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COMMUTANT REPRESENTATIONS OF COMPLETELY BOUNDED MAPS

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1. INTRODUCTION

Let $\mathscr{A}$ and $\mathscr{B}$ be C^*-algebras and let $L : \mathscr{A} \to \mathscr{B}$ be a bounded linear map. If for the maps $L \otimes 1_n : \mathscr{A} \otimes M_n \to \mathscr{B} \otimes M_n$, one has that $\sup_n \|L \otimes 1_n\|$ is finite, then L is called *completely bounded* and we let $\|L\|_{\mathrm{cb}}$ denote this supremum. The map L is called *positive* provided that $L(p)$ is positive whenever p is positive, and is called *completely positive* if $L \otimes 1_n$ is positive for all n. It is well-known that every completely positive map is completely bounded, and that for such maps their ordinary norm and cb-norm coincide.

Let $\mathscr{A}$ be a C^*-algebra, let $\mathscr{L}(\mathscr{H})$ be the bounded linear operators on a Hilbert space $\mathscr{H}$, and $\varphi : \mathscr{A} \to \mathscr{L}(\mathscr{H})$ be a completely positive map. Stinespring's representation theorem [7] asserts that given any such map, there is a Hilbert space $\mathscr{K}$, a $*$-homomorphism $\pi : \mathscr{A} \to \mathscr{L}(\mathscr{K})$ and a bounded linear operator $V : \mathscr{H} \to \mathscr{K}$ with $\|\varphi\| = \|V^*V\|$ such that $\varphi(a) = V^*\pi(a)V$ for all a in $\mathscr{A}$. Furthermore, if a certain minimality condition is imposed on the triple $(\pi, V, \mathscr{K})$, then this representation is unique up to unitary equivalence. The goal this paper is to attempt to generalize the above theory to the class of completely bounded maps.

If $L : \mathscr{A} \to \mathscr{L}(\mathscr{H})$ is a completely bounded map, then we show in Section 2 that there exists a Hilbert space $\mathscr{K}$, a $*$-homomorphism $\pi : \mathscr{A} \to \mathscr{L}(\mathscr{K})$, a bounded operator $V : \mathscr{H} \to \mathscr{K}$, and an operator T in the commutant of $\pi(\mathscr{A})$ such that, $L(a) = V^*T\pi(a)V$ for all a in $\mathscr{A}$. We show that such representations are not particularly well-behaved with respect to the cb-norm. Indeed, if one normalizes the above situation by requiring V to be an isometry, then there exist completely bounded maps such that for any such representation $(\pi, V, T, \mathscr{K})$ of L, one has $\|T\| > \|L\|_{\mathrm{cb}}$. Analogously, if one requires instead that $\|L\|_{\mathrm{cb}} = \|V^*V\|$, then one can find completely bounded maps such that for any such representation $(\pi, V, T, \mathscr{K})$ of L, one has $\|T\| > 1$. Also, it is possible to construct two representations, $(\pi_i, V_i, T_i, \mathscr{K}_i)$ of L, with V_i an isometry, $i = 1,2$, such that $\|T_1\| \neq \|T_2\|$, even under very restrictive hypotheses on the representations.

These difficulties lead us to introduce a new norm, $|||\cdot|||$, on the space of completely bounded maps which is more compatible with the order structure induced by the completely positive maps. We show this norm is comparable to the cb-norm and has the properties that for any representation $(\pi, V, T, \mathscr{K})$ of L of the above form with V an isometry, one has $\|T\| \geqslant |||L|||$, and that representations exist where equality is attained.

In Section 3, we turn to the question of uniqueness of these representations. We impose some additional minimality conditions, similar to those for Stinespring's theorem, and prove that a minimal representation always exists. We show that for any two minimal representation $(\pi_i, V_i, T_i, \mathscr{K}_i)$, $i = 1, 2$, that $(\pi_1, \mathscr{K}_1)$ and $(\pi_2, \mathscr{K}_2)$ are unitarily equivalent and that (V_1, T_1) and (V_2, T_2) differ by a densely defined (usually unbounded) similarity. When both the algebra and the Hilbert space are finite dimensional, these results show that a minimal representation of a completely bounded map of the type discussed above is unique up to similarity.

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Given a subset of a Hilbert space, we shall use $[\cdot]$ to denote its linear span. Given a subset $\mathscr{S}$ of a C^*-algebra and a map $L: \mathscr{S} \to \mathscr{B}$ for some other C^*-algebra $\mathscr{B}$, we let $L^*: \mathscr{S}^* \to \mathscr{B}$ be the map defined by $L^*(S^*) = L(S)^*$. When $\mathscr{S} = \mathscr{S}^*$ and $L = L^*$ we call L self-adjoint. Finally, given $\mathscr{S} \subseteq \mathscr{L}(\mathscr{H})$ we let $\mathscr{S}'$ denote its commutant.

2. COMMUTANT REPRESENTATIONS

In this section we prove the existence of certain representations of a completely bounded map from a C^*-algebra into $\mathscr{L}(\mathscr{H})$ and explore some of their properties.

We begin with the following elementary observation about the numerical radius of an operator. Recall that for T in $\mathscr{L}(\mathscr{H})$, the numerical radius, $w(T)$, is defined by

$$w(T) = \sup\{|\langle Tx, x\rangle| : x \in \mathscr{H},\ \|x\| = 1\}.$$

LEMMA 2.1. *Let* $T \in \mathscr{L}(\mathscr{H})$, *then* $w(T) = \sup\{\|\mathrm{Re}(\lambda T)\| : \lambda \in \mathbf{C}, |\lambda| = 1\}$.

Proof. Since $\mathrm{Re}(\lambda T)$ is self-adjoint, $\|\mathrm{Re}(\lambda T)\| = w(\mathrm{Re}(\lambda T))$. From $w(\mathrm{Re}(\lambda T)) \leqslant w(\lambda T) = |\lambda| w(T)$, we have that $\|\mathrm{Re}(\lambda T)\| \leqslant w(T)$ for all $\lambda \in \mathbf{C}$, $|\lambda| = 1$.

On the other hand, if $x \in \mathscr{H}$, then there exists λ, $|\lambda| = 1$, such that $|\langle Tx, x\rangle| = \lambda\langle Tx, x\rangle = \langle \mathrm{Re}(\lambda T)x, x\rangle \leqslant \|\mathrm{Re}(\lambda T)\|$ from which the result follows.

