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ON NORMAL EXTENSIONS OF UNBOUNDED OPERATORS. I

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In the theory of unbounded operators an important question is whether the closure of a symmetric operator is selfadjoint (or the operator itself is essentially selfadjoint). One of the methods approaching this problem exploits some classes of C^∞-vectors like bounded, analytic, quasi-analytic ones etc. The canonical theorem in this matter says that a closed symmetric operator is selfadjoint if and only if an appropriate class of its C^∞-vectors is (linearly) dense in the underlying Hilbert space. This means that any proper candidate for a selfadjoint operator must necessarily have (linearly) dense sets of each of these classes. However it depends on circumstances which class may be handier to deal with: some theorems (of general nature, like on tensor products) can be proved easier using the simplest (and the smallest) class of bounded vectors [12], whilst in checking whether particular operators (for instance, differential operators) are essentially selfadjoint broader classes of vectors (analytic, quasi-analytic etc.) can serve better.

The interplay between symmetric and selfadjoint operators has its counterpart in the case of unbounded normal operators: here are formally normal and normal ones. In [12] we have considered this case with the help of bounded vectors emphasizing their relation to bounded operators which fill up the operator in question. The allied problem to essential normality is when a formally normal operator has a normal extension. Here some phenomenon appears: while a symmetric operator always has a selfadjoint extension, in some larger space eventually, a formally normal need not have any at all [3].

Going one step further one can ask about all these unbounded operators which can be extended to normal ones, in other words about subnormal operators. In [12] using bounded vectors we have characterized *a class* of subnormal operators. However (unlike to what is the case for selfadjoint operators) there are closed subnormal operators having no nontrivial bounded vectors but having a rich enough collection of analytic or quasi-analytic ones (cf. the Example). Our purpose here is to examine *such a class* of subnormals.

Developing the idea of Sz.-Nagy [17] in the case of unbounded operators we adopt for our purpose a dilation theory of positive definite forms over $*$-semi-groups (as proposed in [14]). This leads in a natural way to spectral (integral) representations of these forms (Sections 5, 6) as well as to normality question of formally normal operators (Sections 2, 3 and 4); both these topics may be interesting for their own sake. All the questions result in Sections 7, 8 and 9 where one may find conditions (cf. Remark 7) reminding the well known Halmos characterization [4] of bounded subnormals. The complex moment problem, which is a parallel matter, is considered n the Appendix.

NORMALITY OF FORMALLY NORMAL OPERATORS

1. Preliminaries

We want to collect some simple facts about series needed in the sequel.

(a) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of non-negative numbers and let b be a positive number. Then $\sum_{n=1}^{\infty} a_n = +\infty$ if and only if $\sum_{n=1}^{\infty} a_n b^{1/n} = +\infty$.

(b) Let $\{a_n\}_{n=2}^{\infty}$ be a decreasing sequence of non-negative numbers. If a_1 is an arbitrary non-negative number then, taking any $p \geqslant 2$, we have $\sum_{n=1}^{\infty} a_n = +\infty$ if and only if $\sum_{n=1}^{\infty} a_{pn} = +\infty$.

(c) Let $\{b_n\}_{n=1}^{\infty}$ be a sequence of non-negative numbers such that

$$b_n^2 \leqslant b_k b_l \quad \text{for each } k, l \geqslant 1,\ k+l=2n. \tag{1}$$

Then the following conditions are equivalent (with $b_0 = 1$)

$$\sum_{n=1}^{\infty} b_n^{-1/n} = +\infty \tag{2}$$

$$\sum_{n=1}^{\infty} b_{pn}^{-1/pn} = +\infty \tag{3}$$

$$\sum_{n=1}^{\infty} \frac{b_{n-1}}{b_n} = +\infty \tag{4}$$

$\left(\text{here } \frac{a}{0} = +\infty\right)$.

The conditions (2) and (3) are equivalent by (a) and (b). To prove (2) $\Rightarrow$ (4) one can use the Carleman inequality (cf. [6]).

2. A SINGLE FORMALLY NORMAL OPERATOR

Let $\mathscr{D}$ be an inner product space with the inner product $\langle f, g\rangle$, $f, g \in \mathscr{D}$. Denote by $\mathscr{L}(\mathscr{D})$ the set of all linear operators on $\mathscr{D}$. In the sequel $\mathscr{L}^{\#}(\mathscr{D})$ stands for the space of all operators $N \in \mathscr{L}(\mathscr{D})$ for which there exists an operator $N^{\#} \in \mathscr{L}(\mathscr{D})$ such that $\langle Nf, g\rangle = \langle f, N^{\#}g\rangle$, $f, g \in \mathscr{D}$. Such $N^{\#}$ is uniquely determined. The operation $\#$ makes $\mathscr{L}^{\#}(\mathscr{D})$ an involution algebra with the identity operator I as a unit.

We say that $f \in \mathscr{D}$ is a *quasi-analytic* vector of $N \in \mathscr{L}(\mathscr{D})$ if

$$\sum_{n=1}^{\infty} \|N^n f\|^{-1/n} = +\infty.$$

Denote by $Q(N)$ the set of all such f's.

The class of quasi-analytic vectors is our principal tool. However we are going to use occasionally another class of C^{∞}-vectors, i.e. the analytic ones. Though the latter class is rather smaller than that of quasi-analytic vectors, it has nicer properties, e.g. they form a linear subspace. One says that $f \in \mathscr{D}$ is an *analytic* vector for an operator $N \in \mathscr{L}(\mathscr{D})$ if there is a positive number t such that

$$\sum_{n=1}^{\infty} \frac{t^n}{n!} \|N^n f\| < +\infty.$$

This is the same as to say that

$$\varlimsup_{n\to\infty} \sqrt[n]{\frac{\|N^n f\|}{n!}} < +\infty. \tag{5}$$

Denote by $\mathscr{A}(N)$ the set of all analytic vectors of N. As we already have said $\mathscr{A}(N) \subset Q(N)$.

An operator $N \in \mathscr{L}^{\#}(\mathscr{D})$ is said to be *formally normal* if $\|Nf\| = \|N^{\#}f\|$ for $f \in \mathscr{D}$, which is equivalent to $NN^{\#} = N^{\#}N$. For any $N \in \mathscr{L}^{\#}(\mathscr{D})$ put $\operatorname{Re} N = 2^{-1}(N^{\#} + N)$ and $\operatorname{Im} N = i2^{-1}(N^{\#} - N)$. Now we want to relate the sets of quasi-analytic and analytic vectors of a formally normal operator N to those of $\operatorname{Re} N$ and $\operatorname{Im} N$. We have the following:

PROPOSITION 1. *If $N \in \mathscr{L}^{\#}(\mathscr{D})$ is formally normal then*

$$Q(N) \subset Q(\operatorname{Re} N) \cap Q(\operatorname{Im} N),$$

$$\mathscr{A}(N) \subset \mathscr{A}(\operatorname{Re} N) \cap \mathscr{A}(\operatorname{Im} N).$$

Proof. Since $N^{\#}N = NN^{\#}$ we have

$$2^n\|(\operatorname{Re} N)^n f\| = \left\| \sum_{k=0}^{n} \binom{n}{k} N^k N^{\#(n-k)} f \right\| \leqslant$$

$$\leqslant \sum_{k=0}^{n} \binom{n}{k} \|N^k N^{\#(n-k)} f\| = \|N^n f\| \, 2^n.$$

Thus

$$\|(\operatorname{Re} N)^n f\| \leqslant \|N^n f\|, \quad f \in \mathscr{D}.$$

Similarly

$$\|(\operatorname{Im} N)^n f\| \leqslant \|N^n f\|, \quad f \in \mathscr{D}.$$

Both these inequalities imply the conclusion.

One can provide arguments which show that the second inclusion in Proposition 1 is actually an equality.

Let $\mathscr{A} \subset \mathscr{L}^{\#}(\mathscr{D})$. Define the commutant $\mathscr{A}^{\mathrm{C}}$ of $\mathscr{A}$ in the usual algebraic way, i.e. $\mathscr{A}^{\mathrm{C}} = \{T \in \mathscr{L}^{\#}(\mathscr{D}) : TN = NT,\ N \in \mathscr{A}\}$.

PROPOSITION 2. *Let $N \in \mathscr{L}^{\#}(\mathscr{D})$ be a formally normal operator. Then $\{N, N^{\#}\}^{\mathrm{C}}$ is an involution subalgebra of $\mathscr{L}^{\#}(\mathscr{D})$ and for any M in $\{N, N^{\#}\}^{\mathrm{C}}$,*

$$MQ(N) \subset Q(N) \quad \textit{and} \quad M\mathscr{A}(N) \subset \mathscr{A}(N).$$

Proof. Only the second part of the conclusion requires a proof. Notice that

$$\|N^n M f\|^2 = \langle N^{\#n} N^n f,\ M^{\#} M f \rangle \leqslant \|N^{2n} f\| \, \|M^{\#} M f\|.$$

Now using (a) and (c) in the case of quasi-analytic vectors and (5) together with $\overline{\lim\limits_{n\to\infty}} \left(\frac{(2n)!}{(n!)^2} \right)^{1/n} < +\infty$ in the case of analytic vectors, we get the conclusion.

