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AN IRREDUCIBLE SUBNORMAL OPERATOR WITH INFINITE MULTIPLICITIES

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1. HYPONORMAL OPERATORS

Let H be a separable infinite dimensional Hilbert space and let the operator T on H with the Cartesian representation $T = A + \mathrm{i}B$ be completely hyponormal. Thus,

$$T^*T - TT^* = D \geqslant 0 \tag{1.1}$$

and T has no normal part. Let $A = \mathrm{Re}(T)$ have the spectral resolution $A = \int t\,\mathrm{d}E_t$.

If β is any Borel set on the real line and if $F(t)$ denotes the linear measure of the vertical cross section $\sigma(T) \cap \{z : \mathrm{Re}(z) = t\}$ of $\sigma(T)$, then

$$\pi\,\|E(\beta)DE(\beta)\| \leqslant \int_\beta F(t)\,\mathrm{d}t. \tag{1.2}$$

Also, if $n(t)$ denotes the spectral multiplicity function of A then

$$\pi\,\mathrm{tr}(E(\beta)DE(\beta)) \leqslant \int_\beta n(t)\,F(t)\,\mathrm{d}t. \tag{1.3}$$

See [7] and [8]. (See also [4], p. 539, for an inequality related to (1.3).) In particular, relation (1.2) implies that

$$\pi\|D\| \leqslant \mathrm{meas}_2\,\sigma(T). \tag{1.4}$$

2. INFINITE MULTIPLICITIES

The following is a variation of (1.4).

THEOREM 1. *If $T = A + \mathrm{i}B$ is completely hyponormal then*

$$\pi \max \sigma_{\mathrm{e}}(D) \leqslant \int_{\beta_\infty} F(t)\,\mathrm{d}t, \tag{2.1}$$

where $\sigma_e(D)$ *is the essential spectrum of* D *and* β_∞ *denotes the set of points where* $A = \mathrm{Re}(T)$ *has infinite multiplicity.*

Proof. If α denotes the set of points where A has finite multiplicity it follows from (1.2) and (1.3) that $E(\alpha)D^{1/2}$, hence also $E(\alpha)DE(\alpha)$, is compact; see [3], also [9]. Choose a sequence $\{x_n\}$ of unit vectors converging weakly to 0 and satisfying $(D^{1/2} - (\max \sigma_e(D))^{1/2})x_n \to 0$. (The arrow will denote strong convergence unless otherwise indicated.) Since $E(\alpha) + E(\beta_\infty) = I$, then

$$\max \sigma_e(D) = \lim \|D^{1/2}x_n\|^2 \leqslant \limsup \|E(\alpha)D^{1/2}x_n\|^2 +$$
$$+ \limsup \|E(\beta_\infty)D^{1/2}x_n\|^2 = \limsup \|E(\beta_\infty)D^{1/2}x_n\|^2 \leqslant \|E(\beta_\infty)D^{1/2}\|^2 \leqslant$$
$$\leqslant \pi^{-1} \int_{\beta_\infty} F(t)\,dt,$$

by (1.2), that is, (2.1).

Clearly, Theorem 1 can be regarded as a refinement of the result ([3]) that D is compact whenever A has only finite multiplicities.

THEOREM 2. *Let* $T = A + iB$ *be completely hyponormal and suppose that*

$$\|D\| \in \sigma_e(D) \tag{2.2}$$

and that equality holds in (1.4), *thus*

$$\sigma\|D\| = \mathrm{meas}_2\,\sigma(T). \tag{2.3}$$

Then $E(\beta_\infty) = I$, *so that* A *has uniformly infinite multiplicity. A similar assertion holds also for* $aA + bB$ *whenever* a, b *are real and* $a^2 + b^2 > 0$.

Proof. As above, let α denote the points where A has finite multiplicity. Then

$$\pi\|D\| = \pi \max \sigma_e(D) \leqslant \int_{\beta_\infty} F(t)dt \leqslant$$
$$\leqslant \int_\alpha F(t)dt + \int_{\beta_\infty} F(t)dt \leqslant \mathrm{meas}_2\,\sigma(T) = \pi\|D\|.$$

Hence, by (1.2),

$$\pi\|E(\alpha)DE(\alpha)\| \leqslant \int_\alpha F(t)dt = 0.$$

However, this implies that $E(\alpha) = 0$, and hence $E(\beta_\infty) = I$. In fact, otherwise, $E(\alpha)TE(\alpha)$ would be normal on $E(\alpha)H$, and, as a consequence, $E(\alpha)H$ would be a reducing space of T for which $T|E(\alpha)H$ is normal (see [7], p. 326), in contradiction to the hypothesis that T is completely hyponormal.

