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HILBERT SPACE OPERATORS WITH ONE DIMENSIONAL SELF-COMMUTATORS

KEVIN F. CLANCEY

The intent is to present a report on the theory of (bounded linear) operators T acting on a separable Hilbert space $\mathscr{H}$ whose self-commutators have the form

$$[T^*, T] = T^*T - TT^* = \varphi \otimes \varphi \tag{0.1}$$

for some φ in $\mathscr{H}$. The notation $\varphi \otimes \varphi$ in (0.1) is the standard notation for the one dimensional operator $\varphi \otimes \varphi(f) = (f, \varphi)\varphi$, where $(\, ,)$ is the inner product of $\mathscr{H}$. The condition (0.1) only characterizes those operators with one dimensional self-commutators satisfying $[X^*, X] \geqslant 0$. Of course, if X has one dimensional self-commutator, then one of X or X^* will satisfy the latter semidefinite condition. By demanding (0.1) we have elected T to satisfy $[T^*, T] \geqslant 0$ and this will lead to surprizing differences in the spatial careers of T and T^*.

As every operator theorist knows "the" unilateral shift satisfies (0.1) so that solitary examples from the class of operators having one dimensional self-commutators have been around for sometime. The first general study of the class appeared in 1962 [32]. The main result in [32] is a singular integral model which can be summarized as follows: Suppose $T = H + \mathrm{i}J$ is the Cartesian form ($H = (1/2)[T + T^*]$; $J = (1/2\mathrm{i})[T - T^*]$) of an operator satisfying (0.1) and H is (unitarily equivalent to) the operator $Hf(t) = tf(t)$ acting on $L^2(a, b)$. Then φ is in $L^\infty(a, b)$ and J has the form

$$Jf(t) = \psi(t)f(t) + \frac{\varphi(t)}{2\mathrm{i}}\int_a^b \frac{\overline{\varphi(s)}f(s)}{t - s}\,\mathrm{d}s \tag{0.2}$$

for some real valued ψ in $L^\infty(a, b)$. Without defining carefully the nature of the singular integral (a Cauchy principal value) the reader can formally verify that with H, J as above $T = H + \mathrm{i}J$ satisfies $[T^*, T] = 2\mathrm{i}[H, J] = \varphi \otimes \varphi$.

Until recent times singular integral models such as (0.2) and Cartesian (or polar) decompositions have been the mainstays of the theory of operators having one dimensional self-commutators. The references [1, 4—10, 13, 15, 24—26, 28, 34] are a representative variety of such techniques. Of course, the Cartesian decomposition is not conducive to a natural analytic functional calculus. One of the goals in this paper is to adopt as much as possible the complex point of view.

To preview the modus operandi we offer the following comments. If T satisfies (0.1), then for every complex z there is a unique solution $T_z^{*-1}\varphi$ of $(T-zI)^* x = \varphi$ orthogonal to the kernel of $(T - zI)^*$. The theory as developed in this report is centered around the weakly continuous $\mathscr{H}$-valued function

$$T_z^{*-1}\varphi, \quad z \in \mathbf{C}. \tag{0.3}$$

This function will be appropriately called the global local resolvent. The existence of the global local resolvent (0.3) is a property which sets T^* apart. The vector φ is rarely in the range of $T - zI$ when z belongs to the spectrum of T.

It is easy to use the global local resolvent to obtain a multiplication operator model for T. In fact, the operator

$$Vf = -\bar{\partial}(f, T_z^{*-1}\varphi) \quad \left(\bar{\partial} = (1/2)\left[\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right]\right)$$

is easily seen to be a continuous map of $\mathscr{H}$ to a space $\widetilde{\mathscr{H}}$ of distributions with compact support and, further,

$$VTf = zVf.$$

In other words, V transplants T to "multiplication by z".

One of the main new consequences of the development here is a proof that T is irreducible if and only if the values $T_z^{*-1}\varphi$, $z \in \mathbf{C}$, of the global local resolvent span $\mathscr{H}$.

The discussion is presented in four sections. Section 1 contains the basics. Of necessity this first section has a somewhat pedantic form and the reader may want to ignore the elementary proofs. In particular, the many equivalent versions of irreducibility are provided by way of contrast to the local resolvent criteria developed in Section 3. Section two studies unitary invariants for the class of irreducible operators having one dimensional self-commutators. The latter part of this section considers connections between the well studied principal function invariant and the global local resolvent which is being featured in our development. Section 3 gives the details of the distributional model referred to above. The completeness of the vectors $T_z^{*-1}\varphi$, $z \in \mathbf{C}$, is also established in Section 3. Section 4 gives a brief treatment of the problem of whether φ satisfying (0.1) is cyclic for either of the irreducible operators T or T^*.

1. BASICS

Throughout this section the notation T is reserved for an operator acting on the Hilbert space $\mathscr{H}$ with $[T^*, T] = \varphi \otimes \varphi$. The main topics to be discussed in this section are the question of irreducibility and the basic properties of the global local resolvent. Certain aspects of the discussion have generalizations to broader classes of non-normal operators (e.g. nearly normal and/or seminormal operators); however, the focus here will be on operators with one dimensional self-commutators. Indeed many of the results described below are special to operators satisfying (0.1).

1.1. CRITERIA FOR IRREDUCIBILITY. The notations $T_z = T - zI$ and $T_z^* = T^* - \bar{z}I$, for $z \in \mathbf{C}$, will be used.

PROPOSITION 1. *For all z, w not in the spectrum $\sigma(T)$ of an operator T satisfying* (0.1)

$$T_w^{*-1}T_z^{-1}\varphi = [1 + (T_z^{-1}\varphi, T_w^{-1}\varphi)]\, T_z^{-1}T_w^{*-1}\varphi \tag{1.1}$$

$$T_z^{-1}T_w^{*-1}\varphi = [1 - (T_w^{*-1}\varphi, T_z^{*-1}\varphi)]\, T_w^{*-1}T_z^{-1}\varphi. \tag{1.2}$$

Proof. For all $z, w \in \mathbf{C}$, $T_w^*T_z - T_zT_w^* = \varphi \otimes \varphi$. Thus for $z, w \notin \sigma(T)$

$$T_w^{*-1}T_z^{-1} - T_z^{-1}T_w^{*-1} = T_z^{-1}T_w^{*-1}\varphi \otimes T_z^{*-1}T_w^{-1}\varphi. \tag{1.3}$$

It follows that

$$T_w^{*-1}T_z^{-1}\varphi = T_z^{-1}T_w^{*-1}\varphi + (\varphi, T_z^{*-1}T_w^{-1}\varphi)\, T_z^{-1}T_w^{*-1}\varphi =$$
$$= [1 + (T_z^{-1}\varphi, T_w^{-1}\varphi)]\, T_z^{-1}T_w^{*-1}\varphi$$

and this is (1.1).

Taking adjoints in (1.3) we obtain

$$T_z^{*-1}T_w^{-1} - T_w^{-1}T_z^{*-1} = T_z^{*-1}T_w^{-1}\varphi \otimes T_z^{-1}T_w^{*-1}\varphi$$

and, consequently,

$$T_w^{-1}T_z^{*-1}\varphi = [1 - (T_z^{*-1}\varphi, T_w^{*-1}\varphi)]\, T_z^{*-1}T_w^{-1}\varphi.$$

Interchanging the roles of z, w in this last identity yields (1.2). The proof of the proposition is complete.

The following notations will be employed:

$$\begin{aligned} \mathscr{T}(z, w) &= 1 + (T_z^{-1}\varphi, T_w^{-1}\varphi) \\ \mathscr{T}_*(z, w) &= 1 - (T_w^{*-1}\varphi, T_z^{*-1}\varphi). \end{aligned} \tag{1.4}$$

From (1.2) and (1.3)

$$\mathscr{T}(z, w) = \mathscr{T}_*^{-1}(z, w), \tag{1.5}$$

in particular, $\mathcal{T}(z, w)$ is invertible. Note also that

$$T_w^* T_z T_w^{*-1} T_z^{-1} = I + \varphi \otimes T_z^{*-1} T_w^{-1} \varphi$$

and, therefore,

$$T_w^* T_z T_w^{*-1} T_z^{-1} \varphi = \mathcal{T}(z, w) \varphi.$$

Thus $\lambda = \mathcal{T}(z, w)$ is the only eigenvalue of $T_w^* T_z T_w^{*-1} T_z^{-1}$ other than $\lambda = 1$. In particular, the formula

$$\det [T_w^* T_z T_w^{*-1} T_z^{-1}] = \mathcal{T}(z, w) \tag{1.6}$$

holds for the determinant.

