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DERIVATIONS OF SIMPLE C^*-ALGEBRAS TANGENTIAL TO COMPACT AUTOMORPHISM GROUPS

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0. INTRODUCTION

Recently [1] Kishimoto and the first named author showed that an asymptotic abelian automorphism τ of a simple C^*-algebra $\mathfrak{A}$ with identity gives rise to a multiple tensor product structure in the limit, which allows one to prove that if an action α of a compact abelian group G as $*$-automorphisms of $\mathfrak{A}$ commutes with τ, hen any derivation δ of the algebra $\mathfrak{A}_F$ of G-finite elements into $\mathfrak{A}$, commuting with τ, is automatically tangential to α_G, in the sense that

a) δ is closable, and

b) the closure $\bar{\delta}$ of δ generates a one parameter subgroup of α_G.

Based on this limit tensor product structure, we prove that the above result remains true without the commutativity assumption on the group G. Since every derivation given by a one parameter subgroup of α_G commutes with τ and is defined on $\mathfrak{A}_F$, our result characterizes the Lie algebra of α_G as those derivations of $\mathfrak{A}_F$ into $\mathfrak{A}$ commuting with τ.

The proof is interesting in its own right. It consists of two parts. First, we prove that any automorphism β commuting with τ and leaving the fixed point algebra $\mathfrak{A}^\alpha$ pointwise fixed is indeed an element of α_G. This Galois type result will be shown by adapting the Tannaka duality arguments developed in a work of Araki-Haag-Kastler-Takesaki [2]. We then prove that the derivation δ leaves any α-invariant finite dimensional subspace globally invariant and this allows us to exponentiate δ and obtain a one parameter group $\{\beta_t\}$ of automorphisms of $\mathfrak{A}_F$. In this proof we use the exterior product based on the limit multiple tensor product. Finally we prove that $\{\beta_t\}$ extends to $\mathfrak{A}$, by positivity arguments adapted from [3], and then apply the first result to conclude that $\{\beta_t\} \subset \alpha_G$.

Before proceeding we recall a few essential definitions.

First the C^*-algebra $\mathfrak{A}$ is defined to be asymptotically abelian with respect to the $*$-automorphism τ if

$$\lim_{n\to\infty}\|[x, \tau^n(y)]\| = 0$$

for all pairs $x, y \in \mathfrak{A}$.

Second if G is a compact group and U an irreducible unitary representation of G then the corresponding character χ_U is defined by

$$\chi_U(g) = \mathrm{tr}(U_g^{-1})$$

where tr denotes the normalized trace over the representation space of U. Moreover if α_G is a representation of G as $*$-automorphisms of the C^*-algebra $\mathfrak{A}$ which is strongly continuous, i.e.,

$$\lim_{g\to h}\|\alpha_g(x) - \alpha_h(x)\| = 0$$

for all $x \in \mathfrak{A}$, $h \in G$, then the spectral subspace $\mathfrak{A}^\alpha(U)$ is defined as the norm closed subspace

$$\mathfrak{A}^\alpha(U) = \left\{\int dg\chi_U(g)\alpha_g(x)\ ;\ x \in \mathfrak{A}\right\}.$$

The G-finite elements $\mathfrak{A}_F$ are then defined as the complex linear span of the $\mathfrak{A}^\alpha(U)$ and it follows automatically that $\mathfrak{A}_F$ is a norm dense $*$-subalgebra of $\mathfrak{A}$. Note that the spectral subspace $\mathfrak{A}^\alpha$ corresponding to the trivial representation of G is the fixed point algebra of α, i.e., the set $\{x; \alpha_g(x) = x \text{ for all } g \in G\}$.

Finally a linear operator δ on $\mathfrak{A}$ is called a symmetric derivation if its domain $D(\delta)$ is a $*$-subalgebra of $\mathfrak{A}$, $\delta(x)^* = \delta(x^*)$ for all $x \in D(\delta)$, and

$$\delta(xy) = \delta(x)y + x\delta(y)$$

for all pairs $x, y \in D(\delta)$. The generators of strongly continuous one parameter groups of $*$-automorphisms of $\mathfrak{A}$ are norm closed symmetric derivations.

