J. OPERATOR THEORY
14(1985), 67–85

THE REGGE PROBLEM FOR STRINGS, UNCONDITIONALLY CONVERGENT EIGENFUNCTION EXPANSIONS, AND UNCONDITIONAL BASES OF EXPONENTIALS IN $L^2(-T, T)$

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A string is the interval $[0, +\infty)$ carrying a non-negative measure $\mathrm{d}m$. The function $m(x) = \int_{0-}^{x} \mathrm{d}m$ evaluates the mass of the string supported by $[0, x]$. The point $x = 0$ is assumed to be a point of growth of m, i.e. $m(x) > 0$ for $x > 0$. It is supposed also that the string is obtained from the classical homogeneous string (corresponding to Lebesgue measure $\mathrm{d}x$) by a finite perturbation. The latter means that $\mathrm{d}m = \mathrm{d}x$ on $(a, +\infty)$ for some $a < +\infty$. In what follows

$$a_m \overset{\mathrm{def}}{=} \inf\{a : \mathrm{d}m = \mathrm{d}x \text{ on } (a, +\infty)\}.$$

Given $a > 0$ let $L^2([0, a], \mathrm{d}m)$ denote the Hilbert space of all m-measurable functions f with

$$\|f\|_m^2 = \int_{0-}^{a} |f(x)|^2 \,\mathrm{d}m(x) < +\infty.$$

Every string determines the formal differential operator

$$f \to \frac{\mathrm{d}^2 f}{\mathrm{d}m\mathrm{d}x}$$

defined on the class $\mathbf{D}_0$ of functions f on $\mathbf{R} = (-\infty, +\infty)$ such that

$$f(x) = \begin{cases} f(0) + f^-(0)x & \text{for } x < 0 \\ f(0) + f^-(0)x + \int_0^x \left\{ \int_{0-}^{\xi} g(\eta)\,\mathrm{d}m(\eta) \right\} \mathrm{d}\xi, & x \geqslant 0 \end{cases}$$

with g satisfying $g \in L^2([0, a], dm)$ for every $a > 0$. Clearly $\frac{d^2 f}{dm\,dx} = g$ for such an f. The symbols $f^+(x)$ and $f^-(x)$ denote the right-hand and left-hand derivatives of f respectively.

Fix $a \geqslant a_m$ and let $\sigma(m)$ be the set of all complex numbers k such that the equation

$$(1) \qquad \frac{d^2 y}{dm\,dx} = -k^2 y, \quad y^-(0) = 0, \quad y^+(a) + iky(a) = 0$$

has a non-zero solution $y_a(x, k)$. In fact the set $\sigma(m)$ does not depend on the parameter a when $a \geqslant a_m$ and coincides with the zero-set of an entire function. It can be shown (and we will do it later) that $\sigma(m)$ is disposed in the open upper half-plane $\mathbf{C}_+$. The spectrum $\sigma(m)$ is always symmetric with respect to the imaginary axis because $\overline{y_a(x, k_0)}$ is a non-zero solution of (1) corresponding to $k = -\bar{k}_0$ provided $k_0 \in \sigma(m)$.

The spectra which occur in the eigenfunction problem (1) are described by Arov's theorem [1]:

THEOREM 1. *A closed countable subset σ of $\mathbf{C}_+$ symmetric with respect to the imaginary axis coincides with the spectrum of a problem* (1) *if and only if σ is the zero-set for an entire function F of exponential type with*

$$\int_{\mathbf{R}} (1 + x^2)^{-1} \cdot |F(x)|^{-2} dx < +\infty, \quad \int_{\mathbf{R}} (1 + x^2)^{-1} \log^+ |F(x)|\, dx < +\infty.$$

Given $a \geqslant a_m$ the Regge problem [2], [3] is to determine whether the family $\{y_a(x, k)\}_{k \in \sigma(m)}$ is complete in $L^2([0, a], dm)$ or not. Let

$$T(x) = \int_0^x [m'(s)]^{1/2} ds,$$

i.e. $T(x)$ is the time required for a point perturbation of the end $x = 0$ to reach the point x.

The following result solves the completeness problem which, of course, is of most interest for the critical value $a = a_m + T(a_m)$. It is assumed that the spectrum $\sigma(m)$ is simple, i.e. the associated function F has only simple zeros.

THEOREM 2. *The family $\{y_a(x, k)\}_{k \in \sigma(m)}$ is complete in $L^2([0, a], dm)$ for $a_m \leqslant a \leqslant a_m + T(a_m)$ and is not complete if $a > a_m + T(a_m)$.*

The next step is to investigate in more detail what is going on in the limit case $a = a_m + T(a_m)$. Although $\{y_a(x, k)\}_{k \in \sigma(m)}$ is complete in $L^2([0, a], dm)$ this fact alone does not permit us, of course, to expand any given function $f \in L^2([0, a], \mathrm{d}m)$ in an unconditionally convergent series

$$f(x) = \sum_{k \in \sigma(m)} \alpha_k y_a(x, y), \quad \alpha_k \in \mathbf{C}.$$

Recall that a family of non-zero vectors $\{e_n\}$ in a Hilbert space H is called an unconditional basis in H if every element $x \in H$ can uniquely be decomposed in an unconditionally convergent series $x = \sum_n \alpha_n \cdot e_n$, $\alpha_n \in \mathbf{C}$. The classical G. Köthe—O. Teoplitz theorem says that a complete family $\{e_n\}$ in a Hilbert space forms an unconditional basis iff the following "approximate Parseval identity" holds

$$c \cdot \sum_n |\alpha_n|^2 \cdot \|e_n\|^2 \leqslant \|\sum_n \alpha_n e_n\|^2 \leqslant c^{-1} \cdot \sum_n |c_n|^2 \cdot \|e_n\|^2$$

for some c, $0 < c < 1$, and for every finite sequence of complex numbers $\{\alpha_n\}$.

The unconditional basis problem for $\{y_a(x, k)\}_{k \in \sigma(m)}$ is intimately connected with the same problem for exponentials $\{\mathrm{e}^{ikx}\}_{k \in \sigma(m)}$ in $L^2(0, a)$. In few words the relationship between the problems looks as follows. Given a string m and $a \geqslant a_m$ one can associate with (1) a semigroup $\{Z_t\}_{t \geqslant 0}$ of contractions in an auxiliary Hilbert space K^a so that the characteristic function of $\{Z_t\}_{t \geqslant 0}$ is $S = \theta \cdot B$, where $\theta = \mathrm{e}^{2i(a-a_m)z}$ and B is the Blaschke product in $\mathbf{C}_+$ with the roots placed at the points of $\sigma(m)$. The eigenfunctions $\mathcal{U}_k$ of $\{Z_t\}_{t \geqslant 0}$, and $\mathcal{U}_k^*$ of the conjugate semigroup $\{Z_t^*\}_{t \geqslant 0}$ can easily be expressed in terms of $\{y_a(x, k)\}_{k \in \sigma(m)}$ (see (10) below). On the other hand the semigroup $\{Z_t\}_{t \geqslant 0}$ is unitarily equivalent to the so-called model semigroup $\{\mathfrak{M}_t\}_{t \geqslant 0}$ whose eigenvectors are related to the exponentials via the usual Fourier transform.

