J. OPERATOR THEORY
13(1985), 33–47

AC FUNCTIONS ON THE CIRCLE AND SPECTRAL FAMILIES

EARL BERKSON and T. A. GILLESPIE

1. INTRODUCTION

If $J = [a, b]$ is a compact interval on the real line $\mathbf{R}$, we denote by $\mathrm{AC}(J)$ the Banach algebra of all complex-valued absolutely continuous functions on J with the norm $\|\cdot\|_J$ given by $\|f\|_J = |f(b)| + \mathrm{var}(f, J)$, where $\mathrm{var}(f, J)$ is the total variation of f on J. In [10], [11], [12] Ringrose and Smart introduced the following notion of *well-bounded* operator. An operator T on a Banach space X is said to be *well-bounded* provided that for some compact interval J, T has an $\mathrm{AC}(J)$-functional calculus (that is, a bounded algebra homomorphism ψ of $\mathrm{AC}(J)$ into $\mathscr{B}(X)$, the algebra of bounded operators on X, which sends the identity map to T and the constant 1 to the identity operator I). Ringrose showed that the well-bounded operators on X can be characterized by a certain representation reminiscent of the spectral theorem for self-adjoint operators, but in terms of a (not necessarily unique) function on $\mathbf{R}$ whose values are projections acting, in general, in X^*, the dual space of X [11, Theorem 2-(i) and Theorem 6-(i)]. In order to ensure the existence of a "spectral family" of projections, $E(\cdot)\colon \mathbf{R} \to \mathscr{B}(X)$, such that T has a Stieltjes integral representation $T = \int_{a-}^{b} \lambda \mathrm{d}E(\lambda)$ with $E(\cdot)$ uniquely determined, the special class of well-bounded operators of type (B) was introduced in [4] (formal definitions will be given in § 2). The well-bounded operators of type (B) can be characterized [4, Theorem 4.2] by additionally requiring that the $\mathrm{AC}(J)$-functional calculus in the definition of well-boundedness be weakly compact—that is, map bounded subsets of $\mathrm{AC}(J)$ onto subsets of $\mathscr{B}(X)$ whose closure in the weak operator topology is compact in that topology. (In particular, every well-bounded operator on a reflexive space is automatically of type (B).)

A major limitation on direct applications of the theory of well-bounded operators is that the definition implicitly requires the spectrum of a well-bounded operator to be real (in fact, a subset of J). Moreover, as pointed out in [11, § 8.1], attempts to extend the notion by a functional calculus type of definition to operators with complex spectra (even to operators whose spectra fill out a simple closed arc) face serious obstacles. On the other hand, operators of the form e^{iA}, where A is well-bounded of type (B), have been shown to occur naturally in analysis. For example, if $1 < p < \infty$, then all translation operators on $L^p(G)$ (where G is an arbitrary locally compact abelian group) and all surjective isometries of the Hardy space $H^p(\mathbf{D})$ (where $\mathbf{D}$ is the open unit disc in the complex plane $\mathbf{C}$) are of this form (see [8, Theorem 1] and the discussion in [3]). It is desirable, therefore, to obtain a characterization of such operators in terms of an intrinsic functional calculus, and this we do in § 2. We show (Theorem 2.3 below) that a necessary and sufficient condition for an operator S to be of the form e^{iA}, with A well-bounded of type (B), is that S should have a weakly compact AC($\mathbf{T}$)-functional calculus, where $\mathbf{T}$ is the unit circle $|z| = 1$ in $\mathbf{C}$. Since the trigonometric polynomials are dense in AC($\mathbf{T}$), we call such operators "trigonometrically well-bounded" (formally in Definition 2.18 below).

The proof of sufficiency in Theorem 2.3 has one additional advantage. With obvious adaptations to an AC(J)-functional calculus, it affords a simpler and more self-contained existence proof than hitherto available for the spectral family of projections of a well-bounded type (B) operator. This is discussed briefly at the end of § 2, and should be compared with the existence proofs for the spectral family of a type (B) operator in [4, Theorem 4.2] and [6, Theorem 17.14]. In § 3, the concluding section, a second characterization of trigonometrically well-bounded operators is given in terms of a suitable "unitary-like" Cartesian decomposition (Theorem 3.4).

Finally, it should be mentioned that trigonometrically well-bounded operators were used in [2] to extend the class of type (B) well-bounded operators to a class of operators with complex spectra having a suitable "polar" decomposition (called polar operators). The polar operators include all scalar-type spectral operators and lend themselves to semigroup considerations. In particular, trigonometrically well-bounded operators can be used to generalize Stone's theorem for unitary groups to arbitrary Banach spaces so as to encompass aspects of multiplier theory [2, Theorem 4.20 and 4.47].

2. THE FUNCTIONAL CALCULUS OF TRIGONOMETRICALLY WELL-BOUNDED OPERATORS

We begin this section with a few basic facts about spectral families of projections and well-bounded operators. For a fuller summary of these topics see [2, § 2]. A complete treatment of the essential facts can be found in [6, Part 5].

DEFINITION. A *spectral family* in a Banach space X is a uniformly bounded, projection-valued function $E(\cdot) : \mathbf{R} \to \mathscr{B}(X)$, which is right continuous on $\mathbf{R}$ in the strong operator topology, has a strong left-hand limit at each point of $\mathbf{R}$, and satisfies

(i) $E(\lambda)E(\mu) = E(\mu)E(\lambda) = E(\min\{\lambda, \mu\})$ for $\lambda, \mu \in \mathbf{R}$;

(ii) $E(\lambda) \to 0$ (resp., $E(\lambda) \to I$) in the strong operator topology as $\lambda \to -\infty$ (resp., $\lambda \to +\infty$).

If there is a compact interval $[a, b]$ such that $E(\lambda) = 0$ for $\lambda < a$ and $E(\lambda) = I$ for $\lambda \geqslant b$, we shall say that $E(\cdot)$ is concentrated on $[a, b]$.