PROPOSITION 2. *Let T satisfy* (0.1). *The following are equivalent:*

(i) *T is irreducible.*

(ii) *The only subspace reducing T on which T is a normal operator is the zero subspace.*

(iii) *The smallest subspace $\mathcal{H}_1$, reducing T containing φ is $\mathcal{H}$.*

(iv) $\{T_w^{*-1} T_z^{-1} \varphi : |z|, |w| > \|T\|\}$ *spans* $\mathcal{H}$.

(v) $\{T_z^{-1} T_w^{*-1} \varphi : |z|, |w| > \|T\|\}$ *spans* $\mathcal{H}$.

(vi) $\{T^{*j} T^k \varphi : j, k = 0, 1, 2, \dots\}$ *spans* $\mathcal{H}$.

(vii) $\{T^j T^{*k} \varphi : j, k = 0, 1, 2, \dots\}$ *spans* $\mathcal{H}$.

Proof. It is clear that (i) implies (ii). Assume (ii). If $T = T_1 \oplus T_2$, then $[T^*, T] = [T_1^*, T_1] \oplus [T_2^*, T_2]$. In virtue of the fact that $[T^*, T]$ has rank one, then one of T_1 or T_2 must be normal. Since (ii) is being assumed, then this operator acts on the zero subspace. This completes the proof of the equivalence of (i) and (ii).

The closed linear span of the vectors

$$T^{*n_1} T^{m_1} \dots T^{*n_r} T^{m_r} \varphi; \quad n_1, m_1, \dots, n_r, m_r \geqslant 0 \tag{1.7}$$

is the smallest space $\mathcal{H}_1$ reducing T containing φ.

If $f \perp \mathcal{H}_1$, then $T^* T f = T T^* f$. Thus T restricted to $\mathcal{H}_0 = \mathcal{H} \ominus \mathcal{H}_1$ is normal. Hence (ii) implies (iii). On the other hand if $\mathcal{N}$ is a reducing subspace on which $\mathcal{N}$ is normal, then $T^* T f - T T^* f = (f, \varphi)\varphi = 0, f \in \mathcal{N}$. Consequently, $\varphi \perp \mathcal{N}$ and this implies $\mathcal{N} \perp \mathcal{H}_1$. Thus (iii) implies (ii) and the equivalence of (ii) and (iii) has been established.

The equivalence of (iv) and (vi) and the equivalence of (v) and (vi) are trivial.

From (1.1) and (1.2)

$$T_w^{*-1} T_z^{-1} \varphi = \mathcal{T}(z, w) T_z^{-1} T_w^{*-1} \varphi$$

$$T_z^{-1} T_w^{*-1} \varphi = \mathcal{T}_*(z, w) T_w^{*-1} T_z^{-1} \varphi.$$

These identities make it clear that (iv) is equivalent to (v).

If (vi) holds, then $\mathscr{H}_1 = \mathscr{H}$ and, consequently, (iii) holds. There remains to show that (iii) implies (vi). From (1.1) and (1.2) it is not difficult to conclude that

$$T^j T^{*k}\varphi = \sum_{r=1}^{j} \sum_{s=1}^{k} c_{rs} T^{*s} T^r \varphi$$

$$T^{*k} T^j \varphi = \sum_{r=1}^{j} \sum_{s=1}^{k} d_{rs} T^r T^{*s} \varphi$$

for well determined c_{rs}, d_{rs}. Therefore the vector (1.7) can be written in the form

$$\sum_{r=1}^{m_1+\ldots+m_k} \sum_{s=1}^{n_1+\ldots+n_k} b_{rs} T^{*s} T^r \varphi.$$

This shows (iii) implies (vi) and completes the proof of the proposition.

REMARK. An operator T that satisfies (ii) of the above proposition is said to be "completely non-normal" or "pure". The equivalence of (ii) and (iii) for a general seminormal operator is established in [28].

For X a compact set in the plane we will denote by $\mathscr{R}(X)$ the collection of rational functions with poles off X and by $\mathscr{P}(X)$ the subalgebra of $\mathscr{R}(X)$ consisting of (analytic) polynomials. For later reference we indicate that the notation $R(X)$ (respectively, $P(X)$) will be used for the closure of $\mathscr{R}(X)$ (respectively, $\mathscr{P}(X)$) in the uniform algebra $C(X)$ of continuous complex valued functions on X.

For A an operator on $\mathscr{H}$ and f in $\mathscr{H}$ the notation

$$R(A:f) = \bigvee \{A_z^{-1} f : z \notin \sigma(A)\}$$

will be employed for the smallest closed subspace of $\mathscr{H}$ containing f which is invariant under $r(A)$ for all $r \in \mathscr{R}(\sigma(A))$. Similarly,

$$P(A:f) = \bigvee \{A_z^{-1} f : |z| > \|T\|\}$$

is the smallest invariant subspace for A containing f. As usual f is called a rationally (or analytically) cyclic vector for A in case $R(A:f) = \mathscr{H}$. Similarly, $P(A:f) = \mathscr{H}$ means f is a cyclic vector for A.

PROPOSITION 3. *Assume T is an irreducible operator satisfying* (0.1). *The following statements hold:*

a) *The vector $T\varphi$ belongs to $R(T^*:\varphi)$ if and only if φ is a rationally cyclic vector for T^*.*

b) *The vector $T^*\varphi$ belongs to $R(T:\varphi)$ if and only if φ is a rationally cyclic vector for T.*

Proof. (a) We have the identity

$$TT_z^{*-1}\varphi = T_z^{*-1}T\varphi + (\varphi, T_z^{-1}\varphi)T_z^{*-1}\varphi, \quad z \notin \sigma(T).$$

Thus $T\varphi$ in $R(T^*: \varphi)$ implies $R(T^*: \varphi)$ is T invariant. Since $R(T^*: \varphi)$ is clearly T^* invariant, the irreducibility of T implies $R(T^* : \varphi) = \mathscr{H}$.

The proof of (b) is similar.

REMARK. In a similar manner one can show φ is a cyclic vector for T (respectively, T^*) if and only if $T^*\varphi$ (respectively, $T\varphi$) is in $P(T : \varphi)$ (respectively, $P(T^* : \varphi)$).

1.2. THE GLOBAL LOCAL RESOLVENT. In spite of the contradictory nature of the title of this subsection it will be shown below that the resolvent T_z^{*-1}, $z \notin \sigma(T)$, when "localized" to the vector φ has a globally defined weakly continuous extension $T_z^{*-1}\varphi$, $z \in \mathbf{C}$.

The initial discussion can just as easily be cast for an arbitrary seminormal operator. Suppose S is an operator on $\mathscr{H}$ that satisfies $D = S^*S - SS^* \geqslant 0$. For every complex z

$$\sqrt{D}\sqrt{D} + S_zS_z^* = S_z^*S_z$$

and, consequently, there are unique contractive operator functions $C = C(z)$, $K = K(z)$ satisfying

$$S_z^*C(z) = \sqrt{D}, \quad S_z^* = K(z)\, S_z$$

(1.8)

$$C^*(z)[\operatorname{Ker} S_z^*] = K(z)\,[\operatorname{Ker} S_z^*] = (0),$$

where $\operatorname{Ker} S_z^*$ denotes the kernel of the operator S_z^*.

For any f in $\mathscr{H}$ it follows from (1.8) that

$$S_z^*C(z)f = \sqrt{D}f$$

and, therefore, given $d = \sqrt{D}f$ in the range of $\sqrt{D}$ there is a solution of $S_z^*x = d$, for every $z \in \mathbf{C}$. Moreover, $C(z)f$ represents the unique solution of $S_z^*x = d$ in $[\operatorname{Ker} S_z^*]^\perp$.

The surprizing existence of the global local resolvent $C(z)f$ for $d = \sqrt{D}f$ in the range of $\sqrt{D}$ is in [29]. The use of the contractive operator function $C = C(z)$ to account for the existence of this local resolvent appears in [30]. Further details on the connections between the operator functions $C = C(z)$ and $K = K(z)$ are in [19].

If the above discussion is specialized to the case of an operator T satisfying (0.1) so that $D = \varphi \otimes \varphi$ ($\sqrt{D} = \varphi \otimes \|\varphi\|^{-1}\varphi$), then

$$C(z) = T_z^{*-1}\varphi \otimes \frac{\varphi}{\|\varphi\|} \tag{1.9}$$

where

$$T_z^{*-1}\varphi, \quad z \in \mathbf{C},$$

denotes the unique solution of $T_z^* x = \varphi$ which is orthogonal to $\operatorname{Ker} T_z^*$.

The remainder of this section contains a discussion of the elementary properties of the global local resolvent $T_z^{*-1}\varphi$, $z \in \mathbf{C}$.