THEOREM 2.2. *Let $\mathscr{A}$ be a unital C^*-algebra and let $L: \mathscr{A} \to \mathscr{L}(\mathscr{H})$ be a completely bounded map, then there exists a Hilbert space $\mathscr{K}$, an isometry $V: \mathscr{H} \to \mathscr{K}$, a unital $*$-representation $\pi: \mathscr{A} \to \mathscr{L}(\mathscr{K})$ with $[\pi(\mathscr{A})V\mathscr{H}]$ dense in $\mathscr{K}$, and a unique operator T in $\pi(\mathscr{A})'$, such that*

$$L(a) = V^*T\pi(a)V \quad \text{for all } a \in \mathscr{A}.$$

Furthermore,

$$\|\mathrm{Re}(\lambda L)\|_{\mathrm{cb}} \leqslant w(T) \leqslant \|L\|_{\mathrm{cb}} \leqslant \|T\| \leqslant 2\ \|L\|_{\mathrm{cb}}$$

for all $\lambda \in \mathbf{C}$, with $|\lambda| = 1$.

If $L = L^*$, then $T = T^*$ and $\|T\| = \|L\|_{\mathrm{cb}}$.

Proof. Without loss of generality we may assume that $\|L\|_{\mathrm{cb}} = 1$. By [4, Theorem 2.5] there exist unital completely positive maps $\varphi_i: \mathscr{A} \to \mathscr{L}(\mathscr{H})$ such that the map $\Phi : \mathscr{A} \otimes M_2 \to \mathscr{L}(\mathscr{H}) \otimes M_2$ defined by

$$\Phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} \varphi_1(a) & L(b) \\ L^*(c) & \varphi_2(d) \end{pmatrix}$$

is completely positive. Since compositions of completely positive maps are completely positive, we have that for any λ, with $|\lambda| = 1$, the following map is completely positive,

$$\begin{pmatrix} 1 \\ \lambda \end{pmatrix}^* \Phi\left(\begin{pmatrix} a & a \\ a & a \end{pmatrix}\right)\begin{pmatrix} 1 \\ \lambda \end{pmatrix} = \varphi_1(a) + \varphi_2(a) + \lambda L(a) + \bar{\lambda} L^*(a).$$

Hence, setting $\varphi = (\varphi_1 + \varphi_2)/2$, we have that for any λ, with $|\lambda| = 1$, $\varphi + \mathrm{Re}(\lambda L)$ is completely positive. In particular, $\varphi \pm \mathrm{Re}(L)$ and $\varphi \pm \mathrm{Im}(L)$ are all completely positive.

Now, let $(\pi, V, \mathscr{K})$ be the minimal Stinespring representation of φ. That is, $\mathscr{K}$ is a Hilbert space, $V: \mathscr{H} \to \mathscr{K}$ is an isometry, $\pi: \mathscr{A} \to \mathscr{L}(\mathscr{K})$ is a unital $*$-representation with $[\pi(A)V\mathscr{H}]$ dense in $\mathscr{K}$, and $\varphi(a) = V^*\pi(a)V$.

Note that $(\varphi + \mathrm{Re}(L))/2$ is completely positive and $\varphi - (\varphi + \mathrm{Re}(L))/2 = (\varphi - \mathrm{Re}(L))/2$ is completely positive, i.e., $\varphi \geqslant (\varphi + \mathrm{Re}(L))/2$. By [1, Theorem 1.4.2] there exists a unique positive P in the commutant of π, $P \leqslant 1$, such that $V^*P\pi V = (\varphi + \mathrm{Re}(L))/2$ and, consequently, $\mathrm{Re}\, L = V^*(2P - 1)\pi V$. Setting $H = 2P - 1$, $\|H\| \leqslant 1$, $H = H^*$, and $\mathrm{Re}\, L = V^*H\pi V$. Similarly, there is a K in the commutant of π, $K = K^*$, $\|K\| \leqslant 1$ with $\mathrm{Im}\, L = V^*K\pi V$, Thus, we have that $L = V^*T\pi V$, by setting $T = H + iK$, T is in the comutant of π, and $\|T\| \leqslant 2$.

If T' was another operator in the commutant of π such that $L = V^*T'\pi V$, letting $T' = H' + iK'$ be its Cartesian decomposition, then we would have $(\varphi + \mathrm{Re}\, L)/2 = V^*((1 + H')/2)\pi V$ which would imply that $1 + H' = 2P$ or $H' = H$.

Similarly, $K' = K$ from which it follows that the T obtained in the above fashion is unique, and the first assertions of the theorem are proven.

Note that when $L = L^*$, $K = 0$ and in this case one has $T = H$ so that $\|T\| \leqslant 1$, which establishes the last claim of the theorem.

Since $\varphi \pm \operatorname{Re}(\lambda L)$ are completely positive for any λ, $|\lambda| = 1$, we may apply the above result to $\operatorname{Re}(\lambda L)$. Thus, we obtain a unique T_λ such that $T_\lambda = T_\lambda^*$, $\operatorname{Re}(\lambda L) = V^* T_\lambda \pi V$, and $\|T_\lambda\| \leqslant 1$. But, clearly $\operatorname{Re}(\lambda L) = V^* \operatorname{Re}(\lambda T) \pi V$ and so by uniqueness, $\operatorname{Re}(\lambda T) = T_\lambda$ and hence $\|\operatorname{Re}(\lambda T)\| \leqslant 1$. Thus, by Lemma 2.1, $w(T) \leqslant 1$. Finally, observe that if $L = V^* T \pi V$, then necessarily $\|L\|_{cb} \leqslant \|T\|$ since $L \otimes 1_n = (V \otimes 1_n)^*(T \otimes 1_n)(\pi \otimes 1_n)(V \otimes 1_n)$. Similarly, since $\operatorname{Re}(\lambda L) = V^* \operatorname{Re}(\lambda T) \pi V$, $\|\operatorname{Re}(\lambda L)\|_{cb} \leqslant \|\operatorname{Re}(\lambda T)\| \leqslant w(T)$. This completes the proof of the theorem.

In general, one can have that $\|T\| > \|L\|_{cb}$ for any representation of L of the above form. In fact, the constant 2 in the above theorem is sharp, even when the domain algebra is commutative, as the following example shows. Consider the completely bounded map $L\colon \mathbf{C} \oplus \mathbf{C} \to M_2$ defined by

$$L(\alpha, \beta) = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}.$$

Identifying $M_2 \otimes M_n$ with $M_n \otimes M_2$, the map

$$L \otimes 1_n \colon (\mathbf{C} \oplus \mathbf{C}) \otimes M_n \to M_2 \otimes M_n$$

is given by

$$L \otimes 1_n((\alpha_{i,j}), (\beta_{i,j})) = \begin{bmatrix} 0 & (\alpha_{i,j}) \\ (\beta_{i,j}) & 0 \end{bmatrix},$$

from which it is clear that $\|L \otimes 1_n\| = 1$.

