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# THE PROBLEM OF INTEGRAL GEOMETRY AND INTERTWINING OPERATORS FOR A PAIR OF REAL GRASSMANNIAN MANIFOLDS

I. M. GELFAND, M. I. GRAEV, R. ROŞU

## INTRODUCTION

Let $G_p$, $G_q$ be the Grassmannian manifolds of the $p$ respective $q$ dimensional subspaces of the real linear space $V\,(p<q)$. This paper is concerned with the remarkable integral transform associated with $G_p$ and $G_q$. It can be defined as the transform which associates with each function on $G_p$, a function on $G_q$; namely, if $f$ is a function on $G_p$ and $a\in G_q$, then $(\mathscr{I}f)(a)$ is the value of the integral of the function $f$ on $G_p(a)$— the set of the subspaces $b\in G_p$ which are contained in $a$. When so defining it, we already assume that on each $G_p(a)$ there is a given measure and, by this, a supplementary structure is being introduced on $G_p$. In order to avoid this we shall define the operator $\mathscr{I}$ slightly different: we introduce the function spaces $F_p^\lambda$ on $\tilde{G}_p$ — the manifold of the pairs $(b, \beta)$ where $b\in G_p$ and $\beta$ is a non-oriented volume element in $b$, which satisfy the homogeneity condition $f(b, t\beta) = t^\lambda f(b, \beta)$ for any $t>0$. We shall define (see §1, p. 3) $\mathscr{I}$ as an operator

$$\mathscr{I}: F_p^q \to F_q^p$$

which, for the natural representation of $\mathrm{SL}(V)$ on $F_q^p$ and $F_p^q$, is an intertwining operator (that means, by definition, that it commutes with the operators of the representation). If an Euclidian structure is introduced on $V$ and, by this, we have a measure on each $G_p(a)$ then the definition of the operator coincides with the one at the beginning.

The main purpose of this paper is to explicitly construct an operator $F_q^p\to F_p^q$ which on $\mathrm{Im}\,\mathscr{I}$, the image of $\mathscr{I}$, coincides with the inverse of $\mathscr{I}$ — that is obtaining an inversion formula for $\mathscr{I}$. (We shall consider the case $p+q\leqslant n$, when $\mathscr{I}$ is injective.)

Several approaches on the subject are already known: in [6] it was constructed the operator $\varkappa$ which associates with every function $\varphi\in F_q^p$ and every $b\in G_p$ the $p(q-p)$ differential form $\varkappa_b\varphi$ on the manifold $G_{q-p}$. It was proved that if $\varphi=\mathscr{I}f$

then $\varkappa_b \varphi$ is a closed form on $G_{q-p}$ and, moreover, for any $p(q-p)$-dimensional cycle $\gamma \subset G_{q-p}$

$$\int_\gamma \varkappa_b \varphi = C_\gamma f(b) \tag{1}$$

where $C_\gamma$ is depending only on the homology class of the cycle $\gamma$. For even $q-p$ there are cycles $\gamma$ with $C_\gamma \neq 0$ and then (1) is an inversion formula for $\mathscr{I}$; for odd $q-p$ there are no such cycles. The affine variant of the problem was solved in [1] for $p = 1$ and arbitrary $q$, and was considered in [12] for arbitrary $p$ and $q$. In the projective case the problem was solved in [4] for $p = 1$ and arbitrary $q$.

In this paper the inversion formula for $\mathscr{I}$ is obtained by means of simple standard constructions. We start by defining (see § 2) a sequence of functions $P_k$ $(1 \leqslant k < n-p)$ which by their geometrical intrinsic sense are in themselves important. The function $P_{q-p}$ is crucial in constructing the inversion formula. We then consider the manifold $\mathscr{E}$ of the pairs $(c, \tilde{b})$ where $c \in G_{q-p}$, $\tilde{b} = (b, \beta) \in \tilde{G}_p$ and $b \cap c = 0$. $\mathscr{E}$ is a bundle over $\tilde{G}_p$ having as fibre at $\tilde{b}$ the set $\pi^{-1}(\tilde{b}) = G^b_{q-p}$ consisting of those $c \in G_{q-p}$ with $c \cap b = 0$.

A central achievement of this paper is the construction of the intertwining operator

$$\chi : F^p_q \to \Omega^N(\mathscr{E})$$

where $\Omega^N$ is the space of $N$-densities on $\mathscr{E}$ $(N = p(q-p))$ with the natural action of the representation of $\mathrm{SL}(V)$. This operator is explicitly constructed in §3 by composing several mappings which, on the respective manifolds, commute with the action of the group $\mathrm{SL}(V)$.

Afterwards we obtain the inversion formula for $\mathscr{I}$ in §4 by means of the operator $\chi$. Namely we introduce a class of $N$-dimensional manifolds $C \subset G_{q-p}$, the harmonic manifolds, which play for the densities the same role the $N$-dimensional cycles play for differential forms. It is proved that if $C \subset \pi^{-1}(b)$ is a harmonic manifold, then the following inversion formula is true:

$$\int_C (\chi\varphi)(\cdot\,|\tilde{b}) = \mathrm{Cr}(C) f(\tilde{b}) \tag{2}$$

where the cofficient $\mathrm{Cr}(C)$ is depending only on $C$. We like to point out that, in the same manner, the operator $\varkappa$ can be defined as an intertwining operator (in [6] the operator $\varkappa$ was defined in a different way).

The inversion formula (2) naturally leads us to the notion of permissible complex in $G_q$. Namely, since for $p < q < n-p$ we have $\dim G_p < \dim G_q$, in this case, the knowledge of the function $\mathscr{I}f$ on the whole manifold $G_q$ is over-sufficient for recovering the function $f$. We can assume that the function $\mathscr{I}f$ is given only on a certain submanifold $K \subset G_q$, whose dimension is $k \geqslant \dim G_p$, and recover the function $f$ by means of the function $\mathscr{I}_K f = \mathscr{I}f|_K$. We call the submanifold $K \subset G_q$ a permissible complex if for almost every $b \in G_p$ there is a harmonic manifold $C_b$, such that for any $\varphi \in F_q^p$, $(\chi\varphi)(\cdot|\tilde{b})$ restricted at $C_b$ is uniquely defined by means of $\varphi|_K$. It is true that for permissible complexes, the associated operator $\mathscr{I}_K$ is one to one, but remarkable for them is the fact that the inversion formula for $\mathscr{I}_K$ has the explicit form given by (2) for $C = C_b$.

Using the structure of the operator $\chi$ we define in §5 a large class of permissible complexes. The problem of describing all the permissible complexes is of great interest.

## 1. THE INTERTWINING OPERATOR $\mathscr{I}$

1. PRELIMINARIES. 1° We define the (non-oriented) volume element $\omega$ in the real, $n$-dimensional vector space $V$, as a non-negative function $\omega(v_1, \dots, v_n)$, $v_i \in V$, which for any change $v_i \to \sum_{j=1}^{n} g_{ij}v_j$ $(i = 1, \dots, n)$ is multiplied by $|\det g|$; in particular $\omega(v_1, \dots, v_n) = 0$ for $v_1, \dots, v_n$ linearly dependent vectors. It is obvious that the volume element $\omega$ is uniquely determined by the number $\omega(e_1, \dots, e_n)$, where $\{e_1, \dots, e_n\}$ is a basis in $V$.

We introduce some notations. Let $\omega$ be a volume element in $V$. We then denote by $\omega'$ the volume element in the dual space $V'$ given by the equality

$$\omega'(e^1, \dots, e^n)\omega(e_1, \dots, e_n) = 1$$

where $\{e^i\}$ and $\{e_j\}$ are dual bases respectively in $V'$ and $V$. Next let $L$ be a subspace in $V$ and $\alpha$ a volume element in $L$. We denote by $\omega/\alpha$ the volume element in the factor space $V/L$, determined by the pair $\omega, \alpha$. That is if $\{e_1, \dots, e_k\}$ is a basis in $L$ and $\{e_{k+1}, \dots, e_n\}$ completes it up to a basis in $V$, then let $\tilde{e}_i$ $(k+1 \leqslant i \leqslant n)$ be the projection of $e_i$ on the factor space $V/L$. Then

$$\omega/\alpha(\tilde{e}_{k+1}, \dots, \tilde{e}_n) = \frac{\omega(e_1, \dots, e_n)}{\alpha(e_1, \dots, e_k)}.$$

Finally, if in the vector spaces $L_1, L_2$ there are respectively given the volume elements $\nu_1, \nu_2$, then in the tensorial product space $L_1 \otimes L_2$ we canonically define a volume element which is denoted by $\nu_1 \otimes \nu_2$. That is, if $\{e_i\}, \{f_j\}$ are bases respectively in $L_1$ and $L_2$ then

$$(\nu_1 \otimes \nu_2)(\{e_i \otimes f_j\}) = \nu_1(\{e_i\})\nu_2(\{f_j\}).$$

It is easy to verify that the introduced definitions are correct, that is, they are independent of the choice of the bases in the respective vector spaces. The next lemma follows from the definitions.

LEMMA. *For any $t > 0$, $s > 0$, the following are true:*

$$(1.1') \qquad (t\omega)' = t^{-1}\omega$$

$$(1.1'') \qquad s\omega/t\alpha = s/t\ \omega/\alpha$$

$$(1.1''') \qquad (s v_1) \otimes (t v_2) = s^{\dim L_2} t^{\dim L_1}(v_1 \otimes v_2).$$

2° We mean by $k$ (even) density on the smooth manifold $X$, $\dim X = n > k$, a smooth function $\sigma$ which associates with each pair $(x, h)$ where $x \in X$ and $h$ is a $k$-dimensional subspace of $T_xX$, a volume element in $h$. In other words, $\sigma$ is a function of $x \in X$ and of the vectors $\xi_1, \ldots, \xi_k \in T_xX$, which for any change $\xi_i \to \sum_{j=1}^{k} g_{ij}\xi_j$ $(i = 1, \ldots, k)$ is multiplied by $|\det g|$.

For $k$-densities we can define by the same methods used for $k$ differential forms, the following operations:

1) integration on $k$ dimensional submanifolds in $X$,

2) the pullback $\pi^*$ from $X$ to $Y$ where $\pi : Y \to X$ is a bundle,

3) integration of the $k$-densities on $Y$ denoted by $\pi_*$ on the fibres of the bundle $\pi : Y \to X$ ($\pi_*$ is defined for $k \geqslant l$, $l$ — the fibre's dimension, and it transforms the $k$-densities on $Y$ into $k - l$ densities on $X$).