The remainder of the theorem follows from an application of the first part to the completely hyponormal operator $(a^2+b^2)^{1/2} e^{-i\theta} T$, where $\theta = \arg(a+ib)$.

THEOREM 3. *Let T be completely hyponormal and satisfy* (2.2) *and* (2.3). *In addition, suppose that T has a polar factorization*

$$T = U|T|, \quad U \text{ unitary and } |T| = (T^*T)^{1/2}. \tag{2.4}$$

Then U has uniformly infinite multiplicity.

Proof. The proof is similar to that of Theorem 2 but using formulas analogous to (1.2)—(1.4) and involving the spectral resolution of the unitary operator U; see [10], pp. 192, 198—199. Incidentally, it may be noted that the polar factorization (2.4) holds (for T completely hyponormal) if and only if 0 is not in the point spectrum of T^*.

The assertions of Theorems 2 and 3 above suggest the possibility that a completely hyponormal operator T may be reducible whenever (2.2) and (2.3) are satisfied. In fact, if T is any completely hyponormal operator satisfying (2.3) and if $\|D\|$ is in the point spectrum of D with multiplicity m ($1 \leqslant m \leqslant \infty$) then T can be expressed as $T = R \oplus \left(\bigoplus_{k=1}^{m} T_k\right)$, where R may be absent and the operators $\{T_1, T_2, \ldots\}$ are all unitarily equivalent and have self-commutators of rank one; see [9], [11].

However, in view of the next theorem, relations (2.2) and (2.3) are not sufficient to imply the reducibility of a hyponormal operator.

THEOREM 4. *There exists an irreducible hyponormal (and even subnormal) operator T whose self-commutator D of* (1.1) *satisfies both* (2.2) *and* (2.3).

Such an operator T will be defined below by imposing a slight extra condition on an operator defined in [2], pp. 276 ff. (See also the example of [5].)

3. AN IRREDUCIBLE SUBNORMAL OPERATOR

For completeness the definition and properties of the example given in [2] will be recalled below.

Let $\{\nu_1, \nu_2, \ldots\}$ be positive measures on [0,1] satisfying

(3.1)

(a) $$1 \in \operatorname{supp} \nu_k \quad \text{and} \quad \nu_k(\{1\}) = 0,$$

(b) $$\nu_j \perp \nu_k \text{ if } j \neq k,$$

(c) $$\sum_{k=1}^{\infty} \|\nu_k\| < \infty.$$

Let μ_0 denote arc length on ∂D, where D is the open unit disk, and, for $k \geqslant 1$, let $d\mu_k(re^{i\theta}) = (2\pi)^{-1}d\theta d\nu_k(r)$. Define S_k for $k \geqslant 0$ by

$$(S_k f)(z) = zf(z) \quad \text{on } P^2(\mu_k), \tag{3.2}$$

where $P^2(\mu_k)$ denotes the closure of the polynomials in $L^2(\mu_k)$. Thus, S_0 is the simple unilateral shift on H^2 and, for $k \geqslant 0$, S_k is a subnormal weighted shift satisfying $\sigma(S_k) = \text{cl}\, D$, $\sigma_e(S_k) = \partial D$ and $\text{ind}(S_k - z) = -1$ for $|z| < 1$. (See Theorem 8.16 of [2], pp. 159—160.)

Let $0 = \theta_0 < \theta_1 < \theta_2 < \ldots < 2\pi$ and, for $k \geqslant 1$, put $\Delta_k = \{re^{i\theta}: 0 \leqslant r < 1, \theta_{k-1} < \theta < \theta_k\}$. For $f \in H^2$, put $\hat{f} = (f, f|\Delta_1, f|\Delta_2, \ldots)$. Then $M = \{\hat{f} : f \in H^2\}$ is a subspace of $K = \bigoplus_{k=0}^{\infty} L^2(\mu_k)$. Further, if $H = M + \bigoplus_{k=1}^{\infty} P^2(\mu_k) = \{\hat{f} + (0, g_1, g_2, \ldots): f \in H^2, g_k \in P^2(\mu_k)\}$, then H is a subspace of K.