This new approach to the problem based on the Lax-Phillips scattering theory for unitary groups and originating in earlier papers by B. S. Pavlov has been developed in [4] (see Part IV) to investigate the basis problem for a special class of strings.

In the present paper we exploit the connection indicated above in the direction inverse to one considered in [4]. This yields the following result.

THEOREM 3. *Let Λ be a subset of $\mathbf{C}_+$ invariant under $z \to -\bar{z}$ such that* $\inf\{\operatorname{Im} \lambda : \lambda \in \Lambda\} > 0$. *Suppose that $\{\mathrm{e}^{i\lambda x}\}_{\lambda \in \Lambda}$ is an unconditional basis in $L^2(0, 2d)$ for some $d > 0$. Then there exists a string m with $\sigma(m) = \Lambda$ such that for $a = a_m + d$ the family $\{y_a(x, k)\}_{k \in \sigma(m)}$ forms an unconditional basis in $L^2([0, a], \mathrm{d}m)$.*

Notice that for strings obtained $T(a_m) > 0$ because $d = T(a_m)$.

The proof is based on a refinement of the technique of [4] coupled with M. G. Kreĭn's solution of the inverse spectral problem for strings. We use here the L. de Branges approach to the inverse spectral problem as it is exposed in [5].

The paper is organized as follows. Section 1 contains preliminaries. It deals mainly with the construction of the corresponding functional model. For reader's convenience we present the proof of Theorem 1 in §2. This section also deals with the completeness problem, i.e. with the proof of Theorem 2. The most interesting case here is the case $a = a_m + T(a_m)$ with $T(a_m) > 0$. In § 3 the proof of Theorem 3 is given (see also Theorem 3.1 below).

Theorem 2 is closely related to similar results obtained by M. G. Kreĭn and A. A. Nudel'man in [6], [7], [8]. The papers [6], [8], besides other things, deal with the completeness problem of root elements of the dissipative operator associated with a string which is constrained to satisfy slightly different boundary conditions. The main technical tool used in [6], [8] to prove the corresponding completeness theorem is the well-known criterion of M. S. Livšic, while in the present paper the proof of Theorem 2 is based on the theory of entire functions.

Acknowledgements. I am grateful to the Institute INCREST in Bucharest, Romania for the support of this investigation. I am indebted to L. de Branges for valuable discussions concerning Hilbert spaces of entire functions during the fall of 1982. It is my pleasure to express a gratitude to M. G. Kreĭn and A. A. Nudel'man who turned my attention to some inaccuracies of the preliminary manuscript.

1. THE CONSTRUCTION OF THE FUNCTIONAL MODEL

1.1. THE OPERATOR $\mathfrak{S}$. The string defines a self-adjoint operator in the Hilbert space $\mathbf{M} = L^2([0, +\infty), \mathrm{d}m)$ which can be specified as the restriction of $\mathrm{d}^2/\mathrm{d}m\mathrm{d}x$ to the domain

$$\mathbf{D}(\mathfrak{S}) = \{f \in \mathbf{D}_0 : f^-(0) = 0, \quad \|f\|_{\mathbf{M}} + \|\mathfrak{S}f\|_{\mathbf{M}} < +\infty\}.$$

Given $k \in \mathbf{C}$ denote by $A(x, k)$ the unique solution in $\mathbf{D}_0$ of $\dfrac{\mathrm{d}^2 A}{\mathrm{d}m\mathrm{d}x} = -k^2 A$ satisfying $A^-(0, k) = 0$, $A(0, k) = 1$. It can be obtained as a solution of the following integral equation

$$A(x, k) = 1 - k^2 \int_0^x \left\{ \int_-^t A(s, k)\,\mathrm{d}m(s) \right\} \mathrm{d}t \tag{2}$$

which implies that both $k \to A(x, k)$ and $k \to B(x, k) = -\dfrac{A^+(x, k)}{k}$ are entire functions, in fact of exponential type. Let

$$E(x, k) = A(x, k) - \mathrm{i}B(x, k). \tag{3}$$

Clearly, (see (2)), $E(x, 0) = 1$, $x \in [0, +\infty)$ and the set of zeros of $E^{\#}(a, k) \overset{\text{def.}}{=\!=} \overline{E(a, \bar{k})}$ coincides with the spectrum $\sigma(m)$ of (1).

The functions A, B, E, (the last is called a de Branges function), play an essential role for the spectral representation of $\mathfrak{S}$.

Let Δ be a principal spectral function of $\mathfrak{S}$ which is an increasing odd function on $\mathbf{R}$ completely determined by $\mathfrak{S}$ [5]. Consider the Hilbert space $\mathbf{Z}(\Delta)$ consisting of all functions on $\mathbf{R}$ with

$$\|f\|_{\mathbf{Z}}^2 = \frac{1}{\pi} \int_{\mathbf{R}} |f(\gamma)|^2 \, \mathrm{d}\Delta(\gamma) < +\infty$$

and two orthogonal subspaces $\mathbf{Z}_{\text{even}}(\Delta)$ and $\mathbf{Z}_{\text{odd}}(\Delta)$ there, formed by even and odd functions respectively.

The "even" transform

$$(\hat{f})_{\text{even}}(\gamma) = \int_{0-}^{+\infty} A(x, \gamma) f(x) \, \mathrm{d}m(x)$$

defines a unitary mapping of $\mathbf{M}$ onto $\mathbf{Z}_{\text{even}}(\Delta)$. Accordingly the "odd" transform

$$(\hat{f})_{\text{odd}}(\gamma) = \int_{0}^{+\infty} B(x, \gamma) f(x) \, \mathrm{d}x$$

is a unitary mapping of $L^2([0, +\infty), \mathrm{d}x)$ onto $\mathbf{Z}_{\text{odd}}(\Delta)$.

1.2. DE BRANGES FUNCTIONS. Any entire function E satisfying

$$|E(z)| > |E^{\#}(z)|, \quad z \in \mathbf{C}_+$$

is called a de Branges function. We assume that E satisfies the reality condition

$$E^{\#}(z) = E(-z), \quad z \in \mathbf{C}$$

and that $E(0) = 1$. Let us notice that this class of functions appeared for the first time in 1938 in a paper by M. G. Kreĭn (see the English translation [9], p. 214—260).