A densely defined operator N in a complex Hilbert space $\mathscr{K}$ is said to be *normal* if $\mathscr{D}(N) = \mathscr{D}(N^*)$ and $\|Nf\| = \|N^* f\|$ for $f \in \mathscr{D}(N)$. It is well-known that a normal operator N has a unique spectral measure E on the complex plane $\mathbf{C}$ which represents N in a usual way:

$$N = \int_{\mathbf{C}} z E(\mathrm{d}z).$$

Two densely defined normal operators in $\mathscr{K}$ *commute* if so do their spectral measures.

Suppose we are given a Hilbert space $\mathscr{K}$ and a linear subspace $\mathscr{D}$ of $\mathscr{K}$ which is dense in $\mathscr{K}$. Consider a densely defined operator N in $\mathscr{K}$ such that its domain $\mathscr{D}(N)$ contains $\mathscr{D}$ and $\mathscr{D}$ *reduces* N. Then $M = N|\mathscr{D} \in \mathscr{L}^{\#}(\mathscr{D})$ because $M^{\#} = N^{*}|\mathscr{D}$. On the other hand, given $M \in \mathscr{L}^{\#}(\mathscr{D})$, one can find a densely defined operator N in $\mathscr{K}$ such that its domain $\mathscr{D}(N)$ contains $\mathscr{D}$, $\mathscr{D}$ reduces N and $M = N|\mathscr{D}$. There are two extreme of such closed operators N, namely the smallest $\overline{M}$ = the closure of M in $\mathscr{K}$ and the largest $(M^{\#})^{*}$.

An operator $N \in \mathscr{L}^{\#}(\mathscr{D})$ is said to be *essentially normal* if its closure $\overline{N}$ in $\mathscr{K}$ is normal. We say that two essentially normal operators $M, N \in \mathscr{L}^{\#}(\mathscr{D})$ *essentially commute* if their closures in $\mathscr{K}$ commute. It is important to notice that both essential normality and essential commutativity do not depend on the choice of a Hilbert space $\mathscr{K}(=\overline{\mathscr{D}})$. Consequently we do not specify $\mathscr{K}$ unless this is necessary. However one should be aware of the fact that essential normality, essential commutativity as well as formal normality, as used in this paper, refer to densely defined Hilbert space operators having reducing dense subspace.

The main result of this section is the following:

THEOREM 1. *Let $N \in \mathscr{L}^{\#}(\mathscr{D})$ be a formally normal operator. Suppose we are given a set $Q \subset Q(N)$ such that the set $\{Tf : T \in \{N, N^{\#}\}^{\mathrm{C}}, f \in Q\}$ is linearly dense in $\mathscr{D}$. Then N is essentially normal.*

Proof. Fix an arbitrary complex Hilbert space $\mathscr{K}$ such that $\mathscr{K} = \overline{\mathscr{D}}$. Taking if necessary the set $\{Tf : T \in \{N, N^{\#})^{\mathrm{C}}, f \in Q\}$ instead of Q, we may assume (due to Proposition 2) that Q is already linearly dense in $\mathscr{K}$. Because $N^{\#}N \subset \overline{N}^{*}\overline{N}$, the operator $T = N^{\#}N$ is symmetric. Take $f \in Q(N)$. Then, by (c) of Section 1

$$\sum_{n=1}^{\infty} \|T^{n}f\|^{-1/2n} = \sum_{n=1}^{\infty} \|N^{2n}f\|^{-1/2n} = +\infty.$$

This means that the set of Stieltjes vectors of T is linearly dense in $\mathscr{K}$. Since T is also a positive symmetric operator, its closure $\overline{T}$ is self-adjoint (cf. [6] or [9]). The equality $N^{\#}N = NN^{\#}$ implies that $\overline{T} \subset \overline{N}^{*}\overline{N}$ and $\overline{T} \subset \overline{N}\overline{N}^{*}$. Since these three operators are self-adjoint, $\overline{T} = \overline{N}^{*}\overline{N} = \overline{N}\overline{N}^{*}$. This means that $\overline{N}$ is a normal operator. Consequently N is essentially normal.

REMARK 1. It is known that if $N \in \mathscr{L}^{\#}(\mathscr{D})$ is a formally normal operator then both $\operatorname{Re} N$ and $\operatorname{Im} N$ are symmetric and commute; on the other hand if N is normal then $\operatorname{Re} N$ and $\operatorname{Im} N$ are essentially selfadjoint and essentially commute. Accordingly, someone might suppose that the quickest way to prove essential normality would be to apply any result on essential self-adjointness of commuting symmetric operator (cf. for instance [11], Corollary 3.4 or [8], Section 3). However this way does not seem to be "easy to see"; the reason is that though $(\mathrm{Re}\,N)^- + \mathrm{i}(\mathrm{Im}\,N)^-$ is a *normal* extension of N, the equality

$$(\mathrm{Re}\,N + \mathrm{i}\,\mathrm{Im}\,N)^- = (\mathrm{Re}\,N)^- + \mathrm{i}(\mathrm{Im}\,N)^- \tag{6}$$

does not follow automatically. Thus the essence of Theorem 1 is, among other things, in proving the equality (6) under our circumstances.

3. Commutative pairs of formally normal operators

The following theorem gives some sufficient conditions for commutativity of two formally normal operators.

THEOREM 2. *Let $M, N \in \mathscr{L}^{\#}(\mathscr{D})$ be two formally normal operators such that $M \in \{N, N^{\#}\}^{\mathrm{C}}$. Suppose we are given a set $Q \subset Q(N) \cap Q(M)$ such that the set $\{Tf : T \in \{N, N^{\#}, M, M^{\#}\}^{\mathrm{C}}, f \in Q\}$ is linearly dense in $\mathscr{D}$. Then the essentially normal operators M and N essentially commute.*

Proof. As usually $\mathscr{K}$ stands for the complex Hilbert space such that $\overline{\mathscr{D}} = \mathscr{K}$. It is clear that the set $Q_0 = \{Tf : T \in \{N, N^{\#}, M, M^{\#}\}^{\mathrm{C}}, f \in Q\}$ is invariant for M, $M^{\#}$, N and $N^{\#}$, and so is the linear subspace $\mathscr{D}_0$ spanned by Q_0. Using Proposition 2 and Theorem 1 we can infer that both $(M|\mathscr{D}_0)^-$ and $(N|\mathscr{D}_0)^-$ are normal operators, and consequently $M^- = (M|\mathscr{D}_0)^-$ and $N^- = (N|\mathscr{D}_0)^-$. Thus we can assume that $\mathscr{D}$ is already linearly spanned by Q, taking if necessary Q_0, $\mathscr{D}_0$, $M|\mathscr{D}_0$ and $N|\mathscr{D}_0$ instead of Q, $\mathscr{D}$, M and N.

Now we show that the spectral measure of $\overline{M}$ is the product of the spectral measures of the self-adjoint operators $(\mathrm{Re}\,M)^-$ and $(\mathrm{Im}\,M)^-$ (the same is true for $\overline{N}$). It is well-known that the spectral measure of $\overline{M}$ is precisely the product of the spectral measures of self-adjoint operators $M_1 = 2^{-1}(\overline{M}^* + \overline{M})^-$ and $M_2 = \mathrm{i}\,2^{-1}(\overline{M}^* - \overline{M})^-$. Since $\mathrm{Re}\,M$ and $\mathrm{Im}\,M$ are symmetric and $Q(\mathrm{Re}\,M) \cap Q(\mathrm{Im}\,M)$ is linearly dense in $\mathscr{D}$, $\mathrm{Re}\,M$ and $\mathrm{Im}\,M$ are essentially self-adjoint ([8], Theorem 2). Because $\mathrm{Re}\,M \subset M_1$ and $\mathrm{Im}\,M \subset M_2$, we obtain $M_1 = (\mathrm{Re}\,M)^-$ and $M_2 = (\mathrm{Im}\,M)^-$.

All what we have said so far will help us to see that M and N essentially commute provided so do any two operators A, B of $\{\mathrm{Re}\,M, \mathrm{Re}\,N, \mathrm{Im}\,M, \mathrm{Im}\,N\}$. If follows from Proposition 1 that $Q \subset Q(A) \cap Q(B)$. Since $AB = BA$, we can easily check that $A(B - \mathrm{i})\mathscr{D} \subset (B - \mathrm{i})\mathscr{D}$ and consequently, by Proposition 2,

$$(B - \mathrm{i})Q \subset Q(A) \cap Q(B) \cap Q(A|(B - \mathrm{i})\mathscr{D}). \tag{7}$$

Moreover, since B is essentially self-adjoint $\overline{(B - \mathrm{i})\mathscr{D}} = \mathscr{K}$. Recalling that $\mathscr{D}$ is linearly spanned by Q, we can say that the left hand side of the inclusion (7) is linearly dense in $\mathscr{K}$ and so is each of the sets on the right hand side. Now we are in the position to apply Theorem 6.9, p. 273 of [2] to get the conclusion of our theorem.

REMARK 2. The proof of Theorem 2 would have become shorter, if we could have applied Theorem 6 of Nussbaum's paper [8]. However there is an essential mistake in his proof, which seems to be overlooked, namely the vectors $\sum_i a_i T^{n_i} S^{m_i} x$, where $x \in \mathscr{D}$, need not compose any linear subspace, consequently they can not form a Nussbaum's space $\tilde{\mathscr{D}}$. Fortunately, one can provide with some additional arguments bridging this gap. Furthermore, when M and N are symmetric operators our Theorem 2 implies Theorem 6 of [8].