1. AUTOMORPHISMS OF COMPACT ACTIONS

In this section we establish the first result mentioned in the introduction.

THEOREM 1.1. *Let $\mathfrak{A}$ be a simple C^*-algebra with identity which is τ-asymptotically abelian and let α be a continuous action of a compact group G as $*$-automorphisms of $\mathfrak{A}$ such that* $[\alpha, \tau] = 0$.

If β is a $*$-automorphism of $\mathfrak{A}$ such that $[\beta, \tau] = 0$ and $\beta(x) = x$ for all $x \in \mathfrak{A}^\alpha$, the fixed point algebra of α, then $\beta = \alpha_g$ for some $g \in G$.

The theorem is proved by reducing it to a statement concerning translations on $C(G)$. For this purpose we require a number of preliminary estimates.

LEMMA 1.2. *Let α be a continuous action of the compact group G on the asymptotically abelian system $(\mathfrak{A}, \tau)$.*

It follows that

$$\lim_{n\to\infty} \|[\tau^n(x), \alpha_g(y)]\| = 0$$

for $x, y \in \mathfrak{A}$, uniformly for $g \in G$.

Proof. Suppose $\|x\|, \|y\| \leq 1$. Since G is compact and α is continuous one can, for each $\varepsilon > 0$, choose a finite set $\{g_1, g_2, \ldots, g_m\} \subset G$ such that

$$\inf_{1\leq i\leq m} \|\alpha_g(y) - \alpha_{g_i}(y)\| < \varepsilon/2$$

for all $g \in G$. Moreover by asymptotic abelianness one can choose N such that

$$\|[\tau^n(x), \alpha_{g_i}(y)]\| < \varepsilon/2$$

for all $i = 1, 2, \ldots, m$, and all $n > N$. Combining these estimates gives

$$\|[\tau^n(x), \alpha_g(y)]\| < \varepsilon$$

for all $g \in G$, and all $n > N$.

Next if $\mathfrak{A}$ contains an identity the τ-invariant states form a convex weak*-compact set with extremal points, the τ-ergodic states. Moreover for each τ-ergodic state ω one has

$$\lim_{n\to\infty} \frac{1}{n} \sum_{p=1}^{n} \omega(x\tau^p(y)) = \omega(x)\omega(y)$$

for all $x, y \in \mathfrak{A}$ (see, for example, [4] Section 4.3.2). If $(\mathfrak{A}, \tau)$ is asymptotically abelian it then easily follows that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{p=1}^{n} \omega(x\tau^p(y)z) = \omega(xz)\omega(y)$$

for all $x, y, z \in \mathfrak{A}$.

The next lemma states properties of this type with an additional uniformity in G.

LEMMA 1.3. *Adopt the assumptions of Lemma 1.2 and also assume the existence of a τ-ergodic state ω.*

It follows that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{p=1}^{n} \omega(x\alpha_g(y)z\tau^p(x'\alpha_g(y')z')) = \omega(x\alpha_g(y)z)\omega(x'\alpha_g(y')z')$$

for all $x, y, z, x', y', z' \in \mathfrak{A}$, uniformly for $g \in G$.

Moreover if $[\alpha, \tau] = 0$ *then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{p=1}^{n} \omega(x\tau^p(x')\alpha_g(y\tau^p(y'))z\tau^p(z')) = \omega(x\alpha_g(y)z)\omega(x'\alpha_g(y')z')$$

for all $x, y, z, x', y', z' \in \mathfrak{A}$, *uniformly for* $g \in G$.

Proof. The first limit exists pointwise in g by the preceding discussion. But since G is compact and α continuous uniformity in g is achieved by the same reasoning used in Lemma 1.2.

The second statement follows from the first statement, and asymptotic abelianness. In particular one uses Lemma 1.2.

