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# DISSIPATIONS, DERIVATIONS, DYNAMICAL SYSTEMS, AND ASYMPTOTIC ABELIANNESS

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## 1. INTRODUCTION

Let $(\mathfrak{A}, G, \alpha)$ be a $C^*$-dynamical system where $\mathfrak{A}$ is a simple $C^*$-algebra with identity and $G$ is compact and abelian. Next let $\delta$ be a linear operator from the $G$-finite elements $\mathfrak{A}_F$ of $\mathfrak{A}$ into $\mathfrak{A}$. There have been many recent investigations of this situation with the extra assumption that $\delta$ commutes with $\alpha$ (see, for example, [1] and the references therein). The principal aim of these investigations was to characterize those $\delta$ which generate $C_0$-groups of $^*$-automorphisms of $\mathfrak{A}$, or $C_0$-semigroups of completely positive maps. In this paper we study the same questions without the assumption that $\delta$ and $\alpha$ commute. Instead we assume that $\mathfrak{A}$ is asymptotically abelian with respect to an automorphism $\tau$ and that $\alpha$ and $\delta$ commute with $\tau$. Somewhat surprisingly this latter assumption leads to similar but even stronger conclusions. (Related results have previously been given by Takesaki [8] and Longo and Peligrad [9].) For example if $\delta$ is a $^*$-derivation then $\delta$ automatically vanishes on the fixed point algebra $\mathfrak{A}^\alpha$ of $\alpha$, $\delta$ is closable, and its closure $\bar{\delta}$ generates a group of $^*$-automorphisms $\beta$ which commutes with $\alpha$. Similarly if $\delta$ is a $^*$-dissipation for which $\delta(\mathfrak{A}^\alpha) = \{0\}$ then $\delta$ is closable and $\bar{\delta}$ generates a $C_0$-semigroup of completely positive contractions $\beta$ which commutes with $\alpha$. In both cases $\beta$ acts by multiplication on the spectral subspaces $\mathfrak{A}^\alpha(\gamma)$ of $\mathfrak{A}$, i.e.,

$$\beta_t(x) = \mathrm{e}^{-t\varphi(\gamma)}x, \quad x \in \mathfrak{A}^\alpha(\gamma).$$

If $\delta$ is a $^*$-derivation then $\mathrm{i}\varphi(\gamma) = \lambda(\gamma)$ is a homomorphism of $\hat{G}$, the dual of $G$, into $\mathbf{R}$. If $\delta$ is a $^*$-dissipation then $\varphi$ is a negative definite function.

The key point in deriving these results is the observation that asymptotic abelianness defines a natural topology on the multiple tensor products of $\mathfrak{A}$ with itself. This structure is analyzed in Section 2 and is exploited in Section 3 to obtain the generator results. In Section 3 we also examine almost periodic actions $\alpha$ of the real line $\mathbf{R}$ in place of the action $\alpha$ of $G$.

## 2. ASYMPTOTICALLY ABELIAN SYSTEMS

In this section we consider a $C^*$-algebra $\mathfrak{A}$ which is asymptotically abelian with respect to an automorphism $\tau$. By this we mean

$$\lim_{n\to\infty} \|[x, \tau^n(y)]\| = 0$$

for all pairs $x, y \in \mathfrak{A}$. This concept expresses a certain independence of $x$ and $\tau^n(y)$ for large values of $n$. Another expression of independence is through a tensor product structure and our aim is to relate these two notions. Specifically we show that asymptotic abelianness defines a natural topology on the algebraic tensor products $\mathfrak{A}_2 = \mathfrak{A}\odot\mathfrak{A}$, $\mathfrak{A}_3 = \mathfrak{A}\odot\mathfrak{A}\odot\mathfrak{A}$, etc. of $\mathfrak{A}$ with itself. This topology is introduced by a family of $C^*$-crossnorms which are defined in the following proposition. Note that it is not essential for this result that $\mathfrak{A}$ is simple and we discuss possible generalizations at the end of the section.

PROPOSITION 2.1. *Let $\mathfrak{A}$ be a simple $C^*$-algebra with identity which is asymptotically abelian with respect to a $^*$-automorphism $\tau$*, i.e.,

$$\lim_{n\to\infty} \|[x, \tau^n(y)]\| = 0, \quad x, y \in \mathfrak{A}.$$

*For each $k = 1,2,3, \ldots$ there exists a $C^*$-norm $\|\cdot\|_k$ on the algebraic tensor product $\mathfrak{A}_k = \mathfrak{A}\odot\mathfrak{A}\odot\ldots\odot\mathfrak{A}$ of $k$ copies of $\mathfrak{A}$ such that*

$$\|\textstyle\sum x_i^{(1)} \otimes x_i^{(2)} \otimes \ldots \otimes x_i^{(k)}\|_k = \limsup_{|n|\to\infty} \|\textstyle\sum \tau^{n_1}(x_i^{(1)})\tau^{n_2}(x_i^{(2)}) \ldots \tau^{n_k}(x_i^{(k)})\|$$

*for any $\sum x_i^{(1)} \otimes x_i^{(2)} \otimes \ldots \otimes x_i^{(k)}$ in $\mathfrak{A}_k$ where $|n| = \min\{|n_i - n_j|;\ i \neq j\}$. Moreover if $x \in \mathfrak{A}_k$ and $y \in \mathfrak{A}_l$ one has the cross-norm property*

$$\|x \otimes y\|_{k+l} = \|x\|_k \|y\|_l .$$

*Proof.* We begin by arguing that $\|\cdot\|_k$ defines a $C^*$-seminorm. First define, for each $k$-tuple $n = (n_1, \ldots, n_k)$ of positive integers, a map $T_n$ from the direct product of $k$ copies of $\mathfrak{A}$ into $\mathfrak{A}$ by

$$T_n(x^{(1)}, x^{(2)}, \ldots, x^{(k)}) = \tau^{n_1}(x^{(1)})\, \tau^{n_2}(x^{(2)}) \ldots \tau^{n_k}(x^{(k)}).$$

Since $T_n$ is linear in each coordinate it extends to a linear map, also denoted by $T_n$, from $\mathfrak{A}_k$ into $\mathfrak{A}$. Thus $a \in \mathfrak{A}_k \mapsto \|T_n a\|$ defines a seminorm on the linear space $\mathfrak{A}_k$. Since $\|T_n a\|$ is uniformly bounded in $n$, for each $a \in \mathfrak{A}_k$ one concludes that

$$\|a\|_k = \limsup_{|n|\to\infty} \|T_n a\|$$

also defines a seminorm on the linear space. Next we examine the algebraic properties of the norm.

