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HYPONORMAL OPERATORS ARE SUBSCALAR

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INTRODUCTION

In this paper we construct a universal functional model for hyponormal operators. This shows in particular that every hyponormal operator is subscalar, i.e. is similar to the restriction to an invariant subspace of a (generalized) scalar operator (in the sense of Colojoară-Foiaş [5]).

Let H be a complex (separable) Hilbert space and let $\mathscr{L}(H)$ denote the linear bounded operators on H. Recall that $T \in \mathscr{L}(H)$ is called *subnormal* if there is a Hilbert space K, containing H isometrically, and a normal operator N on K, such that $Nh = Th$, $h \in H$, in other words H is a closed invariant subspace for N and the restriction $N|H$ coincides with T. Interest in subnormal and related classes of operators has risen considerably since S. Brown [3] proved that every subnormal operator has a nontrivial invariant subspace. A larger class of operators related to subnormals is the following: $T \in \mathscr{L}(H)$ is called *hyponormal* if $TT^* \leqslant T^*T$, or equivalently, if $\|T^*h\| \leqslant \|Th\|$ for every $h \in H$. There are classical examples of hyponormal non-subnormal operators, see [7, Chapter 16].

As we shall prove below the distinction between hyponormal and subnormal operators lies only in two degrees of differentiability added to the admissible functional calculus of an extension. More precisely,

THEOREM 1. *Any hyponormal operator is subscalar of order* 2.

A linear bounded operator S on H is called in [5] *scalar of order m* if it possesses a spectral distribution of order m, i.e. if there is a continuous unital morphism of topological algebras

$$U: C_0^m(\mathbf{C}) \to \mathscr{L}(H),$$

such that $U(z) = S$, where as usual z stands for the identical function on $\mathbf{C}$ and $C_0^m(\mathbf{C})$ for the space of compactly supported functions on $\mathbf{C}$, continuously differentiable of order m, $0 \leqslant m \leqslant \infty$. An operator is subscalar if it is the restriction of a scalar operator to an invariant subspace. It is necessary to point out the distinction between scalar and Dunford scalar operators, which are characterized by an integral representation with respect to a spectral measure, [6].

As a matter of fact, the proof of Theorem 1 is constructive and it offers some additional information concerning hyponormal operators and a canonical scalar extension. Let us sketch below this construction.

Let T be a linear bounded operator on H. The starting point is the investigation of the operator $z - T$ on various spaces of H-valued functions. The study of the operator $z - T$ on the space $\mathcal{O}(U, H)$ of analytic H-valued functions on U, $U = \overset{\circ}{U} \subset \mathbf{C}$, led E. Bishop [2] to fundamental results in spectral theory. Among other things he isolated in [2] the *single valued extension property*, which means by definition that the operator $z - T$ acts one to one on $\mathcal{O}(U, H)$ for an arbitrary open subset U of $\mathbf{C}$, and the *property* (β), which requires that $z - T$ should be one to one and with closed range on $\mathcal{O}(U, H)$ for every open set U. The operators with a rich functional calculus, e.g. the scalar operators, as well as their restrictions to invariant subspaces have property (β). The importance of property (β) lies in assuring the natural framework for localizing the analytic functional calculus and the spectrum with respect to each vector $h \in H$, separately, see [12, Chapter 4]. This can be explained as follows.

Let us assume that the operator T has property (β). Let U be an open subset of $\mathbf{C}$ and consider the Fréchet space

$$\mathcal{F}(U) = \mathcal{O}(U, H)/(z - T)\mathcal{O}(U, H).$$

When U runs over the open subsets of $\mathbf{C}$, $\mathcal{F}$, with the natural restriction maps, becomes an analytic Fréchet sheaf on $\mathbf{C}$, which caries all the information, local or global, concerning T. For example, the global sections space $\mathcal{F}(\mathbf{C})$ corresponds to H because of the existence of the analytic functional calculus for T, the operator induced by the multiplication with z on $\mathcal{O}(U, H)$ corresponding in this identification to T. Moreover, it turns out that the local spectrum $\sigma_T(h)$ is the support of the corresponding section $h \in \mathcal{F}(\mathbf{C})$, and so on. This sheaf model appeared in [9] in connection with some decomposability phenomena.