Clearly,

1° $$\|T_z^{*-1}\varphi\| \leqslant 1, \quad z \in \mathbf{C}.$$

Further,

2° $$T_z^{*-1}\varphi \in \mathscr{H}_1, \quad z \in \mathbf{C},$$

where $\mathscr{H}_1$ is the smallest reducing subspace for T containing φ.

In Section 3 we will establish the identity $\mathscr{H}_1 = \bigvee \{T_z^{*-1}\varphi : z \in \mathbf{C}\}$ and, therefore, $\mathscr{H}$ is spanned by $T_z^{*-1}\varphi$, $z \in \mathbf{C}$, whenever T is irreducible. For the present we do not choose to assume this irreducibility.

The space $\operatorname{Ker} T_z^*$ splits

$$\operatorname{Ker} T_z^* = Z_0(z) \oplus Z_1(z)$$

relative to the decomposition $\mathscr{H} = \mathscr{H}_0 \oplus \mathscr{H}_1$, where $Z_i(z) = \operatorname{Ker} T_z^* \cap \mathscr{H}_i$, $i = 0, 1$. On $Z_0(z)$ the operator T_z is the zero operator and $Z_1(z)$ is at most one dimensional. If $Z_1(z) \neq (0)$, then we will denote by e_z the unique element in $Z_1(z)$ that satisfies $(\varphi, e_z) = 1$.

From the identity

$$T_z^* T_z e_z - T_z T_z^* e_z = T_z^* T_z e_z = (e_z, \varphi)\varphi = \varphi$$

we conclude

3° $$T_z^{*-1}\varphi = T_z e_z \quad \text{on } \overline{\pi_0(T_1^*)}$$

where $\pi_0(T_1^*)$ denotes the set of eigenvalues of the operator $T_1^* \equiv T^*|\mathscr{H}_1$ and the bar denotes complex conjugation.

Moreover

$$\|T_z^{*-1}\varphi\|^2 = \|T_z e_z\|^2 = (T_z^* T_z e_z, e_z) = (\varphi, e_z) = 1.$$

Therefore,

4° $$\|T_z^{*-1}\varphi\| = 1 \quad \text{on } \overline{\pi_0(T_1^*)}.$$

The following result was obtained in part in [29]. See also [12, 16].

PROPOSITION 4. *Let T satisfy* (0.1). *The global local resolvent $T_z^{*-1}\varphi$, $z \in \mathbf{C}$, is weakly continuous on $\mathbf{C}$ and strongly continuous off $\{z \in \sigma(T) : \|T_z^{*-1}\varphi\| < 1\}$.*

Proof. Let $\{z_k\}_{k=1}^{\infty}$ in $\mathbf{C}$ have limit z_0. Any weak subsequential limit x_0 of $T_{z_k}^{*-1}\varphi$ satisfies $T_{z_0}^{*}x = \varphi$. Thus $T_z^{*-1}\varphi$ is clearly weakly continuous off $\overline{\pi_0(T_1^*)}$. Further, if $z_0 \in \overline{\pi_0(T_1^*)}$, then by 4°, $\|T_{z_0}^{*-1}\varphi\| = 1$ and $T_{z_0}^{*-1}\varphi$ is the solution of $T_{z_0}^{*}x = \varphi$ of minimum norm. Since any such x_0 satisfies $\|x_0\| \leqslant 1$, then $x_0 = T_{z_0}^{*-1}\varphi$. This shows $T_z^{*-1}\varphi$, $z \in \mathbf{C}$, is weakly continuous on $\mathbf{C}$. It follows from this weak continuity that $\|T_{z_0}^{*-1}\varphi\| \leqslant \underline{\lim}\ \|T_{z_k}^{*-1}\varphi\| \leqslant 1$. Thus $\|T_{z_0}^{*-1}\varphi\| = 1$ implies $\|T_{z_k}^{*-1}\varphi\| \to \|T_{z_0}^{*-1}\varphi\|$ and, therefore, $T_z^{*-1}\varphi$ is strongly continuous off $\{z \in \sigma(T) : \|T_z^{*-1}\varphi\| < 1\}$. The proof of the proposition is complete.

REMARKS. (i) The set $\{z \in \sigma(T) : \|T_z^{*-1}\varphi\| < 1\}$ may be non-empty. In the following section an explicit expression for $\|T_z^{*-1}\varphi\|$ will be given which will make this last statement easy to verify. See also [14].

(ii) The local resolvent $T_z^{-1}\varphi$ behaves in an entirely different manner than $T_z^{*-1}\varphi$. Indeed, there is no solution of $T_z x = \varphi$ on any open set intersecting $\sigma(T)$ [14].

(iii) The local resolvent kernel

$$\mathscr{T}_{*}(z, w) = 1 - (T_w^{*-1}\varphi,\ T_z^{*-1}\varphi)$$

of (1.5) can now be extended to $\mathbf{C}^2$. The kernel $\mathscr{T}_{*}$ is sectionally continuous on $\mathbf{C}^2$ and jointly continuous off

$$\{(z, w) : z \in \sigma(T),\ \|T_z^{*-1}\varphi\| < 1 \quad \text{or} \quad w \in \sigma(T),\ \|T_w^{*-1}\varphi\| < 1\}.$$

Moreover, for w fixed $\mathscr{T}_{*}(w, \cdot)$ is analytic off $\sigma(T)$ and for z fixed $\mathscr{T}_{*}(\cdot, z)$ is coanalytic off $\sigma(T)$.

(iv) A sequence of Hilbert space operators $\{A_n\}_{n=1}^{\infty}$ is said to converge in the strong-* sense to the operator A in case $A_n f \to Af$ and $A_n^* f \to A^* f$ for f in $\mathscr{H}$. If $\{{}_nT\}_{n=1}^{\infty}$ satisfying $[{}_nT^*, {}_nT] = \varphi_n \otimes \varphi_n$ converges in the strong-* sense to an operator T satisfying $[T^*, T] = \varphi \otimes \varphi$, with $\varphi_n \to \varphi$ then the sequence of global resolvents ${}_nT_z^{*-1}\varphi_n$, $z \in \mathbf{C}$ converges weakly to $T_z^{*-1}\varphi$ and, further ${}_nT_z^{*-1}\varphi_n \to T_z^{*-1}\varphi$ off the set $\left\{z : z \in \text{closure}\left[\bigcup_{n=1}^{\infty} \sigma({}_nT)\right],\ \|T_z^{*-1}\varphi\| < 1\right\}$. The proofs of these statements are similar to the proof of Proposition 4. See [12].

2. UNITARY INVARIANTS

This section contains a discussion of unitary invariants for the class of operators with rank one self-commutators. In the first part of the development the unitary invariants are related to the resolvent. Later in the section unitary invariants based on the spectrum are studied.

2.1. INVARIANTS FROM RESOLVENTS. It is not difficult to obtain scalar valued unitary invariants for operators with one dimensional self-commutators.

THEOREM 1. *Let T, S be irreducible operators on Hilbert spaces $\mathscr{H}$, $\mathscr{K}$, respectively, such that $[T^*, T] = \varphi \otimes \varphi$ and $[S^*, S] = \psi \otimes \psi$. The operators T and S are unitarily equivalent if and only if the local resolvent kernels*

$$\mathscr{T}(z, w) = 1 + (T_z^{-1}\varphi,\ T_w^{-1}\varphi); \quad \mathscr{S}(z, w) = 1 + (S_z^{-1}\psi,\ S_w^{-1}\psi)$$

agree in a neighborhood of ∞ in $\mathbf{C}^2$.