If $a = \sum x_i^{(1)} \otimes \ldots \otimes x_i^{(k)}$ then $\|T_n a^* - (T_n a)^*\|$ is bounded by a finite linear combination of terms

$$\prod_{\substack{p=1 \\ p \neq q, r}}^{k} \|x_i^{(p)}\| (\|[\tau^{n_q}(x_i^{(q)}), \tau^{n_r}(x_i^{(r)})]\|).$$

Hence $\|T_n a^* - (T_n a)^*\| \to 0$ as $|n| \to \infty$ and

$$\|a^*\|_k = \|a\|_k.$$

Similarly for $a, b \in \mathfrak{A}_k$ one finds that $\|(T_n ab) - (T_n a)(T_n b)\| \to 0$ as $|n| \to \infty$ and consequently

$$\|ab\|_k \leqslant \|a\|_k \|b\|_k.$$

But since

$$(T_n a^* a) - (T_n a)^*(T_n a) = (T_n a^* a) - (T_n a^*)(T_n a) + ((T_n a^*) - (T_n a)^*)(T_n a)$$

one concludes from the previous observations that $\|(T_n a^* a) - (T_n a)^*(T_n a)\| \to 0$ as $|n| \to \infty$ and hence

$$\|a^* a\|_k = \|a\|_k^2.$$

This completes the proof that $\|\cdot\|_k$ is a $C^*$-seminorm.

The proof that $\|\cdot\|_k$ is in fact a $C^*$-norm depends upon the crossnorm property. The crucial step in deriving this latter property is the following.

LEMMA 2.2. *For each pair* $x, y \in \mathfrak{A}$

$$\lim_{n\to\infty} \|x\tau^n(y)\| = \|x\|\|y\|.$$

*Proof.* Since

$$\|x^*x\,\tau^n(yy^*)\| \leqslant \|x\|\cdot\|x\tau^n(y)\|\cdot\|y\| \leqslant \|x^*x\|\|\cdot\|yy^*\|$$

it suffices to prove the statement for positive $x, y \in \mathfrak{A}$.

Let $x, y \in \mathfrak{A}$ be positive with $\|x\| = 1 = \|y\|$ and assume there is a positive $a \in \mathfrak{A}$ with $\|a\| = 1$ such that $ax = a$. Since $\mathfrak{A}$ is simple and has an identity $\mathbf{1}$ there exist $b_1, b_2, \ldots, b_m \in \mathfrak{A}$ such that

$$z = \sum_{i=1}^{m} b_i^* a b_i \geqslant \mathbf{1}.$$

Setting $c_i = a^{1/2} b_i z^{-1/2}$ one has $xc_i = c_i$ and

$$\sum_{i=1}^{m} c_i^* c_i = \mathbf{1}.$$

But

$$\left\| \sum_{i=1}^{m} c_i^* \tau^n(y)\, c_i - \tau^n(y) \right\| \leqslant \sum_{i=1}^{m} \|c_i\| \cdot \|[\tau^n(y), c_i]\| \xrightarrow[n\to\infty]{} 0.$$

Moreover

$$\sum_{i=1}^{m} c_i^* \tau^n(y) c_i = \sum_{i=1}^{m} c_i^* x\tau^n(y)x c_i \leqslant \|x\tau^n(y)x\| \mathbf{1}.$$

Therefore

$$1 = \widetilde{\lim_{n\to\infty}} \left\| \sum_{i=1}^{m} c_i^* \tau^n(y) c_i \right\| \leqslant \widetilde{\lim_{n\to\infty}} \|x\tau^n(y)x\| \leqslant \lim_{n\to\infty} \|x\tau^n(y)\| \leqslant 1$$

where $\widetilde{\lim}$ denotes the limit over any convergent subsequence. Consequently

$$\lim_{n\to\infty} \|x\tau^n(y)\| = 1.$$

Finally for each $\varepsilon > 0$ one can choose a positive $x_1 \in \mathfrak{A}$ with $\|x_1\| = 1$ and a positive $a \in \mathfrak{A}$ such that $\|x - x_1\| < \varepsilon$ and $ax_1 = x_1$. Then

$$1 \geqslant \|x\tau^n(y)\| \geqslant \|x_1\tau^n(y)\| - \varepsilon$$

and one deduces from the special case above that

$$\lim_{n\to\infty} \|x\tau^n(y)\| = 1.$$

Now we return to the proof of Proposition 2.1.

Let $a \in \mathfrak{A}_k$ and $b \in \mathfrak{A}_l$. Write $n = (n_1, \ldots, n_{k+l})$ as $n = (n', n'')$ where $n' = (n_1, \ldots, n_k)$ and $n'' = (n_{k+1}, \ldots, n_{k+l})$. Then $T_n(a \otimes b) = T_{n'}(a)T_{n''}(b)$ and hence

$$\|a \otimes b\|_{k+l} = \limsup_{|n|\to\infty} \|T_n(a \otimes b)\| \leqslant$$

$$\leqslant \limsup_{|n'|\to\infty\, |n''|\to\infty} \|T_{n'}a\|\|T_{n''}b\| = \|a\|_k \|b\|_l.$$

But if one considers $k + l$-tuples for which $n'' = (m + m_1, \ldots, m + m_l)$ then $T_n(a \otimes b) = T_{n'}(a)\tau^m(T_{m'}(b))$ where $m' = (m_1, \ldots, m_l)$. Therefore

$$\lim_{m\to\infty} \|T_n(a \otimes b)\| = \|T_{n'}(a)\| \, \|T_{m'}(b)\|$$

by Lemma 2.2. Hence

$$\|a \otimes b\|_{k+l} = \limsup_{|n|\to\infty} \|T_n(a \otimes b)\| \geqslant$$

$$\geqslant \limsup_{|n'|,\, |m'|\to\infty} \lim_{m\to\infty} \|T_n(a \otimes b)\| =$$

$$= \limsup_{|n'|,\, |m'|\to\infty} \|T_{n'}(a)\| \, \|T_{m'}(b)\| = \|a\|_k \|b\|_l$$

Combination of these two estimates establishes the general cross-norm property.

It remains to prove that the $\|\cdot\|_k$ are in fact norms. The proof is based upon the following observation.