What happens when the operator T satisfies (β) or the single valued extension property with respect to some other functions space $\mathcal{A}$ instead of $\mathcal{O}$? Of course, the above procedure still works, but it becomes effective only when the initial space H and the operator T can be recuperated in the corresponding quotient space. If this is the case, then the bigger space and the multiplication operator M_z on it will provide an extension of T (i.e. H will be invariant for M_z and $M_z|H = T$) with a functional calculus as rich as allowed by $\mathcal{A}$.

$$\begin{array}{ccc} H & \hookrightarrow & \mathcal{A}(\mathbf{C}, H)/(z - T)\mathcal{A}(\mathbf{C}, H) \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle M_z} \\ H & \hookrightarrow & \mathcal{A}(\mathbf{C}, H)/(z - T)\mathcal{A}(\mathbf{C}, H) \end{array} .$$

This construction is functorial in T and H and has some minimality properties.

For hyponormal operators a L^2-estimate involving $z - T$ and $\bar{\partial}$-derivatives up to the degree 2 (see Proposition 2.1 below) insures that a Sobolev space with respect only to the $\bar{\partial}$ derivatives has all the required properties for $\mathscr{A}$ in the above general scheme.

The paper has two sections:

§1 deals with the preliminaries concerning vector valued function spaces. Some facts from local spectral theory are also recalled.

In §2 the construction of the functional model for hyponormal operators is performed. The properties of this canonical functional model appear in Proposition 2.5.

The paper concludes with some remarks on the applications of the functional model.

1. VECTOR VALUED FUNCTION SPACES

Let z be the coordinate in the complex plane $\mathbf{C}$ and let $\mathrm{d}\mu(z)$, or simply $\mathrm{d}\mu$, denote the planar Lebesgue measure. Fix a complex (separable) Hilbert space H and a bounded (connected) open subset U of $\mathbf{C}$.

We shall denote as usually by $L^2(U, H)$ the Hilbert space of measurable functions $f : U \to H$, such that

$$\|f\|_{2,U} = \left(\int_U \|f(z)\|^2 \mathrm{d}\mu(z)\right)^{1/2} < \infty.$$

The functions $f \in L^2(U, H)$ which are in addition analytic functions in U, i.e. $\bar{\partial} f = 0$, form a closed subspace denoted

$$A^2(U, H) = L^2(U, H) \cap \mathcal{O}(U, H).$$

The orthogonal projection onto this space will be denoted by P.

Similarly $L^\infty(U, H)$ is the Banach space of essentially bounded H-valued functions on U. There is a continuous natural imbedding $L^\infty(U, H) \subset L^2(U, H)$.

Let $\bar{U}$ be the closure in $\mathbf{C}$ of the open set U and let $C^p(\bar{U}, H)$ denote the space of germs on $\bar{U}$ of continuously differentiable functions of order p, $0 \leqslant p \leqslant \infty$.

The integral representation formula with potentials of the (elliptic) operator $\bar{\partial}$ has a remarkable simpler form, known as the *Cauchy-Pompeiu formula* (see for instance [12, Theorem II.3.2] for its vector valued version). We shall use this formula in the case of a bounded disk D:

$$(1) \qquad f(z) = \frac{1}{2\pi \mathrm{i}} \int_{\partial D} f(\zeta)\, \mathrm{d}\zeta/(\zeta - z) - \frac{1}{\pi} \int_D \bar{\partial} f(\zeta)/(\zeta - z)\, \mathrm{d}\mu(\zeta),$$

where $z \in D$ and $f \in C^2(\bar{D}, H)$.

Let us also recall some elementary facts concerning the Cauchy kernel (see for example the introductory chapter to [10]). The function $1/z$ is summable in the neighbourhood of 0,

$$\int_{|z|<1} \mathrm{d}\mu/|z| = 2\pi,$$

and the function

$$g(z) = \int_{\partial D} f(\zeta)/(\zeta - z)\,\mathrm{d}\zeta$$

appearing in (1) is analytic in D and continuous on $\bar{D}$, in particular $g \in A^2(D, H)$ for $f \in C^2(\bar{D}, H)$.

We shall also use the following well known fact.

LEMMA 1.1. *If U, V are bounded connected open sets in* $\mathbf{C}$, *and if V is relatively compact in U, then there is a constant $c > 0$, such that*

$$\|f\|_{\infty,V} \leqslant c\|f\|_{2,U}$$

for every $f \in A^2(U, H)$.