One could try to state mutatis mutandis an analogue of Nussbaum's Theorem 6 for formally normal operators without any explicit appeal to reducing subspaces. However it would take much more complicated (and less clear) form than ours and in fact would follow from ours.

POSITIVE DEFINITE FORMS AND THEIR SHIFT OPERATORS

4. THE SHIFT OPERATORS AS FORMALLY NORMAL ONES

Let $\mathscr{E}$ be a complex linear space and let $\mathscr{S}$ be a commutative involution semigroup with the unit 1. Let $\varphi : \mathscr{S}\times\mathscr{E}\times\mathscr{E} \to \mathbf{C}$ be a map such that for every $s \in \mathscr{S}$, $\varphi(s;\ \cdot,\ -)$ is a hermitian bilinear form on $\mathscr{E}$. Following [14] we call φ simply a *form* over $(\mathscr{S}, \mathscr{E})$. Such a form φ is said to be *positive definite* if for all finite sequences $s_1, \ldots, s_n \in \mathscr{S}$ and $f_1, \ldots, f_n \in \mathscr{E}$

$$\sum_{j,k=1}^{n} \varphi(s_j^* s_k\ ; f_k, f_j) \geqslant 0.$$

A form φ over $(\mathscr{S}, \mathscr{E})$ such that for every $f \in \mathscr{E}$, $\varphi(\cdot\,; f, f)$ is positive definite as a form over $(\mathscr{S}, \mathbf{C})$, i.e.

$$\sum_{j,k=1}^{n} \varphi(s_j^* s_k\ ; f, f) a_k \bar{a}_j \geqslant 0$$

for all finite sequences $s_1, \ldots, s_n \in \mathscr{S}$ and $a_1, \ldots, a_n \in \mathbf{C}$ is called *weakly positive definite*.

Given a positive definite form φ, one can construct [14] an inner product space $\mathscr{D}$, a linear map $V : \mathscr{E} \to \mathscr{D}$ and an involution preserving semigroup homomorphism $\Phi : \mathscr{S} \to \mathscr{L}^{\#}(\mathscr{D})$ such that

$$\varphi(s; f, g) = \langle \Phi(s) Vf, Vg \rangle, \quad f, g \in \mathscr{E}, \quad s \in \mathscr{S} \tag{8}$$

$$\mathscr{D} \text{ is linearly spanned by } \Phi(\mathscr{S}) V\mathscr{E}. \tag{9}$$

Standard arguments show that the above objects are uniquely determined up to unitary equivalence (cf. [7]). We call each $\Phi(s)$ the *shift* operator related to φ at $s \in \mathscr{S}$. Since $\Phi(s^*) = \Phi(s)^{\#}$ and $ss^* = s^*s$, the operator $\Phi(s)$ is formally normal.

Now the following questions appear:

1° when the shift operator $\Phi(s)$ is essentially normal;

2° when two shift operators essentially commute;

3° when for every $s \in \mathscr{S}$, $\Phi(s)$ has a normal extension in $\mathscr{K} = \bar{\mathscr{D}}$.

Our aim here is to give an answer to these questions.

Given a weakly positive definite form φ over $(\mathscr{S}, \mathscr{E})$, denote by $Q_\varphi(s)$ the set of all $f \in \mathscr{E}$ such that

$$\sum_{n=1}^{\infty} \varphi((s^*s)^n;\, f, f)^{-1/2n} = +\infty,$$

and denote by $\mathscr{A}_\varphi(s)$ the set of all $f \in \mathscr{E}$ for which there is $t = t(f) > 0$ such that

$$\sum_{n=1}^{\infty} \varphi((s^*s)^n;\, f, f)^{1/2} \frac{t^n}{n!} < +\infty.$$

It follows from (8) that $f \in Q_\varphi(s)$ (resp. $f \in \mathscr{A}_\varphi(s)$) if and only if $Vf \in Q(\Phi(s))$ (resp. $Vf \in \mathscr{A}(\Phi(s))$, provided φ is positive definite. Also $\mathscr{A}_\varphi(s) \subset Q_\varphi(s)$. Thus we can think of elements of $Q_\varphi(s)$ (resp. $\mathscr{A}_\varphi(s)$) as quasi-analytic vectors (resp. analytic vectors) related to the form φ at s.

PROPOSITION 3. *Let φ be a positive definite form over $(\mathscr{S}, \mathscr{E})$. If $\mathscr{E}$ is linearly spanned by the set $Q_\varphi(s)$ for some $s \in \mathscr{S}$, the shift operator $\Phi(s)$ is essentially normal. Moreover if s and t are two different elements of $\mathscr{S}$ and $\mathscr{E}$ is linearly spanned by $Q_\varphi(s) \cap Q_\varphi(t)$, then $\Phi(s)$ and $\Phi(t)$ essentially commute.*

Proof. Since $f \in Q_\varphi(u)$ if and only if $Vf \in Q(\Phi(u))$, the fact that $\Phi(\mathscr{S}) \subset \Phi(\mathscr{S})^c$ and that (9) holds true allows to derive Proposition 3 directly from Theorem 1 and Theorem 2.

Suppose that φ is a positive definite form such that $\varphi(1; f, f) = 0$ implies $f = 0$. Then $V : \mathscr{E} \to V\mathscr{E}$ is invertible. Let $t \in \mathscr{S}$ be such that

(10) the linear manifold $V\mathscr{E}$ is invariant for $\Phi(t)$.

Taking the restriction $\tilde{S}$ of $\Phi(t)$ to $V\mathscr{E}$ and setting $S = V^{-1}\tilde{S}V$ we can easily check that

$$VS = \Phi(t)V. \tag{11}$$

In other words we have

$$\varphi(s;\, Sf, g) = \langle \Phi(s)VSf, Vg\rangle = \langle VSf, \Phi(s^*)Vg\rangle =$$

$$= \langle \Phi(t)Vf, \Phi(s^*)Vg\rangle = \langle \Phi(st)Vf, Vg\rangle = \varphi(st; f, g).$$

Thus

$$\varphi(s;\, Sf, g) = \varphi(st; f, g), \quad s \in S, \quad f, g \in \mathscr{E}. \tag{12}$$

Going back, we can say that t satisfies (10) if and only if there is $S = S_t \in \mathscr{L}(\mathscr{E})$ such that (11) (or equivalently (12)) holds.

For an arbitrary (not necessarily positive definite) form define $\mathscr{S}_\varphi = \{t \in \mathscr{S} :$: there exists $S = S_t \in \mathscr{L}(\mathscr{E})$ such that (12) holds$\}$. Consequently $\mathscr{S}_\varphi$ is a unital subsemigroup of $\mathscr{S}$. If $\varphi(1; f, f) \neq 0$ for $f \neq 0$ then $S = S(t)$ in (12) is uniquely determined and $t \to S(t)$ is a semigroup homomorphism.

PROPOSITION 4. *Let φ be a weakly positive definite form such that $\overline{\varphi(s; f, g)} = \varphi(s^*; g, f)$. Then for $t \in \mathscr{S}$ and any S corresponding to t via* (12) *we have*

$$SQ_\varphi(s) \subset Q_\varphi(s), \quad S\mathscr{A}_\varphi(s) \subset \mathscr{A}_\varphi(s), \quad s \in \mathscr{S}.$$

Proof. Using the Schwarz'inequality (cf. [14])

$$|\varphi(u^*v; f, f)|^2 \leqslant \varphi(v^*v; f, f)\varphi(u^*u; f, f)$$

we can write

$$\varphi((s^*s)^n;\, Sf, Sf) = \varphi((s^*s)^n t; f, Sf) =$$

$$= \varphi(t^*(s^*s)^n;\, Sf, f) = \varphi(t^*t(s^*s)^n; f, f) \leqslant$$

$$\leqslant \varphi((s^*s)^{2n}; f, f)^{1/2}\varphi((t^*t)^2; f, f)^{1/2}.$$

Using this (and (a) and (b) of Section 1 for the quasi-analytic case) we get the conclusion.

The above two propositions imply the following:

THEOREM 3. *Let φ be a positive definite form over $(S, \mathscr{E})$ such that $\varphi(1; f, f) \neq 0$ for $f \neq 0$. Let $\mathscr{T}$ be a subset of $\mathscr{S}_\varphi$ such that $\mathscr{E}$ is linearly spanned by the vectors $\{S(s)f : s \in \mathscr{S}_\varphi,\ f \in \bigcap_{t \in \mathscr{T}} Q_\varphi(t)\}$. Then the shift operators $\Phi(t)$, $t \in \mathscr{T}$ are essentially normal and essentially commuting. If moreover $\mathscr{E}$ is an inner product space and*

$$\varphi(1; f, f) = \|f\|^2_{\mathscr{E}} \tag{13}$$

then one can think of $\Phi(t)$ as an extension of $S(t)$, $t \in \mathscr{T}$.

In the proof of the second part of the conclusion, the formula (11) plays an essential role.

5. SPECTRAL REPRESENTATIONS OF THE SHIFT OPERATORS

The following result gives a partial answer to the question 3° in Section 4. Call $\chi : \mathscr{S} \to \mathbf{C}$ a *character* of $\mathscr{S}$ if $\chi(st) = \chi(s)\chi(t)$, $\chi(s^*) = \overline{\chi(s)}$, $\chi(e) = 1$. Denote by $\hat{\mathscr{S}}$ the set of all characters of $\mathscr{S}$ and equip it with the topology of pointwise convergence. Define the *Fourier transform* $\hat{s}$ of $s \in \mathscr{S}$ in the usual way, i.e.

$$\hat{s}(\chi) = \chi(s), \quad \chi \in \hat{\mathscr{S}}.$$

Denote by $\mathscr{B}(\hat{\mathscr{S}})$ the Borel σ-algebra of $\hat{\mathscr{S}}$.