Now suppose that $L = V^* T \pi V$, where $(\pi, V, T, \mathscr{K})$ are as in Theorem 2.2. Let $\pi(1,0) = P_1$, $\pi(0,1) = P_2$ so that P_1 and P_2 are orthogonal projections, $P_1 + P_2 = 1$. Also, let e_1 and e_2 be the canonical basis vectors for $\mathbf{C}^2$, and set $x_i = Ve_i$, $i = 1, 2$.

We then have that,

$$1 = \langle L(0, 1)e_2, e_1 \rangle = \langle TP_1 x_1, x_2 \rangle = \langle TP_1 x_1, P_1 x_2 \rangle \leqslant \|T\| \cdot \|P_1 x_2\| \cdot \|P_1 x_1\|.$$

Similarly,

$$1 = \langle L(0, 1)e_1, e_2 \rangle \leqslant \|T\| \cdot \|P_2 x_2\| \cdot \|P_2 x_1\|.$$

Choosing θ_i such that $\cos\theta_i = \|P_1 x_i\|$, for $i = 1,2$, and adding the above inequalities, yields, $2 \leqslant \|T\| \cdot [\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2] = \|T\| \cos(\theta_1 - \theta_2) \leqslant \|T\|$.

The necessity of the constant 2 in the above theorem can be better understood if one takes a closer look at the relationship between the order structure, induced by the completely positive maps, and the $\|\cdot\|_{cb}$-structure on the space of completely bounded maps. We shall show that in some sense these structures, while compatible, are not entirely consistent and we shall introduce a new, equivalent, norm structure, which is consistent.

Let $L: \mathscr{A} \to \mathscr{L}(\mathscr{H})$ be completely bounded, then we set

$$|||L||| = \inf\left\{\|\varphi\| : \begin{pmatrix} \varphi & L \\ L^* & \varphi \end{pmatrix} \text{ is completely positive}\right\}.$$

PROPOSITION 2.3. *Let $L : \mathscr{A} \to \mathscr{L}(\mathscr{H})$ be completely bounded, then $\|L\|_{cb} \leqslant |||L||| \leqslant 2\|L\|_{cb}$ with $\|L\|_{cb} = |||L|||$ when $L = L^*$.*

Proof. By [4, Theorem 2.5], there exist completely positive maps $\varphi_i : \mathscr{A} \to \mathscr{L}(\mathscr{H})$ with $\|\varphi_i\| = \|L\|_{cb}$ for $i = 1,2$ such that $\begin{pmatrix} \varphi_1 & L \\ L^* & \varphi_2 \end{pmatrix}$ is completely positive. Hence,

$$\begin{pmatrix} \varphi_1 & L \\ L^* & \varphi_2 \end{pmatrix} + \begin{pmatrix} \varphi_2 & 0 \\ 0 & \varphi_1 \end{pmatrix} = \begin{pmatrix} \varphi_1 + \varphi_2 & L \\ L^* & \varphi_1 + \varphi_2 \end{pmatrix}$$

is completely positive and it follows that $|||L||| \leqslant 2\|L\|_{cb}$. For the other inequality note that if $\begin{pmatrix} \varphi & L \\ L^* & \varphi \end{pmatrix}$ is completely positive, then for $A \in \mathscr{A} \otimes M_k$ with $\|A\| \leqslant 1$, we have that $\begin{pmatrix} 1 & A \\ A^* & 1 \end{pmatrix}$ is positive in $\mathscr{A} \otimes M_k \otimes M_2$ and hence

$$\begin{pmatrix} \varphi \otimes 1_k(1) & L \otimes 1_k(A) \\ L^* \otimes 1_k(A) & \varphi \otimes 1_k(1) \end{pmatrix}$$

is positive. Thus, $\|L \otimes 1_k(A)\| \leqslant \|\varphi \otimes 1_k(1)\| = \|\varphi\|$, and so $\|L\|_{cb} \leqslant \|\varphi\|$, from which $\|L\|_{cb} \leqslant |||L|||$.

Finally, if $L = L^*$, let φ_i, $i = 1,2$ be such that $\|\varphi_i\| = \|L\|_{cb}$ and

$$\Phi = \begin{pmatrix} \varphi_1 & L \\ L & \varphi_2 \end{pmatrix}$$

is completely positive. Consequently,

$$\Phi' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \Phi \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \varphi_2 & L \\ L & \varphi_1 \end{pmatrix}$$

is completely positive, and so

$$(\Phi + \Phi')/2 = \begin{pmatrix} \varphi & L \\ L & \varphi \end{pmatrix}$$

is completely positive, where $\varphi = (\varphi_1 + \varphi_2)/2$. Since $\|\varphi\| = \|L\|_{cb}$, we have $|||L||| = \|L\|_{cb}$.

REMARK 2.4. We shall show later that 2 is the best constant.

REMARK 2.5. If $L: \mathscr{A} \to \mathscr{B}$ is completely bounded and $\mathscr{B}$ is injective, then one can still define $|||L|||$ as above and the conclusions of Proposition of 2.3 hold. In fact, a necessary and sufficient condition for there to exist a completely positive map φ such that $\begin{pmatrix} \varphi & L \\ L^* & \varphi \end{pmatrix}$ is completely positive is that L belongs to the span of the completely positive maps. Indeed, if $L = \varphi_1 - \varphi_2 + i(\varphi_3 - \varphi_4)$ then $\varphi = \varphi_1 + \varphi_2 + \varphi_3 + \varphi_4$ will do. Thus, $|||\cdot|||$ can always be defined on the span of the completely positive maps between any two C^*-algebras.

An example of [6, Example 2.2] shows that, in general, $|||\cdot|||$ and $\|\cdot\|_{cb}$ need not be equivalent norms on the span of the completely positive maps. It would be interesting to know if they are equivalent if and only if every completely bounded map from $\mathscr{A}$ to $\mathscr{B}$ is in the span of the completely positive maps. It would also be interesting to know if it is possible for these norms to be equivalent with a constant different from 2. We begin with a Radon-Nikodym type theorem.

If $L_{i,j}: \mathscr{A} \to \mathscr{B}$, $i, j = 1, \ldots, n$ are linear maps, then by $(L_{i,j})$ we mean the linear map $\Phi: \mathscr{A} \otimes M_n \to \mathscr{B} \otimes M_n$, defined by $\Phi((a_{i,j})) = (L_{i,j}(a_{i,j}))$. We remark that the following diagram commutes,

$$\begin{array}{ccc} (\mathscr{A} \otimes M_n) \otimes M_k & \xrightarrow{\Phi \otimes 1_k} & (\mathscr{B} \otimes M_n) \otimes M_k \\ \downarrow & & \downarrow \\ (\mathscr{A} \otimes M_k) \otimes M_n & \xrightarrow{(L_{i,j} \otimes 1_k)} & (\mathscr{B} \otimes M_k) \otimes M_n, \end{array}$$

where the vertical arrows are the canonical isomorphisms. In particular, one sees that Φ is completely positive if and only if $(L_{i,j} \otimes 1_k)$ is positive for all k. Throughout the remainder of this paper we shall frequently use this diagram to identify $\Phi \otimes 1_k$ with $(L_{i,j} \otimes 1_k)$.