2. THE SPACES $F_p^\lambda$. Let $G_p(V)$ be the manifold of the $p$ dimensional subspaces of the vector space $V$, $\dim V = n$. We denote by $\tilde{G}_p(V)$ the manifold of the pairs $\tilde{b} = (b, \beta)$ where $b \in G_p(V)$, $\beta$ is a volume element in $b$ (clearly $\tilde{G}_p(V)$ can be viewed as line bundle over $G_p(V)$). We introduce the spaces $F_p^\lambda = F_p^\lambda(V)$ $(p = 1, \ldots, n-1, \lambda \in \mathbf{C})$ of $C^\infty$ functions on $\tilde{G}_p(V)$ which satisfy the homogeneity condition

$$(1.2) \qquad f(b, t\beta) = t^\lambda f(b, \beta)$$

for any $t > 0$. We define a representation of the group $\mathrm{SL}(V)$ on each space $F_p^\lambda$ by:

$$(T(g)f)(\tilde{b}) = f(\tilde{b}g)$$

where $\tilde{b} \to \tilde{b}g$ is the natural action of the element $g \in \mathrm{SL}(V)$ on $\tilde{G}_p(V)$.

Next, we shall give several interpretations for the spaces $F_p^\lambda$ that shall be used later on.

1) $F_p^\lambda(V) \simeq F_{n-p}^\lambda(V')$ where $V'$ is the dual space of $V$. This isomorphism is induced by the one to one and onto mapping $\tilde{G}_p(V) \to \tilde{G}_{n-p}(V')$ which carries $(b, \beta) \in \tilde{G}_p(V)$ to $(\mathrm{Ann}\, b, (\omega/\beta)')$ in $\tilde{G}_{n-p}(V')$, $\omega$ — a fixed volume element in $V$.

2) We consider $F_p^\lambda(V)$ as a $C^\infty$ function space on $G_p(V)$. In order to do this we give an Euclidian structure on $V$ and denote by $\beta_b$ the Euclidian volume element in $b \in G_p(V)$. We obtain the desired interpretation by associating with each function $f \in F_p^\lambda(V)$ the function $\varphi(b) = f(b, \beta_b)$ on $\tilde{G}_p(V)$.

3) Notice that $\tilde{G}_p(V)$ can be interpreted as the manifold of $p$ vectors $u = u_1 \wedge \dots \wedge u_p \neq 0$ of $V$, where we identified $u$ with $-u$. That is, for each pair $(b, \beta) \in \tilde{G}_p(V)$ there is a corresponding $p$-vector $u = u_1 \wedge \dots \wedge u_p$ given, up to multiplication by $-1$, by the conditions: the space spanned by $u_1, \dots, u_p$ is $b$ and $\beta(u_1, \dots, u_p) = 1$. By this, $F_p^\lambda$ can be regarded as the space of $C^\infty$ functions on the manifold of the $p$-vectors $u \neq 0$ which satisfy

$$f(tu) = |t|^{-\lambda} f(u) \quad \text{for any } t \neq 0.$$

From this interpretation we get two more others.

4) We regard $F_p^\lambda$ as the space of $C^\infty$ functions on the manifold $E_{p,n}$ of the $p$-frames in $V$, which satisfy the condition

$$f(gx) = |\det g|^{-\lambda} f(x) \tag{1.3}$$

for any $g \in \mathrm{GL}(p, \mathbf{R})$. That is, as $E_{p,n}$ is a bundle on the manifold of non-zero $p$-vectors, we obtain the desired interpretation associating with each function defined on the manifold of $p$ vectors, its pullback on $E_{p,n}$.

5) In a coordinate system for $V$, any $p$-frame is given by a $p \times n$ matrix of rank $p$. So, $F_p^\lambda$ is interpreted as the $C^\infty$ function space on the manifold of $p \times n$ matrices of rank $p$, which satisfy the condition (1.3), or, equivalently, as the space of $C^\infty$ functions that are even, homogenous, of $-\lambda$ homogeneity on the $p$-minors of the matrix $x$.

6) $F_p^\lambda$ interpreted as the $C^\infty$ function space on the manifold $M_{p,n-p}$ of $p \times (n-p)$ matrices *). That is done by associating with each matrix $u = \|u_i^j\|$ in $M_{p,n-p}$, the $p$-vector $\tau u = v_1 \wedge \dots \wedge v_p$ where

$$v_i = (\delta_i^1, \dots, \delta_i^p, u_i^1, \dots, u_i^{n-p}) \quad i = 1, \dots, p$$

($\delta_j^i$ - Kronecker's symbol). It is obvious that the projection of the set $\tau M_{p,n-p}$ on $G_p(V)$ is an open and dense subset of $G_p(V)$. We obtain the desired interpretation, associating with each function $f$ defined on $\tilde{G}_p(V)$, the function $(\tau^* f)(u) = f(\tau u)$ defined on $M_{p,n-p}$.

NOTE. There is a purely group interpretation of the spaces $F_p^\lambda(V)$. They are interpreted as the $C^\infty$ function spaces on the group $\mathrm{SL}(n, \mathbf{R})$, satisfying the condition $f(ug) = |\det a|^\lambda f(u)$ for any block triangular matrix $g = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$ where $a$ is a $p$-matrix. In this interpretation, the operators of the representation act by right translations.

*) These functions satisfy supplementary conditions of decreasing at infinity, which are not presented here.

3. THE DEFINITION OF THE INTERTWINING OPERATOR $\mathscr{I}$. The main subject of this paper is the intertwining operator

$$\mathscr{I} : F_p^q \to F_q^p \quad (1 \leqslant p < q < n)$$

that is, an operator which commutes with the operators of the representation. This operator has a simple geometrical sense: roughly speaking, $(\mathscr{I}f)(a)$, $a \in G_q(V)$ is equal to the value of the integral of the function $f$ on the set of the $p$ subspaces that are contained in $a$; we shall now give the exact definition of the operator $\mathscr{I}$.

Let $f \in F_p^q$, $\tilde{a} = (a, \alpha) \in \tilde{G}_q(V)$, $\tilde{b} = (b, \beta) \in \tilde{G}_p(a)$. We denote by $\sigma_{\tilde{a}}(\tilde{b})$ the volume element in the tangent space $T_b G_p(a) = b' \otimes a/b$ canonically defined by the volume elements $\alpha$ and $\beta$:

$$\sigma_{\tilde{a}}(\tilde{b}) = \beta' \otimes \alpha/\beta. \tag{1.4}$$

By the homogeneity conditions (1.1) and (1.2) it follows that the product $f(\tilde{b})\sigma_{\tilde{a}}(\tilde{b})$ is independent of the choice of the volume element $\beta$ and, therefore, it defines for any fixed $a$ a $N$-density on $G_p(a)$, where $N = p(q - p)$. We define

$$(\mathscr{I}f)(\tilde{a}) = \int_{G_p(a)} f(\tilde{b})\sigma_{\tilde{a}}(\tilde{b}). \tag{1.5}$$

It follows immediately from the definition that $\mathscr{I}f \in F_q^p$ and that $\mathscr{I}$ is an intertwining operator.

We give now the formula of $\mathscr{I}$ in coordinates when $F_p^q$ and $F_q^p$ are interpreted as function spaces respectively on the manifold of $p \times (n - p)$ matrices and on the manifold of $q \times (n - q)$ matrices. We write $u \in M_{p,n-p}$ and $v \in M_{q,n-q}$ as block matrices: $u = (u_1, u_2)$ where $u_1 \in M_{p,q-p}$, $u_2 \in M_{p,n-q}$ and $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ where $v_1 \in M_{p,n-q}$, $v_2 \in M_{q-p,n-q}$. Then

$$(\mathscr{I}f)(v) = \int_{M_{p,q-p}} f(t, v_1 + tv_2)\mathrm{d}t \tag{1.6}$$

where $\mathrm{d}t = \prod \mathrm{d}t_i^j$.

Our purpose is to obtain inversion formulas for $\mathscr{I}$, that is to construct an inverse operator $\mathscr{I}^{-1}$ on $\operatorname{Im}\mathscr{I}$. As for $q > n - p$ implies $\dim G_p > \dim G_q$, $\operatorname{Ker}\mathscr{I}$, the kernel of the operator $\mathscr{I}$, is non-zero in this case. That is why we shall from now on assume $q \leqslant n - p$.

## 2. THE SEQUENCE OF FUNCTIONS $P_k$

1. DEFINITION OF THE FUNCTIONS $P_k$ IN GEOMETRIC MANNER. By means of simple geometric constructions we shall define a sequence of functions $P_k(\tilde{c}, A|\tilde{h})$, $1 \leqslant k < n - p$, where $\tilde{c} = (c, \gamma) \in \tilde{G}_k(V)$, $\tilde{h} = (h, \eta) \in \tilde{G}_p(V')$, $c \perp h$ and $A = \{A_i\}$, $i = 1, \ldots, p \cdot k$, vectors in the tangent space $T_c G_k(V)$.

The function $P_{q-p}$ plays a central role in the construction of the operator $\chi$.

Let $k$ be fixed. We consider $\mathscr{B}$ the manifold of the pairs $(c, h)$ where $c \in G_k(V)$, $h \in G_p(V')$ and $c \perp h$. We define the mappings $\pi_1 : \mathscr{B} \to G_p(V')$ and $\pi_2 : \mathscr{B} \to G_k(V)$ by $\pi_1(c, h) = h$, $\pi_2(c, h) = c$. We obtain a double fibration

$$\begin{array}{ccc} & \mathscr{B} & \\ {\scriptstyle \pi_1} \swarrow & & \searrow {\scriptstyle \pi_2} \\ G_p(V') & & G_k(V) \, . \end{array}$$

We notice that $\pi_2^{-1}(c) \simeq G_p((V/c)') = G_p(\operatorname{Ann} c)$.