A de Branges function of exponential type is called short if

$$\int_{\mathbf{R}} (1 + \gamma^2)^{-1} |E(\gamma)|^{-2} \, \mathrm{d}\gamma < +\infty.$$

Clearly, any short de Branges function is root free on $\mathbf{C}_+ \cup \mathbf{R}$ and the trivial inequality $\log^- x < x^{-2}$ implies

$$\int_{\mathbf{R}} (1 + y^2)^{-1} \log^- |E(y)| \, dy < +\infty$$

which together with the assumption of finite exponential type of E yields by the Carleman formula that

$$\int_{\mathbf{R}} (1 + y^2)^{-1} \log^+ |E(y)| \, dy < +\infty.$$

The class of all entire functions of exponential type satisfying the last condition is called Cartwright's class $\mathscr{C}$. The basic facts concerning $\mathscr{C}$ can be found in [10], [11].

Let Λ be the set of zeros of a short de Branges function E. Because of the inclusion $E \in \mathscr{C}$ the function E admits the following factorization

$$E(z) = e^{-icz} \cdot \text{v.p.} \prod_{\lambda \in \Lambda} \left(1 - \frac{z}{\lambda}\right), \tag{4}$$

where $c \in \mathbf{R}$ and $\text{v.p.} \prod_{\lambda \in \Lambda} \stackrel{\text{def.}}{=} \lim_{R \to +\infty} \prod_{|\lambda| \leqslant R}$. Let $B(z) = \prod_{\Lambda} (1 - z/\bar{\lambda})(1 - z/\lambda)^{-1}$ be the Blaschke product corresponding to $\bar{\Lambda}$. The function $E^{\#} \cdot E^{-1}$ being bounded in $\mathbf{C}_+$, it follows that $c \geqslant 0$:

$$\frac{E^{\#}(z)}{E(z)} = e^{2icz} \cdot B(z). \tag{5}$$

There exists a nice correspondence between the class of strings under consideration and the class of short de Branges functions. The proof of the following result can be found in [5], Sections 6.3, 6.12.

THEOREM 1.1. *Given a string m and $a > 0$ the function $E(a, z)$ is a short de Branges function of exponential type*

$$T = \int_0^a [m'(s)]^{1/2} \, ds.$$

The function $\Delta(y) = \int_0^y |E(y')|^{-2} \, dy'$ is the principal spectral function of the string with mass function

$$m^0(x) = \begin{cases} m(x) & \text{for } x \leqslant a \\ m(a) + (x - a) & \text{for } x > a. \end{cases}$$

The converse is also true. Given a short de Branges function E, $E(0) = 1$, satisfying the reality condition there exist precisely one number $a > 0$ and precisely one mass function m with $a_m \leqslant a$ such that

$$E(\gamma) = E(a, \gamma).$$

LEMMA 1.2. *Given a string m and $a \geqslant a_m$ we have*

$$E(a, z) = e^{-i(a-a_m)z} \text{ v.p.} \prod_{\lambda \in \Lambda} \left(1 - \frac{z}{\lambda}\right).$$

Proof. Straightforward computations show that $E(a, z) = e^{-i(a-a_m)z} \cdot E(a_m, z)$. So we need to prove the equality only for $a = a_m$. Clearly (4) holds for $E(z) = E(a_m, z)$ with $c \geqslant 0$. If $c > 0$ we can consider an auxiliary short de Branges function $E^{\bullet}(z) = e^{icz}E(a_m, z)$ which by Theorem 1.1 generates the same string m^0 because $|E| = |E^{\bullet}|$, and there exists $a \geqslant a_m$ such that $E^{\bullet}(z) = E(a, z) = e^{-i(a-a_m)z}E(a_m, z)$ which obviously contradicts the assumption $c > 0$. ▨

1.3. THE WAVE EQUATION. Let $\mathbf{N}$ denote the space of all functions on $[0, +\infty)$ with

$$\|f\|_{\mathbf{N}}^2 = \int_0^{\infty} |f'|^2 \mathrm{d}x < +\infty.$$

Being factored by the subspace of constant functions and endowed with the corresponding factor-norm, the space $\mathbf{N}$ becomes a Hilbert space.

The Cauchy problem for the wave equation is defined by

$$\frac{\partial^2 u}{\partial t^2} = \frac{\mathrm{d}^2 u}{\mathrm{d}m\mathrm{d}x}, \qquad u^-(0, t) = 0$$

$$u(x, 0) = u_0(x), \qquad u_t(x, 0) = u_1(x)$$

and the space $\mathbf{E} = \mathbf{N} \oplus \mathbf{M}$ supplied with the norm

$$\|\mathscr{U}\|_{\mathbf{E}} = \left\|\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}\right\|_{\mathbf{E}} = \frac{1}{2}\int_0^{+\infty} |u_0'|^2 \mathrm{d}x + \frac{1}{2}\int_{0-}^{+\infty} |u_1(x)|^2 \mathrm{d}m(x)$$

is a natural Hilbert space of all "data" $\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}$ with finite energy.

THEOREM 1.3. *The operator*

$$\mathscr{L} = \mathrm{i}\begin{pmatrix} 0 & -I \\ -\mathfrak{S} & 0 \end{pmatrix}, \quad \mathbf{D}(\mathscr{L}) = \left\{ \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} : u_0 \in \mathbf{D}(\mathfrak{S}),\ u_1 \in \mathbf{M} \cap \mathbf{N} \right\}$$

is self-adjoint in $\mathbf{E}$.

The proof of the theorem is essentially the proof of self-adjointness of $\mathfrak{S}$ which can be found for example in [2], see also [1] for a partial case.

The operator $\mathscr{L}$ being self-adjoint generates the strongly continuous unitary group $U_t = \exp \mathrm{i}\mathscr{L}t$. Given any data $\mathscr{U}(0) = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in \mathbf{E}$ this group defines $\mathscr{U}(t) = \begin{pmatrix} u_0(x, t) \\ u_1(x, t) \end{pmatrix} = U_t\mathscr{U}(0)$ and $u_0(x, t)$ is the solution of the Cauchy problem. The spectral representation for U_t can be obtained with the help of even and odd transforms. Define an operator $\mathscr{F} : \mathbf{E} \to \mathbf{Z}(\Delta)$ by

$$\mathscr{F}\mathscr{U}(\gamma) = \mathscr{F}\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}(\gamma) = -\mathrm{i}\int_0^{+\infty} u_0' B(x, \gamma)\,\mathrm{d}x + \int_{0-}^{\infty} u_1 A(x, \gamma)\,\mathrm{d}m(x).$$

Then

$$\|\mathscr{U}\|_{\mathbf{E}}^2 = \frac{1}{2\pi}\int_{\mathbf{R}} |\mathscr{F}\mathscr{U}|^2\,\mathrm{d}\Delta$$

and $U_t\mathscr{U} = \mathscr{F}^{-1}\mathrm{e}^{\mathrm{i}t\gamma}\mathscr{F}\mathscr{U}$, $\mathscr{U} \in \mathbf{E}$.