Although spectral families of projections need not arise from projection-valued measures (indeed, one of the advantages of spectral families is their ability to treat conditional convergence, as illustrated in [2, 4.47 (i)]), an integration theory is available for spectral families ([6, Chapter 17]). If $E(\cdot)$ is a spectral family and $J = [\alpha, \beta]$ is a compact interval, then for each $f \in \mathrm{AC}(J)$, $\int_\alpha^\beta f(\lambda)\mathrm{d}E(\lambda)$ exists as a strong limit of Riemann-Stieltjes sums. We shall denote $f(\alpha)E(\alpha) + \int_\alpha^\beta f(\lambda)\mathrm{d}E(\lambda)$ by $\int_J^\oplus f\mathrm{d}E$. The map $f \mapsto \int_J^\oplus f\mathrm{d}E$ is an algebra homomorphism of $\mathrm{AC}(J)$ into $\mathscr{B}(X)$, and

$$\left\|\int_J^\oplus f\mathrm{d}E\right\| \leqslant \|f\|_J \sup\{\|E(\lambda)\| : \lambda \in J\} \quad \text{for } f \in \mathrm{AC}(J). \tag{2.1}$$

The definition of well-bounded operator of type (B) we shall use is that indicated in § 1. Specifically, an operator $T \in \mathscr{B}(X)$ is well-bounded of type (B) if and only if for some compact interval J, T has a weakly compact $\mathrm{AC}(J)$-functional calculus. The following characterization of well-bounded operators of type (B) in terms of spectral families is known (see [6, Theorem 17.14 and Theorem 16.3 (i)]).

2.2. Proposition. *An operator $T \in \mathscr{B}(X)$ is well-bounded of type* (B) *if and only if there is a spectral family $E(\cdot)$ in X such that for some compact interval J, $E(\cdot)$ is concentrated on J and $T = \int_J^\oplus \lambda\mathrm{d}E(\lambda)$. If this is the case, there is a unique spectral family $E(\cdot)$ for which such a compact interval J exists. This unique $E(\cdot)$ is called the spectral family of T.*

For each complex-valued function g on $\mathbf{T}$, we denote by $\tilde{g}: [0, 2\pi] \to \mathbf{C}$ the function given by $\tilde{g}(\theta) = g(e^{i\theta})$ for $\theta \in [0, 2\pi]$. We shall denote $\{g \in \mathbf{C}^\mathbf{T} : \tilde{g} \in \mathrm{AC}[0, 2\pi]\}$ by $\mathrm{AC}(\mathbf{T})$. Thus $\mathrm{AC}(\mathbf{T})$ is a Banach algebra with pointwise operations and the norm given by $\|g\| = \|\tilde{g}\|_{[0,2\pi]}$. In fact, the mapping $g \in \mathrm{AC}(\mathbf{T}) \mapsto \tilde{g}$ is an isometric Banach algebra isomorphism of $\mathrm{AC}(\mathbf{T})$ onto $\{f \in \mathrm{AC}[0, 2\pi] : f(0) = f(2\pi)\}$. This concludes our discussion of basic background material. We come now to our main theorem. For each integer n, let e_n be the element of AC(**T**) given by $e_n(z) = z^n$ for $z \in \mathbf{T}$.

2.3. THEOREM. *Let X be a Banach space and $T \in \mathscr{B}(X)$. A necessary and sufficient condition that T have the form $T = e^{iA}$ with A well-bounded of type* (B) *is that there exist a norm continuous algebra homomorphism Φ of* AC(**T**) *into $\mathscr{B}(X)$ such that:*

(i) *$\Phi(e_0) = I$, the identity operator, and $\Phi(e_1) = T$;*

(ii) *for each bounded subset B of* AC(**T**), *the closure in the weak operator topology of $\Phi(B)$ is compact in the weak operator topology.*

In order not to digress from the main theorem at this juncture, we first take up the necessity argument, since the sufficiency proof will require some lemmas.

Proof of Necessity. By [2, proof of Proposition 3.11], $T = e^{iA_0}$, where A_0 is well-bounded of type (B) and $\sigma(A_0) \subseteq [0, 2\pi]$ ($\sigma(A_0)$ denotes the spectrum of A_0). Let $E(\cdot)$ be the spectral family of A_0. In particular, $E(\cdot)$ is concentrated on $[0, 2\pi]$ (by virtue of [6, Theorem 19.2]), and $A_0 = \int_{[0,2\pi]}^{\oplus} \lambda \, dE(\lambda)$. Thus $T = \int_{[0,2\pi]}^{\oplus} e^{i\lambda} dE(\lambda)$. Let ψ be the norm continuous algebra homomorphism of AC$[0,2\pi]$ into $\mathscr{B}(X)$ given by $\psi(f) = \int_{[0,2\pi]}^{\oplus} f dE$. By [6, Theorem 17.14] for each bounded subset $\mathscr{S}$ of AC$[0, 2\pi]$, w-cl. $\psi(\mathscr{S})$ is compact in the weak operator topology, where "w-cl." signifies closrure in the weak operator topology. The necessity proof is now easily concluded by setting $\Phi(g) = \psi(\tilde{g})$ for each $g \in$ AC(**T**).

We now take up some lemmas needed for the sufficiency proof of Theorem 2.3. The first lemma, a recent result of Fong and Lam [7, proof of Proposition 2.2], provides, from first principles, a key tool for producing the spectral family crucial to the sufficiency proof. In order to make the discussion in this section more self-contained, we shall reproduce the simple Fong-Lam proof of the lemma.

2.4. LEMMA. *Suppose $\mathscr{A}$ is an algebra over* **R** *with identity I, and $\mathscr{K}$ is a subset of $\mathscr{A}$ such that each of $\mathscr{K}$ and $I - \mathscr{K}$ is closed under multiplication. Then every extreme point of $\mathscr{K}$ is an idempotent.*

Proof. Let x be an extreme point of $\mathscr{K}$. Since $\mathscr{K}$ and $I - \mathscr{K}$ are closed under multiplication, $x^2 \in \mathscr{K}$ and $(2x - x^2) \in \mathscr{K}$. Since $x = 2^{-1}\{x^2 + (2x - x^2)\}$, and x is an extreme point, x must be idempotent.

2.5. LEMMA. *Suppose Ψ is a norm continuous homomorphism of* AC(**T**) *into $\mathscr{B}(X)$ such that $\Psi(e_0) = I$. Let $U = \Psi(e_1)$. Then $\sigma(U) \subseteq$* **T**. *If $g \in$* AC(**T**), *and g*

vanishes on a subset of $\mathbf{T}$ *open in the topology of* $\mathbf{T}$ *and containing* $\sigma(U)$, *then* $\Psi(g) = 0$.

Proof. If $z_0 \in \mathbf{C} \setminus \mathbf{T}$, then clearly

$$\Psi((z_0 - e_1)^{-1})(z_0 - U) = (z_0 - U)\Psi((z_0 - e_1)^{-1}) = I.$$

So $\sigma(U) \subseteq \mathbf{T}$. The remaining conclusion can be seen by an elementary argument (based on Liouville's theorem) which follows the first part of the proof of [5, Theorem 1.6, pp. 60, 61].