THEOREM 4. *Let φ be a positive definite form over $(\mathscr{S}, \mathscr{E})$. Suppose that the involution semigroup $\mathscr{S}$ has at most a countable set $\mathscr{T} = \{t_n\}$ of generators and suppose that $\mathscr{E}$ is linearly spanned by $\bigcap_{t \in \mathscr{T}} Q_\varphi(t)$. Then there is a unique spectral measure F in $\mathscr{K} = \overline{\mathscr{D}}$ defined on $\mathscr{B}(\hat{\mathscr{S}})$ such that*

$$\Phi(s)f = \int_{\hat{\mathscr{S}}} \hat{s} \, \mathrm{d}Ff, \quad f \in \mathscr{D},$$

where $\Phi(s)$ is a shift operator at $s \in \mathscr{S}$.

Proof. Denote by $\mathscr{C}$ the Cartesian product $\prod_{t \in \mathscr{T}} \mathscr{C}_t$, every $\mathscr{C}_t = \mathbf{C}$. The Tychonoff topology makes $\mathscr{C}$ a Polish space. Moreover the σ-algebra on cylinder sets of $\mathscr{C}$ and the σ-algebra $\mathscr{B}(\mathscr{C})$ coincide. Define the mapping $j : \hat{\mathscr{S}} \to \mathscr{C}$ by $j(\chi) = \{\chi(t)\}$, $t \in \mathscr{T}$. Now we wish to show that j identifies (algebraically and topologically) the set $\hat{\mathscr{S}}$. Two properties of j are at hand

(14) $j(\hat{\mathscr{S}})$ is closed in $\mathscr{C}$,

(15) j is a homeomorphism onto its image.

To state other properties of j we need some more notations. Using the usual multi-index notations and taking two finite sequences $\alpha = (\alpha_1, \ldots, \alpha_m)$ and $\beta = (\beta_1, \ldots, \beta_n)$ of non-negative integers define a complex valued function $j_{\alpha\beta}$ on $\mathscr{C}$ as $j_{\alpha\beta}(\lambda) = \lambda_1^{\alpha_1} \ldots \lambda_m^{\alpha_m} \bar{\lambda}_1^{\beta_1} \ldots \bar{\lambda}_n^{\beta_n}$, $\lambda \in \mathscr{C}$. For an arbitrary $s \in \mathscr{S}$ there are two multi-indexes α, β such that

$$s = t_1^{\alpha_1} \ldots t_m^{\alpha_m} (t_1^{\beta_1} \ldots t_n^{\beta_n})^* \tag{16}$$

where $\{t_k\}$ are generators. Then

$$\hat{s} = j_{\alpha\beta} \circ j. \tag{17}$$

It follows from Proposition 3 that the essentially normal operators $\{\Phi(t_n)\}$ essentially commute. Let F_n be the spectral measure of $\Phi(t_n)$. Denote by F^∞ the Cartesian product $\prod_n F_n$ of the spectral measures F_n. If s is of the form (16) then

(18) $$\Phi(s)f = \int_{\mathscr{C}} j_{\alpha\beta}\, \mathrm{d}F^\infty f, \quad f \in \mathscr{D}.$$

We want to show that $F^\infty(\mathscr{C} \setminus j(\hat{\mathscr{S}})) = 0$ which allows us to look at the measure F^∞ as one defined on $\hat{\mathscr{S}}$. It is evident that $\lambda \in \mathscr{C} \setminus j(\hat{\mathscr{S}})$ if and only if some $s \in \mathscr{S}$ has two expressions of the form (16) with α, β and γ, δ respectively, such that

(19) $$j_{\alpha\beta}(\lambda) \neq j_{\gamma\delta}(\lambda).$$

In other words, denoting by $A = A_{s,\alpha,\beta,\gamma,\delta}$ the set of all $\lambda \in \mathscr{C}$ such that (19) holds, we can say that $\mathscr{C} \setminus j(\hat{\mathscr{S}})$ is precisely equal to the union of all such A's. Since the number of all these A's is at most countable, it sufficies to show that each $F^\infty(A) = 0$. Having such a $A = A_{s,\alpha,\beta,\gamma,\delta}$ notice that

$$\int_{\mathscr{C}} (j_{\alpha\beta} - j_{\gamma\delta})\, \mathrm{d}F^\infty f = \Phi(s)f - \Phi(s)f = 0, \quad f \in \mathscr{D}.$$

Since $\mathscr{D}$ is invariant for all the integrals $\int_{\mathscr{C}} j_{\alpha\beta}\, \mathrm{d}F^\infty$, we have

$$\int_A |j_{\alpha\beta} - j_{\gamma\delta}|^2\, \mathrm{d}\langle F^\infty(\cdot)f, f\rangle = \left\langle \int_A |j_{\alpha\beta} - j_{\gamma\delta}|^2\, \mathrm{d}F^\infty f, f \right\rangle =$$

$$= \left\langle \left(\int_A (j_{\alpha\beta} - j_{\gamma\delta})\, \mathrm{d}F^\infty\right)^* \int_{\mathscr{C}} (j_{\alpha\beta} - j_{\gamma\delta})\, \mathrm{d}F^\infty f, f \right\rangle = 0, \quad f \in \mathscr{D}.$$

Since A is a Borel set and $|j_{\alpha\beta} - j_{\gamma\delta}|^2 > 0$ on A, the only possibility is $F^\infty(A)f = 0$. Consequently $F^\infty(A) = 0$. This means that F^∞ is supported by $j(\hat{\mathscr{S}})$.

Define the spectral measure F on $\mathscr{B}(\hat{\mathscr{S}})$ as follows

$$F(A) = F^\infty(j(A)), \quad A \in \mathscr{B}(\hat{\mathscr{S}}).$$

Since the support of F^∞ is contained in $j(\hat{\mathscr{S}})$ we have the inversion formula

(20) $$F^\infty(A) = F(j^{-1}(A)), \quad A \in \mathscr{B}(\mathscr{C}).$$

By (18), (20), and (17) we have

$$\Phi(s)f = \int_{\mathscr{C}} j_{\alpha\beta}\,\mathrm{d}F^{\infty}f = \int_{j(\hat{\mathscr{S}})} j_{\alpha\beta}\,\mathrm{d}F^{\infty}f =$$

$$= \int_{\hat{\mathscr{S}}} j_{\alpha\beta}\circ j\,\mathrm{d}Ff = \int_{\hat{\mathscr{S}}} \hat{s}\,\mathrm{d}Ff, \quad f\in\mathscr{D}.$$

Thus we have got a measure F.

The uniqueness of F is forced by the fact that each spectral measure satisfying the conclusion must necessarily, via (20), be a product of spectral measures of $\{\Phi(t_n)\}$, which are always uniquely determined.

THEOREM 5. *Suppose that the semigroup $\mathscr{S}$ and the form φ are as in Theorem* 4. *Then there are (uniquely determined up to unitary equivalence) a Hilbert space $\mathscr{K}$, a linear operator $V:\mathscr{E}\to\mathscr{K}$ and a spectral measure F on $\mathscr{B}(\hat{\mathscr{S}})$ acting in $\mathscr{K}$ such that:*

(i) $V\mathscr{E}\subset\mathscr{D}\left(\int_{\hat{\mathscr{S}}}\hat{s}\,\mathrm{d}F\right)$;

(ii) $\varphi(s;f,g) = \left\langle \int_{\hat{\mathscr{S}}}\hat{s}\,\mathrm{d}FVf,\ Vg\right\rangle$;

(iii) *$\mathscr{K}$ is a closed linear span of the integrals*

$$\int_{\hat{\mathscr{S}}}\hat{s}\,\mathrm{d}FVf,\quad s\in\mathscr{S},\ f\in\mathscr{E}.$$

Proof. Applying Theorem 4 to the shift operators $\{\Phi(s): s\in\mathscr{S}\}$ related to φ, we get immediately $\mathscr{K}$, V and F. Having other $\mathscr{K}_1$, V_1 and F_1, a typical argument (essentially the same as that of [7]) gives us a unitary operator $U:\mathscr{K}\to\mathscr{K}_1$ such that

$$U\int_{\hat{\mathscr{S}}}\hat{s}\,\mathrm{d}Ff = \int_{\hat{\mathscr{S}}}\hat{s}\,\mathrm{d}F_1Uf,\quad f\in\mathscr{D}, \tag{21}$$

($\mathscr{D}$ is a linear span of the integrals $\int_{\hat{\mathscr{S}}}\hat{s}\,\mathrm{d}FVf$, $f\in\mathscr{E}$, $s\in\mathscr{S}$), and

$$UV = V_1. \tag{22}$$

It follows from (21) that

$$\int_{\hat{\mathscr{S}}} \hat{s}\, \mathrm{d}Ff = \int_{\hat{\mathscr{S}}} \hat{s}\, \mathrm{d}\tilde{F}f, \quad s \in \mathscr{S},\ f \in \mathscr{D},$$

where $\tilde{F} = U^{-1}F_1U$. Since the measure F in Theorem 4 is uniquely determined we can check directly that

$$UF = F_1U$$

which gives our conclusion.