1.4. THE SEMIGROUP OF CONTRACTIONS $\{Z_t\}_{t\geqslant 0}$. Fix $a \geqslant a_m$ and consider the subspace $\mathbf{K}^a$ (or briefly $\mathbf{K}$) of all elements of $\mathbf{E}$ such that $|u_0'(x)| + |u_1(x)| = 0$ for $x > a$.

Let $\mathscr{P}_{\mathbf{K}}$ denote the orthogonal projection onto $\mathbf{K}$. It is easy to check that $\mathbf{K} = \mathbf{E} \ominus (\mathscr{D}_- \oplus \mathscr{D}_+)$, where $\mathscr{D}_- = \left\{ \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} : u_1 = u_0',\ u_0'(x) = 0 \text{ for } x \leqslant a \right\}$ and $\mathscr{D}_+ = \left\{ \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} : -u_1 = u_0',\ u_0'(x) = 0 \text{ for } x \leqslant a \right\}$ are the spaces of incoming and outgoing waves correspondingly. The family

$$Z_t \stackrel{\text{def.}}{=\!=} \mathscr{P}_{\mathbf{K}} U_t | K, \quad t \geqslant 0$$

is a strongly continuous semigroup of contractions. Notice that $\mathscr{P}_{\mathbf{K}}\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}$ with $u_0(x) = v_0(x)$, $u_1(x) = v_1(x)$ for $x \leqslant a$ and $v_0(x) = u_0(a)$, $v_1(x) = 0$ for $x > a$.

THEOREM 1.4. *The generator T of the semigroup $z_t = e^{itT}$ is a maximal completely dissipative operator in* $\mathbf{K}$ *with*

$$\mathbf{D}(T) = \left\{\mathscr{P}_{\mathbf{K}}\mathscr{U} : \mathscr{U} = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in \mathbf{D}(\mathscr{L}),\ u_0^+(a) + u_1(a) = 0\right\},$$

$T(\mathscr{P}_{\mathbf{K}}\mathscr{U}) = \mathscr{P}_{\mathbf{K}}\mathscr{L}\mathscr{U}$, *for* $\mathscr{P}_{\mathbf{K}}\mathscr{U} \in \mathbf{D}(T)$.

The adjoint operator T^* is defined on

$$\mathbf{D}(T^*) = \left\{\mathscr{P}_{\mathbf{K}}\mathscr{U} : \mathscr{U} = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in \mathbf{D}(\mathscr{L}),\ u_0^+(a) - u_1(a) = 0\right\},$$

$T^*(\mathscr{P}_{\mathbf{K}}\mathscr{U}) = \mathscr{P}_{\mathbf{K}}\mathscr{L}\mathscr{U}$ *for* $\mathscr{P}_{\mathbf{K}}\mathscr{U} \in \mathbf{D}(T^*)$.

The proof can be found in [4] (see Theorem 2.1 of Part IV). Denote by $\mathscr{U}_k$ and $\mathscr{U}_k^*$ the eigenvectors of T and T^* : $T\mathscr{U}_k = k\mathscr{U}_k$, $T^*\mathscr{U}_k^* = \bar{k}\mathscr{U}_k^*$. An easy computation (using Theorem 1.4) shows that

$$\mathscr{U}_k(x) = \begin{pmatrix} y_a(x, k) \\ iky_a(x, k) \end{pmatrix} \quad \mathscr{U}_k^*(x) = \begin{pmatrix} y_a(x, \bar{k}) \\ iky_a(x, k) \end{pmatrix} \tag{6}$$

for $0 \leqslant x \leqslant a$, where k ranges over the spectrum $\sigma(m)$.

1.5. THE MODEL SEMIGROUP $\{\mathfrak{M}_t\}_{t>0}$. $\mathscr{F}$ transforms $\mathbf{K}^a$ onto the class $I^T(\Delta)$ of entire functions $f \in Z(\Delta)$ of type $\leqslant T$. The relationship between T and a is the following: a is the biggest root of the equation

$$T = \int_0^x [m'(s)]^{1/2}\, ds,$$

which clearly has only one root a if $a > a_m$.

The entire function $E(a, z)$, being a short de Branges function, generates a de Branges space of entire functions. This space $B(E)$ consists precisely of entire functions f satisfying

$$\|f\|^2 = \int_{\mathbf{R}} |f(\gamma)/E(\gamma)|^2\, d\gamma < +\infty$$

$$|f(z)| \leqslant C_f \cdot (\operatorname{Im} z)^{-1/2} |E(z)| \quad \text{if } \operatorname{Im} z > 0 \tag{7}$$

$$|f(z)| \leqslant C_f \cdot (\operatorname{Im} z)^{-1/2} |E^\#(z)| \quad \text{if } \operatorname{Im} z < 0. \tag{8}$$

It turns out that $I^T(\Delta)$ and $B(E(a, z))$ are identical (see [5], Sections 6.3—6.4).

Let now $\mathcal{M}_E$, $\mathcal{M}_E f = f/E$ denote the map which maps $\mathbf{Z}(\Delta)$ isometrically onto $L^2(\mathbf{R}, \mathrm{d}x)$. It follows from (7) that $\mathcal{M}_E f \in H^2_+$ for any $f \in B(E)$ and (8) implies $\mathcal{M}_E f \in SH^2_-$ on $\mathbf{R}$ for such an f with $S = E^\#/E$ inner. Therefore $\mathcal{M}_E B(E) \subset K_S$, where K_s is a "model space" defined as $K_s = H^2_+ \ominus SH^2_+$. In fact $\mathcal{M}_E B(E) = K_s$ because every function from K_s has a pseudo-analytical (in our case usual analytical) continuation in the lower half-plane $\mathbf{C}_-$ to a meromorphic function with simple poles at zeros of E. Multiplied by E this function clearly turns into an entire function belonging to $B(E)$.

Let $\mathcal{T} := \mathcal{M}_E \circ \mathcal{F} : \mathbf{E} \to L^2(\mathbf{R})$ and define the norm in L^2 by $\|f\|^2_{L^2} = \frac{1}{2\pi}\int_{\mathbf{R}} |f|^2 \, \mathrm{d}x$. Then obviously

$$\|\mathcal{U}\|^2_{\mathbf{E}} = \|\mathcal{T}\mathcal{U}\|^2_{L^2}, \quad U_t \mathcal{U} = \mathcal{T}^{-1} e^{it\gamma} \mathcal{T}\mathcal{U}$$

with $\mathcal{T}\mathbf{K}^a = K_s$ and $S = E^\#/E$. Summarizing we get

THEOREM 1.5. *The mapping $\mathcal{T}$ defines a unitary equivalence of the semigroup $(Z_t)_{t \geq 0}$ to the model semigroup*

$$\mathfrak{M}_t f = P_s e^{it\gamma} f, \quad f \in K_s, \ t \geq 0,$$

where P_s denotes the orthogonal projection of H^2_+ onto K_s.