2.6. LEMMA. *Suppose* Ψ *and* U *satisfy the hypotheses of Lemma* 2.5. *If* $\lambda_0 \in \mathbf{T}$ *and* $\sigma(U) = \{\lambda_0\}$, *then* $U = \lambda_0 I$.

Proof. Let $W = \bar{\lambda}_0 U$. For each $F \in \mathrm{AC}(\mathbf{T})$, define $F_{\lambda_0} \in \mathrm{AC}(\mathbf{T})$ by $F_{\lambda_0}(z) = F(\bar{\lambda}_0 z)$ for $z \in \mathbf{T}$, and let $\varphi(F) = \Psi(F_{\lambda_0})$. Then φ is a norm continuous homomorphism of AC(T) into $\mathscr{B}(X)$ with $\varphi(e_0) = I$, $\varphi(e_1) = W$. So it suffices to prove the lemma for the case $\lambda_0 = 1$, which we now consider. For each δ with $0 < \delta < \pi/2$, define $f_\delta \in \mathrm{AC}(\mathbf{T})$ by setting $\tilde{f}_\delta = 0$ on $[0, \delta] \cup [2\pi - \delta, 2\pi]$, $\tilde{f}_\delta(t) = e^{it} - 1$ for $t \in [2\delta, 2\pi - 2\delta]$, $\tilde{f}_\delta(t) = \delta^{-1}(e^{2i\delta} - 1)(t - \delta)$ for $t \in [\delta, 2\delta]$, and $\tilde{f}_\delta(t) = \delta^{-1}(e^{-2i\delta} - 1)(2\pi - \delta - t)$ for $t \in [2\pi - 2\delta, 2\pi - \delta]$. Since f_δ vanishes on an open arc containing 1, we have from Lemma 2.5 that $\Psi(f_\delta) = 0$. Let $f = e_1 - 1$. Then

$$\tilde{f}(t) - \tilde{f}_\delta(t) = \begin{cases} e^{it} - 1, & \text{if } t \in [0, \delta] \cup [2\pi - \delta, 2\pi], \\ 0, & \text{if } t \in [2\delta, 2\pi - 2\delta], \\ e^{it} - 1 - \tilde{f}_\delta(t), & \text{if } t \in [\delta, 2\delta] \cup [2\pi - 2\delta, 2\pi - \delta]. \end{cases}$$

Hence

$$\|f - f_\delta\| = \mathrm{var}(e^{it} - 1, [-\delta, \delta]) + \mathrm{var}(e^{it} - 1 - \tilde{f}_\delta(t), [\delta, 2\delta]) +$$
$$+ \mathrm{var}(e^{it} - 1 - \tilde{f}_\delta(t), [2\pi - 2\delta, 2\pi - \delta]).$$

It follows readily that

$$\|f - f_\delta\| \leqslant \mathrm{var}(e^{it} - 1, [-2\delta, 2\delta]) + \mathrm{var}(\tilde{f}_\delta, [\delta, 2\delta]) +$$
$$+ \mathrm{var}(\tilde{f}_\delta, [2\pi - 2\delta, 2\pi - \delta]).$$

The majorant in this inequality is just

$$4\delta + 2|e^{2i\delta} - 1|.$$

So $\Psi(f_\delta) \to \Psi(f) = U - I$ as $\delta \to 0^+$. Since $\Psi(f_\delta) = 0$ for all δ, the proof of the lemma is finished.

We now complete the proof of Theorem 2.3.

Proof of Sufficiency. For $\lambda \in [0, 2\pi)$ and $0 < \delta < 2^{-1}(2\pi - \lambda)$, let $\mathscr{F}_{\lambda,\delta}$ be the set consisting of all real-valued $f \in AC(\mathbf{T})$ such that $\tilde{f} = 1$ on $[0, \lambda]$, $\tilde{f} = 0$ on $[\lambda + \delta, 2\pi - \delta]$, $\tilde{f}$ is decreasing on $[\lambda, \lambda + \delta]$, and $\tilde{f}$ is increasing on $[2\pi - \delta, 2\pi]$. Let $\mathscr{K}_{\lambda,\delta} = \text{w-cl.}\{\Phi(f): f \in \mathscr{F}_{\lambda,\delta}\}$. Put $\mathscr{K}_{\lambda} = \bigcap_{\delta} \mathscr{K}_{\lambda,\delta}$. Clearly such $\mathscr{F}_{\lambda,\delta}$ is convex, non-empty, and norm-bounded (by the constant 3) in $AC(\mathbf{T})$. It follows that each $\mathscr{K}_{\lambda,\delta}$ is a convex, non-void, weakly compact subset of $\mathscr{B}(X)$. Since $\delta_1 < \delta_2$ implies $\mathscr{K}_{\lambda,\delta_1} \subseteq \mathscr{K}_{\lambda,\delta_2}$, it follows by compactness that $\mathscr{K}_{\lambda}$ is non-void. Obviously $\mathscr{K}_{\lambda}$ is weakly compact and convex. It is easy to see that for each δ, $\mathscr{K}_{\lambda,\delta}$ and $(I - \mathscr{K}_{\lambda,\delta})$ are commutative semigroups. Hence $\mathscr{K}_{\lambda}$ and $(I - \mathscr{K}_{\lambda})$ are commutative semigroups. Moreover, it is a straightforward observation that the set $\mathscr{M}$ defined by

$$\mathscr{M} = \{\Phi(f): f \in AC(\mathbf{T})\} \cup \Big(\bigcup_{\lambda,\delta} \mathscr{K}_{\lambda,\delta}\Big) \tag{2.7}$$

is commutative. We now proceed to show that each set $\mathscr{K}_{\lambda}$ for $\lambda \in [0, 2\pi)$ is a singleton set consisting of a projection operator. First we establish a lemma.