Let us define now a special Sobolev type space. U will be again a bounded open subset of $\mathbf{C}$ and m will be a fixed non-negative integer. We have already used the notation $\bar{\partial}$ for the operator $\partial/\partial\bar{z}$.

The vector valued *Sobolev space* $W^m(U, H)$ with respect to $\bar{\partial}$ and of order m will be the space of those functions $f \in L^2(U, H)$ whose derivatives $\bar{\partial}f, \bar{\partial}^2 f, \ldots, \bar{\partial}^m f$ in the sense of distributions still belong to $L^2(U, H)$. Endowed with the norm

$$\|f\|^2_{W^m} = \sum_{i=0}^{m} \|\bar{\partial}^i f\|^2_{2,U}$$

$W^m(U, H)$ becomes a Hilbert space contained continuously in $L^2(U, H)$. Let us mention that the notation W^m above differs from the usual one, which involves all the derivatives. The associated sheaf of locally W^m-functions will be denoted W^m_{loc}.

If the open set U has in addition a smooth boundary, then a standard approximation procedure shows that $C^\infty(\bar{U}, H)$ is a dense subspace of $W^m(U, H)$, [8, §I.2.6].

We next discuss some facts concerning the local spectral theory of the multiplication operator by z on $W^m(U, H)$. For the beginning let us recall some terminology and facts from [5].

Let X be a complex Banach space and let us denote by T a linear bounded operator on X. Assuming property (β) holds for T there exists for every $x \in X$ a compact set $\sigma_T(x) \subset \mathbf{C}$, called the *local spectrum* of T at x, which is minimal with respect to the following property: there is an analytic function $f \in \mathcal{O}(\mathbf{C} \setminus \sigma_T(x), X)$, such that

$$(z - T)f(z) = x, \quad \text{for } z \in \mathbf{C} \setminus \sigma_T(x).$$

One associates then to an arbitrary closed subset F of $\mathbf{C}$ the *maximal spectral space*

$$X_T(F) = \{x \in X \mid \sigma_T(x) \subset F\}.$$

This is a closed invariant subspace for T and, denoting by $T \mid X_T(F)$ the restricted operator, the following spectral inclusion

$$\sigma(T \mid X_T(F)) \subset F$$

holds.

If T is a scalar operator, i.e. it admits a spectral distribution U, then it has property (β) and the maximal spectral spaces can be computed as follows

$$X_T(F) = \{x \in X \mid U(\varphi)x = 0, \ \varphi \in C_0^\infty(\mathbf{C}), \ \operatorname{supp}(\varphi) \cap F = \emptyset\}. \tag{2}$$

The maximal spectral spaces of a scalar operator have the following global decomposition property, inherited from the existence of partitions of unity with smooth functions: for an arbitrary finite open covering $(U_i)_{i=1}^n$ of $\mathbf{C}$,

$$\sum_{i=1}^{n} X_T(\overline{U}_i) = X.$$

An operator T with property (β) and which satisfies the above condition is said to be decomposable, [5], [12].

A more technical notion which appears in [1] is the following. The operator $T \in \mathscr{L}(X)$ is called *unconditionally decomposable* if it is decomposable and if for any system $(F_k)_{k=1}^n$ of disjoint closed subsets in $\mathbf{C}$ we have

$$\sup\left\{\left\| \sum_{k=1}^{n} e^{i\theta_k} x_k \right\| : 0 \leqslant \theta_k < 2\pi\right\} \leqslant a_T \left\| \sum_{k=1}^{n} x_k \right\|$$

where $x_k \in X_T(F_k)$ and $a_T < \infty$ depends only on T.

Let us come back now to concrete function spaces. Let U be a (connected) bounded open subset of $\mathbf{C}$ and let m be a non-negative integer.

The linear operator M of multiplication by z on $W^m(U, H)$ is continuous and it has a spectral distribution of order m, defined by the relation

$$U(\varphi)f = \varphi f, \quad \varphi \in C_0^m(\mathbf{C}), \ f \in W^m(U, H).$$

The maximal spectral spaces of M are by (2):

$$W^m(U, H)_M(F) = \{f \in W^m \mid \operatorname{supp}(f) \subset F\}. \tag{3}$$

Instead of $\operatorname{supp}(f) \subset F$ we shall write also $f \mid \overline{U} \setminus F = 0$.