In fact, when $\mathscr{T}(z, w) = \mathscr{S}(z, w)$ agree for $\min[|z|, |w|] > M$ *the mapping U defined by*

$$U \sum a_k T_{z_k}^{*-1} T_{w_k}^{-1} \varphi = \sum a_k S_{z_k}^{*-1} S_{w_k}^{-1} \varphi \tag{2.1}$$

defined on the dense collection of linear combinations of $\{T_z^{*-1} T_w^{-1} \varphi : \min[|z|, |w|] > M\}$ *extends to a unitary operator $U: \mathscr{H} \to \mathscr{K}$ which satisfies $UT = SU$.*

The heart of the proof of this theorem is the following:

LEMMA 1. *For z, w, u, v distinct complex numbers not in $\sigma(S)$*

$$\begin{aligned}(S_z^{*-1} S_w^{-1}\psi,\ S_u^{*-1} S_v^{-1}\psi) &= \\ &= \frac{S(z, w)}{(u - w)\overline{(z - v)}} \left[\frac{S(u, z) - S(u, v)}{S(z, u)} - \frac{S(w, z) - S(w, v)}{S(z, w)} \right].\end{aligned} \tag{2.2}$$

Proof. By repeated use of (1.1)

$$(S_z^{*-1} S_w^{-1}\psi, S_u^{*-1} S_v^{-1}\psi) = (S_v^{*-1} S_u^{-1} S_z^{*-1} S_w^{-1}\psi, \psi) = S(z, w)(S_v^{*-1} S_u^{-1} S_w^{-1} S_z^{*-1}\psi, \psi) =$$

$$= \frac{S(z, w)}{u - w} [(S_v^{*-1} S_u^{-1} S_z^{*-1}\psi,\ \psi) - (S_v^{*-1} S_w^{-1} S_z^{*-1}\psi, \psi)] =$$

$$= \frac{S(z, w)}{u - w} \left[\frac{(S_v^{*-1} S_z^{*-1} S_u^{-1}\psi, \psi)}{S(z, u)} - \frac{(S_v^{*-1} S_z^{*-1} S_w^{-1}\psi, \psi)}{S(z, w)} \right] =$$

$$= \frac{S(z, w)}{u - w} \left[\frac{(S_z^{*-1} S_u^{-1}\psi, \psi) - (S_v^{*-1} S_u^{-1}\psi, \psi)}{S(z, u)\, \overline{z - v}} - \frac{(S_z^{*-1} S_w^{-1}\psi, \psi) - (S_v^{*-1} S_w^{-1}\psi, \psi)}{S(z, w)\, \overline{z - v}} \right].$$

This yields (2.2) and completes the proof of the lemma.

Proof of Theorem 1. If $\mathscr{T}(z, w)$ and $\mathscr{S}(z, w)$ agree in a neighborhood of ∞ in $\mathbf{C}^2$, then by Lemma 1 for all pairs (z, w), (u, v) in this neighborhood (the case of non-distinct z, w, u, v follows by continuity)

$$(T_z^{*-1} T_w^{-1}\varphi, T_u^{*-1} T_v^{-1}\varphi) = (S_z^{*-1} S_w^{-1}\psi,\ S_u^{*-1} S_v^{-1}\psi).$$

This makes it clear that the operator U defined by (2.1) is isometric. The rest of the proof is straightforward.

REMARK. For $|z|$, $|w|$ large

$$(T_z^{-1}\varphi,\ T_{\bar{w}}^{-1}\varphi) = \sum_{i,j=0}^{\infty} \frac{(T^i\varphi,\ T^j\varphi)}{z^{i+1}\bar{w}^{j+1}}$$

and

$$(T_z^{-1}\varphi,\ T_z^{-1}\varphi) = \sum_{i,j=0}^{\infty} \frac{(T^i\varphi,\ T^j\varphi)}{z^{i+1}\bar{z}^{j+1}}.$$

Thus the values $\|T_z^{-1}\varphi\|^2$ in a neighborhood of ∞ determine $\mathscr{T}(z, w)$ for large $|z|$, $|w|$. In particular in Theorem 1 the equality $\|T_z^{-1}\varphi\| = \|S_z^{-1}\psi\|$ for $|z|$ large implies $\mathscr{S}(z, w) = \mathscr{T}(z, w)$ for large $|z|$, $|w|$.

The above remark combined with Theorem 1 yields the following:

COROLLARY 1. *Assume T, S are irreducible operators on the Hilbert spaces $\mathscr{H}$, $\mathscr{K}$, respectively, such that $[T^*, T] = \varphi \otimes \varphi$ and $[S^*, S] = \psi \otimes \psi$. The operators S, T are unitarily equivalent if and only if*

$$\|T_z^{-1}\varphi\| = \|S_z^{-1}\psi\|$$

in a neighborhood of ∞.

REMARKS. (i) Theorem 1 remains true if $\mathscr{T}(z, w)$ and $\mathscr{S}(z, w)$ are replaced by $\mathscr{T}_*(z, w)$ and $\mathscr{S}_*(z, w)$. Similar replacements can be made in Corollary 1. Thus S, T are unitarily equivalent if and only if $\|T_z^{*-1}\varphi\| = \|S_z^{*-1}\psi\|$ in a neighborhood of ∞.

(ii) It is possible to give a different proof that $\|T_z^{*-1}\varphi\| = \|S_z^{*-1}\psi\|$ for large $|z|$ implies S and T are unitarily equivalent. This proof appears in [19] and is based on the "curvature" invariant for hermitian holomorphic line bundles [20]. Actually, the argument that $\|T_z^{*-1}\varphi\|^2$ determines $\mathscr{T}_*(z, w)$ in the remark preceding Corollary 1 is similar to methods in [20].

2.2. INVARIANTS ON THE SPECTRUM. If A is an operator on the Hilbert space $\mathscr{H}$ with $[A^*, A]$ trace class, then there is a compactly supported real valued integrable function $g = g_A$ which for all polynomials $p = p(z, \bar{z})$, $q = q(z, \bar{z})$ provides the representation

$$(p, q) = \mathrm{tr}[p(A, A^*),\ q(A, A^*)] =$$

$$= \frac{1}{\pi}\int_{\mathbf{C}} \{\bar{\partial} p \partial q - \partial p \bar{\partial} q\} g_A \, da, \tag{2.3}$$

where a denotes planar Lebesgue measure and "tr" denotes the trace functional. It should be noted that the substitution of the non-commuting variables A, A^* for z, $\bar{z}$, respectively, in p, q can be carried out in any order without affecting the trace of the commutator $[p(A, A^*), q(A, A^*)]$. Thus (2.3) defines a bilinear form on the algebra $\mathbf{C}[z, \bar{z}]$ in the two commuting indeterminants z, $\bar{z}$.

The principal function $g = g_A$ in the representation (2.3) has a significant history. This invariant appears first in [24] where it was used as a method of diagonalizing self-adjoint operators. In particular, the self-adjoint singular integral operator J in (0.2) can be diagonalized in terms of the g of $T = H + iJ$ [24]. The principal function for operators with trace class self-commutator was developed in [24, 26]. The approach to g_A in [24, 26] was through the determinant. The tracial bilinear form of (2.3) was first investigated in [22]. These authors initially represented this form

$$(p, q) = \frac{1}{\pi}\int \{\bar{\partial} p \partial q - \partial p \bar{\partial} q\}\, \mathrm{d}\mu_A,$$

where μ_A is a finite compactly supported signed measure. The measure μ_A was shown to have the form $\mathrm{d}\mu_A = g_A \mathrm{d}a$ in [25].

The determinant form of (2.3) reads

$$\det[e^P e^Q e^{-P} e^{-Q}] = \exp \frac{1}{\pi}\int \{\bar{\partial} p \partial q - \partial p \bar{\partial} q\}\, g_A\, \mathrm{d}a, \tag{2.4}$$

where we have used the notations $P = p(A, A^*)$, $Q = q(A, A^*\}$. Both of the identities (2.3) and (2.4) extend to sufficently smooth functions of z, $\bar{z}$. In particular, these identities remain valid for functions p, q which are analytic or coanalytic on $\sigma(A)$, [8, 22]. See also [3].

The principal function is a sufficiently fine invariant to capture many spectral properties of A. See [8], [22]. The function g_A has also been used to study cyclic vectors for A. See [2], [4] and [16].

In case T is an operator satisfying (0.1) then the principal function satisfies $0 \leqslant g_T \leqslant 1$. In fact, the principal function is a complete unitary invariant for irreducible operators having one dimensional self-commutator. In detail, two irreducible operators T, S on Hilbert spaces $\mathscr{H}$, $\mathscr{K}$, respectively, which satisfy $[T^*, T] = \varphi \otimes \varphi$, $[S^*, S] = \psi \otimes \psi$ are unitarily equivalent if and only if $g_T = g_S$ [24]. Moreover, to every compactly supported g satisfying $0 \leqslant g \leqslant 1$, there corresponds a unique (up to unitary equivalence) irreducible T with one dimensional self-commutator such that $[T^*, T] \geqslant 0$ with $g = g_T$ [7].

It should be noted that many of the earlier developments connected with the principal function were made in terms of Cartesian (or polar) decomposition of T. In particular, the model of [32] is a Cartesian model. Recently in [27, 33, 34] an analytic model for T based on the principal function has been developed. We will say a bit further about this analytic model in Section 3.

The immediate goal is to point out connections between g_T and the global local resolvent $T_\lambda^{*-1}\varphi$. These connections involve the Cauchy transform. Suppose ν is a compactly supported measure on $\mathbf{C}$. The locally integrable function $\hat{\nu}$ defined by

$$\hat{\nu}(z) = \frac{1}{\pi}\int_{\mathbf{C}} \frac{d\nu(\xi)}{\xi - z}$$

is called the Cauchy transform of the measure ν. The Cauchy transform is a basic tool in the study of approximation by rational functions. See, for example, [21] and [23].