2.8. LEMMA. *Let $\lambda \in [0, 2\pi)$ and $E \in \mathscr{K}_{\lambda}$ with $E^2 = E$. Then $T \mid EX$, the restriction of T to EX, satisfies $\sigma(T \mid EX) \subseteq \{e^{i\theta}: 0 \leqslant \theta \leqslant \lambda\}$. Also $\sigma(T \mid (I - E)X) \subseteq \{e^{i\theta}: \lambda \leqslant \theta \leqslant 2\pi\}$.*

Proof. For $z \in \mathbf{C}\setminus\mathbf{T}$, it follows from (2.7) that T and $(z - T)^{-1}$ commute with E. Thus $\sigma(T \mid EX)$ and $\sigma(T \mid (I - E)X)$ are subsets of $\mathbf{T}$. Fix an arbitrary μ such that $\lambda < \mu < 2\pi$. Pick $\delta > 0$ so that $\lambda + \delta < \mu$ and $2\pi - \delta > \mu$. Let $g \in AC(\mathbf{T})$ be such that $\tilde{g}(t) = (e^{i\mu} - e^{it})^{-1}$ for $t \in [0, \lambda + \delta] \cup [2\pi - \delta, 2\pi]$. Then for each $f \in \mathscr{F}_{\lambda,\delta}$, $(e^{i\mu} - e_1)gf = f$. Hence for each $C \in \mathscr{K}_{\lambda,\delta}$, $(e^{i\mu} - T)\Phi(g)C = C$. If, in particular, we take C to be E, the first conclusion of the lemma is established. If $\lambda = 0$, the second conclusion is trivial. Otherwise, fix an arbitrary β so that $0 < \beta < \lambda$. Let $h \in AC(\mathbf{T})$ be such that $\tilde{h}(t) = (e^{i\beta} - e^{it})^{-1}$ for $t \in [\lambda, 2\pi]$. If $\delta > 0$ and $f \in \mathscr{F}_{\lambda,\delta}$, $(e^{i\beta} - e_1)(1 - f)h = (1 - f)$. It follows readily that $(e^{i\beta} - T)\Phi(h)(I - E) = I - E$, and the remaining conclusion of the lemma is apparent.

In order to show that $\mathscr{K}_{\lambda}$ $(0 \leqslant \lambda < 2\pi)$ consists of a single projection, it suffices, in view of the Kreĭn-Milman theorem and Lemma 2.4 to show that any two projections E, F in $\mathscr{K}_{\lambda}$ are equal. If $\lambda > 0$, then it follows from Lemma 2.8 that neither $\sigma(T \mid FX)$ nor $\sigma(T \mid (I - E)X)$ separates the plane. By standard spectral theory (for example, by [6, Theorem 1.29])

$$\sigma(T \mid F(I - E)X) \subseteq \sigma(T \mid FX) \cap \sigma(T \mid (I - E)X).$$

Hence, from Lemma 2.8 we have

$$\sigma(T \mid F(I - E)X) \subseteq \{e^{i\lambda}, 1\}. \tag{2.9}$$

If $\lambda = 0$, then by Lemma 2.8, $\sigma(T \mid FX) \subseteq \{1\}$, and the above reasoning gives (2.9) trivially. Now notice that the set $\mathscr{T}$ of trigonometric polynomials (that is, the linear span of the functions e_n for all integers n) is dense in AC(**T**) in the norm topology. Thus if M is a closed subspace of X invariant under T and T^{-1}, then M is invariant under $\Phi(f)$ for all $f \in$ AC(**T**). Put $Y = F(I - E)X$. It follows from (2.9) and Lemma 2.6 that

$$\text{(2.10)} \qquad \text{if } \lambda = 0,\ Ty = y \quad \text{for all } y \in Y.$$

If $\lambda > 0$ then standard facts about Riesz projections (as in [6, Theorem 1.39]) in conjunction with (2.9) show that there are idempotents P_1, $P_2 \in \mathscr{B}(Y)$ such that: $P_iP_j = 0 \mid Y$ for $i \neq j$; P_1 and P_2 commute with $T|Y$; $P_1 + P_2 = I \mid Y$; and $\sigma(T|P_1Y) \subseteq \{e^{i\lambda}\}$, $\sigma(T|P_2Y) \subseteq \{1\}$. Clearly, P_1Y and P_2Y are invariant under T^{-1}. Thus by Lemma 2.6 $Ty_1 = e^{i\lambda}y_1$ for $y_1 \in P_1Y$, and $Ty_2 = y_2$ for $y_2 \in P_2Y$. Using the density of $\mathscr{T}$ in AC(**T**) with this and with (2.10) we see that for $\lambda \in [0, 2\pi)$ and $0<\delta<(2\pi - \lambda)/2$, $\Phi(f)|Y = I|Y$ for all $f \in \mathscr{F}_{\lambda,\delta}$. Thus $Cy = y$ for $y \in Y$, $C \in \mathscr{K}_{\lambda,\delta}$. In particular, $Ey = y$ for all $y \in Y = F(I - E)X$. Hence $0 = EF(I - E) = F(I - E)$, and so $FE = F$. Similarly, $EF = E$, and we have $E = F$, as required.

Put $\mathscr{K}_\lambda = \{E(\lambda)\}$ for $\lambda \in [0, 2\pi)$, and define $E(\lambda) = 0$ for $\lambda < 0$, $E(\lambda) = I$ for $\lambda \geqslant 2\pi$. We next show that $E(\cdot)$ is a spectral family in X. Clearly for $\lambda \in [0, 2\pi)$, $0 < \delta < (2\pi - \lambda)/2$,

$$\text{(2.11)} \qquad \mathscr{K}_{\lambda,\delta} \subseteq \text{w-cl.}\{\Phi(f): f \in \text{AC}(\mathbf{T}),\ \|f\| \leqslant 3\}.$$

In particular, it follows that $\{\|E(\lambda)\|: \lambda \in \mathbf{R}\}$ is bounded. If $\lambda \in [0, 2\pi)$ and $\{f_\delta\}$ belongs to the Cartesian product $\prod_\delta \mathscr{F}_{\lambda,\delta}$, we claim that $\Phi(f_\delta) \to E(\lambda)$ in the weak operator topology as $\delta \to 0^+$. For each δ, $\Phi(f_\delta)$ belongs to the weakly compact set, w-cl. $\{\Phi(f): f \in \text{AC}(\mathbf{T}), \|f\| \leqslant 3\}$. Hence in order to establish our claim it suffices to show that all weakly convergent subnets of $\{\Phi(f_\delta)\}$ converge weakly to $E(\lambda)$. But since $\mathscr{K}_{\lambda,\delta_1} \subseteq \mathscr{K}_{\lambda,\delta_2}$ if $\delta_1 < \delta_2$, and each $\mathscr{K}_{\lambda,\delta}$ is weakly compact, any weakly convergent subnet of $\{\Phi(f_\delta)\}$ has its weak limit in $\mathscr{K}_\lambda = \{E(\lambda)\}$, which proves the claim. For $\lambda \in [0, 2\pi)$ and $0 < \delta < (2\pi - \lambda)/2$, define $g_{\lambda,\delta} \in \mathscr{F}_{\lambda,\delta}$ by setting $\tilde{g}_{\lambda,\delta}(t) = 1$ for $t \in [0, \lambda] \cup [2\pi - (\delta/2), 2\pi]$, $\tilde{g}_{\lambda,\delta}(t) = 0$ for $t \in [\lambda + \delta, 2\pi - \delta]$, $\tilde{g}_{\lambda,\delta}$ is linear on $[\lambda, \lambda + \delta]$ and $[2\pi - \delta, 2\pi - (\delta/2)]$. If $0 \leqslant \lambda < \mu < 2\pi$ and $0 < \delta_0 < (2\pi - \mu)/2$, then for all sufficiently small positive δ, $\lambda + \delta < \mu$ and $\delta < \delta_0/2$. It is easy to see that for such δ, $g_{\lambda,\delta}g_{\mu,\delta_0} = g_{\lambda,\delta}$. So $\Phi(g_{\lambda,\delta})\Phi(g_{\mu,\delta_0}) = \Phi(g_{\lambda,\delta})$ for all small enough δ. Hence, by the claim just established, $E(\lambda)\Phi(g_{\mu,\delta_0}) = E(\lambda)$, and so $E(\lambda)E(\mu) = E(\lambda)$ for $0 \leqslant \lambda < \mu < 2\pi$. Since the set $\mathscr{M}$ in (2.7) is commutative, the monotonicity requirement (i) in the definition of spectral family is satisfied. To complete the demonstration that $E(\cdot)$ is a spectral family, it remains only to show that $E(\cdot)$ has a strong left-hand limit at each point of $\mathbf{R}$, and $E(\cdot)$ is strongly right continuous at each point of $\mathbf{R}$. By (2.11),