1.6. EIGENVECTORS. It is well-known that $\left\{\frac{S(z)\cdot\sqrt{2\operatorname{Im} k}}{z - k}\right\}_{k \in \sigma(m)}$ is the family of eigenvectors of the generator $\mathbf{A}$ of $\{\mathfrak{M}_t\}_{t \geq 0}$ and $\left\{\frac{\sqrt{2\operatorname{Im} k}}{z - \bar{k}}\right\}_{k \in \sigma(m)}$ is the family of eigenvectors of $\mathbf{A}^*$. Theorem 1.5 implies that

$$\text{(9)} \qquad \mathcal{T}\mathcal{U}_k = \frac{S(\gamma)\cdot\sqrt{2\operatorname{Im} k}}{\gamma - k}, \quad \mathcal{T}\mathcal{U}^*_k = -\frac{\sqrt{2\operatorname{Im} k}}{\gamma - \bar{k}}, \quad k \in \sigma(m)$$

with appropriate coefficients c_k, $y_a(x, k) = c_k \cdot A(x, k)$, where $k \in \sigma(m) \cup \overline{\sigma(m)}$, in (6). The following formula borrowed from [5], p. 234

$$\int_{0-}^{a} A(x, k)A(x, \gamma)\,\mathrm{d}m(x) + \int_0^a B(x, k)B(x, \gamma)\,\mathrm{d}x = \frac{E^\#(k)E(\gamma) - E(k)E^\#(\gamma)}{-2i(\gamma - k)}$$

yields $c_k = \sqrt{8\operatorname{Im} k}\cdot(ikE(k))^{-1}$, $c_{\bar{k}} = \sqrt{8\operatorname{Im} k}(i\bar{k}\cdot\overline{E(k)})^{-1}$ for $k \in \sigma(m)$. Therefore

$$c_{-\bar{k}} = \overline{(c_{-k})} = \sqrt{8\operatorname{Im} k}(-ik\overline{E(-\bar{k})})^{-1} = \sqrt{8\operatorname{Im} k}(-ikE^\#(-k))^{-1} = -c_k$$

as $k \in \sigma(m)$, and we get

$$\mathcal{U}_k(x) = \begin{pmatrix} c_k A(x,k) \\ ik \cdot c_k A(x,k) \end{pmatrix}, \quad \mathcal{U}^*_{-\bar{k}} = \begin{pmatrix} -c_k A(x,k) \\ ikc_k A(x,k) \end{pmatrix}.$$

Finally

(10) $$\frac{1}{2} \cdot \{\mathcal{U}_k + \mathcal{U}^*_{-\bar{k}}\} = \begin{pmatrix} 0 \\ ky_a(x,k) \end{pmatrix}, \quad \frac{1}{2} \{\mathcal{U}_k - \mathcal{U}^*_{-\bar{k}}\} = \begin{pmatrix} y_a(x,k) \\ 0 \end{pmatrix}$$

for $k \in \sigma(m)$.

LEMMA 1.6. 1) *Let $k \neq -\bar{k} \in \sigma(m)$. Then $\mathcal{U}_k \perp \mathcal{U}^*_{-k}$ and therefore*

$$\|\mathcal{U}_k \pm \mathcal{U}^*_{-k}\|^2 = 2.$$

2) *Let $k = ib \in \sigma(m)$ with $b > 0$. Then*

$$\|\mathcal{U}_k \pm \mathcal{U}^*_{-\bar{k}}\|^2 = 2\{1 \mp 2bi S'(ib)\}.$$

Proof. If $k \neq -\bar{k} \in \sigma(m)$ then

$$(\mathcal{U}_k, \mathcal{U}^*_{-\bar{k}})_{\mathbf{E}} = (\mathcal{T}\mathcal{U}_k, \mathcal{T}\mathcal{U}^*_{-\bar{k}})_{L^2(\mathbf{R})} = -\frac{1}{2\pi}\int_{\mathbf{R}} \frac{S(\gamma)}{(\gamma - k)} \cdot \frac{2mk}{\gamma + \bar{k}}\, d\gamma = 0$$

because $g(z) = S(z) \cdot (z-k)^{-1} \in H^2_+$ and $g(-\bar{k}) = 0$.

Let now $k = ib \in \sigma(m)$ with $b > 0$. Then

$$\|\mathcal{U}_{ib} \pm \mathcal{U}^*_{ib}\|^2 = 2\{1 \pm \operatorname{Re}(\mathcal{U}_{ib}, \mathcal{U}^*_{ib})\}.$$

But

$$(\mathcal{U}_{ib}, \mathcal{U}^*_{ib}) = -\frac{2bi}{2\pi i}\int_{\mathbf{R}} \frac{S(\gamma)}{(\gamma - ib)^2}\, d\gamma = -2bi S'(ib).$$

The spectrum $\sigma(m)$ being invariant under $z \to -\bar{z}$, we have $S(z) = \overline{S(-\bar{z})}$ which implies $-2bi S'(ib) \in \mathbf{R}$. ▨

Set $\delta_m = 0$ if S does not have roots on the imaginary axis and let $\delta_m = \sup\{2b|S'(ib)| : ib \in \sigma(m),\ b > 0\}$. Clearly $0 \leqslant \delta_m \leqslant 1$. In case $S = e^{idz}B(z)$, where $d > 0$ and B is a Blaschke product we always have $\delta_m < 1$. Indeed, $\delta_m = \sup\{2b\, e^{-db} \cdot |B'(ib)| : ib \in \sigma(m),\ b > 0\} \leqslant \exp\{-d \inf_{ib \in \sigma} b\} < 1$.

IONAL BASES OF EXPONENTIALS. A subset $\Lambda = \{\lambda_n\}$ of $\mathbf{C}_+$ is called a Carleson-Newman set if

(CN) $$\inf_n \prod_{m \neq n} \left| \frac{\lambda_n - \lambda_m}{\lambda_n - \bar{\lambda}_m} \right| > 0$$

and the last holds iff $\{\lambda_n\}$ is an interpolating sequence for the Hardy algebra H^∞ (see for details [12], [13] or [14]).