We would like to recommend here the recent Berezanskii's book [2] as a helpful reference concerning families of spectral measures and their products.

This theorem implies some *Bochner type representation* for positive definite functions for $*$-semigroups with an at most countable number of generators. To state the results, we call a form μ over $(\mathscr{B}(\hat{\mathscr{S}}), \mathscr{E})$ a *semi-spectral measure* on $\hat{\mathscr{S}}$ if every $\mu(\cdot\,; f, f)$ is a positive scalar measure.

COROLLARY 1. *Suppose that the semigroup $\mathscr{S}$ and the form φ are as in Theorem* 4. *Then there is a unique semi-spectral measure μ on $\hat{\mathscr{S}}$ such that*

(i) $\hat{s} \in L^2(\mu(\cdot\,; f, f)),\quad f \in \mathscr{E},\ s \in \mathscr{S}$,

(ii) $\varphi(s; f, g) = \int_{\hat{\mathscr{S}}} \hat{s}\, \mathrm{d}\mu(\cdot\,; f, g),\quad f, g \in \mathscr{E},\ s \in \mathscr{S}$.

COROLLARY 2. *If in Corollary* 1 $\mathscr{E} = \mathbf{C}$, *then the conclusion takes the following form: there is precisely one positive scalar measure μ on $\hat{\mathscr{S}}$ such that*

(i) $\hat{s} \in L^2(\mu)$,

(ii) $\varphi(s) = \int_{\hat{\mathscr{S}}} \hat{s}\, \mathrm{d}\mu,\quad s \in \mathscr{S}$.

Existence of μ in both Corollaries is an easy consequence of Theorem 5. Because the uniqueness of μ in Corollary 1 can be easily deduced from that in Corollary 2 (via polarization formula), we concentrate on the latter fact.

Proof of uniqueness in Corollary 2. Suppose μ satisfies (i) and (ii). Then using notations of the proof of Theorem 4 we have

$$\varphi(s) = \int_{\hat{\mathscr{S}}} \hat{s}\, \mathrm{d}\mu = \int_{\hat{\mathscr{S}}} j_{\alpha\beta} \cdot j\, \mathrm{d}\mu = \int_{\mathscr{C}} j_{\alpha\beta}\, \mathrm{d}\mu^{\infty} = \int_{\mathbf{C}^n} \lambda^{\alpha}\bar{\lambda}^{\beta}\mu_n(\mathrm{d}\lambda) \tag{23}$$

where α and β have the same lenght n and

$$(24) \qquad \mu_n(A) = \mu^\infty(\{\lambda \in \mathscr{C} : (\lambda_1, \ldots, \lambda_n) \in A\}), \quad A \in \mathscr{B}(\mathbf{C}^n).$$

Fixing n, we can say that, by (23) and (i), μ_n is a representing measure of n-parameter complex moment sequence

$$\{\varphi(t_1^{\alpha_1} \ldots t_n^{\alpha_n}(t_1^{\beta_1} \ldots t_n^{\beta_n})^*)\}_{\alpha\beta}.$$

Call this moment sequence $\{a(\alpha, \beta)\}$. Because of our assumption on quasi-analyticity, we have for each $k = 1, \ldots, n$

$$\sum_{m=1}^{\infty} a(m\delta_k, m\delta_k)^{-1/2m} = \sum_{m=1}^{\infty} \varphi((t_k^* t_k)^m)^{-1/2m} = +\infty$$

where δ_k is the usual zero-one δ-sequence. Thus, according to Appendix, Theorem 12, the moment sequence $\{a(\alpha, \beta)\}$ is determined. In other words each measure μ_n is uniquely determined. Because cylindric sets generate $\mathscr{B}(\mathscr{C})$, (24) implies that μ^∞ is uniquely determined too. Applying the formula $\mu^\infty = \mu \circ j^{-1}$ we get the uniqueness conclusion.

REMARK 3. Notice that if for a particular semigroup $\mathscr{S}$ the condition

$$(25) \qquad j(\hat{\mathscr{S}}) = \mathscr{C}$$

holds (here j and $\mathscr{C}$ are defined as at the very begining of the proof of Theorem 4), then the proof of Theorem 4 can be shortened drastically. Moreover, under the assumption (25) much of the arguments presented in this section work well even in the case of an uncountable set $\mathscr{T}$ of generators. However the conclusions of Theorems 4 and 5, Corollaries 1 and 2 become weaker: namely, instead of $\mathscr{B}(\hat{\mathscr{S}})$ we have to deal with the σ-algebra $\mathscr{Z}(\hat{\mathscr{S}})$ on cylinder sets of $\hat{\mathscr{S}}$ (in the countable case these σ-algebras coincide), and the measures F and μ are defined on $\mathscr{Z}(\hat{\mathscr{S}})$.

REMARK 4. We have two possible ways to relate to φ the spectral measure appearing in the formula (ii) of Theorem 5. One way is just as in the proof of Theorem 5. The other would be to dilate a semispectral measure μ of Corollary 1 in the sense of Naimark. Uniqueness assertion of Corollary 1 says nothing else than these spectral measures coincide up to unitary equivalence. More precisely, if $\mathscr{K}$, V and F are as in Theorem 5 then one can show that

$$(26) \qquad \mathscr{K} = \text{the closed linear span of } F(\mathscr{B}(\hat{\mathscr{S}}))V\mathscr{E}.$$

Moreover Theorem 5 guarantees that the semi-spectral measure defined by

$$(27) \qquad \mu(A; f, g) = \langle F(A)Vf, Vg\rangle, \quad A \in \mathscr{B}(\hat{\mathscr{S}}),\ f, g \in \mathscr{E},$$

uniquely represents φ (in the sense of Corollary 1). On the other hand, if μ is a semi-spectral measure representing φ and $\mathcal{K}_1$, V_1 and F_1 is a Naimark type dilation of μ (i.e. (26) and (27) hold for them), the uniqueness of Naimark dilations (cf. [7]) implies the existence of a unitary operator $U : \mathcal{K} \to \mathcal{K}_1$ such that $UF = F_1U$ and $UV = V_1$. This, by (iii) of Theorem 5, gives the equality

$$\mathcal{K}_1 = \text{the closed linear span of} \int_{\hat{\mathcal{S}}} \hat{s}\, \mathrm{d}F_1 V_1 f, \quad s \in \mathcal{S},\ f \in \mathcal{E}.$$

When $\mathcal{E} = \mathbf{C}$ (cf. Corollary 2), we get a particular model of the Naimark dilation. This is: $\mathcal{K}_1 = L^2(\mu)$; $F_1(A)f = 1_A f$, $f \in \mathcal{K}_1$ and V_1 maps λ into $\lambda 1_{\hat{\mathcal{S}}}$ for $\lambda \in \mathbf{C}$. Then one can prove that

$$\int_{\hat{\mathcal{S}}} \hat{s}\, \mathrm{d}F_1 1_{\hat{\mathcal{S}}} = \hat{s}, \quad s \in \mathcal{S}.$$

This gives us the welcomed conclusion: *$L^2(\mu)$ is the closed linear span of Fourier transforms $\hat{s}$, $s \in \mathcal{S}$.*

6. Weak positive definiteness versus positive definiteness

Notice that for two weakly positive definite forms φ and ψ we have

$$\mathcal{A}_{a\varphi + b\psi}(s) = \mathcal{A}_\varphi(s) \cap \mathcal{A}_\psi(s) \tag{28}$$

where a, b are arbitrary positive numbers. This property helps us to prove a version of Corollary 1.

THEOREM 6. *Let $\mathcal{S}$ have an at most countable sequence $\mathcal{T} = \{t_n\}$ of generators. Suppose that a form φ over $(\mathcal{S}, \mathcal{E})$ is weakly positive definite. Suppose moreover that for each $t \in \mathcal{T}$, $\mathcal{A}_\varphi(t) = \mathcal{E}$. Then the conclusion of Corollary 1 holds true.*

Proof. Applying Corollary 2 to the positive definite form $\varphi(\cdot\,; f, f)$ $(f \in \mathcal{E})$, we get the existence of a unique measure μ_f on $\hat{\mathcal{S}}$ satisfying (i) and (ii) of Corollary 2. Using the polarization formula we define the complex measures

$$\mu(A; f, g) = \frac{1}{4}\{\mu_{f+g}(A) - \mu_{f-g}(A) + \mathrm{i}\mu_{f+\mathrm{i}g}(A) - \mathrm{i}\mu_{f-\mathrm{i}g}(A)\}.$$

Since the measure μ_f is uniquely determined, we have

$$\mu_{af} = |a|^2 \mu_f \quad (a \in \mathbf{C}). \tag{29}$$

This implies that $\mu_f = \mu(\cdot\,; f, f)$, $f \in \mathscr{E}$ and that the form $\mu(A;\, \cdot, -)$ is hermitian symmetric. Now it remains to prove the linearity of $\mu(A;\, \cdot, -)$ with respect to the first variable. To show it is additive, we write

$$\varphi(s; f+g, h) = \varphi(s; f, h) + \varphi(s; g, h), \quad f, g, h \in \mathscr{E}.$$