$\{E(\lambda):\lambda\in\mathbf{R}\}$ is contained in a weakly compact set. From this fact and the monotonicity of the uniformly bounded family $E(\cdot)$, we see from [1, Theorem 1] that for each $\lambda\in\mathbf{R}$, $E(\cdot)$ has strong left-hand and right-hand limits at λ (denoted $E(\lambda^-)$ and $E(\lambda^+)$, respectively). So the proof that $E(\cdot)$ is a spectral family is reduced to showing that for each $\lambda_0\in[0,2\pi)$, $E(\lambda_0)=E(\lambda_0^+)$. Obviously $E(\lambda_0^+)$ commutes with $E(\lambda)$ for all $\lambda\in\mathbf{R}$, and with $\Phi(f)$ for all $f\in\mathrm{AC}(\mathbf{T})$. For $\lambda_0<\mu<2\pi$, $E(\mu)E(\lambda_0^+)=E(\lambda_0^+)$, and so by Lemma 2.8, $\sigma(T|E(\lambda_0^+)X)\subseteq\{e^{i\theta}:0\leqslant\theta\leqslant\mu\}$. So $\sigma(T|E(\lambda_0^+)X)\subseteq$ $\subseteq\{e^{i\theta}:0\leqslant\theta\leqslant\lambda_0\}$. With the aid of Lemma 2.8, we have (as in the reasoning which established (2.9)) that

$$\sigma(T|(I-E(\lambda_0))E(\lambda_0^+)X)\subseteq\{e^{i\lambda_0},1\}. \tag{2.12}$$

Since $E(\mu)E(\lambda_0)=E(\lambda_0)$ for $\mu>\lambda$, $E(\lambda_0^+)E(\lambda_0)=E(\lambda_0)$. So, to complete the proof that $E(\cdot)$ is a spectral family, it suffices to show that $E(\lambda_0^+)E(\lambda_0)=E(\lambda_0^+)$. This step can be accomplished by applying (as in the previous argument proceeding from (2.9)) Riesz projections and Lemma 2.6 to (2.12). Alternatively, we can show $E(\lambda_0^+)E(\lambda_0)=E(\lambda_0^+)$ as follows. Let $Z=(I-E(\lambda_0))E(\lambda_0^+)X$. Notice that Z is invariant under T and T^{-1}, hence under $\Phi(f)$ for all $f\in\mathrm{AC}(\mathbf{T})$. By Lemma 2.5, with $\Psi(f)=\Phi(f)|Z$ for all $f\in\mathrm{AC}(\mathbf{T})$, if $g\in\mathrm{AC}(\mathbf{T})$ vanishes on a set open in $\mathbf{T}$ which contains $\sigma(T|Z)$, then $\Phi(g)$ vanishes on Z. For each δ with $0<\delta<(2\pi-\lambda_0)/2$, pick $f_\delta\in\mathscr{F}_{\lambda_0,\delta}$ so that $\tilde{f}_\delta=1$ on $[0,\lambda_0+2^{-1}\delta]\cup[2\pi-2^{-1}\delta,2\pi]$. By (2.12) $f_\delta=1$ on a relatively open set of $\mathbf{T}$ containing $\sigma(T|Z)$. Hence

$$\Phi(f_\delta)z=z\quad\text{for all } z\in Z. \tag{2.13}$$

Since $\{f_\delta\}_{\delta>0}\in\prod_\delta\mathscr{F}_{\lambda_0,\delta}$, recall that $\{\Phi(f_\delta)\}_{\delta>0}$ must approach $E(\lambda_0)$ in the weak operator topology of $\mathscr{B}(X)$ as $\delta\to0^+$. From this and (2.13) we have that $E(\lambda_0)z=z$ for all $z\in Z=(I-E(\lambda_0))E(\lambda_0^+)X$, whence

$$0=E(\lambda_0)(I-E(\lambda_0))E(\lambda_0^+)=(I-E(\lambda_0))E(\lambda_0^+).$$

So it is now established that $E(\cdot)$ is a spectral family in X.

Since the mapping $g\mapsto\int_{[0,2\pi]}^{\oplus}g\,dE$ is a bounded homomorphism of the algebra $\mathrm{AC}[0,2\pi]$ into $\mathscr{B}(X)$, and $E(\cdot)$ is concentrated on $[0,2\pi]$, the sufficiency proof will be complete once we know that $T=\int_{[0,2\pi]}^{\oplus}e^{i\lambda}\,dE(\lambda)$ by virtue of Proposition 2.2. We now show that this equality holds. Fix arbitrary $x\in X$ and $x^*\in X^*$. The mapping $f\mapsto\langle\Phi(f)x,x^*\rangle$ for $f\in\mathrm{AC}(\mathbf{T})$ is a continuous linear functional. Since $\mathrm{AC}[0,2\pi]$

is linearly isometric with $L^1[0, 2\pi] \oplus \mathbf{C}$, and AC(**T**) with a subspace of AC$[0, 2\pi]$, it is easy to see that there are $\varphi \in L^\infty[0, 2\pi]$ and $c \in \mathbf{C}$ such that

$$(2.14) \qquad \langle \Phi(f)x, x^* \rangle = \int_0^{2\pi} \tilde{f}'\varphi + c\tilde{f}(2\pi) \quad \text{for all } f \in \mathrm{AC}(\mathbf{T}).$$