Let $V : W^m(U, H) \to \bigoplus_0^m L^2(U, H)$ be the operator $V(f) = (f, \bar{\partial} f, \ldots, \bar{\partial}^m f)$. Then V is an isometry which intertwines M and the normal operator $\bigoplus_0^m M_z$, therefore M is a subnormal operator.

Because of (3) the maximal spectral spaces of M corresponding to disjoint closed sets are orthogonal, hence M is unconditionally decomposable.

Let us remark finally that M is not a Dunford spectral operator [6], because the estimate

$$\|(\lambda - z)^{-1} f\|_{W^m} \leqslant \|f\|_{W^m}/\mathrm{dist}(\lambda, \mathrm{supp}(f)), \quad \lambda \notin \mathrm{supp}(f), \tag{4}$$

would imply by Theorem XV.6.7 of [6] that M is a Dunford scalar operator, which contradicts the fact that M is subnormal but non-normal.

2. THE FUNCTIONAL MODEL

This first part of §2 deals with the basic inequality for the proof of Theorem 1. Let T be a linear bounded operator on the Hilbert space H. Then, for a given open bounded subset U of $\mathbf{C}$, $z - T$ acts (linearly and) continuously on the space $W^2(U, H)$. For a fixed $z \in \mathbf{C}$, the adjoint on H will be denoted as usually by $(z - T)^* = \bar{z} - T^*$. On the other hand the adjoint in $W^2(U, H)$ will be denoted by $(z - T)^{\#}$.

PROPOSITION 2.1. *For every bounded disk D in $\mathbf{C}$ there is a constant C_D, such that for an arbitrary operator $T \in \mathscr{L}(H)$ and $f \in W^2(D, H)$ we have*

$$\|(I - P)f\|_{2,D} \leqslant C_D(\|(z - T)^* \bar{\partial} f\|_{2,D} + \|(z - T)^* \bar{\partial}^2 f\|_{2,D}).$$

Let us recall that P denotes the orthogonal projection of $L^2(D, H)$ onto the Bergman space $A^2(D, H)$.

Proof. Let $f_n \in C^\infty(\bar{D}, H)$ be a sequence which approximates f in the norm W^2. Then for a fixed n we have

$$\bar{\partial}(f_n(z) - (z - T)^* \bar{\partial} f_n(z)) = -(z - T)^* \bar{\partial}^2 f_n(z).$$

By the Cauchy-Pompeiu formula (1) one gets

$$f_n(z) - (z - T)^* \bar{\partial} f_n(z) - \frac{1}{2\pi i} \int_{\partial D} (f_n(\zeta) - (\zeta - T)^* \bar{\partial} f_n(\zeta))(\zeta - z)^{-1} \mathrm{d}\zeta =$$

$$= \frac{1}{\pi} \int_D ((\zeta - T)^* \bar{\partial}^2 f_n(\zeta))(\zeta - z)^{-1} \mathrm{d}\mu(\zeta).$$

Let us denote by g_n the first integral in the above formula. Then $g_n \in A^2(D, H)$, hence

$$\|f_n - g_n\|_{2,D} \leqslant \|(z - T)^* \bar{\partial} f_n\|_{2,D} + 4R\|(z - T)^* \bar{\partial}^2 f_n\|_{2,D}$$

where the second integral was majorized as a convolution with a L^1-function and R is the radius of D.

Finally we obtain the inequalities

$$\|f - Pf\|_2 \leqslant \|f - g_n\|_2 \leqslant \|f - f_n\|_2 + \|f_n - g_n\|_2 \leqslant$$

$$\leqslant \|f - f_n\|_2 + \|(z - T)^* \bar{\partial} f_n\|_2 + 4R\|(z - T)^* \bar{\partial}^2 f_n\|_2 ,$$

which prove the proposition when passing to the limit.

COROLLARY 2.2. *If T is hyponormal, then*

$$\|(I - P)f\|_{2,D} \leqslant C_D(\|(z - T)\bar{\partial} f\|_{2,D} + \|(z - T)\bar{\partial}^2 f\|_{2,D}).$$

Proof. This follows from $\|(z - T)^* h\|_2 \leqslant \|(z - T)h\|_2$.

