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ON $K_*(C^*(SL_2(\mathbf{Z})))$ (APPENDIX TO "K-THEORY FOR CERTAIN GROUP C^*-ALGEBRAS" by E. C. LANCE)

TOSHIKAZU NATSUME

The purpose of this note is to generalize the result obtained in [4]. By J. Cuntz's approach to KK-theory, the structure of the proof becomes much clearer. In particular, we calculate the K-groups $K_*(C^*(SL_2(\mathbf{Z})))$ of the group C^*-algebra of $SL_2(\mathbf{Z})$.

1. PROLOGUE

Let G be a countable discrete group, and let H be a subgroup of G. Let $\overline{\lambda}$ denote the unitary representation of G on $\ell^2(G/H)$ induced from the left multiplication.

DEFINITION. The pair (G, H) has property Λ if there exists a one-parameter family (λ_t) of unitary representations of G on $\ell^2(G/H)$ such that

i) $\lambda_0 = \overline{\lambda}$,

ii) $\lambda_1(g)\delta_{\bar{e}} = \delta_{\bar{e}}$ for every $g \in G$,

iii) (λ_t) (considered as a one-parameter family of representations of $C^*(G)$) is a K-homotopy, that is, for each $x \in C^*(G)$, $t \to \lambda_t(x)$ is a continuous path in $B(\ell^2(G/H))$, and $\lambda_t(x) - \overline{\lambda}(x) \in \mathscr{K}(\ell^2(G/H))$,

iv) $\lambda_t(h) = \overline{\lambda}(h)$ for every $h \in H$.

In particular, G has property Λ if $(G, \{e\})$ has property Λ ([4]).

Our main result is the following:

THEOREM A1. *Let $\Gamma = G *_H S$ be the amalgamated product of countable discrete groups G and S along a subgroup H. Assume that (G, H) has property Λ. Then, for every C^*-dynamical system (A, α, Γ), there exists a six-term cyclic exact* *sequence*

$$
\begin{array}{ccccc}
K_0(A \times_{\alpha r} H) & \xrightarrow{\varkappa^1_* - \varkappa^2_*} & K_0(A \times_{\alpha r} G) \oplus K_0(A \times_{\alpha r} S) & \xrightarrow{\varepsilon^1_* + \varepsilon^2_*} & K_0(A \times_{\alpha r} \Gamma) \\
\uparrow & & & & \downarrow \\
K_1(A \times_{\alpha r} \Gamma) & \xleftarrow{\varepsilon^1_* + \varepsilon_*} & K_1(A \times_{\alpha r} G) \oplus K_1(A \times_{\alpha r} S) & \xleftarrow{\varkappa^1_* - \varkappa^2_*} & K_1(A \times_{\alpha r} H),
\end{array}
$$

where $\varkappa^1$ (resp. $\varkappa^2$) is a natural inclusion of $A \times_{\alpha r} H$ into $A \times_{\alpha r} G$ (resp. $A \times_{\alpha r} S$), and ε^1 (resp. ε^2) is a natural inclusion of $A \times_{\alpha r} G$ (resp. $A \times_{\alpha r} S$) into $A \times_{\alpha r} \Gamma$.

In the case $H = \{e\}$, Theorem A1 coincides with Lance's result ([4, Theorem 5.4]).

It is easy to see that if H is a normal subgroup of G, and the quotient group G/H has property A, then (G, H) has property A. Since any countable amenable group has property A ([4, Theorem 2.1]), if H is a normal subgroup of G, and G/H is amenable, then (G, H) has property A.

It is well-known that $SL_2(\mathbf{Z}) \cong \mathbf{Z}_4 *_{\mathbf{Z}_2} \mathbf{Z}_6$. Hence we can apply Theorem A1 to the group $SL_2(\mathbf{Z})$. Since $SL_2(\mathbf{Z})$ is K-amenable ([1]), the natural map $C^*(SL_2(\mathbf{Z})) \to C^*_r(SL_2(\mathbf{Z}))$ induces isomorphisms of K-groups. Thus we get:

Corollary A2.
$$K_0(C^*(SL_2(\mathbf{Z}))) \cong \mathbf{Z}^8,$$
$$K_1(C^*(SL_2(\mathbf{Z}))) = 0.$$

Another example is the following: Let F_n be a free group with n generators $g_1, \ldots, g_n$, and let H be the subgroup generated by $g_1, \ldots, g_k$, $k < n$. Then (G, H) has property A. So that Theorem A1 applies.

In what follows, we give an outline of the proof of Theorem A1 for the case $A = \mathbf{C}$. By the arguments used in [6], we can then prove the theorem for a reduced crossed product when A is unital. For non-unital A, let A^+ denote the C^*-algebra obtained by adjoining a unit to A. Consider the reduced crossed product $A^+ \times_{\tilde{\alpha} r} \Gamma$, where $\tilde{\alpha}$ is the extended action of Γ on A^+. Then $A \times_{\alpha r} \Gamma$ is an ideal of $A^+ \times_{\tilde{\alpha} r} \Gamma$. Constructing first various homomorphisms for $A^+ \times_{\tilde{\alpha} r} \Gamma$, and then restricting them to suitable subalgebras, Theorem A1 for the general case can be proved.

In this note, by tensor product of C^*-algebras we mean the minimal tensor product.

2. TOEPLITZ EXTENSION

Let $\Gamma = G *_H S$ be an amalgamated product. Write $G^* = G \setminus H$, $S^* = S \setminus H$, $\bar{G} = G/H$, $\bar{G}^* = G/H \setminus \{H\}$, $\bar{\bar{G}} = H \setminus G$ and $\bar{\bar{G}}^* = H \setminus G - \{H\}$. We assume that $G \neq H$, $S \neq H$.

Let Γ_1^* be the set of all non-empty words in Γ which end in G^*, and let $\Gamma_1 = \Gamma_1^* \cup H$. Similarly define Γ_2^* as the set of all non-empty words ending in S^*, and let $\Gamma_2 = \Gamma_2^* \cup H$. The left regular representation $\lambda(g)$ preserves $\ell^2(\Gamma_1)$ for $g \in G$, and $\lambda(s)$ $(s \in S)$ preserves $\ell^2(\Gamma_1^*)$. For $g \in G$, denote by $\mu(g)$ the restriction of $\lambda(g)$ on $\ell^2(\Gamma_1)$. For $s \in S$, denote by $\nu(s)$ the operator $\nu(s) = \lambda(s)q(\Gamma_1^*)$, where $q(\Gamma_1^*)$ is the orthogonal projection corresponding to $\Gamma_1^* \subset \Gamma_1$. μ and ν are extended to representations of $\mathfrak{A} = C_r^*(G)$ and $\mathfrak{B} = C_r^*(S)$ respectively. Let $\mathcal{T}$ be the C^*-algebra generated by $\mu(g)$, $\nu(s)$ $(g \in G,\ s \in S)$. Notice that ν is non-unital, and that $q_H = \mu(1) - \nu(1) \in \mathcal{T}$ is the orthogonal projection of $\ell^2(\Gamma_1)$ onto $\ell^2(H)$. Let $\mathcal{I}$ be the two-sided closed ideal generated by q_H in $\mathcal{T}$. Then:

LEMMA A3. ([6, Lemma 1.1], [4, Lemma 3.1]). *There is a homomorphism π from $\mathcal{T}$ onto $C_r^*(\Gamma)$ such that $\pi(\mu(g)) = \lambda(g)$ $(g \in G)$, $\pi(\nu(s)) = \lambda(s)$ $(s \in S)$, and* $\operatorname{Ker}\pi = \mathcal{I}$.