PROPOSITION 2.6. *Let $\pi: \mathscr{A} \to \mathscr{L}(\mathscr{K})$ be a unital $*$-representation, and let $V: \mathscr{H} \to \mathscr{K}$ be a bounded operator, with the span of $[\pi(\mathscr{A})V\mathscr{H}]$ dense in $\mathscr{K}$. Let $T_{i,j}$ be in the commutant of π, and define $L_{i,j}: A \to \mathscr{L}(\mathscr{H})$ by $L_{i,j} = V^* T_{i,j} \pi V$ for $i, j = 1, \ldots, n$. Then $(L_{i,j})$ is completely positive if and only if $(T_{i,j})$ is positive.*

Proof. Assume that $(T_{i,j})$ is positive. Let $(a_{i,j}) \in \mathscr{A} \otimes M_n$ be positive, then $(\pi(a_{i,j}))$ is positive. Since the sets $\{T_{i,j}\}$ and $\{\pi(a_{i,j})\}$ pairwise commute, by [9, Lemma 4.24], $(T_{i,j} \cdot \pi(a_{i,j}))$ is positive. Similarly, if $A_{i,j}$ is in $\mathscr{A} \otimes M_k$ for $i,j=1,\dots$ $\dots, n$ and $(A_{i,j})$ is positive in $(\mathscr{A} \otimes M_k) \otimes M_n$, then since the sets $\{T_{i,j} \otimes 1_k\}$ and $\{\pi \otimes 1_k(A_{i,j})\}$ pairwise commute, $((T_{i,j} \otimes 1_k)\cdot \pi \otimes 1_k(A_{i,j}))_{i,j=1}^n$ is positive in $(\mathscr{L}(\mathscr{K}) \otimes M_k) \otimes M_n$. Hence,

$$((V \otimes 1_k)^*(T_{i,j} \otimes 1_k)\pi \otimes 1_k(A_{i,j})(V \otimes 1_k))_{i,j=1}^n = (L_{i,j} \otimes 1_k(A_{i,j}))_{i,j=1}^n$$

is positive in $(\mathscr{L}(\mathscr{H}) \otimes M_k) \otimes M_n$, from which it follows that $(L_{i,j})$ is completely positive.

In what follows we adopt [,] to indicate matrices.

Conversely, let $[L_{i,j}]$ be completely positive and let $\{e_l\}_{l=1}^n$ be the canonical basis for $\mathbf{C}^n$. Thus, for $\left(\sum_{l=1}^n x_l \otimes e_l\right) \in \mathscr{K} \otimes \mathbf{C}^n$, we have that

$$[T_{i,j}]\left(\sum_{j=1}^n x_j \otimes e_j\right) = \sum_{i,j=1}^n (T_{i,j}x_j) \otimes e_i.$$

Since the span of $\pi(\mathscr{A})V\mathscr{H}$ is dense in $\mathscr{K}$, to prove that $[T_{i,j}]$ is positive it will suffice to consider a vector of the form $\sum_{l=1}^n \left(\sum_{k=1}^m (a_{l,k})Vh_{l,k}\right) \otimes e_l$, where $h_{l,k} \in \mathscr{H}$ and $a_{l,k} \in \mathscr{A}$.

Define $A_l \in \mathscr{A} \otimes M_m$ by

$$A_l = \begin{bmatrix} a_{l,1}, & \dots, & a_{l,m} \\ 0 & \dots\dots & 0 \\ \vdots & & \vdots \\ 0 & \dots\dots & 0 \end{bmatrix},$$

let $\{f_k\}_{k=1}^m$ be the canonical basis for $\mathbf{C}^m$ and set $h_l = \sum_{k=1}^m h_{l,k} \otimes f_k \in \mathscr{H} \otimes \mathbf{C}^m$. We now have that

$$\left\langle [T_{i,j}]\left(\sum_{j=1}^n \sum_{k=1}^m \pi(a_{j,k})Vh_{j,k} \otimes e_j\right), \left(\sum_{j=1}^n \sum_{k=1}^m \pi(a_{j,k})Vh_{j,k} \otimes e_j\right)\right\rangle_{\mathscr{K}\otimes\mathbf{C}^m} =$$

$$= \left\langle [T_{i,j} \otimes 1_m]\left(\sum_{j=1}^n \pi \otimes 1_m(A_j)(V \otimes 1_m)h_j \otimes e_j\right),\right.$$

$$\left.\left(\sum_{j=1}^n \pi \otimes 1_m(A_j)(V \otimes 1_m)h_j \otimes e_j\right)\right\rangle_{\mathscr{K}\otimes\mathbf{C}^m\otimes\mathbf{C}^n} = \left\langle \sum_{i,j=1}^n (T_{i,j} \otimes 1_m)(\pi \otimes 1_m)(A_j)\cdot\right.$$

$$\left.\cdot(V \otimes 1_m)h_j \otimes e_i, \sum_{j=1}^n \pi \otimes 1_m(A_j)(V \otimes 1_m)h_j \otimes e_j\right\rangle_{\mathscr{K}\otimes\mathbf{C}^m\otimes\mathbf{C}^n}.$$

$$\cdot \left\langle \sum_{i,j=1}^{n} (V \otimes 1_m)^* \pi \otimes 1_m(A_i^*)(T_{i,j} \otimes 1_m)\pi \otimes 1_m(A_j)(V \otimes 1_m)h_j, h_i \right\rangle_{\mathscr{H}\otimes\mathbf{C}^m} =$$

$$= \langle [(V \otimes 1_m)^*(T_{i,j} \otimes 1_m)\pi \otimes 1_m(A_i^* A_j)(V \otimes 1_m)]x, x\rangle_{\mathscr{H}\otimes\mathbf{C}^m\otimes\mathbf{C}^n} =$$

$$= \langle [L_{i,j} \otimes 1_m(A_i^* A_j)]x, x\rangle$$

which is positive since $[L_{i,j}]$ is completely positive, with $x = \sum_{i=1}^{n} h_i \otimes e_i$.