Using the polarization formula for the form $\varphi(s, \cdot, -)$ and the integral representation (ii) of Corollary 2 we get

$$\int_{\hat{\mathscr{S}}} \hat{s}\, \mathrm{d}\nu_1 - \int_{\hat{\mathscr{S}}} \hat{s}\, \mathrm{d}\nu_2 + \mathrm{i}\left(\int_{\hat{\mathscr{S}}} \hat{s}\, \mathrm{d}\nu_3 - \int_{\hat{\mathscr{S}}} \hat{s}\, \mathrm{d}\nu_4\right) = 0, \quad s \in \mathscr{S}, \tag{30}$$

where

$$\nu_1 = \mu_{f+g+h} + \mu_{f-h} + \mu_{g-h},$$

$$\nu_2 = \mu_{f+g-h} + \mu_{f+h} + \mu_{g+h},$$

$$\nu_3 = \mu_{f+g+\mathrm{i}h} + \mu_{f-\mathrm{i}h} + \mu_{g-\mathrm{i}h},$$

$$\nu_4 = \mu_{f+g-\mathrm{i}h} + \mu_{f+\mathrm{i}h} + \mu_{g+\mathrm{i}h}.$$

Since the Fourier transform preserves the involution and the measures ν_k, $k = 1, 2, 3, 4$ are positive we can deduce from (30) that

$$\int_{\hat{\mathscr{S}}} \hat{s}\, \mathrm{d}\nu_1 = \int_{\hat{\mathscr{S}}} \hat{s}\, \mathrm{d}\nu_2 \quad \text{and} \quad \int_{\hat{\mathscr{S}}} \hat{s}\, \mathrm{d}\nu_3 = \int_{\hat{\mathscr{S}}} \hat{s}\, \mathrm{d}\nu_4.$$

Each of these integrals represents a positive definite form over $(\mathscr{S}, \mathbf{C})$. For instance

$$\int_{\hat{\mathscr{S}}} \hat{s}\, \mathrm{d}\nu_1 = \varphi(s; f+g+h, f+g+h) + \varphi(s; f-h, f-h) + \varphi(s; g-h, g-h).$$

This form, due to (28), satisfies the assumption of Corollary 2. Since a representing measure in Corollary 2 is uniquelly determined, we infer that $\nu_1 = \nu_2$. Similarly $\nu_3 = \nu_4$. This, in conclusion, implies the required additivity $\mu(A; f+g, h) = \mu(A; f, h) + \mu(A; g, h)$. By the same trick we can prove that $\mu(A; af, g) = a\mu(A; f, g)$, first for $a > 0$, then for $a < 0$ and finally for $a = \mathrm{i}$, which exhausts all the possibilities.

The uniqueness of μ is forced by the uniqueness of $\mu(\cdot, f, f)$.

Taking a Naimark dilation (cf. Remark 4) of the semi-spectral measure μ we get the equality (ii) of Theorem 5. Therefore we have:

COROLLARY 3. *Under the assumption of Theorem 6, φ is a positive definite form over $(\mathscr{S}, \mathscr{E})$ and the conclusion of Theorem 5 holds.*

Having weak positive definiteness instead of positive definiteness, Proposition 4 allows us to derive from Corollary 3 an analogue of Theorem 3.

THEOREM 7. *Let φ be a weakly positive definite form over $(\mathscr{S}, \mathscr{E})$ such that $\overline{\varphi(s;f,g)} = \varphi(s^*;g,f)$ $f,g \in \mathscr{E}$ and $\varphi(1;f,f) \neq 0$ for $f \neq 0$. Suppose that the (at most countable) sequence $\mathscr{T} = \{t_n\}$ of generators of $\mathscr{S}$ is contained in $\mathscr{S}_\varphi$. Moreover suppose that $\mathscr{E}$ is linearly spanned by the vectors $\{S(s)f : s \in \mathscr{S}_\varphi,\ f \in \bigcap_{t\in\mathscr{T}} \mathscr{A}_\varphi(t)\}$. Then φ is positive definite and the shift operators $\Phi(t)$, $t \in \mathscr{T}$, are essentially normal and essentially commuting. Moreover if $\mathscr{E}$ is an inner product space and* (13) *holds, then one can think of $\Phi(t)$ as an extension of $S(t)$, $t \in \mathscr{T}$.*

REMARK 5. Merits of Remark 3 refer to all what have been said in this section too.

REMARK 6. We wish to say some words about the role played by the generators in this context. In [12] we have considered the so-called bounded vectors which in the form set-up can be defined as follows: f is a bounded vector of φ at s (in notation: $f \in \mathscr{B}_\varphi(s)$) if there are non-negative numbers $a = a(f)$ and $c = c(f)$ such that

$$\varphi((s^*s)^n; f, f) \leqslant ac^n$$

or, equivalently,

$$\lim_{n\to\infty} \varphi((s^*s)^n; f, f)^{1/n} \leqslant c.$$

There is a significant difference between bounded vectors from one side and analytic and quasi-analytic ones from the other side. Namely if $f \in \mathscr{B}_\varphi(s) \cap \mathscr{B}_\varphi(t)$ then $f \in \mathscr{B}_\varphi(st)$; in other words if φ is "bounded" on the set of generators, then so is it at all the members of the semigroup. This is not the case for analytic and quasi-analytic vectors. Consider the following example: take $\mathscr{S} = \mathbf{N}$, $\mathscr{E} = \mathbf{C}$, $\varphi(n) = n!$. Then $\mathscr{A}_\varphi(1) = \mathscr{E}$ ($s = 1$ is a generator of the semigroup $\mathscr{S}$) but $Q_\varphi(s) = \{0\}$ for any other $s \neq 0$ and $s \neq 1$. This shows that our assumptions of "quasi-analyticity" at generators are *definite*. However, we have another property: if for some $n, f \in Q_\varphi(s^n)$ or $\mathscr{A}_\varphi(s^n)$, then $f \in Q_\varphi(s)$ or $\mathscr{A}_\varphi(s)$, respectively (to prove this, use results of Section 1). This property can be employed in the case of continuous positive forms on topological semigroups like $\mathbf{R}_+$ (in this case $\{1/n\}_1^\infty$ generates topologically the additive semigroup $\mathbf{R}_+$ and 1 is the n-th semigroup power of $1/n$; quasi-analyticity at 1 implies that at $1/n$).

SUBNORMAL OPERATORS

7. A SINGLE SUBNORMAL OPERATOR

This and the next section contain the main part of our paper. Recall that an operator S (densely defined) in a Hilbert space $\mathcal{H}$ is said to be *subnormal* if there is a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and a (densely defined) normal operator N in $\mathcal{K}$ such that

$$Sh = Nh, \quad h \in \mathcal{D}(S). \tag{31}$$

If the operator S satisfies (31), then S is closable and its closure $\bar{S}$ satisfies (31) too.

In this section we restrict our interest to operators S having invariant domain $\mathcal{E} = \mathcal{D}(S)$. This enables us to think of S as a member of $\mathcal{L}(\mathcal{E})$. Consequently we can apply what we have developed so far.

THEOREM 8. *Suppose S satisfies the following two conditions:*

(i) $\sum_{j,k=0}^{n} \langle S^j f_k, S^k f_j\rangle \geqslant 0$ *for each finite sequence* $f_0, \ldots, f_n \in \mathcal{D}(S)$,

(ii) $\mathcal{D}(S)$ *is a linear subspace spanned by the vectors* $\{S^n f : n \geqslant 0, f \in Q(S)\}$.
Then S is subnormal.

Proof. Take $\mathcal{S} = \mathbf{N}\times\mathbf{N}$ with the coordinatewise addition as the semigroup multiplication and the involution defined as $(m, n)^* = (n, m)$. Define a form φ over $(\mathcal{S}, \mathcal{E})$ by

$$\varphi((m, n); f, g) = \langle S^m f, S^n g\rangle. \tag{32}$$

Then one can show (cf. [17]) that (i) implies the positive definiteness of φ. Denote by $t = (1, 0)$. Then $Q_\varphi(t) = Q(S)$, $t \in \mathcal{S}_\varphi$ and $S(t) = S$. Consequently, we can apply Theorem 3 with $\mathcal{T} = \{t\}$ to get that the shift operator $\Phi(t) \in \mathcal{L}^{\#}(\mathcal{D})$ is an essentially normal extension of $S \in \mathcal{L}(\mathcal{E})$. Take any completion $\mathcal{K}$ of $\mathcal{D}$. Then $N = \Phi(t)^-$ is a normal operator in $\mathcal{K}$. Since $\overline{\mathcal{E}} = \mathcal{H}$ and $\mathcal{E} \subset \mathcal{D} \subset \mathcal{K}$ we can identify $\mathcal{H}$ with the closure of $\mathcal{E}$ in $\mathcal{K}$, to conclude that S is a subnormal Hilbert space operator.

REMARK 7. Conditions (i) and (ii) bear resemblance to the classical characterization of bounded subnormal operators in its original (two-condition) Halmos'version (cf. [4]); condition (i) is just the same as the first of Halmos, condition (ii) is a weaker form of the other. While for bounded operators, due to Bram (cf. [15] or [16] for an elementary proof), the Halmos characterization reduces to the condition (i), in most unbounded cases there is a need of some additional condition (cf. [12] for an intermediate version of (ii)).

Theorem 7 implies the following variation of Theorem 8:

THEOREM 9. *Suppose S satisfies the following two conditions:*

(i) *for each $h \in \mathscr{E}$ and for all $f_0, \ldots, f_n \in \mathscr{E}_h$*

$$\sum_{j,k=0}^{n} \langle S^j f_k, S^k f_j \rangle \geqslant 0,$$

where $\mathscr{E}_h$ is a linear subspace spanned by the vectors $\{S^n h : n \geqslant 0\}$,

(ii) *$\mathscr{E}$ is linearly spanned by the vectors*

$$\{S^n f : n \geqslant 0,\ f \in \mathscr{A}(S)\}.$$

Then S is subnormal.