By taking $f \equiv 1$, we see that $c = \langle x, x^* \rangle$. For any $\lambda \in [0, 2\pi)$, consider the functions $g_{\lambda,\delta}$, $\delta > 0$, defined in the previous monotonicity proof for $E(\cdot)$. Using $g_{\lambda,\delta}$ in (2.14) we have

$$(2.15) \qquad \langle \Phi(g_{\lambda,\delta})x, x^* \rangle = \langle x, x^* \rangle - \delta^{-1} \int_\lambda^{\lambda+\delta} \varphi + 2\delta^{-1} \int_{2\pi-\delta}^{2\pi-(\delta/2)} \varphi.$$

For each λ, $\{g_{\lambda,\delta}\}_{\delta>0} \in \prod_\delta \mathscr{F}_{\lambda,\delta}$; so the left-hand side of (2.15) tends to $\langle E(\lambda)x, x^* \rangle$ as $\delta \to 0^+$. For almost all λ, the term on the right of (2.15) preceded by a minus sign approaches $\varphi(\lambda)$ as $\delta \to 0^+$. Thus we have

$$\langle E(\lambda)x, x^* \rangle = \langle x, x^* \rangle - \varphi(\lambda) + \lim_{\delta \to 0^+} 2\delta^{-1} \int_{2\pi-\delta}^{2\pi-(\delta/2)} \varphi,$$

for almost all $\lambda \in [0, 2\pi)$. Since $\varphi(\lambda) = \alpha - \langle E(\lambda)x, x^* \rangle$ a.e., where α is a constant, and $\tilde{f}(2\pi) = \tilde{f}(0)$ for all $f \in$ AC(**T**), substitution for φ in (2.14) gives

$$(2.16) \quad \langle \Phi(f)x, x^* \rangle = -\int_0^{2\pi} \tilde{f}'(\lambda)\langle E(\lambda)x, x^* \rangle \mathrm{d}\lambda + \tilde{f}(2\pi)\langle x, x^* \rangle, \quad \text{for } f \in \mathrm{AC}(\mathbf{T}).$$

Integration by parts applied to (2.16) gives

$$\langle \Phi(f)x, x^* \rangle = \left\langle \left(\int_{[0,2\pi]}^{\oplus} \tilde{f} \mathrm{d}E \right) x, x^* \right\rangle, \quad \text{for all } f \in \mathrm{AC}(\mathbf{T}).$$

Taking $f = e_1$ completes the proof of Theorem 2.3, since x and x^* are arbitrary.

2.17. Corollary. *Let $T \in \mathscr{B}(X)$. Then T has the form $T = \mathrm{e}^{\mathrm{i}A}$ with A well-bounded of type* (B) *if and only if T is invertible in $\mathscr{B}(X)$, and*

(i) *there is a constant K such that for each trigonometric polynomial $q = \sum_{n=-N}^{N} a_n e_n$, $\|q(T)\| \leqslant K\|q\|_{\mathbf{T}}$, where $q(T) = \sum_{n=-N}^{N} a_n T^n$, and $\|q\|_{\mathbf{T}}$ is the norm of q in* AC(**T**);

(ii) *for each set* $\mathscr{S}$ *of trigonometric polynomials bounded with respect to* $\|\cdot\|_{\mathbf{T}}$, w-cl.$\{q(T): q \in \mathscr{S}\}$ *is compact in the weak operator topology of* $\mathscr{B}(X)$.

Proof. The "only if" part is self-evident from Theorem 2.3. The "if" part also follows readily from Theorem 2.3 by using the density of the trigonometric polynomials in AC(**T**).

In view of Corollary 2.17 we introduce the following terminology.

2.18. Definition. An operator $T \in \mathscr{B}(X)$ will be called *trigonometrically well-bounded* provided there is a well-bounded operator A of type (B) such that $T = e^{iA}$.

As indicated in § 1, the sufficiency proof of Theorem 2.3 can be adapted easily to provide a comparatively simple proof that an operator $A \in \mathscr{B}(X)$ which has a weakly compact AC(J)-functional calculus for some compact interval J, can be represented in the form $A = \int_J^{\oplus} \lambda dG(\lambda)$, where $G(\cdot)$ is an appropriate spectral family in X concentrated on J. We conclude with some comments on this adapted proof. For simplicity, we may take $J = [0, 1]$ without loss of generality, and we denote the AC(J)-functional calculus by $f \mapsto f(A)$. The analogue of Lemma 2.6 for well-bounded operators is that a well-bounded operator (not assumed to be of type (B)) whose spectrum is a singleton is automatically a scalar multiple of I. Once this analogue of Lemma 2.6 is established an existence proof for $G(\cdot)$ adapted from the sufficiency proof of Theorem 2.3 is readily obtained by defining $F_{\lambda,\delta}$, for $\lambda \in [0, 1)$ and $0 < \delta < 1 - \lambda$, to be the set of all decreasing functions $f \in$ AC[0, 1] such that $f = 1$ on $[0, \lambda]$ and $f = 0$ on $[\lambda + \delta, 1]$. Then set

$$K_{\lambda,\delta} = \text{w-cl.}\{f(A): f \in F_{\lambda,\delta}\},$$

$$K_\lambda = \bigcap_\delta K_{\lambda,\delta}.$$

This existence proof for $G(\cdot)$ will be considerably simpler than the sufficiency proof of Theorem 2.3 itself, because one deals with intervals of **R** rather than **T**. In particular, at the stages corresponding to (2.9) and (2.12) the majorant sets will be singletons, and the analogue of Lemma 2.6 applies immediately, eliminating the need for auxiliary constructs such as the Riesz projections. We omit further details except for a discussion of the analogue of Lemma 2.6 mentioned above. This is equivalent to the assertion that a quasinilpotent well-bounded operator Q must be 0. One easy way to see this is as follows. Pick $M > 0$ so that Q has an AC$[-M, M]$ functional calculus. If $f \in$ AC$[-M, M]$ and f vanishes on an interval $(-\rho, \rho)$, then for each $z \in \mathbf{C}$ with $|z| < \rho/2$, let $h_z \in$ AC$[-M, M]$ be defined by setting $h_z(t) = (z-t)^{-1}$ for $t \in [-M, M]$ with $|t| \geq \rho$, and then taking h_z to be linear on $[-\rho, \rho]$.