Given $a > 0$ consider the class $\mathscr{M}_a$ of all entire functions of exponential type with indicator diagram $[0, -ia]$ and satisfying the Muckenhoupt condition (A_2) on the line:

$$(A_2) \qquad \sup_{I\in J}\left(\frac{1}{|I|}\int_I |F|^2 dx\right)\cdot\left(\frac{1}{|I|}\int_I |F|^{-2} dx\right) < +\infty.$$

Here J stands for the family of all intervals. It is well-known that any function F satisfying (A_2) satisfies also

$$\int_{\mathbf{R}} (1+\gamma^2)^{-1}\cdot|F(\gamma)|^2 d\gamma < +\infty, \quad \int_{\mathbf{R}} (1+\gamma^2)^{-1}\cdot|F(\gamma)|^{-2} d\gamma < +\infty$$

which implies $\mathscr{M}_a \subset \mathscr{C}$.

Let σ be a subset of $\mathbf{C}_\delta \overset{\text{def.}}{=} \{z \in \mathbf{C}_+ : \operatorname{Im} z > \delta\}$ for some $\delta > 0$, let B be the Blaschke product corresponding to σ with understanding that $B_\sigma \equiv 0$ if σ is a uniqueness set for H^∞, and let $S = \theta\cdot B = e^{idz}\cdot B$ with $d > 0$. Then $K_S = \operatorname{clos}(K_\theta + K_B)$. It is easy to see that $K_S \ominus K_\theta = \theta\cdot K_B$.

THEOREM 1.7. *The following are equivalent:*

1) $\{e^{i\lambda x}\}_{\lambda\in\sigma}$ *is an unconditional basis in* $L^2(0, d)$;

2) $\sigma \in$ (CN), *the angle between* K_B *and* θK_B *is positive and*

$$K_S = K_B \dotplus \theta K_B;$$

3) $\sigma \in$ (CN) *and there exists a function in* $\mathscr{M}_d$ *with simple zeros whose zero-set is* σ.

See the proof in [4] (Theorem 2, Corollary on p. 231, Theorem 7, Theorem 9, Theorem 1.2 of Part III).

2. THE COMPLETENESS PROBLEM

2.1. PROOF OF THEOREM 1. The first implication is clear because $\sigma(m)$ is the set of roots of $E(a, z)$ satisfying $|E^*(\gamma)| = |E(\gamma)|$ on $\mathbf{R}$, and E is a short de Branges function.

The converse is a direct corollary of Theorem 1.1 provided it is known that F is a short de Branges function. The function F being of exponential type and satisfying

$$(11) \qquad \int_{\mathbf{R}} \frac{1}{1+\gamma^2}\,\frac{d\gamma}{|F(\gamma)|^2} < +\infty,$$

it follows $F \in \mathscr{C}$. We can assume therefore that

$$F(z) = \text{v.p.} \prod_{\lambda \in \sigma(m} \left(1 - \frac{z}{\lambda}\right).$$

Denote by B the Blaschke product corresponding to $\sigma(m)$. Clearly $F/F^{\#} = B$ in $\mathbf{C}_+$. This implies that $F^{\#}$ is a de Branges function which in fact is short because of (11). ▨

2.2. Proof of Theorem 2. The following lemma is borrowed from [4] (see Part IV, Theorem 2.3 there). The proof is given for reader's convenience.

Lemma 2.1. *The family* $\{y_a(x, k)\}_{k \in \sigma(m)}$ *is complete in* $L^2([0, a), \mathrm{d}m)$, $a \geqslant a_m$ *if and only if the joint family* $(\mathscr{U}_k, \mathscr{U}_k^*\}_{k \in \sigma(m)}$ *of eigenvectors of* T *and* T^* *is complete in* $\mathbf{K}^a$.

Proof. Identify $\mathbf{K}^a$ with $\mathbf{N}^a + \mathbf{M}^a$, where

$$\mathbf{N}^a = \{f \in \mathbf{N} : f'(x) = 0,\ x > a\}, \quad \mathbf{M}^a = L^2([0, a], \mathrm{d}m).$$

Suppose $\operatorname{span}\{\mathscr{U}_k, \mathscr{U}_k^* : k \in \sigma(m)\} = \mathbf{K}^a$ and let $g \in \mathbf{M}^a$ with $g \perp y_a(x, k)$, $k \in \sigma(m)$ in $\mathbf{M}^a$. The function $G = \begin{pmatrix} 0 \\ g \end{pmatrix}$ being orthogonal to $\mathbf{N}^a$, we get from (10) that $G \perp \mathscr{U}_k$, $G \perp \mathscr{U}_k^*$ for $k \in \sigma(m)$ and therefore $g = 0$.

Suppose now that $\operatorname{span}\{y_a(x, k) : k \in \sigma(m)\} = \mathbf{M}^a$, and pick any function $g \in \mathbf{N}^a$, $g(a) = 0$ with $g \perp y_a(x, k)$ for $k \in \sigma(m)$ in $\mathbf{N}^a$. Then

$$0 = \int_0^a y_a^+(x, k)\overline{g^+(x)}\,\mathrm{d}x = \int_0^a y_a^+(x, k)\,\mathrm{d}\overline{g(x)} = -k^2 \int_{0-}^a y_a(x, k)\overline{g(x)}\,\mathrm{d}m(x),$$

which implies $g \equiv 0$. It follows that $\operatorname{span}\{y_a(x, k) : k \in \sigma(m)\} = \mathbf{N}^a$ and finally $\operatorname{span}\{\mathscr{U}_k, \mathscr{U}_k^* : k \in \sigma(m)\} = \mathbf{K}^a$ (see (10)). ▨

Since semigroups $\{Z_t\}_{t \geqslant 0}$ and $\{\mathcal{M}_t\}_{t \geqslant 0}$ are unitarily equivalent by Theorem 1.5, the question of completeness of $\{\mathscr{U}_k, \mathscr{U}_k^*\}$ in $\mathbf{K}^a$ reduces to that for the model semigroup $\{\mathcal{M}_t\}_{t \geqslant 0}$, i.e. to the joint completeness problem in $K_s = H_+^2 \ominus SH_+^2$ of

$$\text{(12)} \qquad \left\{\frac{s(z)}{z - k}\right\}_{k \in \sigma(m)}, \quad \left\{\frac{1}{z - k}\right\}_{k \in \sigma(m)},$$

$S(z) = \mathrm{e}^{2\mathrm{i}(a - a_m)z} B(z) = \theta B$ being the characteristic function of $\{\mathcal{M}_t\}_{t \geqslant 0}$.

In case $a = a_m$ we have $S = B$ and it is well-known that each of the above families is complete in K_B (see, for example, [12]).

Suppose that $a - a_m = d > 0$.

LEMMA 2.2. *The families* (12) span K_S, $S = e^{2idz}B = \theta B$ *if and only if* $\{e^{ikx}\}_{k \in \sigma(m)}$ *is complete in* $L^2(0, 2d)$.