If φ is a compactly supported integrable (with respect to planar Lebesgue measure) function, then it is customary to write $\hat{\varphi}$ for $\widehat{\varphi da}$. The Cauchy transform has a natural extension to any distribution with compact support. Since we will have later use for this "generalized" Cauchy transform, we will recall its definition. Let $\mathscr{D}$ be the space of test functions on $\mathbf{C}$, $\mathscr{D}'$ the dual space of distributions on $\mathbf{C}$ and $\mathscr{E}'$ the space of distributions with compact support. For $\nu \in \mathscr{E}'$ one defines $\hat{\nu}$ in $\mathscr{D}'$ by

$$\hat{\nu}(\varphi) = -\nu(\hat{\varphi}).$$

The map $\nu \to \hat{\nu}$ is a continuous map of $\mathscr{E}'$ to $\mathscr{D}'$. Moreover, since for φ in $\mathscr{D}$

$$\bar{\partial}\hat{\varphi} = -\varphi = \widehat{\bar{\partial}\varphi},$$

then in the sense of distributions

$$\bar{\partial}\hat{\nu} = -\hat{\nu}.$$

The injectivity of $\hat{}: \mathscr{E}' \to \mathscr{D}'$ follows from Weyl's lemma which states that the only solutions of $\bar{\partial}\nu = 0$ are entire functions. (See, e.g., [31].)

The following result from [16] describes the Cauchy transform of the principal function.

THEOREM 2. *Let S be an operator such that $D = [S^*, S]$ satisfies $D \geqslant 0$ and $\sqrt{D}$ is trace class. Let $C = C(z)$, $K = K(z)$ be the contractive operator functions satisfying* (1.8). *Then*

$$\operatorname{tr}(\sqrt{D}C^*) = \operatorname{tr}[K, S] = \hat{g}_S. \tag{2.5}$$

In particular, in the sense of distributions

$$g_S = \bar{\partial}\operatorname{tr}[S, K] = -\bar{\partial}\operatorname{tr}(\sqrt{D}C^*). \tag{2.6}$$

We will not present a detailed proof of the above theorem. We offer the following remarks. The operator functions C and K are related by the identity

$$I = C(z)\,C^*(z) + K^*(z)\,K(z) + P(z) \tag{2.7}$$

where $P(z)$ denotes the orthogonal projection onto $\mathrm{Ker}(S_z^*)$. The identity (2.7) is from [19]. Consequently,

$$[K(z), S] = K(z)\,S_z - S_z K(z) =$$

$$= S_z^* - S_z^* K^*(z)K(z) = S_z^* C(z)C^*(z) = \sqrt{D}C^*(z).$$

This gives the first equality in (2.5).

Further, for $z \notin \sigma(S)$, $K(z) = (S-z)^*(S-z)^{-1}$. Thus for $z \notin \sigma(S)$, we may use (2.3) to compute

$$\mathrm{tr}[K(z), S] = \mathrm{tr}[(S-z)^*(S-z)^{-1}, S] =$$

$$= \frac{1}{\pi}\int\limits_{\mathbf{C}} \frac{1}{\xi - z}\, g_S(\xi)\, \mathrm{d}a(\xi).$$

This is the second identity in (2.5) for $z \notin \sigma(S)$.

For the case of an operator T satisfying (0.1), the identity (1.9) allows one to conclude the following:

COROLLARY 2. *Let T be an operator satisfying $[T^*, T] = \varphi \otimes \varphi$. For every rational function r with poles off $\sigma(T)$*

$$(r(T)\varphi, T_z^{*-1}\varphi) = \frac{1}{\pi}\int\limits_{\mathbf{C}} \frac{r(\xi)}{\xi - z}\, g_T(\xi)\, \mathrm{d}a(\xi). \tag{2.8}$$

In particular

$$g_T = -\bar{\partial}(\varphi, T_z^{*-1}\varphi). \tag{2.9}$$

It follows from (2.8) that elements of the form $r(T)\varphi$, $r \in \mathscr{R}(\sigma(T))$, are determined by "testing" against the global local resolvent. In fact, we have

$$-\bar{\partial}(r(T)\varphi, T_z^{*-1}\varphi) = rg.$$

It is our goal to further this point of view of the global local resolvent as a set of test vectors. This is accomplished by first computing $(T_w^{*-1}\varphi, T_z^{*-1}\varphi)$ (see Theorem 3 below). Eventually, this idea will lead to a proof that the span of $T_z^{*-1}\varphi$, $z \in \mathbf{C}$, is $\mathscr{H}_1$ ($= \mathscr{H}$ when T is irreducible).

It is immediate from (1.6) and (2.4) that for $z, w \notin \sigma(T)$

$$\mathscr{T}(z, w) = 1 + (T_z^{-1}\varphi, T_w^{-1}\varphi) = \tag{2.10}$$

$$= \exp\left\{\frac{1}{\pi}\int_{\mathbf{C}} \frac{g_T(\xi)}{(\xi - z)\overline{(\xi - w)}} \, da(\xi)\right\}\cdot$$

Combining this with the identity $\mathscr{T}_*(z, w) = \mathscr{T}^{-1}(z, w)$ one obtains immediately for $z, w \notin \sigma(T)$ the exponential representation

$$\mathscr{T}_*(z, w) = 1 - (T_w^{*-1}\varphi, T_z^{*-1}\varphi) = \exp\left\{-\frac{1}{\pi}\int \frac{g_T(\xi)}{(\xi - z)\overline{(\xi - w)}} \, da(\xi)\right\}\cdot \tag{2.11}$$

The identity (2.11) extends to $\mathbf{C}^2$. This has been done in [12] and, independently, by R. W. Carey and J. D. Pincus. (See the remarks in the introduction to [27].) The precise result is the following:

THEOREM 3. *Let T satisfy* (0.1). *Then for z, w in* $\mathbf{C}$ *the equality* (2.1) *holds. In case $z = w$ and* $\int |\xi - z|^{-2} g_T(\xi) \, da(\xi) = +\infty$, *then the right side of* (2.11) *is taken to be zero.*

The idea behind the proof of Theorem 3 is the following. If the spectrum of T is nowhere dense, then for $z \neq w$ the identity (2.11) can be concluded from the weak continuity of the global local resolvent. For the general case when $z \neq w$ the identity (2.11) is obtained by approximating T in the strong-* sense by a sequence of operators with one dimensional self-commutators whose spectra are nowhere dense. For further details on the proof of Theorem 3 including the interesting and subtle case when $z = w$ we refer to [12]. (See also [14].)

We conclude this section with two consequences of the exponential representation (2.11).

1° The equality

$$(\varphi, T_z^{*-1}\varphi) = \frac{1}{\pi}\int_{\mathbf{C}} \frac{g(\xi)}{\xi - z} \, da(\xi)$$

of Corollary 2 is an immediate consequence of (2.11).

2° For almost every z with respect to the measure $g_T da$ the integral $\int |\xi - z|^{-2} g_T(\xi) \, da(\xi)$ is infinite. Consequently, $\|T_z^{*-1}\varphi\| = 1$ a.e. with respect to $g da$. This fact has been used to develop a "Toeplitz type" model for T in [17].

Further, it is not hard to construct examples where $\int |\xi - z|^{-2} g_T(\xi) \mathrm{d}a(\xi)$ is finite at some $z_0 \in \sigma(T)$. At such a point $\|T_{z_0}^{*-1}\varphi\| < 1$ and the local resolvent $T_z^{*-1}\varphi$ cannot be strongly continuous at $z = z_0$.

3. A DISTRIBUTIONAL MODEL

It was noted in the introduction that it is easy to use the global local resolvent $T_z^{*-1}\varphi$, $z \in \mathbf{C}$, to obtain a multiplication operator model for T on a space of distributions. This model will be studied in greater detail in this section. One of the more interesting consequences of the development is a proof that the irreducibility of T implies the vectors $T_z^{*-1}\varphi$, $z \in \mathbf{C}$ span the Hilbert space.

Throughout this section it will be assumed that the operator T is an irreducible operator on the Hilbert space $\mathscr{H}$ and that $[T^*, T] = \varphi \otimes \varphi$. The notations

$$(3.1) \qquad \mathscr{V}_1 = \bigvee \{T_z^{*-1}\varphi : z \in \mathbf{C}\}; \quad \mathscr{V}_0 = \mathscr{H} \ominus \mathscr{V}_1$$

will be employed. The space $\mathscr{V}_1$ is T^*-invariant and $\mathscr{V}_0$, which will eventually be shown to be zero, is T-invariant.

The following proposition provides in a simple manner a distribution model for T.