LEMMA 2.3. *Let $\mathfrak{A}_i$, $i = 1,2, \dots, k$ be $C^*$-algebras with identity and $\mathfrak{B} = \mathfrak{A}_1 \odot \odot \mathfrak{A}_2 \odot \dots \odot \mathfrak{A}_k$ the algebraic tensor product of the $\mathfrak{A}_i$. If $\alpha$ is a $C^*$-seminorm on $\mathfrak{B}$ and $\mathfrak{J} = \{x \in \mathfrak{B} ; \alpha(x) = 0\}$ then $\mathfrak{J} \neq \{0\}$ if, and only if, $\mathfrak{J}$ contains a non-zero elementary product $x_1 \otimes x_2 \otimes \dots \otimes x_k$.*

*Proof.* First note that as $\alpha$ is a $C^*$-seminorm $\mathfrak{J}$ is a two-sided ideal of $\mathfrak{B}$. Next assume that $\mathfrak{J}$ is non-zero and let $n_0$ be the largest positive integer such that

$$\sum_{i=1}^{n} x_i^{(1)} \otimes \dots \otimes x_i^{(k)} \in \mathfrak{J} \setminus \{0\}$$

implies $n \geqslant n_0$. Then choose

$$a = \sum_{i=1}^{n_0} x_i^{(1)} \otimes \dots \otimes x_i^{(k)} \in \mathfrak{J} \setminus \{0\}.$$

In particular each term $x_i^{(1)} \otimes \dots \otimes x_i^{(k)} \neq 0$. Therefore, by multiplying with $x_1^{(1)} \otimes \dots \otimes x_1^{(k)*}$ if necessary, we may suppose that $x_1^{(j)} \geqslant 0$ for all $j$. Let $y \in \mathfrak{A}_j$ with $[y, x_1^{(j)}] = 0$ then setting $b = \mathbf{1} \otimes \dots \otimes y \otimes \dots \otimes \mathbf{1}$, with $y$ in the $j$-th position, one has

$$[b, a] = \sum_{i=2}^{n_0} x_i^{(1)} \otimes \dots \otimes [y, x_i^{(j)}] \otimes \dots \otimes x_i^{(n)} \in \mathfrak{J}.$$

Hence each term of the sum must be zero, by definition of $n_0$, i.e., $[y, x_i^{(j)}] = 0$ for $i = 2, \dots, n_0$. Thus the $x_i^{(j)}$, $i = 1,2, \dots, n_0$, belong to a maximal abelian $C^*$-subalgebra $\mathfrak{C}_j$ of $\mathfrak{A}_j$. Hence

$$\mathfrak{J} \cap \mathfrak{C}_1 \odot \dots \odot \mathfrak{C}_k \neq \{0\}.$$

Thus we have reduced the proof to the case that the $\mathfrak{A}_i$ are abelian.

Let $\mathfrak{D}$ be the completion of $\mathfrak{B}/\mathfrak{J}$ with respect to the $\alpha$-seminorm and $\Pi$ the canonical quotient map of $\mathfrak{B}$ into $\mathfrak{D}$. If $\chi$ is a character of $\mathfrak{D}$ then $x^{(j)} \in \in \mathfrak{A}_j \to \chi(\Pi(\mathbf{1} \otimes \dots \otimes x^{(j)} \otimes \dots \otimes \mathbf{1}))$ defines a character $\chi_j$ of $\mathfrak{A}_j$. Thus there is a map $\varphi$ of the spectrum $\Omega$ of $\mathfrak{D}$ into $\Omega_1 \times \dots \times \Omega_k$ where $\Omega_j$ is the spectrum of $\mathfrak{A}_j$. This map $\varphi$ is continuous and hence $\varphi(\Omega)$ is compact.

If $\varphi(\Omega) = \Omega_1 \times \dots \times \Omega_k$ and

$$a = \sum_{i=1}^{n} x_i^{(1)} \otimes \dots \otimes x_i^{(k)} \in \mathfrak{J}$$

then $\Pi(a) = 0$ and $\chi(\Pi(a)) = 0$ for all characters $\chi$. But then by a simple inductive argument beginning with $k = 2$ one deduces that $a = 0$ and hence $\mathfrak{J} = \{0\}$ which is a contradiction.

If, however, $\varphi(\Omega) \neq \Omega_1 \times \ldots \times \Omega_k$ there are non-zero $x^{(j)} \in \mathfrak{A}_j$ such that $(\mathrm{supp}\, x^{(1)}) \times (\mathrm{supp}\, x^{(2)}) \times \ldots \times (\mathrm{supp}\, x^{(k)}) \cap \varphi(\Omega) = \emptyset$, i.e., $\chi(\Pi(x^{(1)} \otimes \ldots \otimes x^{(k)})) = 0$ for all $\chi \in \Omega$. Thus $x^{(1)} \otimes \ldots \otimes x^{(k)} \in \mathfrak{J}$.

REMARK. An alternative proof of this lemma can be based on Exercise 3 of Section IV.4 in [3].

Now let us return to the proof of Proposition 2.1.

Let $\mathfrak{J}_k$ be the two-sided ideal of $\mathfrak{A}_k$ associated with the $C^*$-seminorm $\|\cdot\|_k$, i.e.,

$$\mathfrak{J}_k = \{a \in \mathfrak{A}_k \,;\, \|a\|_k = 0\}.$$

Then $\|\cdot\|_k$ is a norm if, and only if, $\mathfrak{J}_k = \{0\}$. But if $\mathfrak{J}_k \neq \{0\}$ there is a non-zero product $a = x^{(1)} \otimes \ldots \otimes x^{(k)} \in \mathfrak{J}_k$ by Lemma 2.3. Then

$$0 = \|x^{(1)} \otimes \ldots \otimes x^{(k)}\|_k = \|x^{(1)}\| \|x^{(2)} \otimes \ldots \otimes x^{(k)}\|_{k-1} = \prod_{i=1}^{k} \|x^{(i)}\|$$

by iteration of the cross-norm property and the identification $\|\cdot\|_1 = \|\cdot\|$. Therefore $a = 0$, which is a contradiction, and hence $\|\cdot\|_k$ is a norm.

This completes the proof of Proposition 2.1.

For applications it is also essential to have continuity properties of product states. In the simplest case, $k = 2$, the continuity follows from the existence of a minimal $C^*$-norm on $\mathfrak{A}_2$ (see, for example, [2], Section 1.22 or [3], Section IV.4). But a similar reasoning is valid in the general case [7].

PROPOSITION 2.4. *Let $\mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n$ be $C^*$-algebras and let $\alpha_0$ be the $C^*$-tensor product norm on $\mathfrak{M}_n = \mathfrak{B}_1 \odot \mathfrak{B}_2 \odot \ldots \odot \mathfrak{B}_n$ defined by*

$$\alpha_0(a) = \sup \|(\pi_1 \otimes \pi_2 \otimes \ldots \otimes \pi_n)(a)\|$$

*for $a \in \mathfrak{M}_n$ where $\pi_i$ runs over all representations of $\mathfrak{B}_i$.*

*It follows that $\alpha_0$ is the minimum $C^*$-norm among all $C^*$-norms $\alpha$ on $\mathfrak{M}_n$.*

The proof of this proposition for the case $n = 2$ can be found in [2], Theorem 1.22.6, or in [3], Theorem 4.19. The general case is covered by Li, [7], Theorem 1.7.