Proof. We have

$$K_S \ominus K_\theta = \theta \cdot K_B = \theta \cdot \text{span}\left\{\frac{B(z)}{z - \bar{k}} : k \in \sigma(m)\right\} = \text{span}\left\{\frac{S(z)}{z - \bar{k}} : k \in \sigma(m)\right\}.$$

Therefore it is clear that the families (12) span K_S iff the family of orthogonal projections $P_\theta\left(\frac{1}{z - \bar{k}}\right)$, $k \in \sigma(m)$ spans K_θ. The Fourier transform

$$\hat{f}(x) = \int_{\mathbf{R}} e^{-itx} f(t)\, dt$$

maps H^2_+ isometrically onto $L^2(0, +\infty)$. The space K_θ is mapped onto the subspace $L^2(0, 2d)$, while P_θ turns into the multiplication by the indicator $\chi_{(0,2d)}$ of $(0, 2d)$. Finally

$$P_\theta \frac{1}{z - \bar{k}} \overset{\wedge}{\longrightarrow} -2\pi i\, e^{-\bar{k}ix} \chi_{(0,2d)}(x), \quad k \in \sigma(m).$$

The last lemma reduces the joint completeness problem to a special case of completeness problem of exponentials in $L^2(0, 2d)$.

LEMMA 2.3. *Let F be an entire function of exponential type with the width of indicator diagram equal to* $2d > 0$. *Suppose that all roots of F are simple and that*

$$\int_{\mathbf{R}} \frac{1}{1 + y^2} \frac{dy}{|F(y)|^2} < +\infty.$$

Then the family $\{e^{ikx}\}_{F(k)=0}$ *is complete in* $L^2(0, 2d)$.

Proof. The function F belongs to the Cartwright class $\mathscr{C}$ and we can assume without loss of generality that the set σ of all roots of F is contained in $\mathbf{C}_+$. Otherwise we can multiply F by a corresponding Blaschke product to transform all roots in $\mathbf{C}_+$. Multiplying F by an appropriate exponential we may assume that

$$F(z) = e^{idz} \cdot \text{v.p.} \prod_{\lambda \in \sigma} \left(1 - \frac{z}{\lambda}\right)$$

and the indicator diagram of F is $[0, -2id]$.

Suppose that the family $\{e^{i\lambda x}\}_{\lambda \in \Lambda}$ is not complete in $L^2(0, 2d)$. Then there exists a non-zero function $g \in L^2(0, 2d)$ such that

$$G(z) = \int_0^{2d} e^{izx} g(x)\,dx$$

vanishes on σ. Multiplying G by an appropriate Blaschke product we can reflect all the zeros of G in $\mathbf{C}_-$ to $\mathbf{C}_+$ keeping $|G|^2$ invariant on $\mathbf{R}$. So we can assume that G does not have zeros in $\mathbf{C}_-$. Denote

$$\alpha = \inf\{t : t \in \operatorname{supp}(g)\}, \quad \beta = \sup\{t : t \in \operatorname{supp}(g)\}.$$

Since $G \in H^2_+$, we have

$$G(z) = \begin{cases} C_+ e^{i\alpha z} B_G(z) G_e(z) & \text{for } z,\ \operatorname{Im} z \geqslant 0 \\ C_- e^{i\beta z} G_e(z) & \text{for } z,\ \operatorname{Im} z \leqslant 0 \end{cases}$$

where $C_\pm \in \mathbf{C}$, $|C_+| = |C_-| = 1$, B_G stands for the Blaschke product and

$$f_e(z) \overset{\text{def.}}{=} \exp\left\{\frac{1}{\pi i}\int_{\mathbf{R}} \frac{tz+1}{t-z}\cdot\frac{\log f(t)}{1+t^2}\,dt\right\}$$

denotes the outer factor of f. Similarly,

$$F(z) = \begin{cases} C^\bullet_+ B_F(z) F_e(z) & \text{for } z,\ \operatorname{Im} z \geqslant 0 \\ C^\bullet_- e^{2idz} F_e(z) & \text{for } z,\ \operatorname{Im} z \leqslant 0. \end{cases}$$

Consider now an auxiliary entire function $h(z) = e^{i(2d-\beta)z}\cdot\dfrac{G(z)}{F(z)}$ which can be factored as follows

$$h(z) = \begin{cases} \dfrac{C_+}{C^\bullet_+}\, e^{i(2d-(\beta-\alpha))z}\cdot\dfrac{B_G}{B_F}\cdot\dfrac{G_e}{F_e}, & \operatorname{Im} z \geqslant 0 \\[2ex] \dfrac{C_-}{C^\bullet_-}\cdot\dfrac{G_e}{F_e}, & \operatorname{Im} z \leqslant 0. \end{cases} \tag{13}$$

Notice that $I(z) = \exp i(2d - (\beta - \alpha))z \cdot B_G/B_F$ is an inner function in $\mathbf{C}_+$.

Case 1. $I(k) = 0$ for some $k \in \mathbf{C}_+$. Let $H(z) = \dfrac{h(z)}{z-k}$. Then

$$\int_{\mathbf{R}} |H(x)|\,dx = \int_{\mathbf{R}} \frac{|G(x)|}{|x-k|\cdot|F(x)|}\,dx \leqslant \left(\int_{\mathbf{R}} \frac{dx}{|x-k|^2|F(x)|^2}\right)^{1/2}\cdot\left(\int_{\mathbf{R}} |G(x)|^2\,dx\right)^{1/2},$$

which implies that $H \in H^1_+ \cap H^1_- = 0$ (see (13), [14]).

Case 2. $B_G = B_F$ but $G(x_0) = 0$ for some real x_0. Put again $H(z) = \frac{h(z)}{z - x_0}$. Then

$$\int_{|x-x_0|\geq 1} |H(x)|\,dx \leq \left(\int_{|x-x_0|\geq 1} \frac{dx}{|x-x_0|^2|F(x)|^2}\right)^{1/2}\left(\int_{\mathbf{R}} |G(x)|^2\,dx\right)^{1/2} < +\infty$$

and therefore $H \in H^1_+ \cap H^1_- = 0$.

In both cases we get $G = 0$ which contradicts to the assumption $g \neq 0$. We have to consider the last possibility.

Case 3. $B_G = B_F$ and G is root free on the real line. In this case every root of G is simple and G and F have the same set of roots. Therefore $G = e^{ic_1 z}\cdot F\cdot c_2$ for some $c_1 \in \mathbf{R}$, $c_2 \in \mathbf{C}$, $c_2 \neq 0$. We have

$$+\infty = \int_{\mathbf{R}} \frac{|F(y)|}{(1+y^2)^{1/2}|F(y)|}\,dy \leq \left(\int_{\mathbf{R}} \frac{dy}{(1+y^2)|F(y)|^2}\right)^{1/2}\cdot\frac{1}{|c_2|}\cdot\left(\int_{\mathbf{R}} |G(y)|^2\,dy\right)^{1/2}$$

and $G \notin L^2(\mathbf{R})$ which clearly is a contradiction. ▨

I $(= I(\lambda))$ is said to be the completeness interval of $\lambda = \{\lambda_n\}$ if $\{e^{i\lambda_n x}\}$ is complete in L^2 on every interval of length less than I and on no larger interval.