The way to deduce Theorem 9 from Theorem 7 is to observe that the condition (i) implies weak positive definiteness of the appropriate form φ related to S via (32).

Theorem 9 says that in presence of analytic vectors an operator is subnormal if so it is on every of its cyclic subspace $\mathscr{E}_h$, $h \in \mathscr{E}$.

COROLLARY 4. *Let S be such that $\mathscr{E}$ is linearly spanned by the vectors $\{S^n f : n \geqslant 0, f \in \mathscr{A}(S)\}$. Then S is subnormal if and only if for each $h \in \mathscr{E}$, $S|\mathscr{E}_h$ is a subnormal operator in the Hilbert space $\overline{\mathscr{E}}_h$, where $\mathscr{E}_h$ is a linear subspace spanned by the vectors $\{S^n h : n \geqslant 0\}$.*

Corollary 4 presents an unbounded version of the fact which has appeared in a recent paper of Trent [18] (cf. [13] for a more detailed discussion of Trent's result).

8. SUBNORMAL SYSTEMS OF COMMUTING OPERATORS

Using the same tools we can state similar results for a finite or infinite number of operators. A family $\{S_\sigma\}_{\sigma \in \Sigma}$ of (densely defined) operators in a complex Hilbert space $\mathscr{H}$ with the same domain $\mathscr{E} = \mathscr{D}(S_\sigma)$ ($\sigma \in \Sigma$) is said to be a *subnormal system* if there are a Hilbert space $\mathscr{K} \supset \mathscr{H}$ and a family $\mathbf{N} = \{N_\sigma\}_{\sigma \in \Sigma}$ of commuting normal operators in $\mathscr{K}$ such that

$$S_\sigma f = N_\sigma f, \quad f \in \mathscr{E},\ \sigma \in \Sigma. \tag{33}$$

In this section we are interested in systems $\mathbf{S} = \{S_\sigma\}_{\sigma \in \Sigma}$ of densely defined operators having a (common) domain $\mathscr{E} = \mathscr{D}(S_\sigma)$ ($\sigma \in \Sigma$) invariant for each S_σ, $\sigma \in \Sigma$, and such that $S_\sigma S_\rho f = S_\rho S_\sigma f$ for each $f \in \mathscr{E}$ and for all indices $\sigma, \rho \in \Sigma$. Under these assumptions, the subnormal system $\{S_\sigma\}_{\sigma \in \Sigma}$ must necessarily satisfy more than (33); namely

$$\mathbf{S}^\alpha f = \mathbf{N}^\alpha f, \quad f \in \mathscr{E},$$

where $S^\alpha f = \prod S_\sigma^{\alpha(\sigma)} f$ (the same about $N^\alpha f$) and $\alpha : \Sigma \to \mathbf{N}$ is an arbitrary function equal to zero for all but a finite number of σ's. Denote by A the set of all such α's.

THEOREM 10. *Let* $\mathbf{S}$ *satisfy the following two conditions:*
(i) $\sum_{\alpha,\beta \in A} \langle S^\alpha f(\beta), S^\beta f(\alpha) \rangle \geqslant 0$ *for every finitely supported function* $f : A \to \mathscr{E}$,
(ii) $\mathscr{E}$ *is a linear subspace spanned by the set*

$$\{S^\alpha f : \alpha \in A,\ f \in \bigcap_{\sigma \in \Sigma} Q(S_\sigma)\}.$$

Then $\mathbf{S}$ *is a subnormal system.*

THEOREM 11. *Let* $\mathbf{S}$ *satisfy the following two conditions:*
(i) *for each* $h \in \mathscr{E}$ *and for each finitely supported function* $f : A \to \mathscr{E}_h$

$$\sum_{\alpha,\beta} \langle S^\alpha f(\beta), S^\beta f(\alpha) \rangle \geqslant 0,$$

where $\mathscr{E}_h$ *is a linear subspace spanned by the set*

$$\{S^\alpha h : \alpha \in A\},$$

(ii) $\mathscr{E}$ *is a linear subspace spanned by the set*

$$\{S^\alpha f : \alpha \in A,\ f \in \bigcap_{\sigma \in \Sigma} \mathscr{A}(S_\sigma)\}.$$

Then $\mathbf{S}$ *is a subnormal system.*

COROLLARY 5. *Let* $\mathbf{S}$ *be such that* $\mathscr{E}$ *is a linear subspace spanned by the set* $\{S^\alpha f : \alpha \in A, f \in \bigcap_{\sigma \in \Sigma} \mathscr{A}(S_\sigma)\}$. *Then* $\mathbf{S}$ *is a subnormal system if and only if for each* $h \in \mathscr{E}$ *so is* $\mathbf{S}|\mathscr{E}_h$ *(here* $\mathscr{E}_h$ *is as in Theorem* 11*).*

To prove Theorems 10, 11 and Corollary 5 one can use essentially the same tools as those involved in the proofs of Theorems 8, 9 and Corollary 4. Here we point out some essentials. The semigroup $\mathscr{S}$ is just the direct sum of card Σ copies of the semigroup we have used in the single operator case. $\mathscr{S}$ is nothing else but $A \times A$ with the involution $(\alpha, \beta)^* = (\beta, \alpha)$. Then the form φ is as follows:

$$\varphi((\alpha, \beta); f, g) = \langle S^\alpha f, S^\beta g \rangle, \quad \alpha, \beta \in A. \tag{34}$$

The set $\mathscr{T}$ of generators consists of $(\delta_\sigma, 0)$, where δ_σ is the usual δ-function at σ. To prove Theorem 11 we have to use Theorem 7 combined with Remark 5 because in this particular case of the semigroup $\mathscr{S}$, the condition (25) is satisfied. The proof of the "if" part of Corollary 5 requires to check that the condition (i) of Theorem 11 is fulfilled. This can be done using (35) and (36) below.

REMARK 8. Suppose $\boldsymbol{S}$ is a subnormal system and $\boldsymbol{N}$ its normal extension. A look at the (joint) spectral representation of $\boldsymbol{N}$ leads us to the following statements:

(35) $$\mathscr{E} \subset \mathscr{D}(M_1 \dots M_k), \quad \text{where } M_j \in \boldsymbol{N} \cup \boldsymbol{N}^*,$$

(36) $$\text{for all } M_1, M_2 \in \boldsymbol{N} \cup \boldsymbol{N}^* \quad M_1 M_2 f = M_2 M_1 f, \quad f \in \mathscr{E}.$$

This justifies the following definition: the extension $\boldsymbol{N}$ is said to be *minimal* if $\{\boldsymbol{N}^{*\alpha}\boldsymbol{N}^{\beta} f : f \in \mathscr{E}, \alpha, \beta \in A\}$ is linearly dense in $\mathscr{K}$. A question we ought to speak of is the uniqueness of a minimal normal extension. The substantial argument we can provide here with is that of Theorem 5. The conclusion would be roughly the following: *a minimal normal extension of Theorems* 8, 9, 10 *and* 11 *is uniquely determined up to unitary equivalence.*

9. EXAMPLE: THE CREATION OPERATOR

Neither Theorem 8 nor Theorem 9 characterizes subnormal operators completely. However there is an important example which fits nicely in this scope.

Consider the operator (which is also known as the creation one)

$$A_+ = \frac{1}{\sqrt{2}}\left(x - \frac{\mathrm{d}}{\mathrm{d}x}\right)$$

with $\mathscr{D}(A_+) = \mathbb{S}(\mathbf{R})$, the Schwartz space. Recall some properties of the operator A_+. Having the Hermite functions

$$f_n(x) = \mathrm{e}^{\frac{x^2}{2}} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \mathrm{e}^{-x^2}$$

one can figure out the following formulas

(37) $$A_+^n f_m = \left(-\frac{1}{\sqrt{2}}\right)^n f_{m+n}$$

(38) $$\|A_+^n f_0\|^2 = n!\,.$$

In (38) we have the norm of $L^2(\mathbf{R})$. Take $\mathscr{E}$ = the linear span of $\{f_n\}_{n=0}^{\infty}$ and consider $S = A_+|\mathscr{E} \in \mathscr{L}(\mathscr{E})$ (the restriction is possible because, by (37), $\mathscr{E}$ is invariant under A_+). Condition (37) helps us to prove the condition (i) of Theorem 8.

For $h_k = \sum_m \xi_{km} f_m$ we have

$$\Delta = \sum_{k,l} \langle A_+^k h_l, A_+^l h_k \rangle = \sum_{k,l} \sum_{m,n} \xi_{ln} \bar{\xi}_{km} \langle A_+^k f_n, A_+^l f_m \rangle =$$

$$= \sum_{l,n} \sum_{k,m} \xi_{ln} \bar{\xi}_{km} \left(-\frac{1}{\sqrt{2}} \right)^{k+l} \langle f_{n+k}, f_{m+l} \rangle =$$

$$= \sum_{l,n} \sum_{k,m} \eta_{ln} \bar{\eta}_{km} \delta_{m-k,\, n-l} (n+k)!$$

where

$$\eta_{km} = \xi_{km} \left(-\frac{1}{\sqrt{2}} \right)^k 2^{\frac{k+m}{2}}.$$

The quickest way to see that this sum is non-negative is to write it as an integral

$$2\pi \Delta = \sum_{l,n} \sum_{k,m} \eta_{ln} \bar{\eta}_{km} \int_0^{2\pi} e^{i[(m-k)-(n-l)]t}\, dt \int_0^{+\infty} t^{n+k} e^{-t}\, dt =$$

$$= \int_0^{2\pi} \int_0^{+\infty} \Big| \sum_{l,n} \eta_{ln} e^{i(n-l)t} s^{\frac{n+l}{2}} \Big|^2 e^{-s}\, ds\, dt \geqslant 0.$$

To conclude that the operator S is subnormal we can use either Theorem 8 or Theorem 9. Conditions (37) and (38) together help us to check the condition (ii) in either of these theorems.