Note that if $\omega_1, \omega_2, \ldots, \omega_n$ are states on $\mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n$, then the product state $\omega_1 \otimes \omega_2 \otimes \ldots \otimes \omega_n$ on $\mathfrak{M}_n$ satisfies

$$|(\omega_1 \otimes \omega_2 \otimes \ldots \otimes \omega_n)(a)| \leqslant \alpha_0(a) \leqslant \alpha(a).$$

The first inequality follows from the definition of $\alpha_0$ and the second is valid for any $C^*$-norm $\alpha$ on $\mathfrak{M}_n$ by the minimality of $\alpha_0$. Thus in the setting of Proposition 2.1 one has

$$|\omega^{\otimes k}(a)| \leqslant \|a\|_k$$

for all states $\omega$ on $\mathfrak{A}$ and all $a \in \mathfrak{A}_k$. This will be used in the next section.

We conclude this section with three remarks on extensions of Proposition 2.1.

REMARKS 1. Simplicity of $\mathfrak{A}$ is not essential in Proposition 2.1; it suffices that $\mathfrak{A}$ is $\tau$-simple. The only change in the proof occurs for Lemma 2.2 which must be replaced by

$$\limsup_{n\to\infty}\|x\tau^n(y)\| = \|x\|\cdot\|y\|.$$

This can be proved along the previous lines but an alternative proof is as follows. Let $\omega$ be an extremal $\tau$-invariant state, choose $b^{(1)}, b^{(2)}, \ldots, b^{(k)}$, such that $\omega(b^{(i)*}b^{(i)}) \neq 0$, and set $b = b^{(1)} \otimes b^{(2)} \otimes \ldots \otimes b^{(k)}$. Then

$$(*) \qquad |\omega((T_n b)^*(T_n a)(T_n b))| \leqslant \|T_n a\|\omega((T_n b)^*(T_n b))$$

for all $a \in \mathfrak{A}_k$. But since $\omega$ is extremal $\tau$-invariant, and $(\mathfrak{A}, \tau)$ is asymptotic abelian, the mean value of

$$(**) \qquad n \in \mathbf{Z}^k \mapsto \omega((T_n b)^*(T_n a)(T_n b))$$

exists and is given by $(\varphi^{(1)} \otimes \varphi^{(2)} \otimes \ldots \otimes \varphi^{(n)})(a)$ where $\varphi^{(i)}$ is the linear functional over $\mathfrak{A}$ defined by $\varphi^{(i)}(x) = \omega(b^{(i)*}xb^{(i)})$. Therefore it follows from $(*)$ that

$$|(\omega^{(1)} \otimes \omega^{(2)} \otimes \ldots \otimes \omega^{(k)})(a)| \leqslant \limsup_{|n|\to\infty}\|T_n a\|$$

where $\omega^{(i)}$ denotes the state over $\mathfrak{A}$ defined by $\omega^{(i)}(x) = \varphi^{(i)}(x)/\varphi^{(i)}(\mathbf{1})$.

Now consider the case $k = 2$ with $a = x^*x \otimes y^*y$. One has

$$\omega^{(1)}(x^*x)\omega^{(2)}(y^*y) \leqslant \limsup_{|n|\to\infty}\|x^*x\tau^n(y^*y)\| \leqslant \limsup_{|n|\to\infty}\|x\tau^n(y)\|^2 \leqslant \|x\|^2\|y\|^2.$$

But if $\mathfrak{A}$ is $\tau$-simple the $b^{(i)}$ can be chosen such that $\omega^{(1)}(x^*x) \simeq \|x\|^2$ and $\omega^{(2)}(y^*y) \simeq \|y\|^2$. Consequently

$$\limsup_{|n|\to\infty}\|x\tau^n(y)\| = \|x\|\cdot\|y\|.$$

Finally remark that by weak $*$-approximation (see, for example, [6], Lemme 11.2.1) the above estimates give an independent proof that products of states $\omega_1, \omega_2, \ldots, \omega_k$

over $\mathfrak{A}$ satisfy the continuity property

$$|(\omega_1 \otimes \omega_2 \otimes \ldots \otimes \omega_k)(a)| \leqslant \|a\|_k .$$

This proof does not involve the minimal $C^*$-norm.

2. If $\mathfrak{A}$ is $\tau$-simple and if in addition there exists a $\tau$-invariant factor state $\omega$ over $\mathfrak{A}$ the foregoing conclusion can be strengthened. In this case the function (**) converges pointwise as $|n| \to \infty$ (see, for example, [4], Example 4.3.24) and one deduces from (*) that

$$|(\omega^{(1)} \otimes \omega^{(2)} \otimes \ldots \otimes \omega^{(k)})(a)| \leqslant \liminf_{|n|\to\infty} \|T_n a\|.$$

Therefore by specializing again to the case $k = 2$ with $a = x \otimes y$ and $x, y \geqslant 0$ one concludes that

$$\|x\| \cdot \|y\| \leqslant \liminf_{|n|\to\infty} \|x\tau^n(y)\| \leqslant \limsup_{|n|\to\infty} \|x\tau^n(y)\| \leqslant \|x\| \cdot \|y\|.$$

This estimate together with the first remark in the proof of Lemma 2.2 allows one to conclude that

$$\lim_{|n|\to\infty} \|x\tau^n(y)\| = \|x\| \cdot \|y\|,$$

i.e., Lemma 2.2 remains valid.

Finally a simple elaboration of the above argument gives the estimate

$$\frac{(\omega^{(1)} \otimes \omega^{(2)} \otimes \ldots \otimes \omega^{(k)})(c^* a^* a c)}{(\omega^{(1)} \otimes \omega^{(2)} \otimes \ldots \otimes \omega^{(k)}(c^* c)}, \leqslant \liminf_{|n|\to\infty} \|T_n a\|^2$$

for all $c \in \mathfrak{A}_k$. Therefore

$$\alpha_k(a^* a) \leqslant \liminf_{|n|\to\infty} \|T_n a\|^2$$

where $\alpha_k$ denotes the minimal $C^*$-norm on $\mathfrak{A}_k$. Note that this estimate does not follow automatically from minimality of $\alpha_k$ because the limit inferior does not necessarily define a norm on $\mathfrak{A}_k$.