The proof is similar to that of [4, Lemma 3.1].

We claim that $\mathcal{I}$ is isomorphic to $C_r^*(H) \otimes \mathcal{K}(\ell^2(\bar{\Gamma}_1))$, where $\bar{\Gamma}_1 = \Gamma_1/H$.

Let $\{g_i\}$, $\{s_j\}$ be representatives of G, S respectively. By convention $g_0 = e$, $s_0 = e$. Then each element w of Γ_1 is uniquely written in the form

$$w = s_{j_1} g_{i_1} s_{j_2} g_{i_2} \dots s_{j_n} g_{i_n} h,$$

where $i_1 \neq 0, \dots, i_n \neq 0$, $j_2 \neq 0, \dots, j_n \neq 0$ and $h \in H$. The mapping $w \to (s_{j_i} g_{i_1} \dots s_{j_n} g_{i_n}, h)$ induces an isometric isomorphism

$$v: \ell^2(\Gamma_1) \to \ell^2(\bar{\Gamma}_1 \times H) \simeq \ell^2(\bar{\Gamma}_1) \otimes \ell^2(H).$$

It is not difficult to see that

$$\operatorname{ad}(v)(\mathcal{I}) = \mathcal{K}(\ell^2(\bar{\Gamma}_1)) \otimes C_r^*(H).$$

In particular, the map $p: \mathcal{K}(\ell^2(\bar{\Gamma}_1)) \otimes C_r^*(H) \to \mathcal{T}$ is given by

$$p(e(w', w) \otimes \lambda(h)) = \sigma(w')\,(\mu(h)q_H)\,\sigma(w^{-1}),$$

where, for $w'' = s_{j_1} g_{i_1} \dots s_{j_n} g_{i_n}$,

$$\sigma(w'') = \nu(s_{j_1})\,\mu(g_{i_1}) \dots \nu(s_{j_n})\,\mu(g_{i_n}),$$

and $e(w', w)$ are the natural matrix units.

Thus we get an extension of $C_r^*(\Gamma)$ by $\mathscr{K} \otimes C_r^*(H)$:

$$0 \to \mathscr{K}(\ell^2(\overline{\Gamma}_1)) \otimes C_r^*(H) \xrightarrow{p} \mathscr{T} \xrightarrow{\pi} C_r^*(\Gamma) \to 0.$$

Therefore, to prove Theorem A2, we only have to study $K_*(\mathscr{T})$ and the maps p and π.

REMARK. So far, we have not used the assumption for (G, H).

3. CUNTZ'S APPROACH TO KK-THEORY

In this section we summarize the basic facts about Cuntz's approach to KK-theory that are used in the later section. We use the notations of [2].

For C^*-algebras A, B, the group $KK(A, B)$ is defined as the group consisting of all homotopy classes of prequasihomomorphisms from A into $\mathscr{K} \otimes B$, where $\mathscr{K}$ is the C^*-algebra of all compact operators on a Hilbert space of countably infinite dimension.

LEMMA A4. *Let* $(\alpha, \overline{\alpha})$, $(\beta, \overline{\beta})\colon A \to E \triangleright J \to \mathscr{K} \otimes B$ *be prequasihomomorphisms. Assume that* $\alpha(x)\beta(y) = 0$, $\overline{\alpha}(x)\overline{\beta}(y) = 0$ *for arbitrary* $x, y \in A$. *Then*

1) $(\alpha + \beta, \overline{\alpha} + \overline{\beta})$ *is a prequasihomomorphism* $A \to E \triangleright J \to \mathscr{K} \otimes B$,

2) $[\alpha + \beta, \overline{\alpha} + \overline{\beta}] = [\alpha, \overline{\alpha}] + [\beta, \overline{\beta}]$ *in* $KK(A, B)$.

Proof. 1) is obvious. We show 2). First, notice that the class $[\alpha + \beta, \overline{\alpha} + \overline{\beta}]$ is represented by the following prequasihomomorphism

$$\left(\begin{pmatrix} \alpha+\beta & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \overline{\alpha}+\overline{\beta} & 0 \\ 0 & 0 \end{pmatrix}\right) : A \to \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} \triangleright \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} \to \begin{pmatrix} \mathscr{K}\otimes B & 0 \\ 0 & \mathscr{K}\otimes B \end{pmatrix} \subset \mathscr{K} \otimes B.$$

On the other hand, $[\alpha, \overline{\alpha}] + [\beta, \overline{\beta}]$ is represented by

$$\left(\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \begin{pmatrix} \overline{\alpha} & 0 \\ 0 & \overline{\beta} \end{pmatrix}\right) : A \to \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} \triangleright \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} \to \begin{pmatrix} \mathscr{K}\otimes B & 0 \\ 0 & \mathscr{K}\otimes B \end{pmatrix} \subset \mathscr{K} \otimes B.$$

Put

$$\alpha_t = \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} C_t & S_t \\ -S_t & C_t \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & \beta \end{pmatrix}\begin{pmatrix} C_t & -S_t \\ S_t & C_t \end{pmatrix}$$

and

$$\overline{\alpha}_t = \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} C_t & S_t \\ -S_t & C_t \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & \overline{\beta} \end{pmatrix}\begin{pmatrix} C_t & -S_t \\ S_t & C_t \end{pmatrix},$$

where $C_t = \cos(\pi/2)t$ and $S_t = \sin(\pi/2)t$.

We can show that α_t, $\overline{\alpha}_t$ are actually homomorphisms from A into $M_2(E)$, and that $(\alpha_t, \overline{\alpha}_t)$ defines a prequasihomomorphism

$$A \to M_2(E) \rhd M_2(J) \to M_2(\mathcal{K} \otimes B) \simeq \mathcal{K} \otimes B.$$

It is clear that $(\alpha_t, \overline{\alpha}_t)$ is a homotopy connecting

$$\left(\begin{pmatrix}\alpha+\beta & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}\overline{\alpha}+\bar{\beta} & 0\\ 0 & 0\end{pmatrix}\right) \text{ with } \left(\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}, \begin{pmatrix}\overline{\alpha} & 0\\ 0 & \overline{\beta}\end{pmatrix}\right).$$

Let $(\alpha, \overline{\alpha})\colon A \to E \rhd J$ be a prequasihomomorphism from A into J. Then $(\alpha, \overline{\alpha})$ induces a homomorphism $(\alpha/\overline{\alpha})_*\colon K_*(A) \to K_*(J)$ as follows.