PROPOSITION 5. *Let T be an irreducible operator satisfying* (0.1). *The linear operator $V : \mathscr{H} \to \mathscr{E}'$ defined by*

$$(3.2) \qquad Vf = -\bar{\partial}(f, T_z^{*-1}\varphi)$$

is continuous from $\mathscr{H}$ to $\mathscr{E}'$. The kernel of V is the space $\mathscr{V}_0$ of (3.1). *Moreover,*

$$(3.3) \qquad VTf = zVf, \quad f \in \mathscr{H},$$

where zVf denotes the usual multiplication of the distribution Vf by the $\mathscr{C}^\infty$ function $u(z) = z$.

Proof. If $\{f_n\}$ is a sequence in $\mathscr{H}$ with $f_n \to f$, then $(f_n, T_z^{*-1}\varphi)$ converges uniformly to $(f, T_z^{*-1}\varphi)$ on $\mathbf{C}$. Therefore, the continuity of V is clear. The identity

$$(Tf, T_z^{*-1}\varphi) = (f, \varphi) + z(f, T_z^{*-1}\varphi)$$

immediately implies (3.3).

Using Weyl's lemma we learn that $Vf = 0$ implies $(f, T_z^{*-1}\varphi)$ is entire. Since this function vanishes at ∞ it must be zero. In orther words $Vf = 0$ implies $f \perp T_z^{*-1}\varphi$, $z \in \mathbf{C}$. This shows $\operatorname{Ker} V = \mathscr{V}_0$ and completes the proof.

REMARKS. (i) It is clear that for every f the distribution Vf is supported on $\sigma(T)$.

(ii) The map $V:\mathscr{H} \to \mathscr{E}'$ is a weak sequentially continuous map. In fact, if f_n converges weakly to f in $\mathscr{H}$, then $(f_n, T_z^{*-1}\varphi) \to (f, T_z^{*-1}\varphi)$ pointwise boundedly on C. This implies $Vf_n \to Vf$ in $\mathscr{E}'$.

If h is analytic in a neighborhood of the spectrum of the operator T, then directly from (2.8)

$$Vh(T)\varphi = hg_T. \tag{3.4}$$

The next result computes V on $\mathscr{V}_1$.

PROPOSITION 6. *For* $w \in \mathbf{C}$,

$$VT_w^{*-1}\varphi = \left[\frac{1 - \overline{(T_w^{*-1}\varphi, T_z^{*-1}\varphi)}}{z - w}\right] g_T(z) = \frac{\overline{\mathscr{T}_*(z, w)}}{z - w}\, g_T(z). \tag{3.5}$$

Proof. By definition

$$VT_w^{*-1}\varphi = -\bar{\partial}(T_w^{*-1}\varphi, T_z^{*-1}\varphi) =$$

$$= \bar{\partial}[1 - (T_w^{*-1}\varphi, T_z^{*-1}\varphi)] = \bar{\partial} \exp\left\{-\frac{1}{\pi}\int \frac{g_T(\xi)}{(\xi - z)\overline{(\xi - w)}}\, da(\xi)\right\}$$

where the last equality is (2.11). It is straightforward to verify on $\mathbf{C}\setminus\{w\}$

$$\bar{\partial} \exp\left\{-\frac{1}{\pi}\int \frac{g_T(\xi)}{(\xi - z)\overline{(\xi - w)}}\, da(\xi)\right\} = \exp\left\{-\frac{1}{\pi}\int \frac{g_T(\xi)}{(\xi - z)\overline{(\xi - w)}}\, da(\xi)\right\} \frac{g_T(z)}{\overline{z - w}}$$

in the sense of distributions. This shows (3.5) holds on $\mathbf{C}\setminus\{w\}$, in particular, the two sides of (3.5) differ by a distribution supported on $\{w\}$ of order one. Thus

$$VT_w^{*-1}\varphi = \frac{\overline{\mathscr{T}_*(z, w)}}{z - w}\, g_T(z) + \alpha\delta_w + \beta\bar{\partial}\delta_w + \gamma\partial\delta_w, \tag{3.6}$$

where α, β, γ are constants and δ_w the unit point mass measure at w. (See, for example, [31, Theorem 6.25].) The proof will be completed by showing α, β, γ are zero. Using Weyl's lemma one learns that the (distributional) Cauchy transform of both sides of (3.6) agree modulo an entire function. Behavior at infinity implies this entire function must be zero. Therefore, in the sense of distributions

$$(T_w^{*-1}\varphi, T_z^{*-1}\varphi) = \frac{1}{\pi}\int \frac{\overline{\mathscr{T}_*(\xi, w)}}{(\xi - w)(\xi - z)}\, da(\xi) - \frac{\alpha}{\pi(z - w)} - \beta\delta_w + \gamma\widehat{\partial\delta_w}. \tag{3.7}$$

The next step is to multiply (3.7) by the $\mathscr{C}^\infty$ function $u_w(z) = z - w$. Before doing this we note the identity

$$(z-w)\frac{1}{\pi}\int \frac{\mathscr{T}_*(\xi, w)}{(\overline{\xi - w})(\xi - z)} g_T(\xi)\, da(\xi) =$$

$$= -\frac{1}{\pi}\int \frac{\mathscr{T}_*(\xi, w)}{(\overline{\xi - w})} g_T(\xi)\, da(\xi) + \frac{1}{\pi}\int \frac{\xi - w}{\overline{\xi - w}} \frac{\mathscr{T}_*(\xi, w)}{\xi - z} g_T(\xi) da(\xi).$$

This identity, combined with the continuity as a function of z of the last integral, implies

$$\lim_{z\to w}(z-w)\frac{1}{\pi}\int \frac{\mathscr{T}_*(\xi, w)}{(\overline{\xi - w})(\xi - z)} g_T(\xi)\, da(\xi) = 0. \tag{3.8}$$

So that when (3.7) is multiplied by $(z-w)$ we obtain that the distribution $\gamma u_w \widehat{\partial \delta_w}$ is represented by a continuous function. However,

$$u_w \widehat{\partial \delta_w}(u) = \frac{1}{\pi}\int \frac{u(\xi)}{\overline{\xi - w}}\, da(\xi), \quad u \in \mathscr{D},$$

and we conclude $\gamma = 0$. Now if we use (3.8) along the obvious identity $\lim_{z\to w}(z-w)(T_w^{*-1}\varphi, T_z^{*-1}\varphi) = 0$, we learn $\alpha = 0$. Finally, once we know α, γ are zero then $\beta = 0$ follows from (3.7). This completes the proof.

Proposition 6 provides a representation of VT^* on $\mathscr{V}_1$. More precisely, we have the following:

PROPOSITION 7. *For f in $\mathscr{V}_1$*

$$VT^*f = \bar{z}Vf + (f, T_z^{*-1}\varphi)g_T = \bar{z}Vf + \widehat{Vf}g_T. \tag{3.9}$$

Proof. Assume first that $f = T_w^{*-1}\varphi$ for some fixed $w \in \mathbf{C}$. Then

$$VT^*T_w^{*-1}\varphi = V\varphi + \bar{w}VT_w^{*-1}\varphi =$$

$$= g_T + \bar{w}\left[\frac{1 - (T_w^{*-1}\varphi, T_z^{*-1}\varphi)}{\overline{z - w}}\right] g_T = \bar{z}VT_w^{*-1}\varphi + (T_w^{*-1}\varphi, T_z^{*-1}\varphi)\, g_T,$$

where the penultimate equality is from (3.5). This shows (3.9) holds for $f = T_w^{*-1}\varphi$ and by linearity (3.9) continues to hold for finite linear combinations of $\{T_w^{*-1}\varphi : w \in \mathbf{C}\}$. Now if $f_n \to f, f_n \in \mathscr{V}_1$, then $VT^*f_n \to VT^*f$, $\bar{z}Vf_n \to \bar{z}Vf$ and $(f_n, T_z^{*-1}\varphi)g_T \to (f, T_z^{*-1}\varphi)g_T$ in the topology of $\mathscr{E}'$. Thus (3.9) holds on $\mathscr{V}_1$. This completes the proof.

We are now in a position to establish the following:

THEOREM 4. *Let T be an irreducible operator on $\mathscr{H}$ with self-commutator $[T^*, T] = \varphi \otimes \varphi$. The values $T_z^{*-1}\varphi$, $z \in \mathbf{C}$, of the global local resolvent span $\mathscr{H}$.*

Proof. Relative to the decomposition $\mathscr{H} = \mathscr{V}_0 \oplus \mathscr{V}_1$ the operator T has the 2×2-matrix form

$$T = \begin{bmatrix} T_0 & X \\ 0 & T_1 \end{bmatrix}$$

and the self-commutator has the forms

$$[T^*, T] = \begin{bmatrix} [T_0^*, T_0] - XX^* & T_0^*X - XT_1^* \\ X^*T_0 - T_1X^* & [T_1^*, T_1] + X^*X \end{bmatrix} = \tag{3.10}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & \varphi \otimes \varphi \end{bmatrix}.$$

The proof will be complete when it is shown that $[T_1^*, T_1] = \varphi \otimes \varphi$. In fact, this condition combined with (3.10) implies $X = 0$. Since the operator T is assumed irreducible, then $\mathscr{V}_0$ must be zero.