Since A_+ is contained in the closure of S (use the density of $\mathcal{E}$ in $\mathbb{S}(\mathbf{R})$ and the continuity of A_+ in the $\mathbb{S}(\mathbf{R})$-topology), we infer that A_+ itself is subnormal. Let us remind that the formal adjoint of A_+ is the annihilation operator

$$A = \frac{1}{\sqrt{2}} \left(x + \frac{d}{dx} \right).$$

Since $\|Af_n\| < \|A_+ f_n\|$, neither A_+ is normal nor A is subnormal.

As usual it is interesting to ask about the L^2-model of a normal extension of A_+. From what is in the fundamental paper of Bargmann [1] follows that the (minimal) normal extension of A_+ is unitarily equivalent to the operator of multiplication by z in the Hilbert space $L^2\left(\mathbf{C}, \frac{1}{\pi} \exp(-|z|^2)\, dx\, dy\right)$. This model can be used to show that the closure of A_+ has no nonzero bounded vector.

APPENDIX: THE COMPLEX MOMENT PROBLEM

First we want to clear up the determinancy question of the multiparameter complex moment problem appearing in the proof of Corollary 2. Using the multi-index notation, we say that the sequence $\{a(\alpha, \beta)\}$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$ is the *n-parameter complex moment sequence* if there is a finite positive measure μ on $\mathbf{C}^n$ such that

$$a(\alpha, \beta) = \int_{\mathbf{C}^n} \lambda^\alpha \bar{\lambda}^\beta \mu(\mathrm{d}\lambda). \tag{39}$$

This sequence is said to be determinate if the measure is unique.

THEOREM 12. *If $\{a(\alpha, \beta)\}$ is an n-parameter complex moment sequence and if*

$$\sum_m a(m\delta_k, m\delta_k)^{-1/2m} = +\infty, \quad k = 1, \ldots, n \tag{40}$$

where $\delta_k = (0, \ldots, 0, 1, 0, \ldots, 0)$ with the k-th coordinate 1, then the sequence is determinate.

Proof. There is a natural one-to-one correspondence between n-parameter complex moment sequences and $2n$-parameter real moment sequences, which preserves determinancy (but does not preserve quasi-analyticity). Take such a $2n$-parameter real moment sequence $\{b(\gamma)\}$, $\gamma = (\gamma_1, \ldots, \gamma_{2n})$. Condition (40) guarantees that each of the one-parameter moment sequences $\{b(0, \ldots, 0, n, 0, \ldots, 0)\}_n$ satisfies the classical Carleman condition

$$\sum_{m=1}^{\infty} b(0, \ldots, 0, 2m, 0, \ldots, 0)^{-1/2m} = +\infty,$$

which implies that all these sequences are determinate. Then, by Theorem 10 of [8] (or Theorem 3 of [10]), the $2n$-parameter sequence $\{b(\gamma)\}$ is determined itself. Finally the related n-parameter complex moment sequence $\{a(\alpha, \beta)\}$ is determined too.

In turn, our Corollary 2 can be applicable to the solvability (and determinancy) of the n-parameter complex moment problem.

COROLLARY 6. *Let $\{a(\alpha, \beta)\}$ be a sequence of complex numbers such that*

$$\sum_{\substack{\alpha,\beta \\ \gamma,\delta}} a(\alpha + \delta, \beta + \gamma)\xi(\alpha, \beta)\overline{\xi(\gamma, \delta)} \geq 0 \tag{i}$$

for all finite sequences $\{\xi(\alpha, \beta)\}$ of complex numbers,

$$\sum_{m=1}^{\infty} a(m\delta_k, m\delta_k)^{-1/2m} = +\infty, \quad k = 1, \ldots, n. \tag{ii}$$

Then there exists a unique finite non-negative measure μ on C^n such that (39) is satisfied. Moreover all the polynomials in λ and $\bar{\lambda}$ are dense in $L^2(\mu)$.

REMARK 9. There is a paper of Kilpi [5] concerning the (one-parameter) complex moment problem. He characterizes (Satz 5) complex moment sequences in terms of positiveness of appropriate linear functionals (the M. Riesz approach). He also relates (Satz 7) complex moment sequences to formally normal operators having normal extensions. Unfortunately the part of Satz 7 which begins with "Es sei bemerkt..." is false.

REFERENCES

1. BARGMANN, V., On a Hilbert space of analytic functions and an associated integral transform. I, *Comm. Pure Appl. Math.*, **14**(1961), 187–214.
2. BEREZANSKII, YU. M., *Selfadjoint operators in spaces of infinite number of variables* (Russian), Naukowa Dumka, Kiev, 1978.
3. CODDINGTON, E. A., Formally normal operators having no normal extension, *Canad. J. Math.*, **17**(1965), 1030–1040.
4. HALMOS, P. R., Normal dilations and extensions of operators, *Summa Brasil. Math.*, **2**(1950), 125–134.
5. KILPI, Y., Über das komplexe Momentenproblem, *Ann. Acad. Sci. Fenn. Ser. A* I *Math.*, **236**(1957).
6. MASSON, D.; MCCLARY, W. K., Classes of C^∞-vectors and essential selfadjointness, *J. Functional Analysis*, **10**(1972), 19–32.
7. MLAK, W. Conditionally positive definite functions on linear spaces, *Ann. Polon. Math.*, **42**(1983), 187–239.
8. NUSSBAUM, A. E., Quasi-analytic vectors, *Ark. Mat.*, **6**(1967), 179–191.
9. NUSSBAUM, A. E., A note on quasi-analytic vectors, *Studia Math.*, **33**(1969), 305–309.
10. PETERSEN, L. C., On the relation between the multidimensional moment problem and the one-dimensional moment problem, *Math. Scand.*, **51**(1982), 361–366.
11. SLINKER, S. P., On commuting selfadjoint extensions of unbounded operators, *Indiana Univ Math. J.*, **27**(1978), 629–636.
12. STOCHEL, J.; SZAFRANIEC, F. H., Bounded vectors and formally normal operators, in "*Dilation theory, Toeplitz operators, and other topics*", Birkhäuser, 1983, pp. 363–370.
13. STOCHEL, J.; SZAFRANIEC, F. H., A characterization of subnormal operators, in *Spectral theory of linear operators and related topics*, Birkhäuser, 1984, pp. 261–263.
14. SZAFRANIEC, F. H., Boundedness of the shift operator related to positive definite forms: an application to moment problems, *Ark. Mat.*, **19**(1981), 251–259.
15. SZAFRANIEC, F. H., Dilations on involution semigroups, *Proc. Amer. Math. Soc.*, **66**(1977), 30–32.
16. SZAFRANIEC, F. H., Subnormals in C^*-algebras, *Proc. Amer. Math. Soc.*, **84**(1982), 533–534.

17. Sz.-Nagy, B., *Extensions of linear transformations in Hilbert spaces which extend beyond this space*, F. Ungar, New York, 1960.
18. Trent, T. T., New conditions for subnormality, *Pacific J. Math.*, **93**(1981), 459–464.

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Note added in proofs. A recent inspection of this paper has inspired us with proving a *version* of our Theorem 2. This is

THEOREM 2′. *Let $M, N \subset \mathscr{L}^{\#}(\mathscr{D})$ be two formally normal operators such that $M \in \{N, N^{\#}\}^{c}$. Suppose we are given two sets $\mathfrak{Q}_1 \subset \mathfrak{Q}(M)$ and $\mathfrak{Q}_2 \subset \mathfrak{Q}(M)$ such that each of the sets $\{Tf : T \in \{M, M^{\#}\}^{c}, f \in \mathfrak{Q}_1$ and $\{Tf : T \in \{N, N^{\#}\}^{c}, f \in \mathfrak{Q}_2\}$ spans $\mathscr{D}$. Then the essentially normal operators M and N essentially commute.*

However, when replacing quasi-analytic vectors by analytic ones both these version coincide. We intend to present this in detail in the subsequent papes "Commuting symmetric operators and normality".

Theorem 2 entails the answer to the question 2° of Section 4. This answer is given by a part of Proposition 3. The new version of Theorem 2 impacts Proposition 3 as well as all its consequences (Theorems 3, 4, 5, 10 and Corollary 1). In particular Theorem 10 would now take a stronger form.

THEOREM 10′. *Let $\mathbf{S}$ satisfy the condition* (i) *of Theorem* 10 *and the condition*
(ii′) *for each $\sigma \in \Sigma$, $\mathscr{E}$ is a linear subspace spanned by the set $\{S^{\alpha} f : \beta \in A, f \in \mathfrak{Q}(S_{\sigma})\}$.*
Then $\mathbf{S}$ is a subnormal system.