr{K}$ and put $\mathscr{M} = \bigvee_{P \in \mathscr{P}} P\mathscr{K}$. If $\mathscr{N}$ is a $\mathscr{P}$-invariant neutral subspace such that $\mathscr{N} \supset \mathscr{M}^0$ consider $\mathscr{K} = \mathscr{L}_1 \oplus \mathscr{L}_2 \oplus \mathscr{L}_3$ the J-reduction of $\mathscr{K}$ by $\mathscr{N}$ and suppose that $\mathscr{M}$ is a pseudo-regular subspace. Then $\bigvee_{P \in \mathscr{P}} P_{22}$ does exist.*

Conversely, if $\mathscr{M} = \mathscr{M}^0$ and $\bigvee_{P \in \mathscr{P}} P_{22}$ exists then the subspace $\mathscr{M}$ is pseudo-regular.

Proof. Let us suppose that $\mathscr{N} = \mathscr{M}^0$. Since $\mathscr{M}$ is $\mathscr{P}$-invariant it follows that $\mathscr{M}^{[\perp]}$ is also $\mathscr{P}$-invariant, hence $\mathscr{M}^0$ is $\mathscr{P}$-invariant. It follows, by factorizing with $\mathscr{M}^0$, that we can assume that $\mathscr{K} = \mathscr{L}_2$ and $\mathscr{M}$ is regular (see [5, Remark 3.8]) and now the existence of $\bigvee_{P \in \mathscr{P}} P_{22}$ follows simply by Lemma 3.1.

Conversely, assuming the existence of $\bigvee_{P \in \mathscr{P}} P_{22}$, let us denote by $\mathscr{R}$ the regular non-negative subspace $\bigvee_{P \in \mathscr{P}} P_{22}$. Since $\mathscr{M} \subset \mathscr{M}^{0[\perp]} = \mathscr{L}_1 + \mathscr{L}_2$ it follows that for every $P \in \mathscr{P}$ we have (3.1) and

$$P\mathscr{K} = P\mathscr{M} \subset P(\mathscr{L}_1 + \mathscr{L}_2). \tag{3.3}$$

On the other hand, for every $\xi_1 \in \mathscr{L}_1$ and $\xi_2 \in \mathscr{L}_2$ we have

$$P(\xi_1 + \xi_2) = P_{11}\xi_1 + P_{12}\xi_2 + P_{22}\xi_2 \in \mathscr{M}^0[+]\mathscr{R}, \quad P \in \mathscr{P},$$

whence by (3.3) it follows

$$P\mathscr{K} \subset \mathscr{M}^0[+]\mathscr{R}, \quad P \in \mathscr{P},$$

i.e. $\mathscr{M} \subset \mathscr{M}^0[+]\mathscr{R}$. Since $\mathscr{M}^0[+]\mathscr{R}$ is a pseudo-regular non-negative subspace then we can apply [5, Corollary 4.9] to prove that $\mathscr{M}$ is a pseudo-regular subspace.

Assume now that $\mathcal{M}$ is a pseudo-regular subspace and let $\mathcal{N}$ be a $\mathcal{P}$-invariant neutral subspace such that $\mathcal{N} \supset \mathcal{M}^0$. It is easy to see that one can assume $\mathcal{P} = (P_K)_{K \in \mathcal{F}(\mathcal{P})}$. Since $\mathcal{N} \supset \mathcal{M}^0$ it follows $J\mathcal{N} \supset J\mathcal{M}^0$ hence, if we put $\mathcal{L} = (\mathcal{N} \oplus J\mathcal{N})^{[\perp]}$ and $\mathcal{S} = (\mathcal{M}^0 \oplus J\mathcal{M}^0)^{[\perp]}$, then $\mathcal{L} \subset \mathcal{S}$. Denoting by $P_{\mathcal{L}}$ the selfadjoint projection on the regular subspace $\mathcal{L}$ and by $P_{\mathcal{S}}$ the selfadjoint projection on the regular subspace $\mathcal{S}$ it follows $P_{\mathcal{L}} P_{\mathcal{S}} = P_{\mathcal{S}} P_{\mathcal{L}} = P_{\mathcal{L}}$. Now according to Lemma 3.1 and the above proved facts we have