There remains to show $[T_1^*, T_1] = \varphi \otimes \varphi$. To this end let f be in $\mathscr{V}_1$. Then

$$V[T_1^*T_1 f - T_1T_1^*f] = \bar{z}VT_1 f + (T_1 f, T_z^{*-1}\varphi)g - zVT_1^*f =$$

$$= |z|^2Vf + z(f, T_z^{*-1}\varphi)g + (f, \varphi)g - |z|^2Vf - z(f, T_z^{*-1}\varphi)g = V(f, \varphi)\varphi,$$

where we have used (3.3), (3.9), the invariance of $\mathscr{V}_1$ under T^* and the fact that V is zero on $\mathscr{V}_0$. This shows $V[T_1^*, T_1] = V\varphi \otimes \varphi$ and since V is one to one on $\mathscr{V}_1$, we can conclude $[T_1^*, T_1] = \varphi \otimes \varphi$. The proof is complete.

REMARKS. 1° Theorem 4 provides a further condition for irreducibility which can be added to the seven equivalent conditions of Proposition 2. Namely,

(viii) $$T_z^{*-1}\varphi,\ z \in \mathbf{C},\ \text{spans } \mathscr{H}.$$

Of course, (viii) can be expressed in many ways. For example, (viii) is equivalent to the condition that V defined on $\mathscr{H}$ by $Vf = -\bar{\partial}(f, T^{*-1}\varphi)$ is one-to-one.

2° It is easy to conclude from Theorem 4 that when T is irreducible, then $\{T_z^{*-1}\varphi : z \in \sigma(T)\}$ spans the space. Indeed, if there is an f with $(f, T_z^{*-1}\varphi) = 0$, $z \in \sigma(T)$, then by analyticity of $(f, T_z^{*-1}\varphi)$ on $\mathbf{C}\setminus\sigma(T)$, we can conclude $f \perp T_z^{*-1}\varphi$, $z \in \mathbf{C}$. Thus $f = 0$. In fact, when T is irreducible $\{T_{z_n}^{*-1}\varphi : n = 1,2, \ldots\}$ will span $\mathscr{H}$ for any sequence $\{z_n\}_{n=1}^{\infty}$ which is dense in $\sigma(T)$.

3° If T is irreducible, then the image of V in $\mathscr{E}'$ can be made into a Hilbert space $\widetilde{\mathscr{H}}$ in a trivial manner by setting

$$(Vf,\ Vg) = (f, g).$$

The operator V is unitary from $\mathscr{H}$ to $\widetilde{\mathscr{H}}$ and T has the multiplication model $VTf = zVf$ on $\widetilde{\mathscr{H}}$.

4° In [27, 33, 34] an analytic model for operators with rank one self-commutator is presented. We cannot fully explain this model here. One of the nice features of [27] is that given a g with compact support satisfying $0 \leqslant g \leqslant 1$, there is realized an irreducible operator with rank one self-commutator such that $g = g_T$. Moreover, the operator T is given as $Tf(z) = zf(z)$ on a Hilbert space of functions supported on the spectrum of T with inner product described in terms of g. For further details of this explicit model the reader is referred to [27]. It should be noted that the question of realizing the model through V of Proposition 5 as well as the result in Theorem 4 are not considered in [27].

4. THE CYCLIC NATURE OF THE RANGE OF THE SELF-COMMUTATOR

The discussion in this section centers around the problem of deciding when the vector φ satisfying $[T^*, T] = \varphi \otimes \varphi$ is a (rationally) cyclic vector for the irreducible operators T or T^*. The problems are taken up in natural subsections.

4.1. THE CASE FOR T^*. As "the" unilateral shift capably demonstrates the vector φ in the range of the self-commutator of an irreducible operator satisfying (0.1) may fail to be cyclic for T^*. More generally, a result from [11] (see also [18]) shows that if the self-adjoint operator $|T| = (T^*T)^{1/2}$ when restricted to the space $\bigvee\{|T|^j\varphi : j = 0,1,2, \ldots\}$ has a singular component (in its spectral measure), then φ is not cyclic for T^*. In the positive direction we can establish the following:

PROPOSITION 8. *Let T be an irreducible operator satisfying $[T^*, T] = \varphi \otimes \varphi$ such that the spectrum $\sigma(T)$ is nowhere dense. The vector φ is rationally cyclic for T^*.*

Proof. Suppose $(f, T_z^{*-1}\varphi)=0$ for all $z \notin \sigma(T)$ (equivalently $(f, (T^*-\xi)^{-1}\varphi)=0$ for all $\xi \notin \sigma(T^*)$). By continuity and the assumption that $\sigma(T)$ is nowhere dense, we conclude $(f, T_z^{*-1}\varphi) = 0$, $z \in \mathbf{C}$. Theorem 4 implies $f = 0$. This shows $R(T^* : \varphi) = \mathscr{H}$ and completes the proof.

REMARKS 1. The condition $\sigma(T)$ is nowhere dense appearing in the last proposition is not necessary for $R(T^* : \varphi) = \mathscr{H}$. A collection of (singular integral model) examples of operators with $\sigma(T)$ having interior where $R(T^* : \varphi) = \mathscr{H}$ is described in [13]. Another such example is provided by the bilateral weighted shift

$$(B\{f_j\}_{j=-\infty}^{\infty})_k = \begin{cases} (1/2) f_{k+1} & k \leqslant 0 \\ f_{k+1} & k > 0 \end{cases}$$

acting on the space ℓ_2. Here $[B^*, B] = (3/4)e_0 \otimes e_0$ where $e_0 = \{\delta_{j0}\}_{j=-\infty}^{\infty}$ and $\sigma(T) = \{\xi : 1/2 \leqslant |\xi| \leqslant 1\}$. In this case $R(B^* : e_0) = \ell_2$. Note here that $P(B^* : : e_0) \neq \ell_2$.

2. Suppose T is an irreducible operator with $[T^*, T] = \varphi \otimes \varphi$ and such that the planar Lebesgue measure of the essential spectrum $\sigma_e(T)$ is zero. For each z in $\sigma(T)\setminus\sigma_e(T)$ the space $\operatorname{Ker} T_z^*$ is one dimensional. Let e_z be the unique element in $\operatorname{Ker} T_z^*$ satisfying $(\varphi, e_z) = 1$. This notation is consistent with that used in Section 1. In particular, as noted in 3° of Section 1 $T_z^{*-1}\varphi = T_z e_z$. Under the hypothesis that $\sigma_e(T)$ has measure zero

$$\{e_z : z \in \sigma(T)\setminus\sigma_e(T)\}$$

spans $\mathscr{H}$. See [13] and [18]. Further, the vector function $z \to e_z$ is automatically analytic on $\sigma(T)\setminus\sigma_e(T)$. See [12]. These last two facts supply a second proof of Theorem 4 for the case $\sigma_e(T)$ has measure zero. In fact, if $f \perp T_z^{*-1}\varphi$, $z \in \sigma(T)\setminus\sigma_e(T)$, then $0 = \bar\partial(f, T_z^{*-1}\varphi) = \bar\partial(f, T_z e_z) = -(f, e_z)$ on $\sigma(T)\setminus\sigma_e(T)$. Since e_z, $z \in \sigma(T)\setminus\sigma_e(T)$, spans $\mathscr{H}$, then $f = 0$.

3. It has been conjectured that if T is irreducible with $[T^*, T] = \varphi \otimes \varphi$, then T^* has a cyclic vector [18]. This conjecture has been verified in several cases including the case discussed in the preceding remark when $\sigma_e(T)$ has measure zero [13, 18]. The result in Proposition 8 gives further evidence (at least when $\mathbf{C}\setminus\sigma(T)$ is connected) for the truth of this conjecture.

We remark that if T is irreducible with $[T^*, T] = \varphi \otimes \varphi$, then T^* is "2-cyclic". More precisely, there is a ψ in $\mathscr{H}$ such that the smallest T^*-invariant subspace containing φ and ψ is $\mathscr{H}$. The proof of this (unsatisfactory) remark uses Theorem 4 and the techniques of [18]. No further details will be presented here.

4.2. THE CASE FOR T. In this final subsection we will briefly examine cases where φ is "rationally" cyclic for T. Nothing will be said about non-cyclic cases; moreover, the results formulated are not the strongest possible. For further results on the cyclic nature of φ under T we refer to [16, 27].