LEMMA 1.4. *Adopt the assumptions of Theorem* 1.1 *and further assume* $g \mapsto \alpha_g$ *is faithful. Let* ω *be* α τ*-ergodic state and let* $\mathscr{C} \subset C(G)$ *denote the linear span of the set*

$$\{g \in G \mapsto \omega(x\alpha_g(y)z)\ ;\ x, y, z \in \mathfrak{A}\}.$$

It follows that $\mathscr{C}$ *is norm dense in* $C(G)$.

Proof. Both $\mathscr{C}$ and its closure $\bar{\mathscr{C}}$ are self-adjoint. Moreover since $\mathfrak{A}$ is simple the representation π_ω associated with ω is faithful. Also α is faithful by assumption. Therefore $\mathscr{C}$ separates points of G. Finally we show that $\bar{\mathscr{C}}$ is closed under pointwise multiplication, and hence it is equal to $C(G)$ by the Stone-Weierstrass theorem. For this it suffices to show that $\omega_{x,y,z}\omega_{x',y',z'} \in \bar{\mathscr{C}}$ where

$$\omega_{x,y,z}(g) = \omega(x\alpha_g(y)z).$$

But this follows from Lemma 1.3 because

$$\omega_{x,y,z}(g)\omega_{x',y',z'}(g) = \lim_{n\to\infty} \frac{1}{n} \sum_{p=1}^{n} \omega_{x\tau^p(x'),\, y\tau^p(y'),\, z\tau^p(z')}(g).$$

Now we are prepared to prove the theorem.

Proof of Theorem 1.1. Let ω be a τ-ergodic state. Since $[\beta, \tau] = 0$ it follows that $\omega \circ \beta$ is also τ-ergodic. But if $x \in \mathfrak{A}^\alpha$ then $(\omega \circ \beta)(x) = \omega(\beta(x)) = \omega(x)$, because β leaves $\mathfrak{A}^\alpha$ pointwise invariant. Hence by [2] Theorem II.1 there is an $h \in G$ such that $\omega \circ \beta = \omega \circ \alpha_h$. Let $\beta' = \beta \circ \alpha_h^{-1}$. Then β' satisfies the same assumptions as β but in addition $\omega \circ \beta' = \omega$. Now if we can show that $\beta' \in \alpha_G$ then $\beta \in \alpha_G$. Therefore we may effectively assume $\omega \circ \beta = \omega$. We may also assume without loss of generality that α is faithful.

Next let $\mathscr{C}$ denote the set of functions introduced in Lemma 1.4. Then $\mathscr{C} \subset L^2(G)$ and

$$(\omega_{\beta(z),y,\beta(x)}, \omega_{\beta(x'),y',\beta(z')})_{L^2} =$$

$$= \int dg\omega(\beta(x^*)\alpha_g(y^*)\beta(z^*))\omega(\beta(x')\alpha_g(y')\beta(z')) =$$

$$= \int dg \lim_{n\to\infty} \frac{1}{n} \sum_{p=1}^{n} \omega(\beta(x^*\tau^p(x'))\alpha_g(y^*\tau^p(y'))\beta(z^*\tau^p(z'))) =$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{p=1}^{n} \omega(\beta(x^*\tau^p(x'))P(y^*\tau^p(y'))\beta(z^*\tau^p(z')))$$

where

$$P(y^*\tau^p(y')) = \int dg\alpha_g(y^*\tau^p(y')) \in \mathfrak{A}^\alpha$$

and we have used both Lemma 1.3 and the assumption $[\beta, \tau] = 0$. But since ω is β-invariant and $\mathfrak{A}^\alpha$ is pointwise β-invariant it follows that

$$(\omega_{\beta(z),y,\beta(x)}, \omega_{\beta(x'),y',\beta(z')})_{L^2} =$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{p=1}^{n} \omega(x^*\tau^p(x')P(y^*\tau^p(y'))z^*\tau^p(z')) =$$

$$= (\omega_{z,y,z}, \omega_{x',y',z'})_{L^2}.$$

Therefore one can define an isometric linear operator b on $L^2(G)$ by first setting

$$b\left(\sum_{i=1}^{p} \omega_{x_i,y_i,z_i}\right) = \sum_{i=1}^{p} \omega_{\beta(x_i),y_i,\beta(z_i)}$$

and then extending by continuity. But b is invertible, because β is invertible, and hence b is unitary.