$$\sup_{P \in \mathcal{P}} \|P_{\mathcal{S}} P P_{\mathcal{S}}\|_J < +\infty,$$

hence

$$\sup_{P \in \mathcal{P}} \|P_{\mathcal{L}} P P_{\mathcal{L}}\|_J = \sup_{P \in \mathcal{P}} \|P_{\mathcal{L}}(P_{\mathcal{S}} P P_{\mathcal{S}}) P_{\mathcal{L}}\|_J \leqslant C \sup_{P \in \mathcal{P}} \|P_{\mathcal{S}} P P_{\mathcal{S}}\|_J,$$

hence, again by Lemma 3.1, $\bigvee_{P \in \mathcal{P}} P_{\mathcal{L}} P \mathcal{L}$ is a regular subspace. ▨

We also record without proof the following simple fact:

3.5. LEMMA. *Let $P \in \mathcal{B}(\mathcal{K})$ be such that $P^2 = P$ and $P^{\#} = I - P$. If $\mathcal{M}$ is a semi-definite P-invariant subspace then $\mathcal{M}$ is a neutral subspace.*

4. PSEUDO-REGULAR SPECTRAL FUNCTIONS

A selfadjoint operator $A \in \mathcal{B}(\mathcal{K})$ is called *definitizable* if there exists a polynomial p such that $[p(A)\xi, \xi] \geqslant 0$, $\xi \in \mathcal{K}$. The non-real part $\sigma_0(A)$ of the spectrum of A is a finite set which lies symmetrically with respect to the real axis. We denote the Riesz-Dunford projection corresponding to $\sigma_0(A)$ by E_0.

A definitizable operator A has a spectral function, i.e. there exists a finite set $c(A) \subset \mathbf{R}$ such that, if $\mathfrak{B}(A)$ denotes the Boolean algebra of subsets of $\mathbf{R}$ generated by the closed and open intervals whose endpoints do not belong to $c(A)$, there exists a homomorphism E from $\mathfrak{B}(A)$ into a Boolean algebra of mutually commuting selfadjoint projections in $\mathcal{K}$ such that:

(1) $E(\mathbf{R}) = I - E_0$;

(2) $AE(\Delta) = E(\Delta)A, \quad \Delta \in \mathfrak{B}(A)$;

(3) $\sigma(A)E(\Delta)\mathcal{K} \subset \bar{\Delta}, \quad \Delta \in \mathfrak{B}(A)$;

(4) $E(\Delta)$ is a definite projection for every $\Delta \in \mathfrak{B}(A)$ with $\Delta \cap c(A) = \emptyset$.

The set $c(A)$ of *critical points* of A can be caracterized thus

(5) $\alpha \in c(A)$ iff the subspace $E(\Delta)\mathcal{K}$ is indefinite for all $\Delta \in \mathfrak{B}(A)$ such that $\alpha \in \Delta$.

Detailed proofs of the above mentioned facts can be found in [7] and [11].

Let α be a critical point of the definitizable operator A. If λ and ρ are two real numbers such that $\lambda < \alpha < \rho$ and $[\lambda, \rho] \cap c(A) = \{\alpha\}$ then one can introduce the following subspaces (see [11]):

$$\mathscr{S}_{\alpha,+} = \bigvee \{E(\Delta)\mathscr{K} \mid \Delta \in \mathfrak{B}(A),\ \Delta \subset [\lambda, \rho],\ E(\Delta) \text{ is positive}\},$$

$$\mathscr{S}_{\alpha,-} = \bigvee \{E(\Delta)\mathscr{K} \mid \Delta \in \mathfrak{B}(A),\ \Delta \subset [\lambda, \rho],\ E(\Delta) \text{ is negative}\}.$$

Concerning these subspaces it is easy to see that $\mathscr{S}_{\alpha,+}$ is a non-negative subspace, $\mathscr{S}_{\alpha,-}$ is a non-positive subspace, $\mathscr{S}_{\alpha,+}[\perp]\mathscr{S}_{\alpha,-}$ and if one denotes $\mathscr{S}_\alpha = E([\lambda, \rho])\mathscr{K} \cap (\mathscr{S}_{\alpha,+} + \mathscr{S}_{\alpha,-})^{[\perp]}$ then $\mathscr{S}_\alpha = \bigvee_{n \in \mathbf{N}} \ker(A - \alpha I)^n$ [11, Proposition II. 5.1 and II. 5.2].