Let $\psi(\tilde{h})$ be an arbitrary smooth function on $\tilde{G}_p(V')$ which satisfies the condition $\psi(h, t\eta) = t^n \psi(h, \eta)$ for any $t > 0$. Then $\psi(\tilde{h})\sigma_{(\tilde{V}')}(\tilde{h})$ is independent of the choice of the volume element and defines a $p(n-p)$ density on $G_p(V')$. Consider on $\mathscr{B}$ the pullback of this density, that is, the $p(n-p)$ density $\tau = \pi_1^*(\psi\sigma_{\tilde{V}'})$ given by the following equality:

$$\tau(c, h, B_1, \ldots, B_r) = (\psi(\tilde{h})\sigma_{\tilde{V}'}(\tilde{h})((D\pi_1)B_1, \ldots, (D\pi_1)B_r)$$

where $B_i \in T_{c,h}\mathscr{B}$ and $r = p(n-p)$.

We fix $A = \{A_i\}$ an arbitrary system of $p \cdot k$ vectors in the tangent space $T_c G_k(V)$ and let $\bar{A} = \{\bar{A}_i\}$ where $\bar{A}_i \in T_{c,h}\mathscr{B}$ are vectors in the preimage of $A_i$. Then in $T_{c,h}\mathscr{B}$ we choose $B = \{B_i\}$ a system of $p(n-k-p)$ tangent vectors to the fibre of the bundle $\pi_2$, that is $B_i \in T_h G_p((V/c)')$. Then, for fixed $c$, $\tilde{h}$ and $A$, $\tau(c, h, A, B)$ is independent of the choice of the vectors $\bar{A}_i$ and defines a volume element in $T_h G_p(\operatorname{Ann} c)$. On the other hand $\psi(\tilde{h})\sigma_{\operatorname{Ann} c, (\omega/\gamma)'}(\tilde{h})$ is also a volume element in $T_h G_p(\operatorname{Ann} c)$ hence $\tau(c, h, \bar{A}, B)$ and $(\psi(\tilde{h})\sigma_{\operatorname{Ann} c, (\omega/\gamma)'}(\tilde{h}))(B)$ differ by a multiplicative factor which is independent of $B$.

DEFINITION. We define $P_k(\tilde{c}, A|\tilde{h})$ by the equality

$$\pi_1^*(\psi(\tilde{h})\sigma_{\tilde{V}}(\tilde{h}))_{(c, h, \bar{A}, B)} = P_k(\tilde{c}, A|\tilde{h})(\psi(\tilde{h})\sigma_{\operatorname{Ann} c, (\omega/\gamma)'}(\tilde{h}))(B). \tag{2.1}$$

It is obvious that $P_k$ is independent of the choice of $\psi$. It follows from the definition that the function $P_k$ is invariant for the action of the group $\mathrm{SL}(V)$ on the manifold of the triplets $(\tilde{c}, A|\tilde{h})$. From (2.1) we have

$$P_k(\tilde{c}, A|\cdot) \in F_p^{-k}((V/c)').$$

(Rigourously $P_k(\tilde{c}, A|\cdot)$ belongs to some extension of $F_p^{-k}((V/c)')$ consisting of piecewise smooth functions.)

2. EXPLICIT FORMULA FOR THE FUNCTIONS $P_k(\tilde{c}, A|\tilde{h})$.

PROPOSITION. *Let* $\tilde{c} = (c, \gamma)$, $\tilde{h}=(h,\eta)$, $A=\{A_1,\ldots,A_{p\cdot k}\}$ *where* $A_i \in T_cG_k(V)= \mathrm{Hom}(c, V/c)$. *Then*

$$P_k(\tilde{c}, A|\tilde{h}) = |\det\|\langle v^i, A_s v_j\rangle\|\,| \tag{2.2}$$

*where* $\{v_j\}$, $\{v^i\}$ *are some unitary bases respectively in* $c$ *and* $h \in G_p((V/c)')$, *that is* $\eta(\{v^i\}) = \gamma(\{v_j\}) = 1$ *(the right side is independent of their choice).*

We point out that if in $\tilde{c}$ and $\tilde{h}$ there are respectively given the frames $z = (z_1, \ldots, z_n)$ and $\xi = (\xi^1, \ldots, \xi^p)$ then (2.2) can be written as

$$P_k(z, A|\xi) = |\det\|\langle \xi^i, A_s z_j\rangle\|\,| \quad (z\xi' = 0). \tag{2.3}$$

*Proof.* Both sides of (2.2) are invariant for the action of the group $\mathrm{SL}(V)$. Therefore, as the group acts transitively on the manifold of pairs $(\tilde{c}, \tilde{h})$ it suffices to prove (2.2) for a certain pair $\tilde{c} = \tilde{c}_0$, $\tilde{h} = \tilde{h}_0$. We assume that there are given dual coordinate systems in $V$ and $V'$. On $G_p(V')$ we define local coordinates $(\xi, \eta)$

$$\xi = \|\xi_i^j\|_{\substack{i=1,\ldots,p,\\ j=1,\ldots,k}} \quad \eta = \|\eta_i^j\|_{\substack{i=1,\ldots,p\\ j=1,\ldots,n-p-k}};$$

for each pair $(\xi, \eta)$ there is a corresponding $p$-subspace spanned by the row vectors of the $p\times k$ matrix $h = (\xi, I_p, \eta)$. At the same time we consider $F_p^\lambda(V')$ as space of functions of $(\xi, \eta)$. Analogously we can define on $G_k(V)$ local coordinates

$$a = \|a_i^j\|_{\substack{i=1,\ldots,k,\\ j=1,\ldots,p}} \quad b = \|b_i^j\|_{\substack{i=1,\ldots,k\\ j=1,\ldots,n-k-p}};$$

for each pair $(a, b)$ there is a corresponding $k$-subspace spanned by the row vectors of the matrix $(I_k, a, b)$. As the ortogonality condition of $c = (I_k, a, b)$ and $h = (\xi, I_p, \eta)$ is written $a + \xi' + b\eta' = 0$, we can choose as coordinates on $\mathscr{B}$ the triplet $(a, b, \eta)$. In these coordinates, we have

$$\pi_1(a, b, \eta) = (-a' - \eta b', \eta).$$

Then if we fix $c_0 = (I_k, 0, 0)$ and $h_0 = (0, I_p, 0)$ we find that in $(c_0, h_0)$ the differential of the mapping $\pi_1$ is written

$$(D\pi_1)(A, B, \eta) = (-A', \eta).$$

By standard computation we get the equality (2.2) for $(c, h) = (c_0, h_0)$.

### 3. THE INTERTWINING OPERATOR $\chi$

1. THE SPACE $\Omega^N(\mathscr{E})$. We shall denote by $\mathscr{E} = \mathscr{E}_{q-p}$ the manifold of the pairs $(c, \tilde{b})$ where $c \in G_{q-p}(V)$, $\tilde{b} = (b, \beta) \in \tilde{G}_p(V)$ and $b \cap c = 0$; and by $\Omega^N(\mathscr{E})$, $N = p(q-p)$, the space of $N$-densities $\tau(c, b)$ on $\mathscr{E}$ satisfying the condition

$\tau(c, (b, t\beta)) = t^q\tau(c, (b, \beta))$ for any $t > 0$. As $SL(V)$ acts naturally on $\mathscr{E}$, we can define on $\Omega^N(\mathscr{E})$ a representation of $SL(V)$ by translation operators.

In this paragraph we shall construct an intertwining operator

$$\chi : F_q^p \to \Omega^N(\mathscr{E})$$

which is crucial in the inversion formula. We shall obtain the inversion formula for $\mathscr{I}$ in §4.

2. The Intertwining Operators $R_p^\lambda$. The construction of $\chi$ will be done in several steps. We start by constructing the intertwining operators

$$R_p^\lambda(V, \omega) : F_p^\lambda(V) \to F_q^{n-\lambda}(V').$$

(Recall that $F_p^{n-\lambda}(V') \simeq F_{n-p}^{n-\lambda}(V)$.) In order to do this we introduce the function $U^{(\lambda)}(\tilde{h}, \tilde{b})$ on the manifold of the pairs $(\tilde{h}, \tilde{b})$ where $\tilde{h} = (h, \eta) \in \tilde{G}_p(V')$, $\tilde{b} = (b, \beta) \in \tilde{G}_p(V)$, defined by the equality

$$U^{(\lambda)}(\tilde{h}, \tilde{b}) = C_{p,n}(\lambda)|\det\|\langle f^i, e_j\rangle\|| \quad (\lambda \in \mathbf{C}). \tag{3.1}$$

Here $\{f^i\}$, $\{e_j\}$ are unitary bases respectively in $h$ and $b$ (the right side is independent of their choice),

$$C_{p,n}(\lambda) = \pi^{-p(n-p)}\Gamma_p\left(\frac{\lambda+n}{2}\right)/\Gamma_p\left(\frac{\lambda+p}{2}\right) \tag{3.2}$$

where $\Gamma_p(s) = \prod_{k=0}^{p-1}\Gamma\left(s - \frac{k}{2}\right)$. We point out that

$$U^{(\lambda)}(h, t\eta; b, s\beta) = (ts)^{-\lambda}U^{(\lambda)}(h, \eta; b, \beta) \tag{3.3}$$

for any $t > 0$, $s > 0$.

Let $f \in F_p^\lambda$. From (1.1), (1.2) and (3.3) it follows that the product $U^{(\lambda-n)}(\tilde{h}, \tilde{b})f(\tilde{b})\sigma_{\tilde{V}}(b)$, is independent of $\beta$. Here $\tilde{V} = (V, \omega)$ and $\sigma_{\tilde{V}}(\tilde{b}) = \beta' \otimes \omega/\beta$. Therefore for any fixed $\tilde{h}$, it defines a $p(n-p)$ density on $G_p(V)$. By definition:

$$(R_p^\lambda(\tilde{V})f)_{(\tilde{h})} = \int_{G_p(V)} U^{(\lambda-n)}(\tilde{h}, \tilde{b})f(\tilde{b})\sigma_{\tilde{V}}(\tilde{b}). \tag{3.4}$$

The integral (3.4) is convergent for $\operatorname{Re}\lambda > n-1$. In order to make it have sense for $\operatorname{Re}\lambda \leqslant n-1$, we give an Euclidian structure on $V$ and regard $F_p^\lambda$ as $C^\infty$ function space on $G_p(V)$ (see §1 p. 2). Then the operator $R_p^\lambda$ is written:

$$(R_p^\lambda f)_{(h)} = \int_{G_p(V)} U^{(\lambda-n)}(h, b)f(b)\sigma(b) \tag{3.5}$$

where $U_0^\lambda(h, b) = U^{(\lambda)}(h, \eta_h; b, \beta_b)$, $\sigma(b) = \sigma_{\tilde{V}}(b, \beta_b)$ and $\eta_h$, $\beta_b$ are the Euclidian volume elements respectively in $h$ and $b$.