We have $(z - t)h_z(t)f(t) = f(t)$ for $t \in [-M, M]$, $|z| < \rho/2$. Thus, for $0 < |z| < \rho/2$, $h_z(Q)f(Q) = (z - Q)^{-1}f(Q)$. However $\{h_z : |z| < \rho/2\}$ is a bounded subset of AC$[-M, M]$. So $(z - Q)^{-1}f(Q)$ has a removable singularity at $z = 0$. By Liouville's theorem $f(Q) = 0$. For each positive integer n, let $f_n \in$ AC$[-M, M]$ be obtained by taking $f_n(t) = t$ for $|t| \leqslant n^{-1}$, and making f_n constant on each of the two remaining subintervals of $[-M, M]$. Since $f_n(t) - t$ vanishes for $|t| < n^{-1}$, $f_n(Q) = Q$. But $\|f_n\|_{[-M,M]} = 3/n$ for all n. Hence $Q = 0$. A shorter, but more sophisticated, proof that $Q = 0$ can be argued as follows. Clearly from the AC-functional calculus, $\|e^{itQ}\| = O(|t|)$, $t \in \mathbf{R}$, $|t| \geqslant 1$. Moreover, $(1 - e^{iQ})$ is quasinilpotent. By a theorem of Gelfand and Hille [13, pg. 6], $(e^{iQ} - I)^2 = 0$. But $e^{iQ} - I = QC$, where C is invertible and commutes with Q. So $Q^2 = 0$, and for all $z \in \mathbf{C}$, $(I + zQ) = (I - zQ)^{-1}$. Hence $Q(I - zQ)^{-1} = Q(I + zQ) = Q$. But the AC-functional calculus of Q shows that $\|Q(I - ikQ)^{-1}\| \to 0$ as the positive integer k approaches ∞.

3. CARTESIAN DECOMPOSITION FOR TRIGONOMETRICALLY WELL-BOUNDED OPERATORS

We begin this section with two propositions which are essentially contained in [2].

3.1. PROPOSITION. *If $T \in \mathscr{B}(X)$ is trigonometrically well-bounded, then there is a unique well-bounded operator A of type* (B) *such that: $T = e^{iA}$; $\sigma(A) \subseteq [0, 2\pi]$; and $\sigma_p(A)$, the point spectrum of A, does not contain 2π.*

Proof. By [2, Proposition 3.15 and proof of Proposition 3.11].

DEFINITION. The unique A in Proposition 3.1 will be denoted by $\arg T$.

3.2. PROPOSITION. *Suppose $T = e^{iA_0}$, where A_0 is well-bounded of type* (B) *with spectral family $E_0(\cdot)$. Let $E(\cdot)$ be the spectral family of* $\arg T$. *If k and m are integers such that $2\pi k < \min \sigma(A_0)$ and $2\pi m \geqslant \max \sigma(A_0)$, then*

$$\sup\{\|E(\lambda)\| : \lambda \in \mathbf{R}\} \leqslant 1 + 4(m - k)\,[\sup\{\|E_0(\lambda)\| : \lambda \in \mathbf{R}\}]^2.$$

Proof. Use [2, proof of Proposition 3.11].

3.3. LEMMA. *Suppose $W \in \mathscr{B}(X)$, and W has a representation of the form $W = D_1 + iD_2$, where D_1, D_2 are commuting well-bounded operators of type* (B). *Then this representation is unique.*

Proof. Suppose $W = B_1 + iB_2$ is a second such representation. For $t > 0$, $e^{tW} = e^{tB_1}e^{itB_2}$. It follows by [2, Lemma 4.3] that e^{tB_1}, tB_2 are well-bounded of type (B). Hence by [2, Proposition 3.8], $D_j e^{tB_1} = e^{tB_1} D_j$ for $j = 1,2$, and $t \geqslant 0$. If we differentiate this equation with respect to t, and set $t = 0$, we see that $\{B_1, B_2, D_1, D_2\}$ is a commutative family. Since the operators in this family have real spectra and $B_1 + iB_2 = D_1 + iD_2$, standard Gelfand theory shows that there are quasinilpotents $N_1, N_2 \in \mathscr{B}(X)$ such that $B_1 = D_1 + N_1$, $B_2 = D_2 + N_2$. By [2, Lemma 4.1], $N_1 = N_2 = 0$.

DEFINITION. If W satisfies the hypotheses of Lemma 3.3, we shall denote the nniquely determined operators D_1, D_2 by $\mathrm{Re}(W)$ and $\mathrm{Im}(W)$, respectively.

We come now to the Cartesian decomposition theorem for trigonometrically well-bounded operators.

3.4. THEOREM. *Let $T \in \mathscr{B}(X)$. Then T is trigonometrically well-bounded if and only if there are commuting well-bounded operators A, B of type* (B) *such that*

$$T = A + iB, \tag{3.5}$$

and

$$A^2 + B^2 = I. \tag{3.6}$$

If this is the case, let $E_1(\cdot)$, $E_2(\cdot)$, $E_3(\cdot)$ be the spectral families of $\mathrm{Re}(T)$, $\mathrm{Im}(T)$, *and* $\arg T$, *respectively, and let*

$$s_j = \sup\{\|E_j(\lambda)\| : \lambda \in \mathbf{R}\}, \quad \textit{for } j = 1, 2, 3.$$

Then

$$s_3 \leqslant 1 + C_1(s_1 s_2)^2, \tag{3.7}$$

and

$$\max\{s_1, s_2\} \leqslant C_2 s_3^2, \tag{3.8}$$

where C_1 and C_2 are absolute constants.

Proof. (Note: It can be seen from the proof which follows that the constants C_1, C_2 could be taken to be 200 and 7, respectively.) Suppose first T is trigonometrically well-bounded. Then

$$T = \int_{[0,2\pi]}^{\oplus} e^{i\lambda}\, dE_3(\lambda).$$

Let

$$A = \int_{[0,2\pi]}^{\oplus} \cos\lambda\, dE_3(\lambda), \quad B = \int_{[0,2\pi]}^{\oplus} \sin\lambda\, dE_3(\lambda).$$

Put $X_1 = E_3(\pi)X$ and

$$X_2 = \{E_3(2\pi) - E_3(\pi)\}X = \{I - E_3(\pi)\}X.$$

Since the cosine function is strictly monotone on each of the intervals $[0, \pi]$ and $[\pi, 2\pi]$, and the additive inverse of a type (B) operator is of type (B) (see [9, Proposition 2.2.6]), it follows from [2, Lemma 4.3] and the direct sum decomposition $A = (A|X_1) \oplus (A|X_2)$, that A is well-bounded of type (B), and the spectral family $E_1(\cdot)$ of A satisfies

$$\sup\{\|E_1(\lambda)\| : \lambda \in \mathbf{R}\} \leqslant 4s_3^2.$$

Similar reasoning applies to B on writing

$$X = E_3(\pi/2)X \oplus \{E_3(3\pi/2) - E_3(\pi/2)\}X \oplus \{E_3(2\pi) - E_3(3\pi/2)\}X.$$

The spectral family $E_2(\cdot)$ of B then satisfies

$$\sup\{\|E_2(\lambda)\| : \lambda \in \mathbf{R}\} \leqslant 7s_3^2.$$

Since it is clear that $AB = BA$, and (3.5), (3.6) hold, the "only if" assertion and (3.8) are established.