To motivate the following definition, we recall the relation between norm and order in $\mathscr{L}(\mathscr{H})$. Namely, for $T \in \mathscr{L}(\mathscr{H})$, $\|T\| \leqslant 1$ if and only if $\begin{pmatrix} 1 & T \\ T^* & 1 \end{pmatrix}$ is a positive operator [2, p. 162].

COROLLARY 2.7. *Let $L\colon \mathscr{A} \to \mathscr{L}(\mathscr{H})$ be completely bounded, let $\varphi\colon \mathscr{A} \to \mathscr{L}(\mathscr{H})$ be completely positive, and let $(\pi, V, \mathscr{K})$ denote the minimal Stinespring representation of φ. Then $\begin{pmatrix} \varphi & L \\ L^* & \varphi \end{pmatrix}$ is completely positive if and only if there exists T in the commutant of π, such that $L = V^* T \pi V$ and $\|T\| \leqslant 1$.*

Proof. Let $L = V^* T \pi V$, $\|T\| \leqslant 1$ and T in the commutant of π. Since $\begin{pmatrix} 1 & T \\ T^* & 1 \end{pmatrix}$ is positive, by Theorem 2.6,

$$\begin{pmatrix} V^*\pi V & V^* T \pi V \\ V^* T^* \pi V & V^* \pi V \end{pmatrix} = \begin{pmatrix} \varphi & L \\ L^* & \varphi \end{pmatrix}$$

is completely positive.

Conversely, if $\begin{pmatrix} \varphi & L \\ L^* & \varphi \end{pmatrix}$ is completely positive, then as in the proof of Theorem 2.2, we can construct $T \in \pi(\mathscr{A})'$ such that $L = V^* T \pi V$. Again applying Theorem 2.6 we have that $\begin{pmatrix} 1 & T \\ T^* & 1 \end{pmatrix}$ must be positive and hence, $\|T\| \leqslant 1$.

PROPOSITION 2.8. *Let $L\colon \mathscr{A} \to \mathscr{L}(\mathscr{H})$ be a completely bounded map. Then there exists $\varphi\colon \mathscr{A} \to \mathscr{L}(\mathscr{H})$ with $\|\varphi\| = \|\|L\|\|$, such that $\begin{pmatrix} \varphi & L \\ L^* & \varphi \end{pmatrix}$ is completely positive. Furthermore, one may choose φ such that, in addition, $\varphi(1) = \|\|L\|\| \cdot 1$.*

Proof. Recall the BW-topology of [1]; in this topology the space of completely bounded maps from $\mathscr{A}$ to $\mathscr{L}(\mathscr{H})$ with norm bounded by some constant is compact. Let

$$\mathscr{S}_n = \left\{ \varphi : \begin{pmatrix} \varphi & L \\ L^* & \varphi \end{pmatrix} \text{ is completely positive and } \|\varphi\| \leqslant \|\|L\|\| + 1/n \right\},$$

then it is easily checked that each $\mathcal{S}_n$ is non-empty, closed and hence compact in the BW-topology. Hence, $\mathcal{S} = \bigcap_{n=1}^{\infty} \mathcal{S}_n$ is non-empty and if $\varphi \in \mathcal{S}$ then $\|\varphi\| = |||L|||$ and $\begin{pmatrix} \varphi & L \\ L^* & \varphi \end{pmatrix}$ is completely positive.

Given such a φ, let $\varphi(1) = P$, let $Q = |||L||| - P$, and let $\gamma: \mathcal{A} \to \mathcal{L}(\mathcal{H})$ be any unital completely positive map. Setting $\psi(a) = \varphi(a) + Q^{1/2}\gamma(a)Q^{1/2}$ yields a completely positive map such that $\psi(1) = |||L|||\cdot 1$ with $\begin{pmatrix} \psi & L \\ L^* & \psi \end{pmatrix}$ completely positive.

Combining Proposition 2.8 with Corollary 2.7 yields two representation theorems for completely bounded maps.

THEOREM 2.9. *Let $L: \mathcal{A} \to \mathcal{L}(\mathcal{H})$ be a completely bounded map, then there exists a Hilbert space $\mathcal{K}$, a $*$-homomorphism $\pi: \mathcal{A} \to \mathcal{L}(\mathcal{K})$, a bounded operator, $V: \mathcal{H} \to \mathcal{K}$, and a contraction, $T \in \pi(\mathcal{A})'$ such that $[\pi(\mathcal{A})V\mathcal{H}]$ is dense in $\mathcal{K}$, $\|V^*V\| = |||L|||$, and $L = V^*T\pi V$.*

THEOREM 2.10. *Let $L: \mathcal{A} \to \mathcal{L}(\mathcal{H})$ be a completely bounded map, then there exists a Hilbert space $\mathcal{K}$, a $*$-homomorphism $\pi: \mathcal{A} \to \mathcal{L}(\mathcal{K})$, an isometry $V: \mathcal{H} \to \mathcal{K}$ and an operator $T \in \pi(\mathcal{A})'$ such that $[\pi(\mathcal{A})V\mathcal{H}]$ is dense in $\mathcal{K}$, $\|T\| = |||L|||$, and $L = V^*T\pi V$.*

We shall refer to a representation of the form given by Theorem 2.9 as a *commutant representation* of L and to a representation of the form given by Theorem 2.10 as a *commutant representation with isometry* of L. In either case we shall denote it by $(\pi, V, T, \mathcal{K})$.

We now present an example to show that even for bounded linear functionals on a C^*-algebra, it is possible for $|||L||| \neq \|L\|_{cb}$. It is well-known that if L is a bounded linear functional, then L is completely bounded and $\|L\|_{cb} = \|L\|$.

Consider the linear functional $L: M_2 \to \mathbf{C}$ defined by

$$L\begin{pmatrix} a & b \\ c & d \end{pmatrix} = b,$$

so that $\|L\| = 1$ and

$$L^*\begin{pmatrix} a & b \\ c & d \end{pmatrix} = c.$$

Let $\{E_{i,j}\}$ be the standard matrix units for M_2, and suppose that $\varphi: M_2 \to \mathbf{C}$ is a map such that $\begin{pmatrix} \varphi & L \\ L^* & \varphi \end{pmatrix}$ is completely positive.

Since $\begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix}$ is positive, we would have that $\begin{pmatrix} \varphi(E_{11}) & 1 \\ 1 & \varphi(E_{22}) \end{pmatrix}$ is a positive matrix in M_2. Consequently, $2 \leqslant \varphi(E_{11}) + \varphi(E_{22}) = \varphi(1)$, so that $\|\varphi(1)\| \geqslant 2$. Hence $|||L||| \geqslant 2$ so by Proposition 2.3 we have that $|||L||| = 2$.