Proof of Theorem 1. Let T be a hyponormal operator on the Hilbert space H. Let us consider an arbitrary bounded open subset U of $\mathbf{C}$ and the quotient space

$$\mathscr{H}(U) = W^2(U, H)/\overline{(z - T)W^2(U, H)}$$

endowed with the Hilbert space norm which identifies it with $\operatorname{Ker}(z - T)^{\#}$. The class of a vector f or an operator A on this quotient will be denoted by $\tilde{f}$, respectively $\tilde{A}$. Note that M, the multiplication operator with z on $W^2(U, H)$, leaves invariant the range of $z - T$, hence $\tilde{M}$ is well defined. Moreover, the spectral distribution U of M commutes with $z - T$, therefore $\tilde{M}$ is still a scalar operator of order 2, with $\tilde{U}$ as spectral distribution.

Let V be the operator $V(h) = \widetilde{1 \otimes h}$, from H into $\mathscr{H}(U)$, denoting by $1 \otimes h$ the constant function h. Then

$$VT = \tilde{M}V. \tag{5}$$

Indeed, $VTh = \widetilde{1 \otimes Th} = \widetilde{z \otimes h} = \tilde{M}(\widetilde{1 \otimes h}) = \tilde{M}Vh$. In particular the range of V is an invariant subspace for $\tilde{M}$.

LEMMA 2.3. *Let D be a bounded disk which contains $\sigma(T)$. Then the operator $V : H \to \mathscr{H}(D)$ is one to one and has closed range.*

Proof. We have to prove the following assertion: if $h_n \in H$ and $f_n \in W^2(D, H)$ are sequences such that

$$\lim_n \|(z - T)f_n + 1 \otimes h_n\|_{W^2} = 0, \tag{6}$$

then $\lim h_n = 0$.

The assumption (6) implies $\lim(\|(z - T)\bar{\partial} f_n\|_2 + \|(z - T)\bar{\partial}^2 f_n\|_2) = 0$, which, in view of Corollary 2.2, shows that $\lim \|(I - P)f_n\|_2 = 0$. Then by (5)

$$\lim \|(z - T)Pf_n + 1 \otimes h_n\|_{2,D} = 0,$$

which by Lemma 1.1 insures that

$$\lim \|(z - T)Pf_n + 1 \otimes h_n\|_{\infty,D'} = 0,$$

where D' is a relatively compact domain in D, still containing $\sigma(T)$.

Let us denote by

$$\Phi : \mathcal{O}(D', H) \to H$$

the analytic functional calculus map associated to T. Then by the continuity of Φ (see for instance [12, Proposition III.8.13] for this form of the functional calculus) there is a constant $a > 0$, such that

$$\|h_n\| := \|\Phi((z - T)Pf_n + 1 \otimes h_n)\| \leqslant a\|(z - T)Pf_n + 1 \otimes h_n\|_{\infty,D'} .$$

Consequently $\lim \|h_n\| = 0$ and this concludes the proof of Lemma 2.3.

This also concludes the proof of Theorem 1, because the range of V is, in the conditions of Lemma 2.3, by (5), a closed invariant subspace for the scalar operator $\widetilde{M}$.

The next result establishes an analogue of the single valued extension property for the space W^2 and the hyponormal operator T.

LEMMA 2.4. *Let D be an arbitrary bounded disk in* $\mathbf{C}$. *Then the operator*

$$z - T : W^2(D, H) \to W^2(D, H)$$

is one to one.

Proof. Let $f \in W^2(D, H)$ be such that $(z - T)f = 0$. Then $f = Pf \in A^2(D, H)$ by Corollary 2.2. Because T has the single valued extension property, e.g. as a subscalar operator, then we infer from $(z - T)f = 0$ that $Pf = 0$, that is $f = 0$. Q.E.D.

Let us remark that if $\lim_n \|(z - T)f_n\|_{W^2} = 0$, then we cannot obtain by the same method more than $\lim \|f_n\|_2 = 0$.

A reformulation of Lemma 2.4 above is the following:

$$\operatorname{supp}((z - T)f) = \operatorname{supp}(f)$$

for every $f \in W^2(D, H)$.

Indeed one computes the support of f, respectively of $(z - T)f$, by restricting these functions to small disks contained in D.

Let us return to the functional model. If U_1 and U_2 are two bounded open sets in $\mathbf{C}$ which contain $\sigma(T)$, then the corresponding spaces $\mathscr{H}(U_1)$ and $\mathscr{H}(U_2)$ coincide, because $z - T$ is invertible for $z \in U_1 \setminus \sigma(T)$, respectively $z \in U_2 \setminus \sigma(T)$. In fact they are both isomorphic with the universal Fréchet space

$$W_T^2(H) = W_{\mathrm{loc}}^2(\mathbf{C}, H)/\overline{(z-T)W_{\mathrm{loc}}^2(\mathbf{C}, H)}$$

which depends only on T and H.