3. An alternative version of Proposition 2.1 can be proved assuming $\mathfrak{A}$ is $H$-simple where $H$ is an amenable group of *-automorphisms of which contains $\tau$. Arguing as in Remark 1 above one finds that

$$\sup_{h\in H} \limsup_{|n|\to\infty} \|x\tau^n \circ h(y)\| = \|x\| \cdot \|y\|$$

and then Proposition 2.1 is valid with the norm $\|\cdot\|_k$ on $\mathfrak{A}_k$ defined by

$$\|\sum x_i^{(1)} \otimes \ldots \otimes x_i^{(k)}\|_k = \sup_{h_i \in H} \limsup_{|n| \to \infty} \|\sum \tau^{n_1} \circ h_1(x_i^{(1)}) \ldots \tau^{n_k} \circ h_k(x_i^{(k)})\|.$$

## 3. GENERATION THEOREMS

After these preliminaries on asymptotic abelianness we turn to the discussion of derivations and dissipations. The following theorem concerns the action $\alpha$ of a compact abelian group $G$ as *-automorphisms of the $C^*$-algebra $\mathfrak{A}$. Since $G$ is compact its dual $\hat{G}$ is discrete and we use $\mathfrak{A}^\alpha(\gamma)$ to denote the spectral subspaces of $\alpha$ corresponding to $\gamma \in \hat{G}$, i.e.,

$$\mathfrak{A}^\alpha(\gamma) = \{x\,;\, \alpha_g(x) = (\gamma, g)x,\ g \in G\}.$$

We also use $\mathfrak{A}^\alpha$ to denote the fixed point algebra $\mathfrak{A}^\alpha(0)$.

THEOREM 3.1. *Let $\mathfrak{A}$ be a simple $C^*$-algebra with identity which is asymptotically abelian with respect to a *-automorphism $\tau$ and let $\alpha$ be a continuous action of a compact abelian group $G$ as *-automorphisms of $\mathfrak{A}$ which satisfies $\alpha_g \circ \tau = \tau \circ \alpha_g$, $g \in G$. Further let $D$ be a $\tau$-invariant dense *-subalgebra such that $\mathfrak{A}^\alpha \subset D$ and $D$ is the linear span of the subspaces $D \cap \mathfrak{A}^\alpha(\gamma)$, $\gamma \in \hat{G}$. Finally let $\delta : D \mapsto \mathfrak{A}$ be a linear operator satisfying $\delta(x^*) = \delta(x)^*$ and $\delta \circ \tau(x) = \tau \circ \delta(x)$ for all $x \in D$.*

*The following conditions are equivalent:*

1. *$\delta$ is a derivation,*

2. *$\delta$ is closable and its closure $\bar{\delta}$ generates a $C_0$-group $\beta$ of *-automorphisms of $\mathfrak{A}$.*

*If these conditions are satisfied then there exists a homomorphism $\lambda$ of $\hat{G}$ into* **R** *such that $\delta(x) = \mathrm{i}\lambda(\gamma)x$, for $x \in \mathfrak{A}^\alpha(\gamma)$, and $\beta$ is a one-parameter subgroup of $\alpha_G$.*

*If further $\delta(\mathfrak{A}^\alpha) = \{0\}$ then the following conditions are equivalent:*

1′. *$\delta$ is a dissipation,* i.e., *$\delta(x^*x) \leqslant x^*\delta(x) + \delta(x)^*x$ for all $x \in D$,*

2′. *$\delta$ is closable and its closure $\bar{\delta}$ generates a $C_0$-semigroup $\beta$ of completely positive contractions.*

*If these conditions are satisfied there exists a negative definite function $\varphi$ on $\hat{G}$ such that $\delta(x) = \varphi(\gamma)x$, for $x \in \mathfrak{A}^\alpha(\gamma)$, and a convolution semigroup $\mu_t$ on $C(G)$ such that*

$$\beta_t(x) = \int \mathrm{d}\mu_t(g)\alpha_g(x)$$

*and*

$$\mu_t(f) = \sum_{\gamma \in \hat{G}} e^{-t\varphi(\gamma)} \hat{f}(\gamma)$$

*for all* $f \in C_0(G)$.

*Proof.* First consider the derivation statement. The implication $2 \Rightarrow 1$ is standard and we concentrate on the converse $1 \Rightarrow 2$. Since $D(\delta) \supset \mathfrak{A}^\alpha$ it follows that $\delta | \mathfrak{A}^\alpha$ is bounded (see, for example, [4], Corollary 3.2.23). Next since $\mathfrak{A}^\alpha$ is $\tau$-invariant it follows that

$$\|\delta(\tau^{n_1}(u_1)\tau^{n_2}(u_2) \ldots \tau^{n_k}(u_k))\| \leq \|\delta | \mathfrak{A}^\alpha\|$$

for any unitaries $u_j \in \mathfrak{A}^\alpha$. Therefore in the limit $|n| \to \infty$

$$\left\| \sum_{i=1}^{k} u_1 \otimes \ldots \otimes \delta(u_i) \otimes \ldots \otimes u_k \right\|_k \leq \|\delta | \mathfrak{A}^\alpha\|$$

and hence

$$\left\| \sum_{i=1}^{k} \mathbf{1} \otimes \ldots \otimes u_i^* \delta(u_i) \otimes \ldots \otimes \mathbf{1} \right\|_k \leq \|\delta | \mathfrak{A}^\alpha\|.$$

Therefore setting $u_1 = u_2 = \ldots = u_k = u$ one finds

$$k\,|\omega(u^*\delta(u))| \leq \|\delta | \mathfrak{A}^\alpha\|$$

for any $k$ and any state $\omega$ of $\mathfrak{A}$, by the remark following Proposition 2.4. Consequently $\omega(u^*\delta(u)) = 0$ for any state $\omega$ of $\mathfrak{A}$, which implies $u = 0$. But any element of $\mathfrak{A}^\alpha$ can be written as a linear combination of four unitaries and hence $\delta | \mathfrak{A}^\alpha = 0$.

Now if $x, y \in \mathfrak{A}^\alpha(\gamma) \cap D$ then $x\tau^n(y^*) \in \mathfrak{A}^\alpha$ and

$$\|\delta(x\tau^n(y^*))\| = 0 = \|\delta(x)\tau^n(y^*) + x\tau^n(\delta(y^*))\|.$$

Therefore in the limit $n \to \infty$ one finds

$$(*) \qquad \|\delta(x) \otimes y^* + x \otimes \delta(y^*)\|_2 = 0.$$

It readily follows that

$$\delta(x) = i\lambda(\gamma)x$$

for all $x \in \mathfrak{A}^\alpha(\gamma) \cap D$ where $\lambda(\gamma) \in \mathbf{R}$.

Next for non-zero $x \in \mathfrak{A}^\alpha(\gamma_1) \cap D$ and $y \in \mathfrak{A}^\alpha(\gamma_2) \cap D$ one has $x\tau^n(y) \in \mathfrak{A}^\alpha(\gamma_1 + \gamma_2) \cap D$. But since $\|x\tau^n(y)\| \to \|x\|\|y\|$ as $n \to \infty$ the element $x\tau^n(y)$ is non-zero for large $n$. Therefore $\lambda(\gamma_1 + \gamma_2) = \lambda(\gamma_1) + \lambda(\gamma_2)$ by the derivation property of $\delta$. Thus $\lambda$ is a group homomorphism of $\hat{G}$ into $\mathbf{R}$.