First, add units to A and E and extend $(\alpha, \overline{\alpha})$ to a prequasihomomorphism $(\tilde{\alpha}, \tilde{\overline{\alpha}})\colon A^+ \to E^+ \rhd J$ with $\tilde{\alpha}, \tilde{\overline{\alpha}}$ being unital. For a unitary $u \in M_n(A^+)$ the unitary $\tilde{\alpha}(u)\tilde{\overline{\alpha}}(u^*)$ is contained in $M_n(J)$, hence it defines an element of $K_1(J)$. The correspondence $[u]_1 \to [\tilde{\alpha}(u)\tilde{\overline{\alpha}}(u^*)]_1$ defines a well-defined homomorphism

$$(\alpha/\overline{\alpha})_*\colon K_1(A) \to K_1(J).$$

$(\alpha/\overline{\alpha})_*\colon K_0(A) \to K_0(J)$ is defined by taking suspension. It is not difficult to see that $(\alpha/\overline{\alpha})_*$ depends only on the homotopy class $[\alpha, \overline{\alpha}]$.

Let $(\alpha, \overline{\alpha})$, $(\beta, \overline{\beta})$ be prequasihomomorphisms from A into $\mathcal{K} \otimes B$, and let $(\alpha, \overline{\alpha}) + (\beta, \overline{\beta})$ be represented by $(\gamma, \overline{\gamma})$. Then

$$(\gamma/\overline{\gamma})_* = (\alpha/\overline{\alpha})_* + (\beta/\overline{\beta})_*\colon K_*(A) \to K_*(B).$$

For prequasihomomorphisms $(\alpha, \overline{\alpha})\colon A \to E_1 \rhd J_1 \to \mathcal{K} \otimes B$, $(\beta, \overline{\beta})\colon B \to E_2 \rhd J_2 \to \mathcal{K} \otimes C$, the Kasparov product $[\beta, \overline{\beta}]\,[\alpha, \overline{\alpha}]$ is defined (cf. [2]). Assume that $[\beta, \overline{\beta}]\,[\alpha, \overline{\alpha}]$ is represented by $(\gamma, \overline{\gamma})\colon A \to E_3 \rhd J_3 \to \mathcal{K} \otimes C$. Then we have that

$$(\beta/\overline{\beta})_*(\alpha/\overline{\alpha})_* = (\gamma/\overline{\gamma})_*.$$

As we have seen above, an element $[\alpha, \overline{\alpha}] \in KK(A, B)$ induces a homomorphism $(\alpha/\overline{\alpha})_*\colon K_*(A) \to K_*(B)$. We say that $[\alpha, \overline{\alpha}] \in KK(A, B)$ is invertible if there exists $[\beta, \overline{\beta}] \in KK(B, A)$ such that $[\beta, \overline{\beta}]\,[\alpha, \overline{\alpha}] = 1_A$, $[\alpha, \overline{\alpha}]\,[\beta, \overline{\beta}] = 1_B$, where 1_A (resp. 1_B) is the unit of the ring $KK(A, A)$ (resp. $KK(B, B)$). If this is the case, $(\alpha/\overline{\alpha})_*$ is an isomorphism with the inverse $(\beta/\overline{\beta})_*$.

4. VARIOUS HOMOMORPHISMS

In what follows, we assume that (G, H) has property A with a homotopy $(\lambda_t$. ite $\mathcal{K}$ for $\mathcal{K}(\ell^2(\overline{\Gamma}_1))$.

Notice that $\{g_i^{-1}\}$, $\{s_j^{-1}\}$ are regarded as representatives of $\overline{\overline{G}}$, $\overline{S}$, respectively. Using these fixed representatives, we may regard every quotient space by H (e.g. Γ_1) as a subset of Γ.

Each element of Γ is uniquely written in the form wg with $w \in \overline{\Gamma}_2 = \Gamma_2/H$ and $g \in G$. Each element of Γ_1^* is uniquely written in the form wh with $w \in \overline{\Gamma}_2$ and $h \in G^*$. Thus we get identifications $\Gamma \simeq \overline{\Gamma}_2 \times G$ and $\Gamma_1^* \simeq \overline{\Gamma}_2 \times G^*$. Notice that G, G^* are identified with $H \times \overline{\overline{G}}$, $\overline{G}^* \times H$, respectively.

Let $\{\delta(w, g, g')\}$ be the canonical orthonormal basis, where $(w, g, g') \in \overline{\Gamma}_2 \times G \times \overline{\overline{G^*}} \simeq \Gamma \times \overline{\overline{G^*}}$. Similarly, for $(w, h, h') \in \overline{\Gamma}_2 \times \overline{\overline{G^*}} \times G \simeq \Gamma_1^* \times G$, $\{\delta(w, h, h')\}$ denotes the canonical orthonormal basis.

Put

$$u(\delta(w, g, g')) = \sum_{k \in G^*} \langle \lambda_1(gg')\delta(g'^{-1}), \delta(k)\rangle \, \delta(w, k, k^{-1}gg'),$$

where $(w, k, k^{-1}gg') \in \overline{\Gamma}_2 \times \overline{G^*} \times G$.

Using property ii) of Definition, we see that u extends to an isometry from $\ell^2(\Gamma \times \overline{\overline{G^*}})$ onto $\ell^2(\Gamma_1^* \times \overline{\overline{G}})$, and that its adjoint is given by

$$u^*(\delta(w, h, h')) = \sum_{k \in G^*} \langle \delta(h), \lambda_1(hh')\delta(k)\rangle \, \delta(w, hh'k, k^{-1}).$$

Since $\ell^2(\Gamma_1^* \times \overline{\overline{G}})$ is a closed subspace of $\ell^2(\Gamma_1 \times \overline{\overline{G}})$, we regard u as an isometry into $\ell^2(\Gamma_1 \times \overline{\overline{G}})$.

For $x \in \mathscr{T}$, $\pi(x) \otimes 1 \in B(\ell^2(\Gamma)) \otimes B(\ell^2(\overline{\overline{G^*}}))$. Let ψ be the representation of $\mathscr{T}$ on $\ell^2(\Gamma_1 \times \overline{\overline{G}})$ defined by

$$\psi(x) = u(\pi(x) \otimes 1)u^*.$$

Define also a representation $\overline{\psi}$ by

$$\overline{\psi}(x) = x \otimes 1 \in B(\ell^2(\Gamma_1)) \otimes B(\ell^2(\overline{\overline{G}})) \quad \text{for } x \in \mathscr{T}.$$

Making use of property iv), we see that

$$\psi(x) = \overline{\psi}(x) \quad \text{for } x = v(s) \ (s \in S).$$

LEMMA A5. ([4, Lemma 4.2]). *For* $x \in \mathscr{T}$,

(1) $\psi(x) \in M(\mathscr{K} \otimes \mathfrak{A})$,

(2) $\psi(x) - \overline{\psi}(x) \in \mathscr{K} \otimes \mathfrak{A}$.