Let us fix a bounded disk D which contains $\sigma(T)$ and let us endow $W_T^2(H)$ with the corresponding Hilbert space structure. Because the operators induced by M, respectively $I \otimes T$ on $W_T^2(H)$ coincide, we shall denote $\widetilde{M}$ also by $\widetilde{T}$, depending on the context.

PROPOSITION 2.5. *Let T be a hyponormal operator on the Hilbert space H. Let $W_T^2(H)$ denote the above Hilbert quotient space, with the induced operator $\widetilde{T}$ on it and the natural embedding $V : H \to W_T^2(H)$. Then*

a) *$\widetilde{T}$ is a scalar operator of order 2 with $\sigma(\widetilde{T}) \subset \sigma(T)$.*

b) *The linear span of the vectors $\{\widetilde{U}(\varphi)Vh : \varphi \in C_0^\infty(\mathbf{C}), h \in H\}$ is dense in $W_T^2(H)$.*

c) *If $A \in \mathscr{L}(H)$ and T, S is a pair of hyponormal operators on H, such that $AT = SA$, then A induces an operator $\widetilde{A} \in \mathscr{L}(W_T^2(H), W_S^2(H))$ with the property $\widetilde{A}\widetilde{T} = \widetilde{S}\widetilde{A}$.*

d) *If f is an analytic function in a neighbourhood of $\sigma(T)$, then*

$$Vf(T) = f(\widetilde{T})V,$$

where $f \mapsto f(T)$ is the functional calculus morphism.

Proof. a) follows from the fact that $W_T^2(H)$ can be represented as a quotient of $W^2(U, H)$, with U an arbitrary open neighbourhood of $\sigma(T)$, and consequently $\sigma(\widetilde{M}) \subset \sigma(M|W^2(U, H)) \subset \overline{U}$.

b) is a consequence of the density of $C^\infty(\overline{D})$ in $W^2(D)$.

c) is derived directly from the definitions.

d) is a general property of the analytic functional calculus, [12, Corollary III.9.11].

COROLLARY 2.6. *With the notations of the above proposition*

$$\partial\sigma(T) \subset \sigma(\widetilde{T}) \subset \sigma(T).$$

Proof. Since $\partial\sigma(T)$ is in the approximate point spectrum of T, it is also contained in the spectrum of the extension $\widetilde{T}$.

In particular this corollary shows that, exactly as for subnormal operators, the spectrum $\sigma(T)$ is obtained from $\sigma(\widetilde{T})$ by filling some bounded connected components of $\mathbf{C} \setminus \sigma(\widetilde{T})$.

REMARKS. 1) The problem of scalar extensions of operators is treated in its full generality in [11], in analogy with the theory of subnormal operators. The major distinction between the two cases is the following: while the minimal normal extension of a subnormal operator is unique, the minimality, as it is stated in Proposition 2.5.b) above, does not insure the uniqueness of the scalar extension of a subscalar operator.

2) Though $f(T)$ is not necessarily a hyponormal operator, when f is an analytic function in some neighbourhood of $\sigma(T)$, the assertion d) in the above proposition shows that $f(T)$ is still a subscalar operator.

3) The operator $\widetilde{T}$ is normal whenever T is normal. Indeed, in this case $I \otimes T^*$ commutes with $z - T$, hence $\widetilde{T^*}$ is well defined and $\widetilde{T}^* = \widetilde{T^*}$.

4) Because the main facts of the local spectral theory of a hyponormal operator (for instance property (β), the estimates for the local resolvents, and so on, see [4, Chapter I]) are hereditary, they can be derived directly from the same properties of the scalar extension $\widetilde{T}$. The verification of these properties for the operator $\widetilde{T}$ is completely analogous with that for normal operators and uses the fact that $\widetilde{T}$ has a continuous functional calculus with second order continuous differentiable functions with respect only to the operator $\bar{\partial}$.

5) An intriguing question is whether S. Brown's techniques [3] can produce, via the above Theorem 1, invariant subspaces for hyponormal operators. The closest result in this direction is Proposition 2.5 in [1], but which requires that the scalar extension should be unconditionally decomposable. We ignore if $\widetilde{T} = \widetilde{M}$ has this property when T is hyponormal.

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