Conversely, suppose $T = A + iB$, where A and B are commuting type (B) operators satisfying (3.6). We first show that T is trigonometrically well-bounded. Clearly $\sigma(A), \sigma(B) \subseteq [-1, 1]$ by (3.6). Since arccos $\in$ AC$[-1, 1]$, we can set

$$D = \int_{[-1,1]}^{\oplus} (-\arccos \lambda)\, \mathrm{d}E_1(\lambda).$$

Thus,

$$A = \cos D. \tag{3.9}$$

Let $X_1 = E_2(0)X$ and $X_2 = \{I - E_2(0)\}X$, so that $X = X_1 \oplus X_2$. The subspaces X_1 and X_2 are invariant under $T, A, B, D, E_1(\cdot)$, and $E_2(\cdot)$, by [6, Theorem 16.3 (ii)]. It is easy to see that

$$\sigma(B|X_1) \subseteq [-1, 0], \quad \sigma(B|X_2) \subseteq [0, 1]. \tag{3.10}$$

From (3.6) and (3.9)

$$[\cos(D|X_j)]^2 + (B|X_j)^2 = I|X_j, \quad \text{for } j = 1, 2. \tag{3.11}$$

Since $(-\arccos)$ is strictly increasing on $[-1, 1]$, D is well-bounded of type (B), and its spectral family $H(\cdot)$ satisfies

$$\sup\{\|H(\lambda)\| : \lambda \in \mathbf{R}\} = s_1. \tag{3.12}$$

Observe that by (3.11)

$$(B|X_1)^2 = [\sin(D|X_1)]^2. \tag{3.13}$$

The commutativity of B and D, (3.10), and the fact that $\sigma(D)$ is obviously a subset of $[-\pi, 0]$, allow us to use standard Banach algebra considerations in conjunction with (3.13) to infer that

$$B|X_1 = \sin(D|X_1) + Q_1, \tag{3.14}$$

where $Q_1 \in \mathscr{B}(X_1)$ is quasinilpotent. Since D is well-bounded of type (B), so is $\sin D$ (by similar reasoning to that employed in the "only if" part of the theorem's proof).

Moreover, the restriction of a well-bounded operator of type (B) to an invariant subspace is also of type (B). Applying [2, Lemma 4.1] to (3.14), we see that $Q_1 = 0$. Applying (3.14) and (3.9) to (3.5), we now have

$$T|X_1 = \cos(D|X_1) + \mathrm{i}\sin(D|X_1).$$

Hence

$$T|X_1 = \mathrm{e}^{\mathrm{i}D|X_1}. \tag{3.15}$$

Similarly we can obtain

$$T|X_2 = \mathrm{e}^{-\mathrm{i}D|X_2}, \tag{3.16}$$

and so $T = \mathrm{e}^{\mathrm{i}V}$, where

$$V = D|X_1 \oplus (-D|X_2) \tag{3.17}$$

corresponding to the direct sum decomposition $X = X_1 \oplus X_2$. Since each of the direct summands on the right of (3.17) is well-bounded of type (B), V is well-bounded of type (B), and T is trigonometrically well-bounded. This settles the "if" assertion of the theorem. Moreover, since $\sigma(D) \subseteq [-\pi, 0]$, (3.17) shows that $\sigma(V) \subseteq [-\pi, \pi]$. If we let $P(\cdot)$ denote the spectral family of V, then from (3.17), (3.12), and the definition of the subspaces X_1, X_2, we have

$$\sup\{\|P(\lambda)\|: \lambda \in \mathbf{R}\} \leqslant 5s_1s_2. \tag{3.18}$$

Application of Proposition 3.2 to (3.18) shows (3.7) with $C_1 = 200$, and completes the proof of the theorem.

The work of the first author was supported by the National Science Foundation.
The work of the second author was supported by a grant from the Carnegie Trust for the Universities of Scotland.

REFERENCES

1. BARRY, J. Y., On the convergence of ordered sets of projections, *Proc. Amer. Math. Soc.*, **5**(1954), 313–314.
2. BENZINGER, H.; BERKSON, E.; GILLESPIE, T. A., Spectral families of projections, semigroups, and differential operators, *Trans. Amer. Math. Soc.*, **275**(1983), 431–475.
3. BERKSON, E., On spectral families of projections in Hardy spaces, *Proc. Japan Acad. Ser. A Math. Sci.*, **58**(1982), 436–439.
4. BERKSON, E.; DOWSON, H. R., On uniquely decomposable well-bounded operators, *Proc. London Math. Soc. (3)*, **22**(1971), 339–358.
5. COLOJOARĂ, I.; FOIAŞ, C., *Theory of generalized spectral operators*, Gordon and Breach, New York, 1968.

6. Dowson, H. R., *Spectral theory of linear operators*, London Math. Soc. Monographs, No. 12, Academic Press, New York, 1978.
7. Fong, C. K.; Lam, L., On spectral theory and convexity, *Trans. Amer. Math. Soc.*, **264**(1981), 59–75.
8. Gillespie, T. A., A spectral theorem for L^p translations, *J. London Math. Soc. (2)*, **11**(1975), 499–508.
9. Ralph, D. J., Semigroups of well-bounded operators and multipliers, Thesis, Univ. of Edinburgh, 1977.
10. Ringrose, J. R., On well-bounded operators, *J. Austral. Math. Soc.*, **1**(1960), 334–343.
11. Ringrose, J. R., On well-bounded operators. II, *Proc. London Math. Soc. (3)*, **13**(1963), 613–638.
12. Smart, D. R., Conditionally convergent spectral expansions, *J. Austral. Math. Soc.*, **1**(1960), 319–333.
13. Stone, M. H., On a theorem of Pólya, *J. Indian Math. Soc.*, **12**(1948), 1–7.

EARL BERKSON
Department of Mathematics,
University of Illinois,
Urbana, Illinois 61801,
U.S.A.

T. A. GILLESPIE
Department of Mathematics,
University of Edinburgh,
James Clerk Maxwell Building,
Edinburgh EH9 3JZ,
Scotland.

Received June 20, 1983.