For a fixed $f$, the integral (3.5) is convergent in the domain $\mathrm{Re}\,\lambda > n - 1$ and is $\lambda$-analytic in this domain. For $\mathrm{Re}\,\lambda \leqslant n - 1$ we define $R_p^\lambda f$ as the analytic continuation in $\lambda$. It is easy to verify that this definition is independent of the choice of the Euclidian structure on $V$.

It follows from the definition that in any point in whose neighbourhood $R_p^\lambda(\tilde{V})$ is analytic (such points will be called regular for the operators $R_p^\lambda(\tilde{V})$), it carries the spaces $F_p^\lambda(V)$ into $F_p^{n-\lambda}(V')$ and is an intertwining operator.

NOTE. If we change $\omega$ by $t\omega$ ($t > 0$), then, $\sigma_{V,\omega}(\tilde{b})$ is multiplied by $t^p$, so from (3.4) we have

$$R_p^\lambda(V, t\omega) = t^p R_p^\lambda(V, \omega) \quad \text{for any } t > 0.$$

Elementary computation leads us to the formula of $R_p^\lambda(V, \omega)$ in coordinates, when $F_p^\lambda(V)$ is interpreted as the $C^\infty$ function space on the $p \times (n-p)$ matrices $u = \|u_i^j\|$ and analogously $F_p^{n-\lambda}(V')$ is interpreted as the $C^\infty$ function space on the matrices $(n-p) \times p$, $\xi = \|\xi_j^i\|$. Namely

$$(3.6) \qquad (R_p^\lambda(\tilde{V})f)_{(\xi)} = C_{p,n}(\lambda - n) \int\limits_{M_{p,n-p}} f(u)|I_p + u\xi|^{\lambda-n} \mathrm{d}u$$

where $\mathrm{d}u = \prod \mathrm{d}u_i^j$, $I_p$ being the $p$ identity matrix and we denote by $|\cdot|$, the absolute value of the determinant of the respective matrix.

## 3. REGULARITY CONDITIONS FOR THE OPERATORS $R_p^\lambda(\tilde{V})$ AND THE INVERSION FORMULA.

PROPOSITION 1. *For $p > 1$, $R_p^\lambda(\tilde{V})$ is $\lambda$ regular whenever $\lambda \neq (p-1) - k$, $k = 0, 1, \ldots$ ; for $p = 1$, $R_p^\lambda(\tilde{V})$ is $\lambda$ regular whenever $\lambda \neq 2k$, $k = 0, 1, \ldots$ .*

*Proof.* It is known (see for instance [10], [11]) that the generalized function $\dfrac{|\det x|^\lambda}{\Gamma_p\left(\frac{\lambda+2}{2}\right)}$ on the manifold of $p$-matrices is a $\lambda$-entire function. Therefore, the singular points of $R_p^\lambda$ as function of $\lambda$, coincide with the singular points of the function $\Gamma_p\left(\frac{\lambda}{2}\right)$. Hence our statement follows immediately.

PROPOSITION 2. *If $\lambda$ and $n - \lambda$ are regular points for the operators $R_p^\lambda$ then $R_p^{n-\lambda}(\tilde{V}')\circ R_p^\lambda(\tilde{V})$ is the identity operator on $F_p^\lambda(V)$.*

COROLLARY. *With the same assumption on $\lambda$, the mappings*

$$R_p^\lambda(V) : F_p^\lambda(V) \to F_p^{n-\lambda}(V')$$

*and*

$$R_p^{n-\lambda}(\tilde{V}') : F_p^{n-\lambda}(V') \to F_p^\lambda(V)$$

*are isomorphisms.*

*Proof.* We shall make use of the following statement: any intertwining operator which maps $F_p^\lambda(V)$ on itself is proportional to the identity operator $E$. By this statement we have $R_p^{n-\lambda}(\tilde{V}')\circ R_p^\lambda(V) = cE$. So we have to show that $c = 1$. In order to do this we give respectively on $V$ and $V'$, dual inner products ( , ) and denote by $\mathrm{SO}(n)$ the subgroup of the group $\mathrm{SL}(V)$ which consists of those transformations which preserve the inner product. In $F_p^\lambda(V)$ and $F_p^{n-\lambda}(V')$ there is respectively a unique vector, up to a multiplicative constant, which is invariant for $\mathrm{SO}(n)$. Namely the functions $f(b, \beta) = |\det\|(v_i, v_j)\||^{-\lambda/2}$ respectively $\varphi(h, \eta) = |\det\|(v^i, v^j)\||^{-((n-\lambda)/2)}$ where $\{v_i\}$ and $\{v^j\}$ are unitary basis respectively in $b \in G_p(V)$ and $h \in G_p(V')$. Hence $R_p^\lambda(\tilde{V})f = S_\lambda\varphi$, $R_p^{n-\lambda}(\tilde{V}')\varphi = S_{n-\lambda}f$ and it suffices to show that $S_\lambda = S_{n-\lambda} = 1$. For that we use the matriceal interpretation of $F_p^\lambda(V)$ and $F_p^{n-\lambda}(V')$ where we have

$$f(u) = |I_p + uu'|^{-\lambda/2}, \quad \varphi(\xi) = |I_p + \xi'\xi|^{-\frac{n-\lambda}{2}}.$$

Thus the equality $R_p^\lambda f = S_\lambda\varphi$ becomes

$$S_\lambda|I_p + \xi'\xi|^{-\frac{n-\lambda}{2}} = C_{p,n}(\lambda - n)\int_{M_{p,n-p}} |I_p + uu'|^{-\lambda/2}|I_p + u\xi|^{\lambda-n}\mathrm{d}u.$$

Computing for $\xi = 0$, we get

$$S_\lambda = C_{p,n}(\lambda - n)\int_{M_{p,n-p}} |I_p + uu'|^{-\lambda/2}\mathrm{d}u.$$

As $\int_{M_{p,n-p}} |I_p + uu'|^{-\lambda/2}\mathrm{d}u = [C_{p,n}(\lambda - n)]^{-1}$ (see Appendix 2 following this point), we have $S_\lambda = 1$.

APPENDIX 1. The explicit formula of the generalized function $\dfrac{|\det x|^\lambda}{\Gamma_p\left(\frac{\lambda+p}{2}\right)}$ for $\lambda = -p - k$, $k = 0, 1, \ldots$ (see [10], [11]).

Let $\delta(x)$ be the delta function on $M_p$-the manifold of $p$-matrices $(\delta, f) = f(0)$ for any test function $f$. As it is known $\delta(x) = (2\pi)^{-p^2}\mathscr{F}(\mathbf{1})$, where $\mathscr{F}$ is the Fourier transform defined on the test functions by the equality

$$(\mathscr{F}f)(x) = \int f(x)\mathrm{e}^{i\cdot\mathrm{tr}(y'x)}\mathrm{d}y \quad \mathrm{d}y = \prod \mathrm{d}y_{ij},$$

$\mathbf{1}$ being the function identically equal to 1.

Motivated by the definition of the delta function $\delta(x)$, we introduce the distribution $\gamma(x)$ by the equality

$$\gamma(x) = (2\pi)^{-p^2}\mathscr{F}(\operatorname{sgn}(\det x)).$$

Unlike for $\delta(x)$, the support of the distribution $\gamma$ is the whole manifold $M_p$. Considering $\delta^{(k)}(x) = \mathscr{D}^k\delta(x)$, $\gamma^{(k)}(x) = \mathscr{D}^k\gamma(x)$, $k = 0, 1, \ldots$ where $\mathscr{D} = \det \frac{\partial}{\partial x} = \det\left\|\frac{\partial}{\partial x_{ij}}\right\|$, we have with these notations:

$$\left.\frac{|\det x|^\lambda}{\Gamma_p\left(\frac{\lambda+p}{2}\right)}\right|_{\lambda=-p-2k} = \frac{(-1)^{p\cdot k}\cdot\pi^{p^2/2}}{2^{2kp}\Gamma_p\left(k+\frac{p}{2}\right)}\,\delta^{(2k)}(x)$$

$$\left.\frac{|\det x|^\lambda}{\Gamma_p\left(\frac{\lambda+p}{2}\right)}\right|_{\lambda=-p-2k-1} = \frac{(-i)^{(2k+1)p}\cdot\pi^{p^2/2}}{2^{(2k+1)}\Gamma_p\left(k+\frac{p+1}{2}\right)}\,\gamma^{(2k+1)}(x).$$

APPENDIX 2. Let us make the computations for the formulas

$$I^\lambda_{p,m} = \int\limits_{M_{p,m}} |I_p + xx'|^{-\lambda/2}\mathrm{d}x = \pi^{\frac{pm}{2}}\frac{\Gamma_p\left(\frac{\lambda-m}{2}\right)}{\Gamma_p\left(\frac{\lambda}{2}\right)}.$$

For $p = 1$ we have $I^\lambda_{1,m} = \int(1 + x_1^2 + \ldots + x_m^2)^{-\lambda/2}\mathrm{d}x_1 \ldots \mathrm{d}x_m$. This integral is easily computed by passing to spherical coordinates:

$$I^\lambda_{1,m} = \pi^{m/2}\frac{\Gamma\left(\frac{\lambda-m}{2}\right)}{\Gamma\left(\frac{\lambda}{2}\right)}.$$

Suppose now that $p > 1$. We write $x$ as $x = \begin{pmatrix} y \\ z \end{pmatrix}$ where $z$ is the last row of the matrix $x$. The matrix $I_m + xx'$ is positively defined; we consider $u = (I_m + y'y)^{1/2}$, $v = zu^{-1}$. Then we have $I_m + x'x = I_m + y'y + z'z = u(I_m + v'v)u$ and it follows that $|I_m + x'x| = |I_m + v'v|\cdot|I_m + y'y|$. Therefore, using the evident equality $|I_p + xx'| = |I_m + x'x|$ we have $|I_p + x'x| = (1 + vv')|I_{p-1} + yy'|$. And as $\mathrm{d}x =$

$= \mathrm{d}y\,\mathrm{d}z = |I_{p-1} + yy'|^{1/2}\,\mathrm{d}y\,\mathrm{d}v$ we conclude that

$$I_{p,m} = \int (1 + vv')^{-\lambda/2}\,\mathrm{d}v \int |I_{p-1} + yy'|^{-\frac{\lambda-1}{2}}\,\mathrm{d}y = I^{\lambda}_{1,m} I^{\lambda-1}_{p-1,m}.$$