For each $t \in \mathbf{R}$ let $g(t)$ be the function on $\hat{G}$ defined by $g(t)(\gamma) = \exp\{it\lambda(\gamma)\}$ for $\gamma \in \hat{G}$. Then $t \mapsto g(t)$ is a continuous group of characters of $\hat{G}$ and hence it is a continuous one-parameter subgroup of $G$. Let $\beta_t = \alpha_{g(t)}$ and $\delta_\beta$ the generator of $\beta$. It follows that $\delta = \delta_\beta$ on $D$. But $D$ is $\beta$-invariant and hence a core for $\delta_\beta$. Thus $\delta$ is closable and $\bar{\delta} = \delta_\beta$. This completes the proof of the derivation statement.

Now consider the dissipation statement. Since a completely positive semigroup $\beta$ is strongly positive, i.e.,

$$\beta_t(x^*x) \geqslant \beta_t(x)^*\beta_t(x),$$

its generator is a dissipation. Hence $2' \Rightarrow 1'$ and it remains to prove the converse.

$1' \Rightarrow 2'$. Let $x_i \in \mathfrak{A}^\alpha(\gamma) \cap D$ be non-zero. Then $x_1\tau^{n_2}(x_2^*) \in \mathfrak{A}^\alpha$ and $\tau^{n_2}(x_2^*)\tau^{n_3}(x_3) \in \mathfrak{A}^\alpha$ and

$$\delta(x_1\tau^{n_2}(x_2^*)\tau^{n_3}(x_3)) = x_1\tau^{n_2}(x_2^*)\tau^{n_3}(\delta(x_3)) = \delta(x_1)\tau^{n_2}(x_2^*)\tau^{n_3}(x_3)$$

oy Lemma 1.1 of [1]. Therefore, by taking a limit $|n| \to \infty$, one concludes that

$$x_1 \otimes x_2^* \otimes \delta(x_3) = \delta(x_1) \otimes x_2^* \otimes x_3,$$

i.e.,

$$x_1 \otimes \delta(x_3) = \delta(x_1) \otimes x_3.$$

It immediately follows that

$$\delta(x) = \varphi(\gamma)x$$

for all $x \in \mathfrak{A}^\alpha(\gamma) \cap D$ where $\varphi(\gamma) \in \mathbf{C}$.

Next for $x_i \in \mathfrak{A}^\alpha(\gamma_i) \cap D$, $i = 1, 2, \ldots, k$ set $x = \sum_i x_i$ then

$$\delta(x^*)x + x^*\delta(x) - \delta(x^*x) = \sum_{i,j=1}^{k} M_{ij}x_i^*x_j \geqslant 0$$

where $M = \{M_{ij}\}$ is the matrix with coefficients

$$M_{ij} = \overline{\varphi(\gamma_i)} + \varphi(\gamma_j) - \varphi(\gamma_j - \gamma_i).$$

This demonstrates that $\varphi$ satisfies $\mathfrak{A}$-valued inequalities analogous to the $\mathbf{C}$-valued inequalities which characterize negative definite functions. Our next aim is to prove that the former imply the latter and then the existence of the convolution semigroup $\mu$ follows from the standard results of harmonic analysis [5]. Similar arguments have been used in [1] and [10].

LEMMA 3.2. *Adopt the assumptions of Proposition* 3.1. *If* $a_n = \sum \tau^{n_1}(x_i^1)\tau^{n_2}(x_i^2) \ldots \ldots \tau^{n_k}(x_i^k) \geq 0$ *for all (large)* $n = (n_1, n_2, \ldots, n_k)$ *then* $a = \sum x_i^1 \otimes x_i^2 \otimes \ldots \otimes x_i^k \geq 0$.

*Proof.* Define $T_n$ as in the proof of Proposition 2.1. Then

$$T_n(a - a^*) = T_n a - T_n a^* = (T_n a)^* - T_n a^*$$

converges to zero as $|n| \to \infty$. Thus $a = a^*$ and hence $a$ is positive if, and only if,

$$\left\| \mathbf{1} - \frac{a}{\|a\|_k} \right\|_k \leq 1.$$

But since $T_n a$ is positive

$$\left\| \mathbf{1} - \frac{T_n a}{\|T_n a\|} \right\| \leq 1$$

and the desired result follows by taking an appropriate limit.

Now if one replaces $x_i$ by $\tau^{n_i}(x_i)$ in the inequality preceding the lemma one obtains

$$\sum_{i,j=1}^{k} M_{ij} \tau^{n_i}(x_i)^* \tau^{n_j}(x_j) \geq 0.$$

Hence it follows from Lemma 3.2 that

$$(*) \qquad \sum_{i,j=1}^{k} M_{ij} X_i^* X_j \geq 0$$

in $\mathfrak{A}_k$ where $X_i = \mathbf{1} \otimes \mathbf{1} \otimes \ldots \otimes x_i \otimes \ldots \otimes \mathbf{1}$ with the $x_i$ occurring in the $i$-th position.

Now if $y_i \in \mathfrak{A}^\alpha(\gamma_i) \cap D$ is non-zero one can choose states $\omega_i$ over $\mathfrak{A}$ such that $\omega_i(y_i) \neq 0$. Then if $\lambda_i \in \mathbf{C}$ and one sets $x_i = \lambda_i y_i / \omega_i(y_i)$ and evaluates $(*)$ in the state $\omega_1 \otimes \omega_2 \otimes \ldots \otimes \omega_k$ on $\mathfrak{A}_k$ one obtains

$$\sum_{i,j=1}^{k} M_{ij} \bar{\lambda}_i \lambda_j + \sum_{i=1}^{k} M_{ii} |\lambda_i|^2 \left( \frac{\omega_i(y_i^* y_i)}{|\omega_i(y_i)|^2} - 1 \right) \geq 0.$$

Therefore

$$\sum_{i,j=1}^{k} M_{ij} \bar{\lambda}_i \lambda_j \geq 0$$

by application of the subsequent lemma. But this last family of inequalities is equivalent to negative-definiteness of $\varphi$.