Next, let $N = \bigoplus_{k=0}^{\infty} N_{\mu_k}$, where N_{μ_k} is the normal operator defined for $f \in L^2(\mu_k)$ by $(N_{\mu_k} f)(z) = zf(z)$. Then H is invariant under N and $S = N|H$ is subnormal with N as its minimal normal extension. The operator S is irreducible, its spectrum is the closed unit disk and, if $|z| < 1$, $S - z$ is semi-Fredholm and $\text{ind}(S - z) = -\infty$.

An important inequality ([2], p. 276) which will be needed later is

$$\|f\|^2_{\mu_k} \leqslant \|f\|^2_{\mu_0} \|\nu_k\|, \quad k = 0,1,2, \ldots . \tag{3.3}$$

4. PROOF OF THEOREM 4

Choose $r_1, r_2, \ldots$ so that $0 < r_1 < r_2 < \ldots \to 1$ and let the $\nu_k(r)$ satisfy, in addition to (a), (b) and (c) of (3.1), the restrictions

$$\text{supp}\, \nu_k \subset [r_k, 1], \quad k = 1,2, \ldots . \tag{4.1}$$

Then define the vectors $\{x_1, x_2, \ldots\}$ in H of Section 3 by

$$x_n = c_n(\overbrace{0,0, \ldots, 0}^{n},1,1, \ldots), \quad \text{where } c_n = \left(\sum_{k=n}^{\infty} \|\nu_k\|\right)^{-1/2}. \tag{4.2}$$

A simple calculation shows that each x_n is a unit vector.

It will next be shown that

$$(S^*S - I)x_n \to 0. \tag{4.3}$$

By (4.1),

$$\|Sx_n\|^2 = c_n^2 \sum_{k=n}^{\infty} \int |z|^2 d\mu_k \geqslant c_n^2(1 - r_n^2) \sum_{k=n}^{\infty} \|\nu_k\| = 1 - r_n^2.$$

Since $\|S\| = 1$ and $\|x_n\| = 1$, it is clear that $(1 - S^*S)^{1/2} x_n \to 0$ and hence (4.3).

Next, note that

$$x_n \to 0 \text{ (weakly).} \tag{4.4}$$

For, if $x = (f, f|\Delta_1 + g_1, f|\Delta_2 + g_2, \dots)$ is arbitrary in H, then

$$(x, x_n) = c_n \sum_{k=n}^{\infty} \int (f|\Delta_k + g_k) d\mu_k. \tag{4.5}$$

In view of the Schwarz inequality

$$\left|\int (f|\Delta_k + g_k) d\mu_k\right| \leqslant \left(\int |(f|\Delta_k + g_k)|^2 d\mu_k\right)^{1/2} \left(\int d\mu_k\right)^{1/2}.$$

Relation (4.5) and another application of the Schwarz inequality then yield

$$|(x, x_n)| \leqslant c_n \left(\sum_{k=n}^{\infty} \int |(f|\Delta_k + g_k)|^2 d\mu_k\right)^{1/2} \left(\sum_{k=n}^{\infty} \|\nu_k\|\right)^{1/2} =$$

$$= \left(\sum_{k=n}^{\infty} \int |(f|\Delta_k + g_k)|^2 d\mu_k\right)^{1/2} \to 0,$$

in view of (4.2) and

$$\|x\|^2 = \int |f|^2 d\mu_0 + \sum_{k=1}^{\infty} \int |(f|\Delta_k + g_k)|^2 d\mu_k < \infty. \tag{4.6}$$

This proves (4.4).

Consequently, one also has $S^*x_n \to 0$ (weakly). However, it will be necessary to prove that the S^*x_n converge even strongly to 0, that is,

$$S^*x_n \to 0. \tag{4.7}$$

To see this, let x be an arbitrary vector in H, as above, and let the vectors x_n ($n = 1,2, \dots$) be defined by (4.2). Then

$$(x, S^*x_n) = (Sx, x_n) = \sum_{k=n}^{\infty} \int z(f|\Delta_k + g_k) c_n d\mu_k = c_n \sum_{k=n}^{\infty} \int zf|\Delta_k d\mu_k.$$

By the Schwarz inequality

$$|(x, S^*x_n)| \leqslant c_n \sum_{k=n}^{\infty} \left(\int |(zf|\Delta_k)|^2 d\mu_k\right)^{1/2} \|\nu_k\|^{1/2}. \tag{4.8}$$