It follows that

$$I^{\lambda}_{p,m} = I^{\lambda}_{1,m}\cdot I^{\lambda-1}_{1,m} \ldots I^{\lambda-p-1}_{1,m} = \pi^{\frac{pm}{2}} \frac{\Gamma_p\left(\frac{\lambda-m}{2}\right)}{\Gamma_p\left(\frac{\lambda}{2}\right)}.$$

REMARK. By $|I_p + xx'| = |I_m + x'x|$ we have that $I^{\lambda}_{p,m} = I^{\lambda}_{m,p}$. Therefore we find the following interesting formula for $\Gamma_p$

$$\frac{\Gamma_p\left(\frac{\lambda-m}{2}\right)}{\Gamma_p\left(\frac{\lambda}{2}\right)} = \frac{\Gamma_m\left(\frac{\lambda-p}{2}\right)}{\Gamma_m\left(\frac{\lambda}{2}\right)}.$$

4. THE OPERATORS $\hat{u} : F_p(V) \to F^{\lambda+\mu}_p(V)$. If $f \in F_p(V)$, $u \in F^{-\mu}_p(V')$ then $u\cdot(R^{\lambda}_p(\tilde{V})f) \in F^{n-\lambda-\mu}_p(V')$. We associate with the function $u$ the operator $\hat{u} : F^{\lambda}_p(V) \to$ $\to F^{\lambda+\mu}_p(V)$ defined by the equality

$$\hat{u}f = R^{n-\lambda-\mu}_p(\tilde{V}')(u\cdot R^{\lambda}_p(\tilde{V})f). \tag{3.7}$$

We shall call $u$ the symbol of the operator $\hat{u}$ and $\mu$ its order. The given definition still makes sense when $u$ is a piecewise smooth function. In this case $\hat{u}f$ will be a function from a certain extension of the space $F^{\lambda+\mu}_p(V)$.

We now describe the case when $\hat{u}$ is a differential operator. Let $F^{\lambda}_p(V)$ be interpreted as the function space on the $p\times(n-p)$ matrices of rank $p$ and respectively $F^{\lambda}_p(V')$ as function space on the $p\times(n-p)$ matrices $\xi$. If $\mu$ is a non-negative even number then there is an invariant finite dimensional subspace $\Phi^{-\mu}_p(V')$ which consists of the homogenous polynomials of order $\mu$ of the $p$-minors of the matrix $\xi$.

PROPOSITION 3. *If $u(\xi) \in \Phi^{-\mu}_p(V')$ where $\mu$ is an arbitrary non-negative even number, then $\hat{u}$ is a differential operator of order $p\cdot\mu$, given, up to a multiplicative constant, by the equality*

$$\hat{u} = u\left(\frac{\partial}{\partial x}\right)$$

*(that is $\hat{u}$ is determined by changing, in the formula of $u$, the elements $\xi_{ij}$ with the operators $\frac{\partial}{\partial x_{ij}}$).*

*Proof.* If $f \in F_p^\lambda(V)$ and $\varphi(\xi) = (R_p(\tilde{V})f)(\xi)$, then considering the chosen interpretation for the spaces $F_p^\lambda$, $\hat{u}$ is written like this:

$$(\hat{u}f)(x) = C_{p,n}(-\lambda-\mu)\int_\Omega |x\xi'|^{-\lambda-\mu}u(\xi)\varphi(\xi)\sigma(\xi) \tag{3.8}$$

where $\sigma(\xi)$ is a $p\times(n-p)$ density in whose explicit formula we take no interest, and the integral is computed on an arbitrary section of the boundle $E_{p,n} \to G_p(V')$. From Proposition 2 we obtain

$$f(x) = C_{p,n}(-\lambda)\int_\Omega |x\xi'|^{-\lambda}\varphi(\xi)\sigma(\xi). \tag{3.9}$$

We apply the operator $u\left(\frac{\partial}{\partial x}\right)$ to both sides of the equality (3.9). To write explicitly the obtained formula, we define on the manifold of $p$-matrices $z = \|z_{ij}\|$ the operator $\frac{\partial}{\partial z} = \det\left\|\frac{\partial}{\partial z_{i,j}}\right\|$. It is obvious that

$$u\left(\frac{\partial}{\partial x}\right)|x\xi'|^{-\lambda} = u(\xi)\left(\frac{\partial}{\partial z}\right)^\mu |z|^{-\lambda}\Big|_{z=u\xi'}.$$

On the other hand, it is known (see for instance [10]) that

$$\left(\frac{\partial}{\partial z}\right)^\mu |z|^{-\lambda} = C_{\lambda,\mu}|z|^{-\lambda-\mu}$$

where $C_{\lambda,\mu}$ is a numerical constant. This is how we obtain

$$\left(u\left(\frac{\partial}{\partial x}\right)f\right)(x) = C_{p,n}(-\lambda)C_{\lambda,\mu}\int_\Omega |x\xi'|^{-\lambda-\mu}u(\xi)\varphi(\xi)\sigma(\xi). \tag{3.10}$$

Combining (3.9) and (3.10) we get the statement of the proposition.

For us, the main example of an $\hat{u}$ operator is given by the following.

DEFINITION 1. We shall denote by $P(c, A)$ the operator

$$\hat{P}(\tilde{c}, A) : F_p^p(V/c) \to F_p^q(V/c)$$

given by the symbol $\hat{P}(c, A|\tilde{h}) = P_{q-p}(\tilde{c}, \hat{A}\tilde{h})$ (see §2). If $c \in G_{q-p}(V)$ is given by the frame $z = (z_1, \ldots, z_{q-p})$, then we write $\hat{P}(z, A)$ instead of $\hat{P}(\tilde{c}, A)$.

By the proof of the previous proposition, and by Proposition 1, §2 we have (with the obvious notation):

PROPOSITION 4. *If $z = (z_1, \ldots, z_{q-p})$ and $A$ are such that*

$$\det\|\langle \xi^i, A_s z_j\rangle\| \geqslant 0 \tag{3.11}$$

*for any p-vector* $\xi = (\xi^1, \ldots, \xi^p)$ *then* $\hat{P}(z, A)$ *is a differential operator of order* $N = p(q-p)$.

REMARK. The function $\det\|\langle\xi^i, A_s z_j\rangle\|$ is a homogenous polynomial of degree $q-p$ in each of the vectors $\xi^i$ and, consequently, the inequality (3.11) is possible only for even $q-p$.

5. DEFINITION OF THE OPERATOR $\chi$. Let $\tilde{c} = (c, \gamma) \in \tilde{G}_{q-p}(V)$. We define the injective mapping

$$\theta_{\tilde{c}}: \tilde{G}_p(V/c) \to \tilde{G}_q(V)$$

given by the equality $\theta_{\tilde{c}}(a, \alpha) = (a_1, \alpha_1)$ where $a_1 \in G_q(V)$ is the preimage of $a \in G_p(V/c)$ and $\alpha_1$ is defined by the equality $\alpha_1/\gamma = \alpha$. We shall associate with every function $\varphi \in F_q^p(V)$ and every $\tilde{c} \in \tilde{G}_{q-p}(V)$, the function $\varphi_{\tilde{c}} \in F_p^p(V/c)$ given by the equality

$$\varphi_{\tilde{c}}(\tilde{a}) = \varphi(\theta_{\tilde{c}}\tilde{a}) \quad \tilde{a} \in \tilde{G}_p(V/c). \tag{3.12}$$

Let $\mathscr{E}$ be the manifold of the pairs $(c, \tilde{b})$ we have introduced in Subsection 1, $c \in G_{q-p}(V)$, $\tilde{b} = (b, \beta) \in \tilde{G}_p(V)$ and $b \cap c = 0$. We construct the mappings $\rho: \mathscr{E} \to G_{q-p}(V)$ and $\pi: \mathscr{E} \to G_p(V)$ by the equalities $\rho(c, \tilde{b}) = c$, $\pi(c, \tilde{b}) = \tilde{b}$.

DEFINITION 2. Let $\varphi \in F_q^p(V)$. For any $\tilde{c} = (c, \gamma) \in \tilde{G}_{q-p}(V)$, $\tilde{b} = (b, \beta) \in \tilde{G}_p(V)$ with $b \cap c = 0$ and any $A = \{A_1, \ldots, A_N\}$, $A_i \in T_{c,b}\mathscr{E}$, $N = p(q-p)$, we define:

$$(\chi\varphi)_{(c,\tilde{b};A)} = (\hat{P}(\tilde{c}, \rho A)\, \varphi_{\tilde{c}})(b \oplus c/c, \beta \oplus \gamma/\gamma) \tag{3.13}$$

where $\rho A = \{\rho A_i\}$, $\rho A_i \in T_c G_{q-p}(V)$.

We point out that the right side is independent of the choice of the volume element $\gamma$ in $c$, as when changing $\gamma$ in $t\gamma$ $(t > 0)$, $\varphi_{\tilde{c}}$ is multiplied by $t^p$ and $\hat{P}(\tilde{c}, \rho A)$ is multiplied by $t^{-p}$. It follows from the definition that $\chi\varphi$ is an $N$-density on $\mathscr{E}$. It is easy to verify that $(\chi\varphi)(c, (b, t\beta), A) = t^q(\chi\varphi)(c, (b, \beta), A)$; so $\chi$ determines a mapping

$$\chi: F_q^p(V) \to \Omega^N(\mathscr{E})$$

where $\Omega^N(\mathscr{E})$ is the space of $N$ densities on $\mathscr{E}$ we defined in Subsection 1.

PROPOSITION 5. *$\chi$ is an intertwining operator.*

*Proof.* We obtain $\chi$ as a composition of several mappings

$$\varphi \to \varphi_{\tilde{c}} \to \psi \to P_\psi \to \chi\varphi$$

where $\psi = R_p^\lambda(\tilde{V}/c)\varphi_{\tilde{c}}$. In the above sequence, each element is a function whose domain is a manifold on which the group SL($V$) acts naturally, for instance $\varphi_{\tilde{c}}$ is a function defined on the manifold of the pairs $(\tilde{c}, \tilde{a})$ where $\tilde{c} = (c, \gamma) \in \tilde{G}_{q-p}(V)$, $\tilde{a} = (a, \alpha) \in \tilde{G}_p(V/c)$, $\psi$ is a function on the manifold of the pairs $(\tilde{c}, \tilde{h})$ where $\tilde{c} = (c, \gamma) \in \tilde{G}_{q-p}(V)$, $\tilde{h} = (h, \eta) \in \tilde{G}_p(\operatorname{Ann} c)$, and so on. From the definition of these mappings we see that on the respective manifold, they commute with the action of the group SL($V$).