It is immediate from (2.11) that

$$VT^*\varphi = [\bar z + g_T(z)]g_T(z) \tag{4.1}$$

or, equivalently,

$$(T^*\varphi, T_z^{*-1}\varphi) = \frac{1}{\pi}\int \frac{[\bar\xi + \hat g_T(\xi)]}{\xi - z} g_T(\xi)\, \mathrm{d}a(\xi). \tag{4.2}$$

The identities (4.1) and (4.2) have interesting forms when g_T is a characteristic function χ_G of some Borel set G in $\mathbf{C}$. Let Δ be any open disc centered at $z = 0$

containing $\sigma(T)$ (for example, $\Delta = \{\xi : |\xi| \leqslant 2\|T\|\}$). Then $\hat{\chi}_\Delta(\xi) = -\bar{\xi}$, for $\xi \in \sigma(T)$. Equations (4.1) and (4.2), respectively, can be rewritten in the forms

$$VT^*\varphi = -\hat{\chi}_{\Delta\setminus G} g_T, \tag{4.3}$$

$$(T^*\varphi, T_z^{*-1}\varphi) = -\frac{1}{\pi}\int \frac{\hat{\chi}_{\Delta\setminus G}(\xi)}{\xi - z} g_T(\xi)\, da(\xi). \tag{4.4}$$

The identity (4.3) roughly states that the vector "φ is $R(G)$ cyclic for T". To be more precise: If G is bounded Borel set in $\mathbf{C}$, then $R(G)$ denotes the closure in $C(\bar{G})$ of the functions of the form

$$\hat{u}(z) = \frac{1}{\pi}\int \frac{u(\xi)}{\xi - z}\, da(\xi),$$

where u is a bounded measurable function having compact support such that $u \equiv 0$ on G. The algebra $R(G)$ is the analogue of $R(X)$ for non closed X. It is easily seen that $R(\bar{G}) \subset R(G)$ and that the two possible interpretations of $R(G)$ agree when G is closed. See [21].

When $g_T = \chi_G$, we will set

$$\tilde{\mathscr{R}}(T : \varphi) = \{f \in \mathscr{H} : Vf | G \in R(G) | G\}$$

and $\tilde{R}(T: \varphi)$ to be the closure of $\tilde{\mathscr{R}}(T: \varphi)$. Note $r(T)\varphi$ is in $\tilde{\mathscr{R}}(T : \varphi)$ for all r in $\mathscr{R}(\sigma(T))$. Thus $R(T : \varphi) \subset \tilde{R}(T : \varphi)$.

PROPOSITION 9. *Let T be an irreducible operator satisfying $[T^*, T] = \varphi \otimes \varphi$ and assume the principal function g_T is the characteristic function of the bounded measurable set G. Then φ is $R(G)$-cyclic for T in the sense that $\tilde{R}(T : \varphi) = \mathscr{H}$.*

Proof. The subspace $\tilde{R}(T : \varphi)$ is clearly T-invariant. Equation (4.3) shows $T^*\varphi \in \tilde{R}(T: \varphi)$. Using this fact with (3.9) one can conclude by induction that $T^{*k}\varphi$ is in $\tilde{R}(T : \varphi)$, $k \geqslant 0$. Therefore, $T^j T^{*k}\varphi$ is in $\tilde{R}(T : \varphi)$, $j, k \geqslant 0$ and, by (vii) of Proposition 2, we can conclude $\tilde{R}(T : \varphi) = \mathscr{H}$. This completes the proof.

REMARK. In general Proposition 9 is the best one can hope for in terms of φ being a rationally cyclic vector. There are examples of operators T satisfying $[T^*, T] = \varphi \otimes \varphi$ with g_T a characteristic function; however, $R(T : \varphi) \neq \mathscr{H}$. See the final example of [16].

If g_T is a characteristic function of a closed set with finite perimeter, then we can show $R(T : \varphi) = \mathscr{H}$. Recall that a closed set $X \subset \mathbf{C}$ is said to have finite perimeter in case

$$X = \bigcap_{n=1}^{\infty} X_n, \tag{4.5}$$

where X_n, $n \geqslant 1$ is a sequence of compact sets with $X_{n+1} \subset X_n$ with boundary ∂X_n of X_n a finite disjoint union of simple closed analytic curves, moreover,

$$\sup_n |\partial X_n| < \infty,$$

where $|\partial X_n|$ denotes the length of ∂X_n.

The prototypical example of a set of finite perimeter is

$$X = \bar{\Delta}_1 \setminus \bigcup_{n=1}^{\infty} \Delta_n, \tag{4.6}$$

where Δ_1 is the open unit disc, $\Delta_n = \Delta(z_n : r_n)$ is the open disc centered at z_n of radius r_n. The discs Δ_n are chosen such that

$$\bar{\Delta}_n \cap \bar{\Delta}_m = \varnothing \quad (m \neq n); \quad \sum r_n < +\infty.$$

If the set X in (4.6) has no interior, then X is called a Swiss cheese.

THEOREM 5. *Let T be an irreducible operator with $[T^*, T] = \varphi \otimes \varphi$ such that $g_T = \chi_X$ where X is a closed set of finite perimeter. Then φ is a rationally cyclic vector for T.*

Proof. By Proposition 3 we need only establish $T^*\varphi$ is in $R(T:\varphi)$. Assume X is given by (4.5) and let Δ be a closed disc centered at $z = 0$ which contains X in its interior. Without loss of generality it can be assumed X_{n+1} is in the interior of X_n. Set $\Omega_n = \Delta \setminus X_n$ and u_n to be the analytic function

$$u_n(\xi) = \frac{1}{\pi} \int_{\Omega_n} \frac{1}{\lambda - \xi} \, da(\lambda) = \hat{\chi}_{\Omega_n}(\xi) \tag{4.7}$$

defined in a neighbourhood of $\sigma(T)$. Using (2.8) we obtain

$$(u_n(T)\varphi, T_z^{*-1}\varphi) = \frac{1}{\pi} \int \frac{\hat{\chi}_{\Omega_n}(\xi)}{\xi - z} g_T(\xi) \, da(\xi), \quad z \in \mathbf{C}.$$

Thus any weak limit f of $u_n(T)\varphi$ will satisfy

$$(f, T_z^{*-1}\varphi) = \frac{1}{\pi} \int \frac{\hat{\chi}_{\Delta \setminus X}(\xi)}{\xi - z} g_T(\xi) \, da(\xi), \quad z \in \mathbf{C}.$$

Combining this last identity with (4.4), we obtain

$$(f, T_z^{*-1}\varphi) = (-T^*\varphi, T_z^{*-1}\varphi), \quad z \in \mathbf{C}$$

and since $T_z^{*-1}\varphi$, $z \in \mathbf{C}$, spans $\mathscr{H}$ this leads to $T^*\varphi = f \in R(T:\varphi)$.

There remains only to show that $u_n(T)\varphi$ has a weak limit. This will follow once we show there is an M such that

$$\|u_n(T)\| \leqslant M, \quad n \geqslant 1. \tag{4.8}$$

To this end note

$$u_n(T) = \frac{1}{2\pi i} \int_{\partial X_{n-1}} \frac{u_n(\xi)}{(\xi - T)} \mathrm{d}\xi, \tag{4.9}$$

where ∂X_{n+1} is positively oriented with respect to the interior of X_{n+1}. Substitution of the integral form (4.7) for $u_n(\xi)$ into (4.9) and using Fubini's theorem leads to the identity

$$u_n(T) = -\frac{1}{\pi} \int_{\Omega_n} \frac{1}{\xi - T} \mathrm{d}a(\xi).$$

Applying Green's lemma we have

$$u_n(T) = \frac{1}{2\pi i} \int_{\partial\Omega_n} \bar{\xi}(\xi - T)^{-1} \mathrm{d}\xi,$$

where $\partial\Omega_n = \partial\Delta \cup \partial X_n$ is oriented positively with respect to Ω_n. Since $\frac{1}{2\pi i} \int_{\partial\Omega_n} T^*(\xi - T)^{-1} \mathrm{d}\xi = 0$, we have

$$u_n(T) = \frac{1}{2\pi i} \int_{\partial\Omega_n} K(\xi)\, \mathrm{d}\xi$$

where $K(\xi) = (\xi - T)^*(\xi - T)^{-1}$ is the contractive operator function satisfying $T_z^* = K(\xi)T_\xi$. It is now clear that

$$\|u_n(T)\| \leqslant \frac{|\partial\Delta| + \sup|\partial X_n|}{2\pi}.$$

This is (4.8) and completes the proof.

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KEVIN F. CLANCEY
Department of Mathematics,
University of Georgia,
Athens, Georgia, 30602,
U.S.A.

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