The spectral function E is called *regular at the critical point* α if there exist real numbers λ, ρ such that $[\lambda, \rho] \cap c(A) = \{\alpha\}$ and the family $\{E(\Delta) \mid \Delta \in \mathfrak{B}(A), \Delta \subset [\lambda, \rho]\setminus\{\alpha\}\}$ of selfadjoint projections is bounded, equivalently $\mathscr{S}_{\alpha,+}$ and $\mathscr{S}_{\alpha,-}$ are regular subspaces [11, Proposition II 5.6]. Correspondingly we say that the spectral function E is *pseudo-regular at the critical point* α if there exist real numbers λ, ρ such that $[\lambda, \rho] \cap c(A) = \{\alpha\}$ and $\mathscr{S}_{\alpha,+}$ and $\mathscr{S}_{\alpha,-}$ are pseudo-regular subspaces. It is easy to see that this definition does not depend on the choice of λ and ρ. Naturally, the spectral function is called *pseudo-regular* if it is pseudo-regular at each critical point.

In order to approach our main result we also need the following:

4.1. LEMMA. *Let A be a definitizable operator with spectral function E and $\mathscr{N}$ an A-invariant neutral subspace. If J is a* f.s. *on $\mathscr{K}$, $\mathscr{K} = \mathscr{L}_1 \oplus \mathscr{L}_2 \oplus \mathscr{L}_3$ is the J-reduction of $\mathscr{K}$ by $\mathscr{N}$ and P_2 is the selfadjoint projection on the regular subspace $\mathscr{L}_2$ then*

(i) *$P_2A \mid \mathscr{L}_2$ is a definitizable operator;*

(ii) *$c(P_2A \mid \mathscr{L}_2) \subset c(A)$;*

(iii) *$P_2E(\Delta) \mid \mathscr{L}_2$ is the spectral function of $P_2A \mid \mathscr{L}_2$.*

Proof. (i) Put $A_{22} = P_2A \mid \mathscr{L}_2 \in \mathscr{B}(\mathscr{L}_2)$. Evidently A_{22} is a selfadjoint operator. If p is a definitizing polynomial of A then since $A(\mathscr{L}_1 + \mathscr{L}_2) \subset \mathscr{L}_1 \oplus \mathscr{L}_2$ and $\mathscr{L}_1[\perp]\mathscr{L}_2$ we have

$$0 \leqslant [p(A)\xi_2, \xi_2] = [p(A_{22})\xi_2, \xi_2], \quad \xi_2 \in \mathscr{L}_2,$$

i.e. p is also a definitizing polynomial for A_{22}.

(ii) follows by the above proved fact and the caracterization of $c(A)$ given in [11, II.3].

(iii) If $\lambda \in \rho(A)$ (the resolvent set of A) then it is easy to see that $\lambda \in \rho(A_{22})$ and $P_2(A - \lambda I)^{-1} \mid \mathscr{L}_2 = (A_{22} - \lambda I)^{-1}$.

Since the spectral function can be written as the strong limit of integrals over the resolvent operator (see [11, Theorem II. 3.1]) the assertion (iii) follows. ▨

4.2. THEOREM. *If the spectral function E of the definitizable operator A is pseudo-regular then A has an invariant maximal non-negative subspace which is pseudo-regular.*

Proof. Let $c(A) = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and assume $\alpha_1 < \alpha_2 < \ldots < \alpha_n$. We can choose $\lambda_1, \lambda_2, \ldots, \lambda_{n+1} \in \mathbf{R}$ such that

$$-\|A\| - 1 = \lambda_1 < \alpha_1 < \lambda_2 < \ldots < \lambda_n < \alpha_n < \lambda_{n+1} = \|A\| + 1,$$

and consider $E_i = E([\lambda_i, \lambda_{i+1}])$, $i = \overline{1, n}$. Then a maximal non-negative subspace $\mathscr{M}$ is A-invariant iff $\mathscr{M}$ can be written

$$\mathscr{M} = \mathscr{M}_0 [+] \mathscr{M}_1 [+] \ldots [+] \mathscr{M}_n,$$

where $\mathscr{M}_i = E_i \mathscr{M}$ is $E_i A \mid E_i$-invariant and $E_i \mathscr{K}$-maximal non-negative subspace, $i = \overline{0, n}$. Since by [11, Proposition I. 3.2] and Lemma 3.5 $\mathscr{M}_0$ is a neutral subspace and also the pseudo-regularity of $\mathscr{M}$ is equivalent to the pseudo-regularity of $\mathscr{M}_i$ for any $i = \overline{1, n}$, it follows that we can assume $\sigma(A) \subset \mathbf{R}$ and $c(A)$ consists of a single point α.