DEFINITION 3. We shall denote by $(\chi\varphi)(c, A|\tilde{b})$ the restriction of the $N$-density $\varphi$ at the fibre $\pi^{-1}(\tilde{b}) = G^b_{q-p}(V)$ of the bundle $\pi: \mathscr{E} \to G_p(V)$

$$(\chi\varphi)_{(c, A|\tilde{b})} = (\chi\varphi)_{(c, \tilde{b}, A)}\big|_{\pi^{-1}(b)} \tag{3.14}$$

(where $G^b_{q-p}(V)$ is the manifold of the subspaces $c \in G_{q-p}(V)$ which satisfy $c \cap b = 0$).
For a fixed $\tilde{b} \in \tilde{G}_p(V)$, $(\chi\varphi)_{(c, A, \tilde{b})}$ is a $N$-density on $G^b_{q-p}(V)$.

6. LEMMA FOR PASSING TO SUBSPACES $W \subset V$. In the definition of the operator $\chi$, the framework was the vector space $V$. In order to stress this, hereafter we shall use $\chi^V$ instead of $\chi$. Recall that the dimension of $V$ satisfies $n = \dim V \geqslant p + q$.

LEMMA. *Let $W \subset V$ be an arbitrary subspace of $V$, $\dim W = m \geqslant p + q$ and let $\varphi^W$ be the restriction of the function $\varphi \in F^p_q(V)$ at $G_q(W)$ (that means $\varphi^W \in F^p_q(W)$). Then for any triplet $(c, A, \tilde{b})$ where $c \in G_{q-p}(W)$, $A = \{A_i\}$, $A_i \in T_cG_{q-p}(W)$, $b \in \tilde{G}_p(W)$ we have*

$$(\chi^V\varphi)(c, A|\tilde{b}) = (\chi^W\varphi^W)(c, A|\tilde{b}). \tag{3.15}$$

*Proof.* We can assume without loss of generality that in the coordinate system of $V$, the space $W$ is given by the equations $x^{m+1} = \ldots = x^n = 0$ and $c$ is the space spanned by the vectors $v_1, \ldots, v_{q-p}$ where $v_i = (\delta^1_{i+p}, \ldots, \delta^n_{i+p})$ ($\delta^j_i$ — Kronecker's symbol). We identify $V/c$ with the subspace spanned by $(x^1, \ldots, x^n)$ where $x^{p+1} = \ldots = x^q = 0$. We regard $F^q_p(V/c)$ and $F^{n-q}_p((V/c)')$ as function spaces respectively on the manifold of $p \times (n-q)$ matrices $u$ and on the manifold of $(n-q) \times p$ matrices $\xi$ (where $u$ and $\xi$ are local coordinates respectively for the Grassmannians $G_p(V/c)$ and $G_p((V/c)')$. We write them as block matrices:

$$u = (u_1, u_2), \quad \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \quad \text{where } u_1, \xi_1' \in M_{p, n-q}\,;\; u_2, \xi_2' \in M_{p, n-m}.$$

Considering this the Grassmannian $G_p(W/c) \subset G_p(V/c)$ is given by the equation $u_2 = 0$; thus $\varphi_{\tilde{c}}^W$ is a function of $u_1$, given by the equation

$$\varphi_{\tilde{c}}^W(u_1) = \varphi_{\tilde{c}}(u_1, 0).$$

Using the notation $\psi(\xi_1, \xi_2) = (R_p^p(\widetilde{V/c})\varphi_{\tilde{c}})(\xi_1, \xi_2)$, $\varphi^W(\xi_1) = (R_p^p(\widetilde{W/c})\varphi_{\tilde{c}}^W)(\xi_1)$ we prove that

$$\varphi^W(\xi_1) = \alpha \int \psi(\xi_1, \xi_2)\mathrm{d}\xi_2 \quad \text{where } \alpha = \frac{C_{p,m-q+p}(\lambda)}{C_{p,n-q+p}(\lambda)}\Big|_{\lambda=-q} \tag{3.16}$$

and as usually $\mathrm{d}\xi_2$ is $\prod \mathrm{d}(\xi_2)_{ij}$. Indeed we have

$$\varphi_{\tilde{c}}(u_1, u_2) = C_{p,n-q+p}(\lambda) \int |I_p + u_1\xi_1 + u_2\xi_2|^\lambda \psi(\xi_1, \xi_2)\mathrm{d}\xi_1\mathrm{d}\xi_2 \Big|_{\lambda=-q}$$

which gives us for $u_2 = 0$:

$$\varphi_{\tilde{c}}^W(u_1) = C_{p,n-q+p}(\lambda) \int |I_p + u_1\xi_1|^\lambda \left(\int \psi(\xi_1, \xi_2)\mathrm{d}\xi_2\right)\mathrm{d}\xi_1 \Big|_{\lambda=-q}. \tag{3.17}$$

On the other hand we have

$$\psi_{\tilde{c}}^W(u_1) = C_{p,m-q+p}(\lambda) \int |I_p + u_1\xi_1|^\lambda \psi(\xi_1)\mathrm{d}\xi_1|_{\lambda=-q}. \tag{3.18}$$

Combining (3.17) and (3.18) we get the equality (3.16). We shall now proceed to prove the equality (3.15). Presuming that the assumptions of the lemma are fulfilled, it is easy to verify that $P(c, A|\xi_1, \xi_2)$ is independent of $\xi_2$ and $P(c, A|\xi_1, \xi_2) = P^W(c, A, \xi_1)$. Next, $b \in G_p(V)$ is given by the matrix $v = (v_1, v_2)$ and the condition $b \in G_p(W)$ becomes $v_2 = 0$. We find that

$$(\chi^V\varphi)(c, A|\tilde{b}) = C_{p,n-q+p}(\lambda) \int |I_p + u_1\xi_1|^\lambda P^W(c, A|\xi_1)\psi(\xi_1, \xi_2)\mathrm{d}\xi_1\,\mathrm{d}\xi_2 \Big|_{\lambda=-q}.$$

(3.16) shows us that the right side is equal to $(\chi^W\varphi^W)(c, A|\tilde{b})$ and the proof is ended.

## 4. THE INVERSION FORMULA FOR THE OPERATOR $\mathscr{I}$

1. Main Lemma. We shall denote by $\tilde{\mathscr{B}}$ the manifold of the pairs $(c, \tilde{h})$ where $c \in G_{q-p}(V)$, $\tilde{h} = (h, \eta) \in \tilde{G}_p(V')$ and $c \perp h$. We shall associate with each

function $\varphi \in F_q(V)$ the function $\tilde{\varphi}$ on $\tilde{\mathscr{B}}$ given yb the following equality

$$\tilde{\varphi}(c, \tilde{h}) = (R_p^p(V/c, \omega/\gamma)\varphi_{c,\gamma})(\tilde{h}) . \tag{4.1}$$

Here $\gamma$ is a fixed volume element in $c$ and $\varphi_{c,\gamma} \in F_p^p(V/c)$ is given by the equality (3.12) (easy computation shows us that the right side is independent of the choice of $\gamma$).

LEMMA. *If $\varphi = \mathscr{I}f$ where $f \in F_p^q(V)$, then for any $(c, \tilde{h}) \in \tilde{\mathscr{B}}$ we have*

$$\tilde{\varphi}(c, \tilde{h}) = \alpha_{p,q}(R_p^p(V, \omega)f)(\tilde{h}) \tag{4.2}$$

*where* $\alpha_{p,q} = \pi^{p(q-p)}\Gamma_p(p/2)/\Gamma_p(q/2)$.

*Proof.* It suffices to show that the statement of the lemma is true for a certain fixed $c = c_0$. On the manifolds $G_p(V)$, $G_q(V)$ and $G_p(V')$ we respectively introduce local coordinates $u = (u_1, u_2)$, $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ and $\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$ where $u_1, \xi_1' \in M_{p,q-p}$; $u_2, \xi_2' \in M_{p,n-q}$, $v_1 \in M_{p,n-q}$, $v_2 \in M_{q-p,n-q}$ and we regard $F_p^q(V)$, $F_q^p(V)$ and $F_p^{n-q}(V')$ as function spaces respectively on the manifolds of the matrices $u$, $v$, $\xi$ (see Subsection 2 of Section 1).

In this interpretation the function $\tilde{f} = R_p^q(\tilde{V})f$ is given by the equality

$$\tilde{f}(\xi_1, \xi_2) = C_{p,q}(\lambda - n)\int |I_p + u_1\xi_1 + u_2\xi_2|^{\lambda-n} f(u_1, u_2)\, du_1\, du_2\Big|_{\lambda = q} \tag{4.3}$$

(see Subsection 2 of Section 2). Next we give $c \in G_{q-p}(V)$ using the $(q-p)\times n$ matrix $c = (a, I_{q-p}, b)$ where $a \in M_{q-p,p}$, $b \in M_{q-p,n-q}$. Suppose $c_0 = (0, I_{q-p}, 0)$; then the orthogonality condition for $\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$ and $c_0$ is $\xi_1 = 0$. It follows from the definition of $\tilde{\varphi}(c, \tilde{h})$ that for $c = c_0$, $\tilde{\varphi}$ is a function of $\xi_2$ given by

$$\tilde{\varphi}(b, \xi_2) = C_{p,n-(q-p)}(\lambda - n)\int |I_p + v_1\xi_2|^{\lambda-n}\varphi\begin{pmatrix} v_1 \\ 0 \end{pmatrix} dv_1\Big|_{\lambda = q} . \tag{4.4}$$

Combining this with the formula of $\varphi$ as function of $f$ (see 1.6) we get

$$\tilde{\varphi}(c_0, \xi_2) = C_{p,n-(q-p)}(\lambda - n)\int |I_p + v_1\xi_2|^{\lambda-n} f(t, v_1)\, dt\, dv_1\Big|_{\lambda = q} . \tag{4.5}$$

Therefore $\tilde{\varphi}(c_0, \xi_2) = \alpha_{p,q}\tilde{f}(0, \xi_2)$ where

$$\alpha_{p,q} = \frac{C_{p,n-(q-p)}(\lambda - n)}{C_{p,n}(\lambda - n)}\Big|_{\lambda = q} = \pi^{p(q-p)}\frac{\Gamma_p(p/2)}{\Gamma_p(q/2)} .$$

Hence our lemma is proved.