3. THE UNIQUENESS PROBLEM

Recall that if $\varphi: \mathscr{A} \to \mathscr{L}(\mathscr{H})$ is a completely positive map, and we are given two Stinespring representations of φ, $(\pi_i, V_i, \mathscr{K}_i)$, $i = 1, 2$, which are both minimal, that is $[\pi_i(\mathscr{A})V_i\mathscr{H}]$ is dense in $\mathscr{K}_i$, then these representations will be unitarily equivalent. This means that there exists a unitary $U: \mathscr{K}_1 \to \mathscr{K}_2$ such that $UV_1 = V_2$ and $U\pi_1 = \pi_2 U$. Thus, the minimal Stinespring representation is unique, up to unitary equivalence.

In this section, we develop a uniqueness criterion for the commutant representation of a completely bounded map. The best uniqueness results that we are able to achieve are modulo a densely defined (possibly unbounded) similarity.

Suppose that $L: \mathscr{A} \to \mathscr{L}(\mathscr{H})$ is completely bounded and that $(\pi_i, V_i, T_i, \mathscr{K}_i)$, $i = 1, 2$ are two commutant representations of L. We call these representations *similar* provided that there exists a bounded, invertible operator $S: \mathscr{K}_1 \to \mathscr{K}_2$ such that $SV_1 = V_2$, $S\pi_1 = \pi_2 S$ and $ST_1 = T_2 S$, and we call these representations *unitarily equivalent* when S is a unitary. Given a commutant representation of L, $(\pi, V, T, \mathscr{K})$ we call the completely positive map, $\varphi = V^*\pi V$ the *associated completely positive map*. Note that by the definition of a commutant representation, $(\pi, V, \mathscr{K})$ will always be the unique minimal Stinespring representation of φ.

We note that by Corollary 2.7 and Proposition 2.8, that φ is the associated completely positive map of some commutant representation $(\pi, V, T, \mathscr{K})$ of L if and only if $\begin{pmatrix} \varphi & L \\ L^* & \varphi \end{pmatrix}$ is completely positive and $\|\varphi\| = |||L|||$. We call the collection of such φ, the set of *dominating maps* for L.

PROPOSITION 3.1. *The map which assigns to each commutant representation of L its associated completely positive map is a one-to-one correspondence between the unitary equivalence classes of commutant representations and the set of dominating maps for L. Furthermore, this latter set is convex and compact in the* BW-*topology*.

Proof. First, note that if two commutant representations of L are unitarily equivalent then their associated completely positive maps are equal, and thus the map is well-defined.

Conversely, if two commutant representations, $(\pi_i, V_i, T_i, \mathscr{K}_i)$, $i = 1,2$, have associated completely positive maps which are equal, then $(\pi_i, V_i, \mathscr{K}_i)$ are two minimal Stinespring representations of the same completely positive map. Consequently, there exists a unitary $U: \mathscr{K}_1 \to \mathscr{K}_2$ such that $UV_1 = V_2$, and $U\pi_1 = \pi_2 U$. This implies that $(\pi_1, V_1, U^*T_2U, \mathscr{K}_1)$ is also a commutant representation of L, so that $V_1^*[T_1 - U^*T_2U]\pi_1(a)V_1 = 0$ for all $a \in \mathscr{A}$. Hence, $T_1 = U^*T_2U$ and the two commutant representations are unitarily equivalent.

It is easy to see that the dominating set is convex and closed in the BW-topology and hence is compact since the unit ball is compact in this topology.

There is a certain type of degeneracy in the commutant representation which we wish to avoid, namely, the operator T may not act on the whole space. The following shows that we may assume that it does. Here $\mathrm{N}(T)$ denotes the kernel of T.

PROPOSITION 3.2. *Let $L: \mathscr{A} \to \mathscr{L}(\mathscr{H})$ be a completely bounded map, then there exists a commutant representation $(\pi, V, T, \mathscr{K})$ of L, such that* $\mathrm{N}(T) \cap \mathrm{N}(T^*) = \{0\}$.

Proof. We note that $\mathrm{N}(T) \cap \mathrm{N}(T^*) = \{0\}$ if and only if $[T\mathscr{K} + T^*\mathscr{K}]$ is dense in $\mathscr{K}$.

Let $(\pi, V, T, \mathscr{K})$ be any commutant representation of L and let P be the projection onto the closure of $[T\mathscr{K} + T^*\mathscr{K}]$, which we call $\mathscr{M}$. It is easily seen that $\mathscr{M}$ is invariant and hence reduces $\pi(A)$, so that $P \in \pi(\mathscr{A})'$. Furthermore $TP = PT = T$, and hence $L = V^*T\pi V = (PV)^*(PTP)(P\pi P)(PV)$.

Thus, if we define $\pi_1: \mathscr{A} \to \mathscr{L}(\mathscr{M})$ by $\pi_1 = P\pi P$, then $(\pi_1, PV, PTP, \mathscr{M})$ is the desired commutant representation of L.

Any commutant representation of L, $(\pi, V, T, \mathscr{K})$ for which $\mathrm{N}(T) \cap \mathrm{N}(T^*) = \{0\}$ will be called a *faithful commutant representation* of L. The associated completely positive map, $\varphi = V^*\pi V$, will be called a *faithful dominating map* for L.

THEOREM 3.3. *Let $L: \mathscr{A} \to \mathscr{L}(\mathscr{H})$ be completely bounded, then the set of faithful dominating maps for L is a non-empty convex set.*

Furthermore, if $(\pi_i, V_i, T_i, \mathscr{K}_i)$, $i = 1, 2$, are faithful commutant representations of L, then there exists a unitary $U: \mathscr{K}_1 \to \mathscr{K}_2$ such that $U\pi_1 = \pi_2 U$.

Proof. By Proposition 3.2, we know that the set of faithful dominating maps is non-empty. Thus, let φ_i, $i = 1,2$ be faithful dominating maps, $(\pi_i, V_i, T_i, \mathscr{K}_i)$, $i = 1,2$ the associated faithful commutant representations and let $0 < t < 1$.

Setting $\varphi = t\varphi_1 + (1 - t)\varphi_2$, we have that φ is a dominating map and thus has an associated commutant representation $(\pi, V, T, \mathscr{K})$. We wish to show that $\mathrm{N}(T) \cap \mathrm{N}(T^*) = \{0\}$.

Since, $\varphi_1 \leqslant t^{-1}\varphi$ and $\varphi_2 \leqslant (1 - t)^{-1}\varphi$, we have by [1, Theorem 1.4.2] positive operators $P_i \in \pi(\mathscr{A})'$ such that $\varphi_i = V^*P_i\pi V$, $i = 1,2$. Also, since $\varphi = t\varphi_1 + (1 - t)\varphi_2 = V^*[tP_1 + (1 - tP_2)]\pi V$, we have that $tP_1 + (1 - t)P_2 = 1_{\mathscr{K}}$. Let $R_i = P_i^{1/2}$, for $i = 1, 2$.