Next remark that

$$(b(\omega_{x,y,z}\, \omega_{x',y',z'}))(g) =$$

$$= b\left(\lim_{n\to\infty} \frac{1}{n} \sum_{p=1}^{n} \omega_{x\tau^p(x'),\, y\tau^p(y'),\, z\tau^p(z')}\right)(g) =$$

(1)

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{p=1}^{n} \omega_{\beta(x)\tau^p(\beta(x')),\, y\tau^p(y'),\, \beta(z)\tau^p(\beta(z'))}(g) =$$

$$= (b\omega_{x,y,z})(g)(b\omega_{x',y',z'})(g).$$

It then follows that b is multiplicative on $C(G)$. To establish this first define the multiplication operators $m(f)$, $f \in C(G)$ by $m(f)k = fk$, $k \in L^2(G)$. Then the map $f \mapsto m(f)$ is an isometric isomorphism of $C(G)$ onto its image. We assert that

$$bm(f)b^* = m(bf), \quad f \in C(G). \tag{2}$$

By density of $\mathscr{C}$ in $L^2(G)$ it suffices to show

$$bm(f)b^*k = m(bf)k, \quad f \in C(G), k \in \mathscr{C}. \tag{3}$$

We first show this for $f \in \mathscr{C}$. By linearity it suffices to do this for $f = \omega_{x,y,z}$, $k = \omega_{x',y',z'}$. But then by (1) we have

$$bm(\omega_{x,y,z})b^*\omega_{x',y',z'} = bm(\omega_{x,y,z})\omega_{\beta^{-1}(x'),y',\beta^{-1}(z')} =$$

$$= b(\omega_{x,y,z}\omega_{\beta^{-1}(x'),y',\beta^{-1}(z')}) = (b\omega_{x,y,z})\omega_{x',y',z'},$$

which proves (3) for $f \in \mathscr{C}$. In particular we have for $f \in \mathscr{C}$,

$$\|bf\|_\infty = \|m(bf)\|_\infty = \|bm(f)b^*\|_\infty = \|m(f)\|_\infty = \|f\|_\infty,$$

so b is isometric on $\mathscr{C}$, and hence on $C(G)$. But then (3) is immediate from the density of $\mathscr{C}$ in $C(G)$. Hence (2) follows. Therefore if $f_1, f_2 \in C(G)$, then $f_2 \in L^2(G)$, and we have $b(f_1f_2) = b(m(f_1)f_2) = bm(f_1)b^*bf_2 = m(bf_1)bf_2 = (bf_1)(bf_2)$, proving that b is multiplicative on $C(G)$.

Finally let γ denote right translation on $L^2(G)$, i.e., for each $h \in G$ one has $(\gamma_h f)(g) = f(gh)$. Then

$$(\gamma_h\omega_{x,y,z})(g) = \omega_{x,\alpha_h(y),z}(g).$$

Hence

$$(b\gamma_h\omega_{x,y,z})(g) = (b\omega_{x,\alpha_h(y),z})(g) = \omega_{\beta(x),\alpha_h(y),\beta(z)}(g) =$$

$$= (\gamma_h\omega_{\beta(x),y,\beta(z)}(g) = (\gamma_h b\omega_{x,y,z})(g).$$

Consequently $b\gamma_h = \gamma_h b$ for all $h \in G$, on $L^2(G)$. It now follows by [2] Lemma A.3 that b is a left translation, i.e., there is a $k \in G$ such that $(bf)(g) = f(k^{-1}g)$ for all $f \in C(G)$. Therefore

$$\omega(x\alpha_{k^{-1}}(y)z) = (b\omega_{x,y,z})(e) = \omega(\beta(x)y\beta(z)) = \omega(x\beta^{-1}(y)z)$$

where we have used β-invariance of ω. Since this holds for all $x, y, z \in \mathfrak{A}$, and $\mathfrak{A}$ is simple, it follows that $\beta = \alpha_k$.