COROLLARY. *The image of the operator* $\mathscr{I}$, $\operatorname{Im}\mathscr{I}$, *is the set of those functions* $\varphi \in F_q^p(V)$ *which satisfy the condition* $\tilde{\varphi}(c_1, \tilde{h}) = \tilde{\varphi}(c_2, \tilde{h})$ *for any* $(c_1, \tilde{h})$ *and* $(c_2, \tilde{h})$ *of* $\mathscr{B}$.

NOTE. An equivalent characterization for $\operatorname{Im}\mathscr{I}$ was given in [6].

2. THE INVERSION FORMULA. Let $C \subset G_{q-p}(V)$ be a $N = p(q-p)$ dimensional submanifold. By the Crofton symbol of $C$ we shall mean the function $\mathrm{Cr}_C(h)$ on $G_p(V')$ which assigns to $h$ the number of subspaces $c \in C$ which are orthogonal to $h$. It is easy to see that $\mathrm{Cr}_C$ is almost everywhere finite. $C$ will be called non-degenerate if $\operatorname{supp} \mathrm{Cr}_C$ contains an open subset and will be called non-singular if $\operatorname{supp}\mathrm{Cr}_C = G_p(V')$; we say $C$ is harmonic if $\mathrm{Cr}_C(h) = \mathrm{const} \neq 0$ almost everywhere. We shall denote this contains by $\mathrm{Cr}_C$ and shall call it the Crofton number. We shall denote by $\widehat{\mathrm{Cr}}_C$ the operator $\widehat{\mathrm{Cr}}_C : F_p^q(V) \to F_p^q(V)$ given by the symbol $\mathrm{Cr}_C(h)$ (see Subsection 4 of Section 3). We notice that if $C$ is harmonic then $\widehat{\mathrm{Cr}}_C = \mathrm{Cr}(C)\cdot E$ where $E$ is the identity operator.

THEOREM. *Let* $\tilde{b} = (b, \beta) \in \tilde{G}_p(V)$ *and* $C$ *a non-singular submanifold of* $G_{q-p}^b(V)$ *whose dimension is* $N = p(q-p)$. *Then if* $f \in F_p^q(V)$ *and* $\varphi = \mathscr{I}f$

$$\int_C (\chi\varphi)(\cdot\,|\tilde{b}) = (\widehat{\mathrm{Cr}}_C f)(\tilde{b}). \tag{4.6}$$

*In particular, if C is a harmonic submanifold, then*

$$\int_C (\chi\varphi)(\cdot\,|\tilde{b}) = \mathrm{Cr}(C)\cdot f(\tilde{b}). \tag{4.7}$$

3. PROOF OF THE THEOREM. According to the definition of $\widehat{\mathrm{Cr}}_C$, we have $\widehat{\mathrm{Cr}}_C f = R_p^{n-q}(\tilde{V}')(\mathrm{Cr}_C \tilde{f})$ where $\tilde{f} = R_p^q(\tilde{V})f$, that is

$$(\widehat{\mathrm{Cr}}_C f)(\tilde{b}) = \int_{G_p(V')} U^{(-q)}(\tilde{b}, \tilde{h})\mathrm{Cr}_C(h)\tilde{f}(\tilde{h})\sigma_{\tilde{V}'}(\tilde{h}). \tag{4.8}$$

We shall consider the submanifold $\mathscr{B}_c \subset \mathscr{B}$ consisting of the pairs $(c, h)$ where $c \in C$, $h \in G_p(V')$ and $h \perp c$. We define the projections $\pi_1 : \mathscr{B}_c \to G_p(V')$ and $\pi_2 : \mathscr{B}_c \to C$ by $\pi_1(c, h) = h$ and $\pi_2(c, h) = c$. Let $U = \pi_1(\mathscr{B}_c)$; then we can consider that in (4.8) the integration is done on $U$. For almost every point $h \in U$, the preimage $\pi_1^{-1}(h)$ consists of a finite number of points $\mathrm{Cr}_C(h) \neq 0$; therefore (4.8) can be represented as an integral on the manifold $\mathscr{B}_c$:

$$(\widehat{\mathrm{Cr}}_C f)(\tilde{b}) = \int_{\mathscr{B}_c} \pi_1^*(U^{(-q)}(\tilde{b}, \tilde{h})\tilde{f}(\tilde{h})\sigma_{\tilde{V}'}(\tilde{h})). \tag{4.9}$$

According to the Main Lemma we have $\tilde{\tilde{f}}(\tilde{h}) = \alpha_{p,q}^{-1}\tilde{\varphi}(c, h)$, where $c \in C$ is an arbitrary element orthogonal to $h$. Hence

$$(4.10) \qquad (\mathrm{Cr}_C f)(\tilde{b}) = \alpha_{p,q}^{-1} \int_{\mathscr{B}_c} \pi_1^*(U^{(-q)}(\tilde{b}, \tilde{h})\tilde{\varphi}(c, h)\sigma_{\tilde{V}'}(\tilde{h})).$$

We shall compute this integral as an iterated integral in which we first integrate on the fiber and then on the basis of the bundle $\pi_2 : \mathscr{B}_c \to C$. According to (2.1) we have:

$$(\widehat{\mathrm{Cr}}_C f)(\tilde{b}) = \alpha_{p,q}^{-1} \int_C \left( \int_{\pi_2^{-1}(c)} U^{(-q)}(\tilde{b}, \tilde{h})P(c, A|\tilde{h})\tilde{\varphi}(c, \tilde{h})\sigma_{(\tilde{V}/c)'}(\tilde{h}) \right).$$

It follows from the definition of the operator $\chi$ that the first integral is equal to $\alpha_{p,q}(\chi\varphi)(c, A|\tilde{b})$. So

$$(\widehat{\mathrm{Cr}}_C f)(\tilde{b}) = \int_C (\chi\varphi)(\cdot\,|\tilde{b}).$$

NOTE. For each harmonic submanifold there is a corresponding application

$$\sigma : \Omega^N(\mathscr{E}) \to F_p^q(V)$$

(where $F_p^q(V)$ is a certain extension of $F_p^q(V)$), defined by the equality

$$\sigma(\tau)(\tilde{b}) = \int_C \tau(\cdot\,|\tilde{b}).$$

This application is not an intertwining operator on the whole $\Omega^N(\mathscr{E})$ but according to the proof, its restriction at the subspace $\chi\mathscr{I}(F_p^q) \subset \Omega^N(\mathscr{E})$ is an intertwining operator.

REMARK. In the same manner, for any $k \leqslant q - p$ we can define the operator

$$\chi_k : \operatorname{Im}\mathscr{I} \to \Omega^{p \cdot k}(\mathscr{E}_k)$$

where $\mathscr{E}_k$ — the manifold of the pairs $(c_k, \tilde{b})$, $c_k \in G_k(V)$ and $\tilde{b} \in G_p(V)$.

That is, let $\varphi \in \operatorname{Im}\mathscr{I}$ and $\tilde{\varphi}(c, \tilde{h})$ the function on $\tilde{\mathscr{B}}$ given by (4.1). We define the function $\tilde{\varphi}_k(c_k, \tilde{h})$ on the manifold of the pairs $(c_k, \tilde{h})$ where $c \in G_k(V)$, $\tilde{h} \in \tilde{G}_p(V')$ and $c_k \perp h$, by the equality

$$\tilde{\varphi}_k(c_k, \tilde{h}) = \tilde{\varphi}(c, \tilde{h})$$

where $c \in G_{q-p}(V)$ is a certain subspace satisfying $c_k \subset c$, $c \perp h$. According to the Main Lemma the given definition is correct. By definition

$$\chi_k \varphi = R_p^{q-n-k}(\operatorname{Ann} c_k, (\omega/\gamma_k)')(P_k(\tilde{c}_k, A|\tilde{h})\tilde{\varphi}(c_k, \tilde{h}))$$

where $P_k$ is given by (2.1).

We underline that for $k < q - p$ the operators are defined on the subspace $\operatorname{Im} \mathscr{I} \subset F_q^p$.

Using the operators $\chi_k$ we can construct other inversion formulas for the operator $\mathscr{I}$.

Appendix. On the Operator $\varkappa$. For any $\tilde{b} \in \tilde{G}_p(V)$ there was defined in [6] the operator $\varkappa_{\tilde{b}}$ which associates with a function $\varphi \in F_q^p(V)$ a $N$-differential form on $G_{q-p}^b(V)$, $(N = p(q-p))$. It was proved that if $\varphi = \mathscr{I}f$, $f \in F_p^q(V)$, then $\varkappa_{\tilde{b}}\varphi$ is a closed form on $G_{q-p}^b(V)$ and we have for any $N$-dimensional cycle $\gamma \subset G_{q-p}^b(V)$:

$$\int_\gamma \varkappa_b \varphi = C_\gamma f(\tilde{b}) \tag{4.11}$$

where $C_\gamma$ is depending only on the homology class of $\gamma$. Moreover, for even $q-p$ there are cycles $\gamma$ for which $C_\gamma \neq 0$ and for these cycles (4.11) gives the inversion formula. There are no such cycles for odd $q-p$.

The operator $\varkappa$ can be defined analogously to $\chi$ as intertwining operator. In order to do that, we must replace, in the definitions given in this paper, the non--oriented volume element by an oriented volume element. Therefore we shall regard $\tilde{G}_p(V)$ as the manifold of the pairs $(b, \beta)$ where $b \in G_p(V)$ and $\beta$ is an oriented volume element in $b$. Instead of $F_p^\lambda(V)$ we must introduce a larger class of spaces, $F_p^{\lambda,\varepsilon}(V)$ $(\varepsilon = 0, 1)$ the spaces of $C^\infty$ functions on $\tilde{G}_p(V)$ satisfying the condition $f(b, t\beta) = |t|^\lambda \operatorname{sgn}^\varepsilon t \cdot f(b, \beta)$ for any $t \neq 0$ (we notice that $F_p^{\lambda,0} \simeq F_p^\lambda$). If in the definition of $\chi$ we change $P(\tilde{c}, A|\tilde{h}) = |\det\|\langle\eta^i, A_s v_j\rangle\||$ with $Q(\tilde{c}, A|\tilde{h}) = \det\|\langle\eta^i, A_s v_j\rangle\|$ then instead of $\chi$ we get the intertwining operator

$$\varkappa\colon F_q^p(V) \to \Omega_0^N(\mathscr{E})$$

where $\mathscr{E}$ is a bundle over $\tilde{G}_p(V)$ with the fibre at $\tilde{b} = (b, \beta) \in \tilde{G}_p(V)$ equal to $G_{q-p}^b(V)$, and $\Omega_0^N(\mathscr{E})$ is the space of the $N$-differential forms which satisfy the condition $\tau(c, (b, t\beta)) = |t|^q \tau(c, (b, \beta))$ for any $t \neq 0$.