Proof. A routine computation shows that $\bar{\psi}(x) \in M(\mathscr{K} \otimes \mathfrak{A})$. Using property iv) for λ_1, it follows that

$$(q(\bar{\Gamma}_1 \setminus \bar{G}) \otimes 1)(\psi(\mu(g)) - \bar{\psi}(\mu(g))) = 0$$

for $g \in G$, where $q(\bar{\Gamma}_1 \setminus \bar{G})$ denotes the orthogonal projection corresponding to $\bar{\Gamma}_1 \setminus \bar{G} \subset \bar{\Gamma}_1$. Since $q_{\bar{e}} \otimes 1 \in \mathscr{K} \otimes \mathfrak{A}$, it suffices to show that

$$(q(\bar{G}^*) \otimes 1))(\psi(\mu(g_1)) - \bar{\psi}(\mu(g))) \in \mathscr{K} \otimes \mathfrak{A}.$$

Define a unitary U on $\ell^2(\bar{G}^* \times \bar{G})$ by $U(\delta(h, h')) = \delta(h, hh')$, and notice that $U \in M(\mathscr{K} \otimes \mathfrak{A})$. By direct computation on $q(\bar{G}^*) \otimes 1$,

$$U^*((\lambda_1(y) - \bar{\lambda}(g)) \otimes \lambda(g))U = \psi(\mu(g)) - \bar{\psi}(\mu(g)).$$

Hence, by property iii),

$$(q(\bar{G}^*) \otimes 1)(\psi(\mu(g)) - \bar{\psi}(\mu(g))) \in \mathscr{K} \otimes \mathfrak{A}.$$ ▨

Each element of Γ is uniquely written in the form ws with $w \in \bar{\Gamma}_1$, $s \in S$. Therefore we get an identification $v: \ell^2(\Gamma) \rightleftarrows \ell^2(\bar{\Gamma}_1 \times S)$. The latter space is identified with $\ell^2(\Gamma_1 \times \bar{S})$. Define θ and $\bar{\theta}$ by $\theta(x) = v(\pi(x))v^*$ and $\bar{\theta}(x) = x \otimes 1 \in B(\ell^2(\Gamma_1)) \otimes B(\ell^2(\bar{\bar{S}}))$ $(x \in \mathscr{T})$ respectively. Then, by elementary calculations, we get that

$$\theta(\mu(g)) = \bar{\theta}(\mu(g)) \quad \text{for } g \in G,$$

and

$$\theta(v(s)) - \bar{\theta}(v(s)) = q_{\bar{e}} \otimes \lambda(s) \in \mathscr{K} \otimes \mathfrak{B}.$$

It is easy to see that $\bar{\theta}(x) \in M(\mathscr{K} \otimes \mathfrak{B})$.

Thus we get prequasihomomorphisms

$$(\psi, \bar{\psi}): \mathscr{T} \to M(\mathscr{K} \otimes \mathfrak{A}) \rhd \mathscr{K} \otimes \mathfrak{A}$$

and

$$(\theta, \bar{\theta}): \mathscr{T} \to M(\mathscr{K} \otimes \mathfrak{B}) \rhd \mathscr{K} \otimes \mathfrak{B}.$$

REMARK. In the construction of ψ, we used property iv):

$$\lambda_1(h) = \bar{\lambda}(h) \quad \text{for } h \in H.$$

5. PROOF OF THEOREM A1

Let j denote the homomorphism $x \to q_{\bar{e}} \otimes x$ from $\mathscr{T}$ into $\mathscr{K} \otimes \mathscr{T}$. The homomorphisms μ and ν are considered as homomorphisms from $\mathfrak{A} \oplus \mathfrak{B}$ into $\mathscr{T}$, hence they define prequasihomomorphisms

$$(j\mu, 0), (j\nu, 0): \mathfrak{A} \oplus \mathfrak{B} \to M(\mathscr{K} \otimes \mathscr{T}) \rhd \mathscr{K} \otimes \mathscr{T}.$$

Notice that

$$(j\mu/0)_* = \mu_* : \mathrm{K}_*(\mathfrak{A} \oplus \mathfrak{B}) \to \mathrm{K}_*(\mathscr{T}),$$

and

$$(j\nu/0)_* = \nu_* : \mathrm{K}_*(\mathfrak{A} \oplus \mathfrak{B}) \to \mathrm{K}_*(\mathscr{T}).$$

To show that $\mu_* + \nu_*$ is an isomorphism, it suffices to show that $\xi = [j\mu, 0] + [j\nu, 0]$ is an invertible element of $\mathrm{KK}(\mathfrak{A} \oplus \mathfrak{B}, \mathscr{T})$. We claim that $\eta\xi = 1_{\mathfrak{A}\oplus\mathfrak{B}}$, and that $\xi\eta = 1_{\mathscr{T}}$, where $\eta = [\theta, \bar{\theta}] - [\psi, \bar{\psi}]$.

PROPOSITION A6. $\eta\xi = 1_{\mathfrak{A}\oplus\mathfrak{B}}$.

Proof. Notice that $1_{\mathfrak{A}\oplus\mathfrak{B}}$ is represented by the class

$$[\iota_1, 0] + [\iota_2, 0],$$

where $\iota_1 : \mathfrak{A} \to \mathscr{K} \otimes \mathfrak{A}$ (resp. $\iota_2 : \mathfrak{B} \to \mathscr{K} \otimes \mathfrak{B}$) is defined by $\iota_1(x) = q_{\bar{e}} \otimes x$ (resp. $\iota_2(y) = q_{\bar{e}} \otimes y$).

$$\eta\xi = [\theta, \bar{\theta}]\,[j\nu, 0] - [\psi, \bar{\psi}]\,[j\mu, 0],$$

because $\theta\mu = \bar{\theta}\mu$ and $\psi\nu = \bar{\psi}\nu$.

By the definition of product,

$$\eta\xi = [\theta\nu, \bar{\theta}\nu] - [\psi\mu, \bar{\psi}\mu] = -[\bar{\theta}\nu, \theta\nu] - [\psi\mu, \bar{\psi}\mu].$$

Therefore, to see that $\eta\xi = 1_{\mathfrak{A}\oplus\mathfrak{B}}$, it suffices to show that

$$[\bar{\theta}\nu, \theta\nu] + [\iota_2, 0] = 0 \quad \text{in } \mathrm{KK}(\mathfrak{B}, \mathfrak{B}),$$

and

$$[\psi\mu, \bar{\psi}\mu] + [\iota_2, 0] = 0 \quad \text{in } \mathrm{KK}(\mathfrak{A}, \mathfrak{A}).$$

It is easy to see that $(\psi\mu)(x)\iota_1(y) = 0$ for $x, y \in \mathfrak{A}$, and that $(\bar{\theta}\nu)(x')\iota_2(y') = 0$ for $x', y' \in \mathfrak{B}$. Then, by Lemma A4,

$$[\bar{\theta}\nu, \theta\nu] + [\iota_2, 0] = [\bar{\theta}\nu + \iota_2, \theta\nu],$$

and

$$[\psi\mu, \bar{\psi}\mu] + [\iota_1, 0] = [\psi\mu + \iota_1, \bar{\psi}\mu].$$

By direct computation, $\bar{\theta}\nu + \iota_2 = \theta\nu$, hence

$$[\bar{\theta}\nu + \iota_2, \theta\nu] = 0 \quad \text{in } KK(\mathfrak{B}, \mathfrak{B}).$$

Then Proposition A6 follows from the next lemma.