First, we claim that there exists $T_i \in \pi(\mathscr{A})'$ such that $L = V^*R_iT_iR_i\pi V$ for $i = 1,2$. To see this, let $\mathscr{M}$ be the orthocomplement of $\mathrm{N}(P_1)$, and let Q be projection onto $\mathscr{M}$. Since $P_1 \in \pi(\mathscr{A})'$, $Q \in \pi(\mathscr{A})'$ and we let $\pi_1: \mathscr{A} \to \mathscr{L}(\mathscr{M})$ be the $*$-homomorphism defined by $\pi_1 = Q\pi Q$. It is easily checked that $(\pi_1, QR_1V, \mathscr{M})$ is the minimal Stinespring representation of φ_1 and hence the commutant representation associated with φ_1 is of the form $(\pi_1, QR_1V, T_1', \mathscr{M})$. Now set $T_1 = T_1' \oplus 0_{N(P_1)}$, then $T_1 \in \pi(\mathscr{A})'$ and $L = (QR_1V)^* T_1'\pi_1(QR_1V) = V^*R_1T_1R_1\pi V$. The identical argument works for establishing the existence of T_2.

Note also that since φ_1 is faithful, $N(T_1') \cap N(T_1'^*) = \{0\}$ and hence, $N(R_1T_1R_1) \cap N(R_1T_1^*R_1) = \mathscr{M}^\perp = N(R_1) = N(P_1)$, similarly, $N(R_2T_2R_2) \cap \cap N(R_2T_2^*R_2) = N(P_2)$. However, since $(\pi, V, \mathscr{K})$ is the minimal Stinespring representation of φ, we have that $L = V^*T\pi V = V^*R_iT_iR_i\pi V$ implies that $T = R_iT_iR_i$, $i = 1, 2$. Hence, $N(P_1) = N(T) \cap N(T^*) = N(P_2)$, but since $tP_1 + (1-t)P_2 = I$, we also have that $N(P_1) \cap N(P_2) = \{0\}$. Thus, $N(T) \cap N(T^*) = N(P_1) = N(P_2) = \{0\}$, establishing the first part of the theorem.

To establish the second statement, note that $N(P_1) = N(P_2) = \{0\}$ implies that $(\pi, R_i, V, \mathscr{K})$ is a Stinespring representation of φ_i that satisfies the minimality criterion. Hence if $(\pi_i, V_i, \mathscr{K}_i)$ is any minimal Stinespring representation of φ_i, then there exists $U_i: \mathscr{K}_i \to \mathscr{K}$ unitaries such that $U_iR_iV = V_i$, and $U_i\pi_i = \pi U_i$, $i = 1,2$. Consequently, $(\pi_1, V_1, \mathscr{K}_1)$ and $(\pi_2, V_2, \mathscr{K}_2)$ are themselves unitarily equivalent.

We present a simple example to show that the set of faithful dominating maps is distinct from the set of dominating maps. Let $L: \mathbf{C} \oplus \mathbf{C} \to M_2$ be defined by

$$L(\alpha, \beta) = \begin{pmatrix} \alpha & 0 \\ 0 & -\beta/2 \end{pmatrix}$$

and let $\varphi_{r,s}: \mathbf{C} \oplus \mathbf{C} \to M_2$ be defined by

$$\varphi_{r,s}(\alpha, \beta) = \begin{pmatrix} \alpha & 0 \\ 0 & r\alpha + s\beta \end{pmatrix}$$

where $s \geqslant 1/2$, $r \geqslant 0$, and $r + s \leqslant 1$. Note that since $L = L^*$, $1 = \|L\|_{cb} = |||L||| = \|\varphi_{r,s}\|$.

It is now easily seen that for $r > 0$, defining $V_{r,s}: \mathbf{C}^2 \to \mathbf{C}^3$, $\pi: \mathbf{C} \oplus \mathbf{C} \to M_3 = \mathscr{L}(\mathbf{C}^3)$, and $T_{r,s} \in M_3$ by

$$V_{r,s} = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{r} \\ 0 & \sqrt{s} \end{pmatrix}, \quad \pi(\alpha, \beta) = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix},$$

and

$$T_{r,s} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1/2s \end{pmatrix},$$

yields the commutant representation $(\pi, V_{r,s}, T_{r,s}, \mathbf{C}^3)$ associated with $\varphi_{r,s}$. That is, $(\pi, V_{r,s}, \mathbf{C}^3)$ is the minimal Stinespring representation of $\varphi_{r,s}$. Note that none of these are faithful.

Now, if $r = 0$, defining $V_s\colon \mathbf{C}^2 \to \mathbf{C}^2$, $\pi\colon \mathbf{C} \oplus \mathbf{C} \to M_2$ and $T_s \in M_2$ by

$$V_s = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{s} \end{pmatrix}, \quad \pi(\alpha, \beta) = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \quad \text{and} \quad T_s = \begin{pmatrix} 1 & 0 \\ 0 & -1/2s \end{pmatrix}$$

yields the commutant representation of L associated with $\varphi_{0,s}$, which is clearly faithful.

It is not too difficult to check that every dominating map for L is of the above form.

We now turn our attention to the study of similarity of faithful commutant representations. So let $(\pi_i, V_i, T_i, \mathscr{K}_i)$, $i = 1,2$ be two faithful commutant representations of L. Set $\mathscr{M}_i = [\pi_i(\mathscr{A})V_i\mathscr{H}]$ and $\mathscr{N}_i = [T\pi_i(A)V_i\mathscr{H} + T^*\pi_i(\mathscr{A})V\mathscr{H}]$ and note that these define dense subspaces of $\mathscr{K}_i$, $i=1,2$. It is easy to see that for $a_i, b_i, c_i \in \mathscr{A}$ and $x_i, y_i, v_i \in \mathscr{H}$, we have,

$$\langle \textstyle\sum \pi_1(a_i)V_1x_i, \ \sum (T_1\pi_1(b_i)V_1y_i + T_1^*\pi_1(c_i)V_1v_i)\rangle =$$

$$= \langle \textstyle\sum \pi_2(a_i)V_2x_i, \ \sum (T_2\pi_2(b_i)V_2y_i + T_2^*\pi_2(c_i)V_2v_i)\rangle$$

since both inner products are expressible in terms of L and L^*.