Assuming now that the spectral function of A is pseudo-regular at α, we note that $\mathscr{S}^0_{\alpha,+} \subset \mathscr{S}^0_\alpha$ and also $\mathscr{S}^0_{\alpha,-} \subset \mathscr{S}^0_\alpha$ (indeed, these follow by $\mathscr{S}_{\alpha,+} [\perp] \mathscr{S}_{\alpha,-}$ and $\mathscr{S}^0_\alpha = (\mathscr{S}_{\alpha,+} \vee \mathscr{S}_{\alpha,-})^0$). Considering J a f.s. on $\mathscr{K}$, $\mathscr{K} = \mathscr{L}_1 \oplus \mathscr{L}_2 \oplus \mathscr{L}_3$ the J-reduction of $\mathscr{K}$ by $\mathscr{S}^0_\alpha$ (which is an A-invariant neutral subspace) and P_2 the selfadjoint projection on $\mathscr{L}_2$, by the preceding Lemma and Lemma 3.4 it follows that the spectral function of $P_2 A \mid \mathscr{L}_2$ is regular. On the other hand it is clear that A has a pseudo-regular invariant maximal non-negative subspace iff $P_2 A \mid \mathscr{L}_2$ has a pseudo-regular invariant $\mathscr{L}_2$-maximal non-negative subspace, hence we can assume without restricting the generality that E, the spectral function of A, is regular. Assuming this we immediately get the regularity of $\mathscr{S}_\alpha$ the root subspace of A corresponding to α. Consider now $\mathscr{N}_0$ a subspace with the property that it is maximal A-invariant and neutral (it exists by Zorn Lemma) and let $\mathscr{K} = \mathscr{S}_1 \oplus \mathscr{S}_2 \oplus \mathscr{S}_3$ be a J-reduction of $\mathscr{K}$ by $\mathscr{N}_0$. If Q_2 denotes the selfadjoint projection on $\mathscr{S}_2$ then due to the maximality of $\mathscr{N}_0$ the definitizable operator $Q_2 A \mid \mathscr{S}_2$ has no invariant neutral subspace, in particular $(\ker (Q_2 A \mid \mathscr{S}_2 - \alpha Q_2 \mid \mathscr{S}_2))^0 = \{0\}$. By [13, Theorem 2] $Q_2 A \mid \mathscr{S}_2$ has an invariant regular $\mathscr{S}_2$-maximal non-negative subspace, say it is $\mathscr{M}$, hence $\mathscr{M}[+]\mathscr{N}_0$ is a pseudo-regular A-invariant maximal non-negative subspace, which concludes the proof. ▨

The next example shows that in general the converse of the preceding theorem is not true.

4.3. EXAMPLE. Let $\mathscr{H}$ be a separable complex Hilbert space and $(e_k)_{k \in \mathbf{N}}$ an orthonormal basis of it. Denote $\mathscr{K} = \mathscr{H} \oplus \mathscr{H} \oplus \mathscr{H} \oplus \mathscr{H}$ and consider the operator

$$J = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \end{bmatrix}, \quad \text{w.r.t. } \mathscr{K} = \mathscr{H} \oplus \mathscr{H} \oplus \mathscr{H} \oplus \mathscr{H}.$$

If we put $[\xi, \eta] = (J\xi, \eta)$, $\xi, \eta \in \mathscr{K}$ then $(\mathscr{K}, [\cdot,\cdot])$ is a Kreĭn space and J is a f.s. on $\mathscr{K}$.

For every $k \in \mathbf{N}$ consider the vectors

$$f_k = \frac{e_k}{\sqrt{1+k^2}} \oplus 0 \oplus \frac{ke_k}{\sqrt{1+k^2}} \oplus 0, \quad g_k = 0 \oplus \frac{e_k}{\sqrt{1+k^2}} \oplus \frac{ke_k}{\sqrt{1+k^2}} \oplus 0,$$

and let P_k be the selfadjoint projection on $\mathbf{C}f_k$ and Q_k be the selfadjonint projection on $\mathbf{C}g_k$. It is easy to see that P_k is positive, Q_k is negative and $\{P_k, Q_n\}_{k,n\in\mathbf{N}}$ is a commutative family of operators.