LEMMA A7. ([4, Lemma 5.1]). *$\psi\mu + \iota_1$ and $\bar{\psi}\mu$ are $\mathscr{K} \otimes \mathfrak{A}$-homotopic.*

Proof. For $(h, h') \in G \times \bar{\bar{G}}$, put

$$U_t(\delta(h, h')) = \sum_{k \in G} \langle \lambda_t(hh')\delta(h'^{-1}, k)\rangle\, \delta(kk_2, k_1),$$

where (k_2, k_1) is the decomposition of $k^{-1}hh'$ corresponding to $G \simeq H \times \bar{\bar{G}}$. For $(w, h, h') \in \bar{\Gamma}_2^* \times G^* \times \bar{\bar{G}}$, put

$$U_t(\delta(w, h, h')) = \delta(w, h, h').$$

Then U_t extends to a unitary from $\ell^2(\Gamma_1 \times \bar{\bar{G}})$ onto itself. Put

$$\varphi_t(x) = U_t(\mu(x) \otimes 1)U_t^* \quad \text{for } x \in \mathfrak{A}.$$

φ_t is a representation of $\mathfrak{A}$ on $\ell^2(\Gamma_1 \times G) \simeq \ell^2(\bar{\Gamma}_1 \times G)$. By direct computation, we get that

$$\varphi_1 = \psi\mu + \iota_1, \quad \text{and} \quad \varphi_0 = \bar{\psi}\mu.$$

By the argument used in the proof of Lemma A5,

$$\varphi_t(x) - \varphi_0(x) \in \mathscr{K} \otimes \mathfrak{A} \quad \text{for every } t \in [0, 1],\ x \in \mathfrak{A}. \qquad ▨$$

REMARK. In the proof of Lemma A7, we do not need property iv) of (λ_t).

PROPOSITION A8. $\xi\eta = 1_{\mathscr{T}}$.

Proof. First, notice that $1_{\mathscr{T}}$ is represented by

$$(j, 0)\colon \mathscr{T} \to M(\mathscr{K} \otimes \mathscr{T}) \rhd \mathscr{K} \otimes \mathscr{T}.$$

$$\xi\eta = ([j\mu, 0] + [j\nu, 0])([\theta, \bar{\theta}] - [\psi, \bar{\psi}]) =$$

$$= -[\bar{\mu}\psi, \overline{\mu\psi}] - [\bar{\nu}\psi, \overline{\nu\psi}] + [\bar{\mu}\theta, \overline{\mu\theta}] + [\bar{\nu}\theta, \overline{\nu\theta}],$$

where $\bar{\mu}$ (resp. $\bar{\nu}$) is the homomorphism from $M(\mathscr{K} \otimes \mathfrak{A})$ (resp. $M(\mathscr{K} \otimes \mathfrak{B})$) into $M(\mathscr{K} \otimes \mathscr{T})$ which extends the homomorphism

$$1 \otimes \mu : \mathscr{K} \otimes \mathfrak{A} \to \mathscr{K} \otimes \mathscr{T}$$

$$(\text{resp. } 1 \otimes \nu : \mathscr{K} \otimes \mathfrak{B} \to \mathscr{K} \otimes \mathscr{T}).$$

Note that here we need the fact that $\mathfrak{A}$ and $\mathfrak{B}$ are unital.

Since $\overline{\nu\psi} = \bar{\nu}\bar{\psi}$ and $\overline{\mu\theta} = \bar{\mu}\bar{\theta}$,

$$\xi\eta = -[\bar{\mu}\psi, \overline{\mu\psi}] + [\bar{\nu}\theta, \overline{\nu\theta}].$$

For $x \in \mathscr{T}$, define $k : \mathscr{T} \to B(\ell^2(\Gamma_1)) \otimes B(\ell^2(\bar{\bar{\Gamma}}_1))$ by $k(x) = x \otimes q_{\bar{e}}$, where $q_{\bar{e}}$ is the orthogonal projection corresponding to $\{\bar{e}\} \subset \bar{\bar{\Gamma}}_1 = H \diagdown \Gamma_1$. Then $k(x) \in M(\mathscr{K} \otimes \mathscr{T})$, hence it defines a prequasihomomorphism

$$(k, k) : \mathscr{T} \to M(\mathscr{K} \otimes \mathscr{T}) \triangleright \mathscr{K} \otimes \mathscr{T},$$

which represents $0 \in \mathrm{KK}(\mathscr{T}, \mathscr{T})$. Therefore

$$\xi\eta = [\bar{\nu}\theta, \overline{\nu\theta}] + [k, k] - [\bar{\mu}\psi, \overline{\mu\psi}].$$

Since $\overline{\mu\psi} = \overline{\nu\theta} + k$,

$$\xi\eta = [\bar{\nu}\theta + k, \overline{\mu\psi}] - [\bar{\mu}\psi, \overline{\mu\psi}].$$

Thus, to get the conclusion, it suffices to show that

$$[\bar{\mu}\psi, \overline{\mu\psi}] + [j, 0] = [\bar{\nu}\theta + k, \overline{\mu\psi}].$$

By Lemma A4, the left hand side is equal to

$$[\bar{\mu}\psi + j, \overline{\mu\psi}].$$

Therefore the conclusion follows from the next lemma.

LEMMA A9. ([4, Lemma 5.2]). *$\bar{\mu}\psi + j$ is $\mathscr{K} \otimes \mathscr{T}$-homotopic to $\bar{\nu}\theta + k$.*

Proof. We give only a sketch of the proof modelled on that of [4, Lemma 5.2]. For $g \in G$, define $\varphi_t(g) \in B(\ell^2(\bar{\bar{\Gamma}}_1 \times \bar{\bar{\Gamma}}_1))$ by

$$\varphi_t(g)\delta(w, w') = \begin{cases} \delta(w_1, g_2 w') & \text{if } w \in \bar{\Gamma}_1^* \diagdown \bar{G}, \\ \sum_{k \in G} \langle \lambda_t(g)\,\delta(w), \delta(k)\rangle\, \delta(k, k^{-1}gww') & \text{if } w \in \bar{G}, \end{cases}$$

where (w_1, g_2) is the decomposition of gw corresponding to $\Gamma_1 \simeq \overline{\Gamma}_1 \times H$. Then (φ_t) is a homotopy of representations of $\mathfrak{A}$ on $\ell^2(\overline{\Gamma}_1 \times \Gamma_1)$. Moreover $\varphi_t(g) \in \in M(\mathscr{K} \otimes \mathscr{T})$, and $\varphi_t(g) - \varphi_0(g) \in \mathscr{K} \otimes \mathscr{T}$.