Thus we have that the maps $S\colon \mathscr{M}_1 \to \mathscr{M}_2$, $R\colon \mathscr{N}_2 \to \mathscr{N}_1$ defined by

$$S\left(\sum_{i=1}^{n} \pi_1(a_i)V_1x_i\right) = \sum_{i=1}^{n} \pi_2(a_i)V_2x_i,$$

$$R\left(\sum_{i=1}^{n} (T_2\pi_2(b_i)V_2y_i + T_2^*\pi_2(c_i)V_2y_i)\right) = \sum (T_1\pi_1(b_i)V_1y_i + T_1^*\pi_1(c_i)V_1y_i)$$

are well-defined, linear, one-to-one, and onto.

THEOREM 3.4. *Let $L\colon \mathscr{A} \to \mathscr{L}(\mathscr{H})$ be completely bounded, let $(\pi_i, V_i, T_i, \mathscr{K}_i)$, $i = 1,2$, be faithful commutant representations of L, and let $\mathscr{M}_i, \mathscr{N}_i$, R and S be defined as above, then R and S are closable operators with densely defined adjoints. Furthermore, if S^- denotes the closure of S, then the following equations hold:*

(1) $S^-\pi_1(a)m = \pi_2(a)S^-m$, *for all* $m \in \operatorname{Dom}(S^-)$, $a \in \mathscr{A}$

(2) $\pi_1(a)S^*m = S^*\pi_2(a)m$, *for all* $m \in \operatorname{Dom}(S^*)$, $a \in \mathscr{A}$

(3) $S^*T_2S^-m = T_1m$, *for all* $m \in \operatorname{Dom}(S^-)$

(4) $S^*T_2^*S^-m = T_1^*m$, *for all* $m \in \operatorname{Dom}(S^-)$.

Proof. We begin by noting that for $m_1 \in \mathcal{M}_1$ and $n_2 \in \mathcal{N}_2$, $\langle Sm_1, n_2\rangle = \langle m_1, Rn_2\rangle$. Thus, $R \subseteq S^*$, $S \subseteq R^*$ and since R and S have dense domains, R^* and S^* do also and we have that R and S are closable by [5, Theorem VIII.1].

To see the first relation, note that if $m \in \mathrm{Dom}(S)$, then $S\pi_1(a)m = \pi_2(a)Sm$. If $m \in \mathrm{Dom}(S^-)$, by [5, p. 250] we know there exist $m_n \in \mathrm{Dom}(S)$ with $m_n \to m$ and $Sm_n \to S^-m$. Hence $\pi_2(a)S^-m = \lim \pi_2(a)Sm_n = \lim S\pi_1(a)m_n$. We have that $\pi_1(a)m_n \in \mathrm{Dom}(S)$, $\pi_1(a)m_n \to \pi_1(a)m$, $S\pi_1(a)m_n \to \pi_2(a)S^-m$, so that $\pi_1(a)m \in \mathrm{Dom}(S^-)$ with $S^-\pi_1(a)m = \pi_2(a)S^-m$.

The remaining equations follow by similar manipulations.

In the finite dimensional case everything is bounded and considerably more can be said.

COROLLARY 3.5. *Let $\mathcal{A}$ be a finite dimensional C^*-algebra, and let $L: \mathcal{A} \to \mathcal{M}_n$ be a linear map. If $(\pi_i, V_i, T_i, \mathcal{K}_i)$, $i = 1, 2$ are two faithful commutant representations of L, then there exists a bounded invertible $S: \mathcal{K}_1 \to \mathcal{K}_2$ such that $S\pi_1 S^{-1} = \pi_2$, $S^*T_2S = T_1$ and $SV_1 = V_2$.*

The above result yields a complete description of the set of faithful dominating maps in this case. Fix any faithful commutant representation, $(\pi, V, T, \mathcal{K})$ and let,

$$\mathcal{C} = \left\{P \in \pi(\mathcal{A})' : P \text{ is invertible}, \|V^*PV\| = \|L\|, \text{ and } \begin{pmatrix} P & T \\ T^* & P \end{pmatrix} \geqslant 0\right\}.$$

For each $P \in \mathcal{C}$, we let $\varphi_P(a) = V^*P\pi(a)V$.

THEOREM 3.6. *Let $\mathcal{A}$ be a finite dimensional C^*-algebra, let $L: \mathcal{A} \to \mathcal{M}_n$ a linear map, and let $(\pi, V, T, \mathcal{K})$ be any fixed faithful commutant representation of L. The map $P \to \varphi_P$ defines a one-to-one, affine, order isomorphism from $\mathcal{C}$ onto the set of faithful dominating maps for L.*

Proof. First, note that $\|\varphi_P\| = \|\varphi_P(1)\| = \|V^*PV\| = \|L\|$, and that by Proposition 2.6, $\begin{pmatrix} \varphi_P & L \\ L^* & \varphi_P \end{pmatrix}$ is completely positive. Thus, φ_P is a dominating map for L. Since P is invertible, $(\pi, P^{1/2}V, \mathcal{K})$ is a minimal Stinespring representation for φ_P and $L = V^*T\pi V = (P^{1/2}V)^*(P^{-1/2}TP^{-1/2})\pi(P^{1/2}V)$. Hence, $(\pi, P^{1/2}V, P^{-1/2}TP^{-1/2}, \mathcal{K})$ is necessarily the commutant representation associated with φ_P.

Note that $\mathrm{N}(P^{-1/2}TP^{-1/2}) = \mathrm{N}(T)$, $\mathrm{N}(P^{-1/2}T^*P^{-1/2}) = \mathrm{N}(T^*)$ and so $(\pi, P^{1/2}V, P^{-1/2}TP^{-1/2}, \mathcal{K})$ is actually a faithful commutant representation. Thus, φ_P is a faithful dominating map.

The map $P \to \varphi_P$ is clearly affine, one-to-one and order preserving. It remains to show that it is onto.

To this end consider another faithful commutant representation. By Theorem 3.3, we may assume that it has the form $(\pi, V_1, T_1, \mathcal{K})$. Now by Corollary 3.5, there is an invertible operator $S: \mathcal{K} \to \mathcal{K}$ such that $S\pi S^{-1} = \pi$ and $SV = V_1$.

Polar decomposing $S = UR$ we have that U and R are in the commutant of $\pi(A)$, R is invertible and $\varphi_1 = V_1^*\pi V_1 = V^*R\pi RV = \varphi_P$ where $R^2=P$. To see that P belongs to $\mathscr{C}$, we have that $\||L\|| = \|\varphi_1\| = \|V^*PV\|$, and since $\begin{pmatrix} \varphi_1 & L \\ L^* & \varphi_1 \end{pmatrix}$ is completely positive, another application of Proposition 2.6 yields that $\begin{pmatrix} P & T \\ T^* & P \end{pmatrix}$ is positive.

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