It can be shown, by passing to coordinates, that the restriction of $\varkappa_{\tilde{b}}\varphi$ of the differential form $\varkappa\varphi$, $\varphi \in F_q^p(V)$ at the fiber $\pi^{-1}(b)$, $\pi\colon \mathscr{E} \to \tilde{G}_p(V)$ coincides with the differential form introduced in [6].

## 5. PERMISSIBLE COMPLEXES

1. Definition of the Permissible Complex. As for $p < q < n - p$, $\dim G_p(V) < \dim G_q(V)$, in this case, knowing the values of $\mathscr{I}f$ on the whole manifold $G_q(V)$ is over-sufficient for recovering the function $f$; we can assume that the function $\mathscr{I}f$ is given only on a certain submanifold $K \subset G_q(V)$ whose dimension is $k \geqslant \dim G_p(V)$. There arises the problem of recovering the function $f$ by means of the function $\mathscr{I}_K f = \mathscr{I}f \mid K$.

We introduce a class of submanifolds $K \subset G_q(V)$ for which $\mathscr{I}_K$ is one to one and the inversion formula for $\mathscr{I}_K$ follows immediately from the inversion formula for $\mathscr{I}$ we obtained in Section 4.

Definition. The submanifold $K \subset G_q(V)$ will be called a *permissible complex* (more exactly : a *p-permissible complex*) if for almost every $b \in G_p(V)$ there is a harmonic submanifold $C_b \subset G^b_{q-p}(V)$ such as if $\varphi | K = 0$ then $(\chi\varphi)(\cdot | \tilde{b}) | C_b = 0$ for any $\varphi \in F^p_q(V)$.

In other words the restriction of the density $(\chi\varphi)(\cdot | \tilde{b})$ at $C_b$ is uniquely determined by the restriction of $\varphi$ at $K$. It is obvious that in the case of a permissible complex $K$, the function $f \in F^q_p(V)$ is computed from its image $\varphi = \mathscr{I}_K f$ by means of the inversion formula (4.7) for $C = C_b$.

Note. By replacing in the definition the operator $\chi$ by the operator $\varkappa$ and the harmonic manifold by the $p(q-p)$ cycle in $G^b_{q-p}(V)$ we obtain the notion of permissible complex earlier introduced for complex vector spaces and whose complete characterization is given in [3] and [9] for $p = 1$, $q = 2$.

2. Examples of Permissible Complexes. 1°. *The complex $K_C$*. We associate with each harmonic submanifold $C \subset G_{q-p}(V)$ the complex $K_C \subset G_q(V)$ consisting of the subspaces $a \in G_q(V)$ which contain at least one subspace $c \in C$. We notice that $\dim K_C = \dim G_p(V) = p(n-p)$. Indeed the set of the subspaces $a \in G_q(V)$ which contain a fixed subspace $c \in C$, is equivalent to $G_p(V/c)$ and therefore its dimension is $p(n-q)$. It follows that $\dim K_C = p(q-p) + p(n-q) = p(n-p)$. Obviously, the complex $K_C$ is permissible.

We shall give an example of a $K_C$ complex for $p = 1$, $q = 3$. We consider a pair of lines $l_1, l_2$ in $\mathbf{P}^{n-1}$ situated in general position. We denote by $C$ the set of the lines in $\mathbf{P}^{n-1}$ which intersect the given lines. It is obvious that $C$ is a harmonic manifold with the Crofton symbol 1. The complex $K_C$ consists of the set of 2-dimensional planes of $\mathbf{P}^{n-1}$ which intersect the two lines $l_1, l_2$.

2° *Radon complexes.* The manifold $K \subset G_q(V)$ will be called a Radon complex (more exactely: Radon $p$-complex) if for almost every $b \in G_p(V)$ there is a linear subspace $W_b \subset V$ such that $b \subset W_b$, $\dim W_b \geqslant p + q$ and $G_q(W_b) \subset K$. It follows directly from the lemma in Subsection 6 of Section 3, that any Radon complex is permissible.

REMARK. In the definition of the Radon complexes, we can assume that $\dim W_b = p + q$. Then the inversion formula for $\mathscr{I}_K$ is reduced at the inversion formula for the Radon transform in the projective space.

We shall give an example of Radon complex for $p = 1$, $q = 2$. Consider in $\mathbf{P}^{n-1}$ an arbitrary $(n - 3)$-dimensional family of 2-dimensional planes which contains all the points of $\mathbf{P}^{n-1}$. The complex $K$ consists of the set of all the lines which are contained in at least one of the planes of the considered family. It is obvious that, generally, this complex $K$ is not a $K_C$ complex. At the same time, the complex of 2-dimensional planes in Example 1°, is not a Radon complex.

3. $\mathscr{I}$-COMPLEXES. We shall introduce a class of permissible complexes $K \subset G_q(V)$ which contains both the $K_C$ complexes and the Radon complexes.

DEFINITION. The submanifold $K \subset G_q(V)$ will be called a *$\mathscr{I}$-complex* if for almost every $b \in G_p(V)$ there is a subspace $W_b \subset V$ of dimension $\dim W_b \geqslant p + q$ and a harmonic manifold $C_b \subset G_{q-p}(W_b)$ such that $b \subset W_b$ and $K_{C_b} \cap G_q(W_b) \subset K$.

It follows immediately from the lemma in Subsection 6 of Section 3 that any $\mathscr{I}$-complex is permissible and, moreover, the inversion formula is (4.7) for $C = C_b$. It is also obvious that both the $K_C$ complexes and the Radon complexes are $\mathscr{I}$-complexes.

We shall now construct an example of a $\mathscr{I}$-complex which is neither a $K_C$ complex nor a Radon complex. Let $p = 1$, $q = 3$. Consider in $\mathbf{P}^{n-1}$ an arbitrary one-dimensional family $\pi_t$ of $(n - 2)$-dimensional planes which contain all the points of $\mathbf{P}^{n-1}$. In each plane $\pi_t$ we fix an arbitrary pair of lines $l_t^1, l_t^2$ situated in general position. The complex $K$ consists of all the 2-dimensional planes in $\mathbf{P}^{n-1}$ contained in at least one of the planes $\pi_t$ and which intersect each of the corresponding lines $l_t^1$ and $l_t^2$. We notice that $\dim K = n - 1$.

4. THE INTEGRAL TRANSFORM $\mathscr{I}_K$ FOR AN ARBITRARY COMPLEX $K_C$. Let $C \subset G_{q-p}(V)$ be an arbitrary submanifold, not necessary harmonic, $K_C$ the manifold of those $a \in G_q(V)$ which contain at least one subspace $c \in C$. Generally, the complex $K = K_C$ is not permissible and $\operatorname{Ker} \mathscr{I}_K$, the kernel of the associated integral transform $\mathscr{I}_K$, can be non-zero. We shall compute this kernel. Let $M_C = \{h \in G_p(V') \mid \operatorname{Cr}_C(h) = 0\}$.

PROPOSITION 1. $\operatorname{Ker} \mathscr{I}_K, K = K_C$, *consists of the set of those functions* $f \in F_p^q(V)$ *for which* $\operatorname{supp}(R_p^q(\tilde{V})f) \subset \overline{M}_C$. *In particular, the mapping* $\mathscr{I}_K$ *is one to one if and only if $C$ is non-degenerated, that is,* $\operatorname{Cr}_C(h) \neq 0$ *almost everywhere.*

*Proof.* Let $f \in F_p^q(V)$, $\varphi = \mathscr{I}_K f$ and $\tilde{f} = R_p^q(V)f$. We shall assume without any confusion, that the function $\tilde{f}$ is given as a function on $G_p(V')$ and not on the bundle $\tilde{G}_p(V')$ over $G_p(V')$. We denote by $G_p^1$ the set of the subspaces $h \in G_p(V)$ for which $\operatorname{Cr}_C(h) \neq 0$. By the assumptions in the lemma and by the fact that the mapping

$\varphi \to \tilde{\varphi}(c, \tilde{h}) = (R_p^p(\widetilde{V/c})\varphi_{\tilde{c}})(h)$ is one to one, it follows that

$$\varphi \equiv 0 \Leftrightarrow \tilde{f}|\bar{G}_p^1 \equiv 0$$

where $\bar{G}_p^1$ is the closure of $G_p^1$. Hence our statement follows immediately.

Let now $\mathscr{B}_c \subset \mathscr{B}$ be the manifold of the pairs $(c, \tilde{h}) \in \mathscr{B}$ where $c \in C$. We notice that for $c \in C$, $\tilde{\varphi}(c, \tilde{h})$ is determined only by the restriction $\varphi = \mathscr{I}_K f$ of the function $\mathscr{I} f$ at the complex $K$. Therefore by the results in Subsection 1 of Section 4 we have:

PROPOSITION 2. $\mathrm{Im}\,\mathscr{I}_K$, *the image of the application* $\mathscr{I}_K$, *consists of the set of those* $C^\infty$ *functions* $\varphi$ *on* $K$ *which satisfy* $\tilde{\varphi}(c_1, \tilde{h}) = \tilde{\varphi}(c_2, \tilde{h})$ *for any* $(c_1, \tilde{h})$ *and* $(c_2, \tilde{h})$ *in* $\mathscr{B}_c$.

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*I. M. GELFAND*
*Laboratory of Mathematical Methods in Biology, Corpus A, State University of Moscow, Moscow V–234 U.R.S.S.*

*M. I. GRAEV*
*Keldysh Institute of Applied Mathematics, Minsskaya sq. 4, Moscow A – 47, U.R.S.S.*

*R. ROŞU*
*Institute of Mathematics, University of Bucharest, Academiei 14, Bucharest, Romania*

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