For $s \in S$, define $\varphi_t(s)$ by

$$\varphi_t(s)\,\delta(w, w') = \begin{cases} \delta(w_1, s_2 w') & \text{if } w \neq \overline{e} \\ \delta(\overline{e}, sw') & \text{if } w = \overline{e},\ w' \neq e \\ 0 & \text{if } w = \overline{e},\ w' = e, \end{cases}$$

where (w_1, s_2) is the decomposition of sw. φ_t extends to a representation of $\mathfrak{B}$ on $\ell^2(\overline{\Gamma}_1 \times \Gamma_1)$ and $\varphi_t(s) \in M(\mathscr{K} \otimes \mathscr{T})$.

Notice that

$$(\overline{\nu}\theta + k)\,(\mu(g)) = \varphi_0(g), \quad (\overline{\nu}\theta + k)\,(\nu(s)) = \varphi_0(s),$$

$$(\overline{\mu}\psi) + j)\,(\mu(g)) = \varphi_1(g) \quad \text{and} \quad (\overline{\mu}\psi + j)\,(\nu(s)) = \varphi_1(s).$$

For $w \in \Gamma_1^*$ with $w = \ldots s_{j_{-1}} g_{i_0} s_{j_0} g_{i_1} \ldots$, put

$$\varphi_t(w) = \ldots \varphi_t(g_{i_0})\,\varphi_t(s_{j_0}) \ldots .$$

Notice that each element of Γ_1^* is uniquely written in the form shw' with $s \in S$, $h \in \overline{\overline{G}}{}^*$ and $w' \in \overline{\overline{\Gamma}}{}_1^2 = H \diagdown \Gamma_1^2$, where Γ_1^2 is the set of all elements of Γ_1 beginning in S^*. Define $m_t \in B(\ell^2(\overline{\Gamma}_1 \times \Gamma_1))$ on basis vectors by

$$m_t(\delta(\overline{e}, e)) = \delta(\overline{e}, e),$$

$$m_t(\delta(\overline{e}, shw')) = \sum_{k \in \overline{G}^*} \langle \lambda_t(h)\,\delta(h^{-1}), \delta(k)\rangle\, \delta(s_1, s_2 k^{-1} h w') +$$

$$+ \langle \lambda_t(h)\,\delta(h^{-1}), \delta(\overline{e})\rangle\, \delta(\overline{e}, shw'),$$

where (s_1, s_2) is the decomposition of sk, and

$$m_t(\delta(w, w') = \varphi_t(w)\, m_t(\delta(\overline{e}, w')) \quad \text{if } w \in \overline{\Gamma}_1^*.$$

Using the fact that $\lambda_t(h)\delta(\overline{e}) = \delta(\overline{e})$ for $h \in H$, we can check that $m_t\varphi_0(g) = = \varphi_t(g)\, m_t$, and that $m_t\varphi_0(s) = \varphi_t(s)\, m_t$.

By the argument in the proof of [4, Lemma 5.2,] and property iv), it follows that (m_t) is a homotopy of unitaries.

$\Phi_t = m_t(\overline{\nu}\theta + k)m_t^*$ is a homomorphism from $\mathcal{T}$ into $M(\mathcal{K} \otimes \mathcal{T})$, and $\Phi_t(x) - \Phi_0(x) \in \mathcal{K} \otimes \mathcal{T}$. It is clear that (Φ_t) is a homotopy connecting $\Phi_0 = \overline{\nu}\theta + k$ to $\Phi_1 = \overline{\mu}\psi + j$.

This completes the proof of Proposition A8. ▨

REMARK. Since μ is a unital homomorphism, it is easy to obtain $\overline{\mu}$. As for $\overline{\nu}$, we have to be more careful, because $\overline{\nu}$ is not unital. By the action of S on Γ_1^* from the left, Γ_1^* is decomposed into equivalence classes. This decomposition gives us the extension $\tilde{\nu}: B(\ell^2(S)) \to B(\ell^2(\overline{\Gamma}_1))$ of $\nu: \mathfrak{B} \to \mathcal{T}$. Then $1 \otimes \tilde{\nu}$ extends to a homomorphism $\overline{\nu}: B(\ell^2(\overline{\Gamma}_1) \otimes \ell^2(S)) \to B(\ell^2(\overline{\Gamma}_1) \otimes \ell^2(\Gamma_1))$. Since $\mathfrak{B}$ is unital, an element $x \in B(\ell^2(\overline{\Gamma}_1) \otimes \ell^2(S))$ belongs to $M(\mathcal{K} \otimes \mathfrak{B})$ iff $x(e(w, w') \otimes 1)$, $(e(w, w') \otimes 1)x \in \mathcal{K} \otimes \mathfrak{B}$ for arbitrary matrix units $e(w, w')$ of $\mathcal{K}$.

We claim that if $x \in M(\mathcal{K} \otimes \mathfrak{B})$, then $\overline{\nu}(x) \in M(\mathcal{K} \otimes \mathcal{T})$. As $\mathcal{T}$ is unital, it suffices to show that

$$\overline{\nu}(x)(e(w, w') \otimes 1),\ (e(w, w') \otimes 1)\overline{\nu}(x) \in \mathcal{K} \otimes \mathcal{T}.$$

Put $\nu(1) = p$. Since $\overline{\nu}(x)(e(w, w') \otimes (1-p)) = 0$, and $(e(w, w') \otimes (1-p))\overline{\nu}(x) = 0$,

$$\overline{\nu}(x)(e(w, w') \otimes 1) = \overline{\nu}(x)(e(w, w') \otimes p) = \overline{\nu}(x(e(w, w') \otimes 1)) \in \mathcal{K} \otimes \mathcal{T}.$$

Similarly we get

$$(e(w, w') \otimes 1)\overline{\nu}(x) \in \mathcal{K} \otimes \mathcal{T}.$$

We know that $K_*(\mathcal{T})$ is isomorphic to $K_*(\mathfrak{A} \oplus \mathfrak{B})$. It is clear that $\pi_*\mu_* = \varepsilon_*^1$, $\pi_*\nu_* = \varepsilon_*^2$. We have to show that the composition of the maps:

$$K_*(C_r^*(H)) \cong K_*(\mathcal{K} \otimes C_r^*(H)) \to K_*(\mathcal{T}) \cong K_*(\mathfrak{A} \oplus \mathfrak{B}),$$

coincides with $\varkappa_*^1 - \varkappa_*^2$.