Let $(\lambda_k)_{k\in\mathbf{N}}$ and $(\mu_n)_{n\in\mathbf{N}}$ be two sequences of positive numbers such that

$$\sum_{k\in\mathbf{N}} \lambda_k \|P_k\|_J < +\infty, \quad \sum_{n\in\mathbf{N}} \mu_n \|Q_n\|_J < +\infty,$$

and consider the compact operator

$$A = \sum_{k\in\mathbf{N}} \lambda_k P_k - \sum_{n\in\mathbf{N}} \mu_n Q_n .$$

The operator A is positive, in particular it is definitizable and $\mathrm{c}(A) = 0$. We also have $\mathscr{S}_{\alpha,+} = \bigvee_{k\in\mathbf{N}} P_k\mathscr{K}$ and $\mathscr{S}_{\alpha,-} = \bigvee_{n\in\mathbf{N}} Q_n\mathscr{K}$ and the fact that both $\mathscr{S}_{\alpha,+}$ and $\mathscr{S}_{\alpha,-}$ are non-regular non-degenerate (hence non-pseudo-regular) subspaces follows as in the proof of [5, Proposition 4.8]. On the other hand the subspace $\mathscr{M} = \mathscr{H} \oplus \{0\} \oplus \mathscr{H} \oplus \{0\}$ is A-invariant, pseudo-regular and maximal non-negative.

4.4. COROLLARY. *Let A be a definitizable operator such that for every $\alpha \in \mathrm{c}(A)$ the root subspace $\mathscr{S}_\alpha$ is regular. Then the following statements are equivalent:*

(i) *A has an invariant pseudo-regular maximal non-negative subspace.*

(ii) *The spectral function of A is regular.*

(iii) *The spectral function of A is pseudo-regular.*

Proof. (i) $\Rightarrow$ (ii). We first note that one can assume $\sigma(A) \subset R$ and $\mathrm{c}(A) = \{\alpha\}$; the argument is exactly as in the first part of the proof of Theorem 4.2. Now let $\mathscr{M}$ be an A-invariant pseudo-regular maximal non-negative subspace and denote by P_α the selfadjoint projection on the regular subspace $\mathscr{S}_\alpha$. Since $\mathscr{M}$ is P_α-invariant (indeed, $\mathscr{S}_\alpha = \bigcap_{\alpha\in\Delta} E(\Delta)\mathscr{K}$) it follows that $\mathscr{M} = P_\alpha\mathscr{M}[+](I - P_\alpha)\mathscr{M}$. By (3.2) we get $\mathscr{S}_{\alpha,+} \subset (I - P_\alpha)\mathscr{M}$ and $\mathscr{S}_{\alpha,-} \subset (I - P_\alpha)\mathscr{M}^{[\perp]}$ hence

$$(I - P_\alpha)\mathscr{K} = \mathscr{S}_{\alpha,+} \vee \mathscr{S}_{\alpha,-} \subset (I - P_\alpha)\mathscr{M} + (I - P_\alpha)\mathscr{M}^{[\perp]} \subset (I - P_\alpha)\mathscr{K},$$

hence $(I - P_\alpha)\mathscr{K}$ is a regular subspace and the same is true for $(I - P_\alpha)\mathscr{M}^{[\perp]}$. Since these two subspaces are also definite it follows that $\mathscr{S}_{\alpha,+}$ and $\mathscr{S}_{\alpha,-}$ are regular subspaces.

(ii) $\Rightarrow$ (iii) Obvious.

(iii) $\Rightarrow$ (i) Follows by the preceding Theorem. ▨

4.5. REMARK. Similarly as in Example 3.2 it is easy to construct a positive operator A with non-trivial regular root subspace $\mathscr{S}_0$ but with non-regular spectral function. By the above corollary such an operator has no invariant pseudo-regular maximal non-negative subspace.

4.6. REMARK. If A is a definitizable operator such that the condition from the above corollary holds then one can parametrize all the invariant pseudo-regular maximal non-negative subspaces (e.g. by means of [2]).

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