Further,

$$\left(\int |(zf|\Delta_k)|^2 d\mu_k\right)^{1/2} \leqslant \left(\int |f|^2 d\mu_k\right)^{1/2} = \|f\|_{\mu_k} \leqslant \|f\|_{\mu_0} \|\nu_k\|^{1/2}$$

(by (3.3)). However, $\|f\|\mu_0 \leqslant \|x\|$ (cf. (4.6)) and hence, by (4.8) and (4.2),

$$|(x, S^*x_n)| \leqslant c_n\|x\| \sum_{k=n}^{\infty} \|v_k\| = \left(\sum_{k=n}^{\infty} \|v_k\|\right)^{1/2} \|x\|,$$

where x is arbitrary. If one chooses $x = S^*x_n$, then $\|S^*x_n\| \leqslant \left(\sum_{k=n}^{\infty} \|v_k\|\right)^{1/2} \to 0$ and (4.7) is proved.

It is now easy to complete the proof of Theorem 4. Recall that the vectors $\{x_n\}$ of (4.2) are unit vectors, converging weakly to 0, and satisfying (4.3) and (4.7). Hence, if $T = S$ then $\sigma(T)$ is the closed unit disk and $D = T^*T - TT^*$ satisfies $(D-1)x_n \to 0$. Thus, $\|D\| = 1 \in \sigma_e(D)$ and both (2.2) and (2.3) are satisfied.

REMARKS. (i) The example $T = S$ of Section 4 does not have a polar factorization (2.4) where U is unitary. However, if $c = \text{const}$ is chosen so that $|c| \geqslant 1$ (and hence 0 is not in the point spectrum of $(S-c)^*$) then $T = S - c$ does have such a factorization (2.4). Since $T^*T - TT^* = S^*S - SS^*$ then, again, relations (2.2) and (2.3) are satisfied and so, by Theorems 2, 3 and 4, the operators $\text{Re}(T)$, $\text{Im}(T)$ (and, in fact, $aA + bB$ as defined in Theorem 2) and U of (2.4) all have uniformly infinite multiplicities, and T is irreducible.

(ii) It was shown in [2], p. 280, that the subnormal operator S of Section 3 is similar to

$$Q = \bigoplus_{k=0}^{\infty} S_k, \tag{5.1}$$

where the S_k are defined by (3.2). In Section 4, the additional restriction (4.1) was imposed on the operators S_k. As a consequence, the resulting Q satisfies

$$Q^*Q = I + K, \tag{5.2}$$

where K is compact. From this, it follows that

$$Q = W + K_1, \quad \text{where } W \text{ is an isometry and } K_1 \text{ is compact.} \tag{5.3}$$

The implication (5.3) from (5.2) is contained in [1], p. 71, but will be included here for completeness. Let $Q = W|Q|$ be the polar factorization of Q, so that $|Q| = (Q^*Q)^{1/2}$ and W is a partial isometry with initial set the closure of the range of $(Q^*Q)^{1/2}$. Since 0 is not in the point spectrum of Q then W is in fact an isometry. But $|Q| - 1 = (Q^*Q - 1)((Q^*Q)^{1/2} + 1)^{-1}$, which is compact, by (5.2), and so $Q = W|Q|$ has the form (5.3).

In particular, then, if S satisfies the conditions of Section 3 together with the condition (4.1) (hence, in particular, $T = S$ satisfies the hypotheses of Theorem 4) then S is the compact perturbation of an operator similar to an isometry (of infinite multiplicity).

It will remain undecided whether there exists a subnormal operator T satisfying the hypotheses of Theorem 4 and which, in addition, is of the form $T =$ isometry $+$ compact.

(iii) Note that if Q of (5.1) (Q completely subnormal and reducible) is expressed in the Cartesian form $Q = A_0 + iB_0$ and if S satisfying (3.1) and (4.1) (S subnormal and irreducible, hence also completely subnormal) is expressed as $S = A_1 + iB_1$, then each of the operators A_0, B_0, A_1 and B_1 has an absolutely continuous spectrum $[-1, 1]$ (cf. [6], p. 41) of uniformly infinite multiplicity (by Theorem 2). In particular, the operators A_0, B_0, A_1 and B_1 are unitarily equivalent to one another.

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