LEMMA 3.3. *Let $y$ be a non-zero element of the $C^*$-algebra $\mathfrak{A}$. If $0 < \varepsilon < 1$ there exists a state $\omega$ over $\mathfrak{A}$ such that*

$$0 \leqslant 1 - \frac{|\omega(y)|^2}{\omega(y^*y)} \leqslant \varepsilon.$$

*Proof.* The positivity follows from the Schwarz inequality for any state $\omega$. To obtain the other estimate let $U_y$ denote the set

$$U_y = \{z \in \mathbf{C}\,;\, (\mathbf{1} - zy) \text{ is invertible}\}.$$

Since $(\mathbf{1} - zy)$ is invertible for $|z|$ sufficiently small it follows by a standard argument that $U_y$ is an open subset of $\mathbf{C}$. We first claim that $z \in U_y \mapsto \|(\mathbf{1} - zy)^{-1}\|$ is unbounded. If $U_y = \mathbf{C}$ this is the case since $z \in U_y \mapsto (\mathbf{1} - zy)^{-1}$ is analytic and non-constant. To deduce this when $U_y \in \mathbf{C}$ suppose the contrary and then choose $z_n \in U_y$ converging to a $z \notin U_y$. Thus $(\mathbf{1} - z_n y)$ converges to $(\mathbf{1} - zy)$. But since

$$(\mathbf{1} - z_n y)^{-1} - (\mathbf{1} - z_m y)^{-1} = (\mathbf{1} - z_n y)^{-1}(z_n - z_m)y(\mathbf{1} - z_m y)^{-1}$$

it follows from the boundedness hypothesis that $(\mathbf{1} - z_n y)^{-1}$ also converges to a limit $R_z$ as $n \to \infty$. Then $R_z$ must be the inverse of $(\mathbf{1} - zy)$ which gives the contradiction $Z \in U_y$.

Next choose $z \in U_y$ such that $\|(\mathbf{1} - zy)^{-1}\|^{-2} = \varepsilon$ and a state $\omega$ such that

$$\omega((\mathbf{1} - zy)^*(\mathbf{1} - zy)) = \|(\mathbf{1} - zy)^{-1}\|^{-2}.$$

Then

$$1 - 2\mathrm{Re}\, z\omega(y) + |z|^2\omega(y^*y) = \varepsilon.$$

Now if $\omega(y^*y) = 0$ then $\omega(y) = 0$, by the Schwarz inequality, and one has the contradiction $\varepsilon = 1$. Therefore $\omega(y^*y) \neq 0$ and

$$0 \leqslant 1 - \frac{|\omega(y)|^2}{\omega(y^*y)} \leqslant 1 - \frac{(\mathrm{Re}\, z\omega(y))^2}{|z|^2\omega(y^*y)} \leqslant$$

$$\leqslant 1 - \frac{(\mathrm{Re}\, z\omega(y))^2}{|z|^2\omega(y^*y)} + \left(\frac{\mathrm{Re}\, z\omega(y)}{|z|\omega(y^*y)^{1/2}} - |z|\omega(y^*y)^{1/2}\right)^2 = \varepsilon.$$

Now we return to the proof of Theorem 3.1.

At this point we have established that the dissipation $\delta$ satisfies

$$\delta(x) = \varphi(y)x$$

for all $x \in \mathfrak{A}^\alpha(\gamma) \cap D$ where $\varphi(\gamma)$ is a negative definite function over $\hat{G}$. But since $\delta(\mathfrak{A}^\alpha) = \{0\}$ one also has $\varphi(0) = 0$. Therefore $\varphi$ determines a convolution semigroup $\mu_t$, of probability measures on $G$, such that

$$\mu_t(f) = \sum_{\gamma \in \hat{G}} e^{-t\varphi(\gamma)} \hat{f}(\gamma), \quad f \in C(G),$$

where $\hat{f}$ denotes the Fourier transform of $f$ [5]. Thus one can define a $C_0$-semigroup $\beta$ of contractions on $\mathfrak{A}$ by

$$\beta_t(x) = \int d\mu_t(g) \alpha_g(x)$$

and if $x \in \mathfrak{A}^\alpha(\gamma) \cap D$ one has

$$\beta_t(x) = e^{-t\varphi(\gamma)} x.$$

It follows that $\delta = \delta_\beta$, the generator of $\beta$, on $D$. But $D$ is $\beta$-invariant and hence a core for $\delta_\beta$. Therefore $\delta$ is closable and $\bar{\delta} = \delta_\beta$.

Finally it follows from the definition of $\beta$ in terms of the positive measures $\mu_t$ and the action $\alpha$ that the $\beta_t$ are completely positive, and consequently $\delta$ is a complete dissipation.

There is a result comparable to Theorem 3.1 for $G = \mathbf{R}$ if the action $\alpha$ is almost periodic and one makes an additional spectral assumption on $D(\delta)$.

THEOREM 3.4. *Let $\mathfrak{A}$ be a simple $C^*$-algebra with identity which is asymptotically abelian with respect to a $^*$-automorphism $\tau$ and let $\alpha$ be an almost periodic, but non-periodic, action of $\mathbf{R}$ as $^*$-automorphism of $\mathfrak{A}$ which commutes with $\tau$. Further let $D$ be the linear span of the $\alpha$-eigenelements and $\delta : D \mapsto \mathfrak{A}$ a closable linear operator with closure $\bar{\delta}$ satisfying $\delta(x^*) = \delta(x)^*$ and $\delta \circ \tau(x) = \tau \circ \delta(x)$ for all $x \in D$.*

*The following conditions are equivalent:*

1. *$\delta$ is a derivation and $D(\bar{\delta}) \supseteq \mathfrak{A}^\alpha([-\varepsilon, \varepsilon])$ for some $\varepsilon > 0$,*

2. *there exists a $\lambda \in \mathbf{R}$ such that $\bar{\delta} = \lambda\delta_\alpha$ where $\delta_\alpha$ is the generator of $\alpha$.*

*Moreover the following conditions are equivalent:*

1′. *$\delta$ is a dissipation, $D(\bar{\delta}) \supseteq \mathfrak{A}^\alpha([-\varepsilon, \varepsilon])$ for some $\varepsilon > 0$, and*

$$\lim_{\varepsilon \to 0} \|\bar{\delta}|\mathfrak{A}^\alpha([-\varepsilon, \varepsilon])\| = 0,$$

2′. *there exists a continuous negative definite function $\varphi$ on $\mathbf{R}$ such that $\varphi(0)=0$ and $\delta(x) = \varphi(p)x$ for $x \in \mathfrak{A}^\alpha(\{p\})$.*

*If these conditions are satisfied $\bar{\delta}$ generates a $C_0$-semigroup of completely positive contractions $\beta$ and $D(\bar{\delta}) \supseteq \mathfrak{A}^\alpha(K)$ for any compact subset $K$ of* $\mathbf{R}$.

*Proof.* Let $\Gamma$ be the set of $\alpha$-eigenvalues. Since $\alpha \circ \tau = \tau \circ \alpha$ and $\tau$ is asymptotically abelian $\Gamma$ forms a subgroup of $\mathbf{R}$. Since $\alpha$ is not periodic $\Gamma$ is dense in $\mathbf{R}$. Now let $G$ be the dual group of $\Gamma$, as a discrete group. For each $g \in G$ define a linear map $\alpha_g$ from $D$ into $D$ by

$$\alpha_g(x) = \langle g, p \rangle x, \quad x \in \mathfrak{A}^\alpha(\{p\}).$$

It follows that $\alpha_g$ extends to a $^*$-automorphism of $\mathfrak{A}$, denoted again by $\alpha_g$, and that $g \mapsto \alpha_g$ is a continuous action of $G$. Since $G$ is compact we may now apply Theorem 3.1.