As we have seen above, the isomorphism $K_*(\mathcal{T}) \to K_*(\mathfrak{A} \oplus \mathfrak{B})$ is given by $(\theta/\overline{\theta})_* - (\psi/\overline{\psi})_*$. Let $\iota: C_r^*(H) \to \mathcal{K} \otimes C_r^*(H)$ be defined by $\iota(x) = q_{\bar{e}} \otimes x$.

PROPOSITION A10. $(\psi/\overline{\psi})_*\rho_*\iota_* = -\varkappa_*^1$, $(\theta/\theta)_*\rho_*\iota_* = -\varkappa_*^2$.

Proof. It is sufficient to show that

$$[\overline{\psi}, \psi][p, 0][\iota, 0] = [\iota_1, 0][\varkappa^1, 0] \quad \text{in } KK(C_r^*(H), \mathfrak{A}),$$

and

$$[\overline{\theta}, \theta][p, 0][\iota, 0] = [\iota_2, 0][\varkappa^2, 0] \quad \text{in } KK(C_r^*(H), \mathfrak{B}).$$

By direct computation we see that $\psi p\iota = 0$, $\overline{\psi}p\iota = \iota_1\varkappa^1$, $\theta p\iota = 0$ and $\overline{\theta}p\iota = \iota_2\varkappa^2$.

This completes the proof of Theorem A1. ▨

6. $K_*(C^*(SL_2(\mathbf{Z})))$

Let m, n, k be integers, and let k divide m and n. Consider the group $\mathbf{Z}_m *_{\mathbf{Z}_k} \mathbf{Z}_n$. Recall that $K_0(C^*(\mathbf{Z}_l)) \simeq \mathbf{Z}^l$ and $K_1(C^*(\mathbf{Z}_l)) = 0$. The homomorphism $K_0(C^*(\mathbf{Z}_k)) \to \to K_0(C^*(\mathbf{Z}_m))$ is given by the following $m \times k$ matrix

$$\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & \cdot & 1 \\ 0 & & 0 \\ \vdots & & \vdots \\ 0 & & 0 \end{pmatrix} : \mathbf{Z}^k \to \mathbf{Z}^m.$$

By Theorem A1, we have

$$K_0(C_r^*(\mathbf{Z}_m *_{\mathbf{Z}_k} \mathbf{Z}_n)) \simeq \mathbf{Z}^{m+n-k},$$

and

$$K_1(C_r^*(\mathbf{Z}_m *_{\mathbf{Z}_k} \mathbf{Z}_n)) = 0.$$

In particular, since $SL_2(\mathbf{Z}) \cong \mathbf{Z}_4 *_{\mathbf{Z}_2} \mathbf{Z}_6$,

$$K_0(C_r^*(SL_2(\mathbf{Z}))) \simeq \mathbf{Z}^8,$$
$$K_1(C_r^*(SL_2(\mathbf{Z}))) = 0.$$

REMARK. As $\mathbf{Z}_m *_{\mathbf{Z}_k} \mathbf{Z}_n$ are K-amenable ([1]), $K_*(C^*(\mathbf{Z}_m *_{\mathbf{Z}_k} \mathbf{Z}_n)) \simeq \simeq K_*(C_r^*(\mathbf{Z}_m *_{\mathbf{Z}_k} \mathbf{Z}_n))$.

Next we calculate the K-groups for a certain crossed product C^*-algebra by $SL_2(\mathbf{Z})$.

$SL_2(\mathbf{Z})$ acts faithfully on $\mathbf{R}^2$ and on the space of oriented lines through 0, which we identify with S^1. This gives a natural action σ of $SL_2(\mathbf{Z})$ on S^1 which preserves antipodal points. By Theorem A1, we have an exact sequence:

$$\begin{array}{ccccc} K_0(A \times_\sigma \mathbf{Z}_2) & \to & K_0(A \times_\sigma \mathbf{Z}_4) \oplus K_0(A \times_\sigma \mathbf{Z}_6) & \to & K_0(A \times_{\sigma r} SL_2(\mathbf{Z})) \\ \uparrow & & & & \downarrow \\ K_1(A \times_{\sigma r} SL_2(\mathbf{Z})) & \leftarrow & K_1(A \times_\sigma \mathbf{Z}_4) \oplus K_1(A \times_\sigma \mathbf{Z}_6) & \leftarrow & K_1(A \times_\sigma \mathbf{Z}_2), \end{array}$$

where $A = C(S^1)$.

It is not difficult to see that the map

$$K_0(A \times_\sigma \mathbf{Z}_2) \to K_0(A \times_\sigma \mathbf{Z}_4) \oplus K_0(A \times_\sigma \mathbf{Z}_6)$$

is given by the matrix

$$\begin{bmatrix} 2 \\ -3 \end{bmatrix} : \mathbf{Z} \to \mathbf{Z} \oplus \mathbf{Z}.$$

Similarly, $K_1(A \times_\sigma Z_2) \to K_1(A \times_\sigma Z_4) \oplus K_1(A \times_\sigma Z_6)$ is given by

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} : Z \to Z \oplus Z.$$

Therefore we get

$$K_0((S^1) \times_{\sigma r} SL_2(Z)) \simeq Z,$$

$$K_1((S^1) \times_{\sigma r} SL_2(Z)) \simeq Z^2.$$

Since $\sigma(a)$ $(a \in SL_2(Z))$ preserves antipodal points, σ induces an action $\tilde{\sigma}$ of $SL_2(Z)$ on $RP^1 \simeq S^1$. We can show that

$$K_0(C(S^1) \times_{\tilde{\sigma} r} SL_2(Z)) \simeq Z^2,$$

$$K_1(C(S^1) \times_{\tilde{\sigma} r} SL_2(Z)) \simeq Z^2.$$

The computation is left to the reader.

7. EPILOGUE

In this section, certain interesting examples, which led the author to the study of the results in [4], are presented.

i) It is well-known that the fundamental group Γ_g of an orientable closed surface M_g of genus $g \geqslant 2$ is expressed as an amalgamated product of free groups along a cyclic group.

Let $\alpha_1, \beta_1, \ldots, \alpha_{g-1}, \beta_{g-1}$ and α, β be free generators of $S = F_{2g-2}$ and $G = F_2$ respectively. Let H be the subgroup of G generated by $[\alpha, \beta]$. H is identified with the subgroup of S generated by $[\alpha_1, \beta_1] \ldots [\alpha_{g-1}, \beta_{g-1}]$ via

$$[\alpha, \beta] \to [\alpha_1, \beta_1] \ldots [\alpha_{g-1}, \beta_{g-1}].$$

Then we have $\Gamma_g \cong G *_H S$.

The author is interested in the study of reduced crossed product by Γ_g, which is related to the C^*-algebra of foliation ([5]). Although, for the moment, the author does not know whether (G, H) or (S, H) has property A, it seems most likely that neither of them does so. Thus Theorem A1 can probably not be applied to the group Γ_g. However, we have examples which suggest the existeace of a six-term exact sequence.