2. A GENERATION THEOREM

In this section we establish a generation theorem for derivations which commute with τ. This theorem is a direct extension of the first part of Theorem 3.1 of [1] to non-abelian G.

THEOREM 2.1. *Let $\mathfrak{A}$ be a simple C^*-algebra with identity which is τ-asymptotically abelian and let α be a continuous action of a compact group G as $*$-automorphisms of $\mathfrak{A}$ satisfying* $[\alpha, \tau] = 0$. *Further let δ be a symmetric derivation with domain $D(\delta) = \mathfrak{A}_F$, the G-finite elements, satisfying* $[\delta, \tau] = 0$.

It follows that δ is norm-closable and its closure $\bar{\delta}$ generates a strongly continuous one-parameter group of $*$-automorphisms β of $\mathfrak{A}$ which leaves invariant each finite-dimensional α-invariant subspace of $\mathfrak{A}_F$ and which also leaves the fixed point algebra $\mathfrak{A}^\alpha$ pointwise invariant.

Proof. First note that the fixed point algebra $\mathfrak{A}^\alpha \subseteq D(\delta)$ by assumption. It then follows that $\delta(\mathfrak{A}^\alpha) = \{0\}$ by the argument used in the first part of Theorem 3.1 in [1].

Second let $\mathcal{M}(U) \subseteq \mathfrak{A}^\alpha(U)$ be a finite-dimensional α-invariant subspace of $\mathfrak{A}^\alpha(U)$, e.g., if $x \in \mathfrak{A}^\alpha(U)$ the linear span of $\{\alpha_g(x); g \in G\}$ is both finite-dimensional and α-invariant. Now $\mathcal{M}(U) \subseteq \mathfrak{A}_F = D(\delta)$ and we next argue that $\delta(\mathcal{M}(U)) \subseteq \mathcal{M}(U)$. To this end choose k linearly independent elements $x_1, x_2, \ldots, x_k \in \mathcal{M}(U)$, where k is the dimension of $\mathcal{M}(U)$. Since $\mathcal{M}(U)$ is α-invariant the action of α can be represented by $k \times k$ matrices $\{U_{ij}(g)\}$ in the form

$$\alpha_g(x_i) = \sum_{j=1}^{k} U_{ji}(g) x_j .$$

Next for each k-tuple $n = \{n_1, n_2, \ldots, n_k\}$ of positive integers define an element $x(n) \in \mathfrak{A}$ by

$$x(n) = \sum_{\sigma} \operatorname{sign}(\sigma) \tau^{n_1}(x_{\sigma(1)}) \tau^{n_2}(x_{\sigma(2)}) \ldots \tau^{n_k}(x_{\sigma(k)})$$

where the sum is over all permutations σ of the set $\{1, 2, \ldots, k\}$ and $\operatorname{sign}(\sigma)$ denotes the signature of the permutation σ. Now

$$\alpha_g(x(n)) = \sum_{\sigma} \operatorname{sign}(\sigma) \sum_{j_1, \ldots, j_k = 1}^{k} \prod_{i=1}^{k} U_{j_i, \sigma(i)}(g) \tau^{n_i}(x_{j_i})$$

and the sum over σ is zero unless the j_i are distinct, in which case one can write $j_i = \sigma'(i)$ for some permutation σ'. Then

$$\alpha_g(x(n)) = \sum_{\sigma} \operatorname{sign}(\sigma) \sum_{\sigma'} \prod_{i=1}^{k} U_{\sigma'(i), \sigma(i)}(g) \tau^{n_i}(x_{\sigma'(i)}) = \chi(g) x(n)$$

where $\chi(g) = \text{Det}\ (U_{ij}(g))$. Thus $x(n)x(n')^* \in \mathfrak{A}^\alpha$ for all k-tuples n and n'. But $x(n), x(n') \in D(\delta)$ and $\delta(\mathfrak{A}^\alpha) = \{0\}$ hence