1 $\Rightarrow$ 2. It follows from Theorem 3.1 and Condition 1 that $\delta(x) = i\lambda(p)x$ for $x \in \mathfrak{A}^\alpha(\{p\})$ where $\lambda$ is a homomorphism of $\Gamma$ into $\mathbf{R}$. Since $\lambda$ is linear and bounded on $[-\varepsilon, \varepsilon]$ for some $\varepsilon > 0$ it must be continuous. Thus $\lambda(p) = \lambda p$ for some $\lambda \in \mathbf{R}$ and Condition 2 is satisfied.

2 $\Rightarrow$ 1. This is obvious.

1′ $\Rightarrow$ 2′. The existence of a negative definite function $\varphi$ on $\Gamma$ such that $\delta(x) = \varphi(p)x$ for $x \in \mathfrak{A}^\alpha(\{p\})$ follows from Theorem 3.1. Obviously

$$\|\bar{\delta} \mid \mathfrak{A}^\alpha([-\varepsilon, \varepsilon])\| \geq \sup_{|p| \leq \varepsilon} |\varphi(p)|.$$

Thus Condition 1′ implies that $\varphi$ is continuous at the origin when $\Gamma$ is regarded as a subset of $\mathbf{R}$. It then follows from the negative definiteness that $\varphi$ is in fact locally uniformly continuous on $\Gamma$. (This can be deduced from the continuity properties of positive definite functions and Schoenberg's theorem [5]. A direct proof is given in the Appendix.) Therefore $\varphi$ extends by continuity to a negative definite function on $\mathbf{R}$, also denoted by $\varphi$.

2′ $\Rightarrow$ 1′. It follows from the Levy-Khinchin formula [5], and the assumption that $\varphi(0) = 0$, that $\varphi$ has a representation

$$\varphi(y) = ily + qy^2 + \int_{-\infty}^{\infty} d\mu(x) \left[1 - e^{ixy} - \frac{ixy}{1 + x^2}\right] \frac{1 + x^2}{x^2}$$

where $l \in \mathbf{R}$, $q \in \mathbf{R}_+$, and $\mu$ is a positive finite measure on $\mathbf{R} \diagdown \{0\}$. Therefore one can define an operator $\varphi(\delta_\alpha)$ on $D$ by

$$\varphi(\delta_\alpha) = -l\delta_\alpha + q\delta_\alpha^2 - \int_{-\infty}^{\infty} d\mu(x) \left[1 - e^{x\delta_\alpha} - \frac{x\delta_\alpha}{1 + x^2}\right] \frac{1 + x^2}{x^2}.$$

Then $\delta = \varphi(\delta_\alpha)$ on $D$ and $\varphi(\delta_\alpha)$ generates a semigroup $\beta$ as in Theorem 3.1. Again $\bar{\delta} = \varphi(\delta_\alpha)$. *But if* $\|\delta\|_K$ *denotes the norm of* $\delta_\alpha$ *restricted to* $\mathfrak{A}^\alpha(K)$ *then* by splitting the foregoing integral into two parts, $|x| \leqslant T$ and $|x| \geqslant T$, one estimates that

$$\|\varphi(\delta_\alpha)\,|\,\mathfrak{A}^\alpha(K)\| \leqslant |l|\cdot\|\delta\|_K + q\|\delta\|_K^2 +$$
$$+\mu(\mathbf{R})\{T\|\delta\|_K + (1+T^2)\|\delta\|_K^2 e^{T\|\delta\|_K}\} + \mu(\{|x| \geqslant T\})\{2 + \|\delta\|_K/T\}$$

for any $T > 0$. This establishes that $\mathfrak{A}^\alpha(K) \subseteq D(\bar{\delta})$ and it easily follows that

$$\lim_{\varepsilon\to 0}\|\bar{\delta}|\mathfrak{A}^\alpha([-\varepsilon, \varepsilon])\| = 0.$$

## APPENDIX: NEGATIVE DEFINITE FUNCTIONS

In this appendix we give a direct proof of the continuity properties of negative definite functions used in the proof of Theorem 3.4.

*Let $\Gamma$ be a topological abelian group and $\varphi$ a negative definite function on $\Gamma$. If $\varphi$ is continuous at* 0 *then it is continuous on $\Gamma$. Moreover for any open neighbourhood $U$ of* 0 *in $\Gamma$ for which $\varphi|U$ is bounded $\varphi$ is uniformly continuous on $U + p$ for any $p \in \Gamma$.*

*Proof.* By definition $\varphi(0) \geqslant 0$, $\operatorname{Re}\varphi(x) \geqslant \varphi(0)$, $\varphi(x) = \overline{\varphi(-x)}$, and

$$\begin{pmatrix} \varphi(0) & \varphi(0) & \varphi(0) \\ \varphi(0) & 2\operatorname{Re}\varphi(x) - \varphi(0) & \varphi(x) + \overline{\varphi(y)} - \varphi(x-y) \\ \varphi(0) & \overline{\varphi(x)} + \varphi(y) - \varphi(y-x) & 2\operatorname{Re}\varphi(y) - \varphi(0) \end{pmatrix}$$

is non-negative. Evaluating this matrix with the vector $(c_1, -c_1, c_2)$ one obtains

$$2|c_1|^2(\operatorname{Re}\varphi(x) - \varphi(0)) + |c_2|^2(2\operatorname{Re}\varphi(y) - \varphi(0)) +$$
$$+ 2\operatorname{Re}\bar{c}_1c_2(\varphi(0) - \varphi(x) - \overline{\varphi(y)} + \varphi(x-y)) \geqslant 0.$$

Hence

$$|\varphi(0) - \varphi(x) - \overline{\varphi(y)} + \varphi(x-y)|^2 \leqslant 2(\operatorname{Re}\varphi(x) - \varphi(0))(2\operatorname{Re}\varphi(y) - \varphi(0)).$$

Replacing $x$ by $y - z$ and using the triangle inequality one then finds

$$|\varphi(y) - \varphi(z)| \leqslant 2|\varphi(y-z) - \varphi(0)|^{1/2}(2\operatorname{Re}\varphi(y) - \varphi(0))^{1/2} + |\varphi(y-z) - \varphi(0)|$$

which implies the continuity of $\varphi$ at $y$. Moreover if $\varphi|U$ is bounded then it also follows that $\varphi|U + y$ is bounded and $\varphi$ is uniformly continuous on $U + y$.

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