First, notice that on M_g there exists a Riemann metric of constant cuivature -1. Let D^2 be the hyperbolic plane with the Poincaré metric

$$ds^2 = 4|dz|^2/(1 - |z|^2)^2$$

of constant curvature -1. The group $PSL_2(\mathbf{R}) = SL_2(\mathbf{R})/\mathbf{Z}_2$ is identified with the group of all isometries of (D^2, ds^2).

It is well-known that Γ_g is realized as a discrete subgroup of $PSL_2(\mathbf{R})$, and that D^2/Γ_g is equivalent to M_g.

Let T_1D^2 be the unit tangent bundle of D^2. The geodesic flow on T_1D^2 defines a C^ω-foliation $\mathscr{F}$ whose leaf is a weakly stable manifold of the flow. The unit tangent bundle T_1M_g has the form $T_1M_g \simeq T_1D^2/\Gamma_g$. Since $\mathscr{F}$ is invariant under the action of Γ_g on T_1D^2, it descends to a codimension one C^ω-foliation $\mathscr{F}_A$ of T_1M_g, a so-called Anosov foliation.

By [5], we know that the C^*-algebra $C^*(T_1M_g, \mathscr{F}_A)$ is isomorphic to $(C(S^1) \times_{\tau_1} \Gamma_g) \otimes \mathscr{K}$, where τ is a natural action of Γ_g on S^1, which we view as the boundary of the hyperbolic plane, and $\mathscr{K}$ is the elementary C^*-algebra.

On the other hand, the foliation $\mathscr{F}_A$ comes from an action π of the group

$$P = \left\{ \begin{pmatrix} s & t \\ 0 & s^{-1} \end{pmatrix} \in SL_2(\mathbf{R}) \ ; \ s, \ t \in \mathbf{R}, \ s > 0 \right\}$$

on the space T_1M_g, and $C^*(T_1M_g, \mathscr{F}_A) \simeq C(T_1M_g) \times_\pi P$. Then, by the Thom isomorphism, we have

$$K_j(C^*(T_1M_g, \mathscr{F}_A)) \simeq K^j(T_1M_g) \quad (j = 0,1),$$

where the right hand side is the topological K-theory. Thus we get

$$K_*(C(S^1) \times_{\tau r} \Gamma_g) \simeq K^*(T_1M_g).$$

We can see that

$$K^0(T_1M_g) \simeq \mathbf{Z}^{2g+1} \oplus \mathbf{Z}/(2g-2),$$
$$K^1(T_1M_g) \simeq \mathbf{Z}^{2g+1}.$$

Note that, for $C(S^1) \times_{\tau r} \Gamma_g$ we have maps

$$K_*(C(S^1) \times_\tau H) \xrightarrow{\varkappa_*^1 - \varkappa_*^2} K_*(C(S^1) \times_{\tau r} G) \oplus K_*(C(S^1) \times_{\tau r} S) \xrightarrow{\varepsilon_*^1 + \varepsilon_*^2} K_*(C(S^1) \times_{\tau r} \Gamma_g).$$

The maps

$$K_0(C(S^1) \times_\tau H) \to K_0(C(S^1) \times_{\tau r} G) \oplus K_0(C(S^1) \times_{\tau r} S)$$

and

$$K_1(C(S^1) \times_\tau H) \to K_1(C(S^1) \times_{\tau r} G) \oplus K_1(C(S^1) \times_{\tau r} S)$$

are given by the following matrices

$${}^t\begin{pmatrix} 1 & 0 & 0 & -1 & 0 \dots 0 \\ m & 0 & 0 & 2 + 2g - m & 0 \dots 0 \end{pmatrix} : \mathbf{Z}^2 \to \mathbf{Z}^3 \oplus \mathbf{Z}^{2g-1} \quad \text{for some } m \in \mathbf{Z}$$

and

$${}^t\begin{pmatrix} 1 & 0 & 0 & -1 & 0 \dots 0 \\ 0 & 0 & 0 & 0 & 0 \dots 0 \end{pmatrix} : \mathbf{Z}^2 \to \mathbf{Z}^3 \oplus \mathbf{Z}^{2g-1} \text{ respectively.}$$

The computations of these maps are very interesting in themselves, but these are omitted here.

Assume the existence of a six-term exact sequence given in Theorem A1, for $C(S^1) \times_{\tau r} \Gamma_g$. Then from the above calculation, it follows that

$$K_0(C(S^1) \times_{\tau r} \Gamma_g) \simeq \mathbf{Z}^{2g+2} \oplus \mathbf{Z}/2g-2,$$
$$K_1(C(S^1) \times_{\tau r} \Gamma_g) \simeq \mathbf{Z}^{2g+1}.$$

This result coincides with the one above.

PROBLEM. Prove Theorem A1 for the group Γ_g $(g \geqslant 2)$.

ii) Finally we give examples having property Λ. Let Σ_k be a closed non-orientable surface with $k \geqslant 2$ cross-caps. Topologically Σ_2 is the Klein bottle. Then $\pi_1(\Sigma_k)$ is a group with k generators $\alpha_1, \ldots, \alpha_k$ and the single relation $\alpha_1^2 \ldots \alpha_k^2 = 1$ (cf. [7, p. 149]).

Let G and S be the free groups with generators α_1 and $\alpha_2, \ldots, \alpha_k$ respectively. Let H be the subgroup of G generated by α_1^2, and identify it with the subgroup of S generated by $\alpha_2^2 \ldots \alpha_k^2$ via $\alpha_1^2 \to (\alpha_2^2 \ldots \alpha_k^2)^{-1}$. We have that

$$\pi_1(\Sigma_k) \simeq G *_H S.$$

Since G is abelian, (G, H) has property Λ. Therefore, applying Theorem A1, we get

$$K_0(C_r^*(\pi_1(\Sigma_k))) \simeq \mathbf{Z},$$
$$K_1(C_r^*(\pi_1(\Sigma_k))) \simeq \mathbf{Z}^{k-1} \oplus \mathbf{Z}_2.$$

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REFERENCES

1. CUNTZ, J., K-theoretic amenability for discrete groups, preprint, 1982.
2. CUNTZ, J., Generalized homomorphisms between C^*-algebras and KK-theory, preprint, 1982.
3. ELLIOTT, G. A.; NATSUME, T., in preparation.
4. LANCE, E. C., K-theory for certain group C^*-algebras, *Acta Math.*, **151**(1983), 209–230.
5. NATSUME, T.; TAKAI, H., Connes algebras associated to foliated bundles, preprint, 1981.
6. PIMSNER, M.; VOICULESCU, D., K groups of reduced crossed product by free groups, *J. Operator Theory*, **8**(1982), 131–156.
7. SPANIER, E. H., *Algebraic topology*, McGraw-Hill, 1966.

TOSHIKAZU NATSUME
Department of Mathematics,
Saitama University,
Shimo-Okubo, Urawa 338,
Japan.

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