$$\delta(x(n))x(n')^* + x(n)\delta(x(n'))^* = 0.$$

Now taking limits such that $|n_i - n_j|$, $|n_i' - n_j'| \to \infty$ for $i \neq j$ and $|n_i - n_j'| \to \infty$ for all i, j this translates into an identity in the algebraic tensor product $\mathfrak{A}^{2k} := \mathfrak{A}^k \odot \mathfrak{A}^k$ of $2k$ copies of $\mathfrak{A}$ (see [1] and in particular Proposition 2.1). This identity states

$$X_\delta \otimes X^* + X \otimes X_\delta^* = 0$$

where

$$X = \sum_\sigma \text{sign}(\sigma) x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes \ldots \otimes x_{\sigma(k)} \in \mathfrak{A}^k$$

and

$$X_\delta = \sum_\sigma \text{sign}(\sigma) \sum_{i=1}^{k} x_{\sigma(1)} \otimes \ldots \otimes \delta(x_{\sigma(i)}) \otimes \ldots \otimes x_{\sigma(k)} \in \mathfrak{A}^k.$$

Hence if $\omega_1, \omega_2, \ldots, \omega_k$ are states over $\mathfrak{A}$ and $\omega = \omega_1 \otimes \omega_2 \otimes \ldots \otimes \omega_k$ is the corresponding product state over $\mathfrak{A}^k$ one has

$$X_\delta \overline{\omega(X)} + X \overline{\omega(X_\delta)} = 0.$$

Now define product states

$$\omega^{(j)} = \bigotimes_{i \neq j} \omega_i$$

over $\mathfrak{A}^{k-1}$ and consider the maps $\iota \otimes \omega^{(j)} : \mathfrak{A}^k \mapsto \mathfrak{A}$ where ι denotes the identity map from $\mathfrak{A} \mapsto \mathfrak{A}$. It follows that

$$(\iota \otimes \omega^{(j)})(X_\delta)\overline{\omega(X)} = -(\iota \otimes \omega^{(j)})(X)\overline{\omega(X_\delta)} \in \mathcal{M}(U)$$

because the right hand side is a finite linear combination of the x_i. Hence

$$\sum_{j=1}^{k} (-1)^j \omega_j(x_i)\, (\iota \otimes \omega^{(j)})\, (X_\delta)\overline{\omega(X)} \in \mathcal{M}(U).$$

But the left hand side can be explicitly evaluated as a finite linear combination of the x_i plus a term $\delta(x_i)|\omega(X)|^2$. Therefore

$$\delta(x_i)|\omega(X)|^2 \in \mathcal{M}(U)$$

and $\delta(x_i) \in \mathscr{M}(U)$ unless $\omega(X) = 0$ for all possible product states ω. But since

$$\omega(X) = \mathrm{Det}(\omega_i(x_j))$$

this would imply the x_j are linearly dependent, which is a contradiction. Thus $\delta(x_i) \in \in \mathscr{M}(U)$ and $\delta(\mathscr{M}(U)) \subseteq \mathscr{M}(U)$.

Since $\mathscr{M}(U)$ is finite-dimensional one can now define a continuous group $\beta_t = = \exp\{t\delta\}$ of bounded maps of $\mathscr{M}(U)$ into $\mathscr{M}(U)$ by straightforward exponentiation. Moreover for each $x \in \mathfrak{A}^\alpha(U)$ the linear span of $\{\alpha_g(x); g \in G\}$ is finite-dimensional, and α-invariant, and hence β can be defined as a continuous group of bounded maps of $\mathfrak{A}_F$ into $\mathfrak{A}_F$ which maps each $\mathfrak{A}^\alpha(U)$ into itself. Now consider β on a fixed $\mathfrak{A}^\alpha(U)$. The spectral subspace $\mathfrak{A}^\alpha(U)$ is a Banach space and the generator δ_β of the continuous group β of bounded maps of this space is, by construction, everywhere defined, hence bounded, and in fact $\delta_\beta = \delta$ on $\mathfrak{A}^\alpha(U)$. Therefore

$$\|\beta_t(x)\| \leqslant \|x\| \exp\{|t|\, \|\delta\|_U\}$$

for each $x \in \mathfrak{A}^\alpha(U)$, where $\|\delta\|_U$ denotes the norm of δ restricted to $\mathfrak{A}^\alpha(U)$. Next remark that if $x, y \in \mathfrak{A}_F$ then $\beta_s(x)\beta_s(y) \in \mathfrak{A}_F$ and $\beta_{t-s}(\beta_s(x)\beta_s(y))$ is defined. Hence

$$\beta_t(x)\beta_t(y) - \beta_t(xy) = \int_0^t \mathrm{d}s \frac{\mathrm{d}}{\mathrm{d}s} (\beta_{t-s}(\beta_s(x)\beta_s(y)) = 0$$

where the last conclusion follows by explicit differentiation and use of the derivation property. Thus β is a group of $*$-automorphisms of $\mathfrak{A}_F$ and in particular

$$\beta_t(x^*x) \geqslant 0$$

for each $x \in \mathfrak{A}_F$. Now it is not immediate that β is positive on $\mathfrak{A}_F$ because the positive elements of $\mathfrak{A}_F$ are not necessarily of the form x^*x with $x \in \mathfrak{A}_F$. Nevertheless positivity on $\mathfrak{A}_F$ follows by the arguments of [3] Lemma 1.8 (see also [5]). The details are as follows.

Let $\Lambda, \Lambda', \ldots$ denote the finite sets of irreducible unitary representations U of G ordered by inclusion and consider the net $\mathfrak{A}_\Lambda$ of subspaces of $\mathfrak{A}_F$ formed by the linear span of the $\{\mathfrak{A}^\alpha(U); U \in \Lambda\}$. Next define linear operators P_Λ on $\mathfrak{A}$ by

$$P_\Lambda(x) = |\Lambda|^{-1} \int \mathrm{d}g |\chi_\Lambda(g)|^2 \alpha_g(x)$$

for $x \in \mathfrak{A}$ where

$$\chi_\Lambda(g) = \sum_{U \in \Lambda} \mathrm{tr}(U_g^{-1})$$

and tr is the normalized trace on the representation space of U. It follows that P_Λ is positive and $P_\Lambda(\mathbf{1})=\mathbf{1}$. Therefore $\|P_\Lambda\|=1$. Moreover to each Λ there is a $\tilde{\Lambda}$ such that $P_\Lambda(\mathfrak{A})\subseteq\mathfrak{A}_{\tilde{\Lambda}}$ and $P_\Lambda(\mathfrak{A}_{\Lambda'})\subseteq\mathfrak{A}_{\Lambda'\cap\tilde{\Lambda}}$. Finally $P_\Lambda(x)\to x$ for each $x\in\mathfrak{A}_F$. (The proofs of these statements follow from the standard calculations of harmonic analysis which surround the Peter-Weyl theorem (see, for example, [6] Chapter 7). Note that the use of the normalized trace eliminates the cumbersome dimension factors that occur in the usual formalism.)

Now for $\varepsilon>0$ and $x\in\mathfrak{A}_{\Lambda'}$, with $x\geqslant 0$ choose $y=y^*\in\mathfrak{A}_F$ such that $\|x-y^2\|<\varepsilon$. Then using the previous estimates on β together with $\|P_\Lambda\|=1$ one finds

$$\|\beta_t(P_\Lambda(x)-P_\Lambda(y^2))\|\leqslant\varepsilon\sum_{U\in\tilde{\Lambda}}\exp\{|t|\,\|\delta\|_U\}.$$

But since $\beta(z^*z)\geqslant 0$ for $z\in\mathfrak{A}_F$ one also has

$$\beta_t(P_\Lambda(y^2))=|\Lambda|^{-1}\int dg|\chi_\Lambda(g)|^2\beta_t(\alpha_g(y)^2)\geqslant 0.$$

Therefore

$$\beta_t(P_\Lambda(x))\geqslant-\varepsilon\sum_{U\in\tilde{\Lambda}}\exp\{|t|\,\|\delta\|_U\}.$$

As ε is arbitrary this implies $\beta_t(P_\Lambda(x))\geqslant 0$. But $P_\Lambda(x)\in\mathfrak{A}_{\Lambda'\cap\tilde{\Lambda}}\subseteq\mathfrak{A}_{\Lambda'}$ and $P_\Lambda(x)\to x$ in $\mathfrak{A}_{\Lambda'}$. Since β is bounded in restriction to $\mathfrak{A}_{\Lambda'}$ it follows that $\beta_t(P_\Lambda(x))\to\beta_t(x)$ and hence $\beta_t(x)\geqslant 0$.

This establishes that the group β is positive on $\mathfrak{A}_F$. But if $x\in\mathfrak{A}_F$ with $\|x\|\leqslant 1$ one has $\mathbf{1}-x^*x\in\mathfrak{A}_F$ and $\mathbf{1}-x^*x$ is positive. Hence using $\beta_t(\mathbf{1})=\mathbf{1}$ one finds

$$\mathbf{1}\geqslant\beta_t(x^*x)=\beta_t(x)^*\beta_t(x)\geqslant 0.$$

Consequently $\|\beta_t(x)\|\leqslant 1$ and it follows that $\|\beta_t\|=1$. Therefore β extends by continuity to a strongly continuous one-parameter group of $*$-automorphisms of $\mathfrak{A}$.

The invariance properties of β stated in the theorem follow by construction and it remains to identify the generator δ_β of β. Now $\delta_\beta=\delta$ on $\mathfrak{A}_F$, by construction, and $\mathfrak{A}_F$ is β-invariant. Consequently $\mathfrak{A}_F$ is a core of δ_β and $\delta_\beta=\bar{\delta}$ (see, for example, [4] Corollary 3.1.7).

3. THE LIE ALGEBRA OF α_G

Combination of Theorems 1.1 and 2.1 now establish that the Lie algebra associated with α_G consists of the symmetric derivations of $\mathfrak{A}_F$ into $\mathfrak{A}$ which commute with τ.

THEOREM 3.1. *Let $\mathfrak{A}$ be a simple C^*-algebra with identity which is τ-asymptotically abelian and let α be a continuous action of a compact group G as $*$-automorphisms of $\mathfrak{A}$ satisfying $[\alpha, \tau] = 0$. Further let δ be a closed symmetric derivation on $\mathfrak{A}$. The following conditions are equivalent:*

1. *δ generates a one-parameter subgroup β of α_G.*
2. *a. $\mathfrak{A}_F$ is a core for δ,*
   b. $\delta\tau(x) = \tau\delta(x)$ for all $x \in \mathfrak{A}_F$.

Proof. $1 \Rightarrow 2$. Since β is a one-parameter subgroup of α the G-finite elements $\mathfrak{A}_F$ are contained in the domain of the generator δ. But $\mathfrak{A}_F$ is a norm dense $*$-subalgebra which is a α-invariant and hence β-invariant. Therefore $\mathfrak{A}_F$ is a core for δ. Finally since $[\tau, \alpha] = 0$ one must have $[\tau, \beta] = 0$ and this implies $[\tau, \delta] = 0$ on $\mathfrak{A}_F$.

$2 \Rightarrow 1$. It follows from Theorem 2.1 that δ generates a strongly continuous one-parameter group of $*$-automorphisms of $\mathfrak{A}$ which leaves $\mathfrak{A}^\alpha$ pointwise invariant. Moreover β leaves invariant each finite dimensional α-invariant subspace. Now fix $x \in \mathfrak{A}_F$ and consider the α-invariant subspace of $\mathfrak{A}_F$ spanned by $\{\alpha_g(x);\ g \in G\}$. Since this subspace is both finite-dimensional and β-invariant it is also invariant under all powers δ^n of δ. In particular $\delta^n(x) \in \mathfrak{A}_F$ for all n. Then it follows from Conditions 2b that $\tau\delta^n(x) = \delta^n\tau(x)$. Consequently $\tau\beta_t(x) = \beta_t\tau(x)$ for all $x \in \mathfrak{A}_F$, and $[\tau, \beta_t] = 0$ by continuity. Finally it follows from Theorem 1.1 that $\beta_t = \alpha_{g_t}$ for some $g_t \in G$.

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