See the proof of the following result in [15] (Theorem 28).

THEOREM 2.4. *Let $\{\lambda_n\}$ be the zeros of an entire function F of finite type with the width of indicator diagram $2d > 0$ and such that*

$$\int_{\mathbf{R}} \frac{\log^+|F(x)|}{1+x^2}\,dx < +\infty.$$

Then $(0, 2d) = I(\lambda)$.

It follows from Theorem 1.1 and from the well-known fact that the indicator diagram of the canonical product

$$\text{v.p.}\prod_{\lambda\in\Lambda}\left(1 - \frac{z}{\lambda}\right)$$

is $[-Ti, Ti]$ (see [10] or [11]), that $(0, 2d)$ with $d = \int_0^{a_m} [m'(s)]^{1/2}ds$ is the completeness interval for $\{k\}_{k\in\sigma(m)}$. Moreover $\{e^{ikx}\}_{k\in\sigma(m)}$ is complete in $L^2(0, 2d)$ in view of Lemma 2.3. The proof is finished by application of Lemma 2.1 and Lemma 2.2 which permits us to calculate the critical point $a - d = a_m$, $a = a_m + \int_0^{a_m} [m'(s)]^{1/2}\,ds$. ▨

3. UNCONDITIONAL BASES

3.1. The main ingredient of the proof of Theorem 3 is the following.

THEOREM 3.1. *The following are equivalent:*

1) Λ *is a subset of* $\mathbf{C}_\delta$ *(for some* $\delta > 0$*) invariant under the transform* $z \to -\bar{z}$ *and such that* $\{e^{i\lambda x}\}_{\lambda \in \Lambda}$ *forms an unconditional basis in* $L^2(0, 2d)$ *for some* $d > 0$.

2) *There exists a string* m *with* $a = a_m + d$ *and* $\sigma(m) = \Lambda$ *such that the joint family* $\{\mathcal{U}_k, \mathcal{U}_k^* : k \in \sigma(m)\}$ *of the eigenvectors of* T *and* T^* *forms an unconditional basis in* $\mathbf{K}^a$.

First we show how the proof of Theorem 3 may be finished now.

The part 1) ⇒ 2) clearly implies the existence of a string m with $\sigma(m) = \Lambda$. Lemma 1.6 and (10) would imply the desired conclusion if we could prove that $\delta_m = \sup\{2b|S'(ib)| : ib \in \sigma(m)\} < 1$. But $S = e^{idz} \cdot B(z)$ and $d > 0$. It has been noticed in 1.6 that this implies $\delta_m < 1$. ▨

3.2. PROOF OF THEOREM 3.1. 1) ⇒ 2). Let Λ satisfy the hypotheses of 1). By Theorem 1.7 there exists an entire function F in $\mathcal{M}_{2d}$, $F(0) = 1$ with simple zeros whose zero-set is precisely Λ. Since F satisfies the Muckenhoupt condition (A_2) on $\mathbf{R}$ it follows that

$$\int_{\mathbf{R}} \frac{1}{1+y^2} \cdot \frac{dy}{|F(y)|^2} < +\infty$$

which implies by Theorem 1 that Λ is the spectrum of (1) for some string m and $a \geqslant a_m$. To calculate a consider the indicator diagram of F which is $[0, -i2d]$, and the canonical product

$$E(a_m, z) = \text{v.p.} \prod_{\lambda \in \Lambda} \left(1 - \frac{z}{\lambda}\right)$$

whose indicator diagram is equal to $[-iT(a_m), iT(a_m)]$. The zero-sets of $F^{\#}$ and E being identical, the lengths of the indicator diagrams $2d$ and $2T(a_m)$ of F and E must coincide. Therefore $T(a_m) = d > 0$,

$$E^{\#}(a, z) = e^{iT(a_m)z} E^{\#}(a_m, z) = F(z),$$

and the characteristic function of $\{Z_t\}_{t \geqslant 0}$ is

$$S(z) = \frac{E^{\#}(a, z)}{E(a, z)} = e^{2idz} B = \theta \cdot B,$$

where B stands for the Blaschke product with roots in Λ.

By Theorem 1.7 $K_S = K_B + \theta K_B$ and the angle between K_B and $\theta \cdot K_B$ is positive. The families (12) form unconditional bases in their closed linear spans θK_B and K_B because $\Lambda \in$ (CN) [12]. This implies (see (9)) that $\{\mathscr{U}_k, \mathscr{U}_k^* : k \in \Lambda\}$ forms an unconditional basis in $\mathbf{K}^a$.

2) $\Rightarrow$ 1). Let m be a string satisfying the hypotheses of 1). By Theorem 1.5 the semigroup $\{Z_t\}_{t\geqslant 0}$ is unitarily equivalent to the model semigroup $\{\mathfrak{M}_t\}$ with the characteristic function $S = e^{idz} \cdot B$. Clearly the angle between the spaces

$$\theta \cdot K_B = \operatorname{span}\{\mathscr{T}\mathscr{U}_k : k \in \sigma(m)\}, \quad K_B = \operatorname{span}\{\mathscr{T}\mathscr{U}_k^* : k \in \sigma(m)\}$$

(see (9)) must be positive which implies, in particular, that $d > 0$. On the other hand $K_B + \theta \cdot K_B = K_S$ which implies $a = a_m + T(a_m)$ and therefore $d = T(a_m)$ (see Theorem 2). The spectrum $\sigma(m)$ lies in $\mathbf{C}_\delta$, $\delta > 0$. Indeed, the orthogonal projection of $\frac{\sqrt{2\operatorname{Im} k}}{z - \bar{k}}$, $k \in \sigma(m)$ onto K_θ has the norm $(1 - |\theta(k)|^2)^{1/2}$. The space $\theta \cdot K_B$ is the orthogonal complement of K_θ in K_S. Therefore by Pythagora theorem the projection of $\frac{\sqrt{2\operatorname{Im} k}}{z - \bar{k}}$ onto $\theta \cdot K_B$ has the norm $|\theta(k)|$. It follows

$$|\theta(k)| \leqslant \cos\alpha < 1$$

where α denotes the angle between K_B and $\theta \cdot K_B$. It remains to apply Theorem 1.7.

3.3. Examples of unconditional bases of exponentials $\{e^{i\lambda_n x}\}_{n\in\mathbf{Z}}$ in $L^2(0, a)$ with $\overline{\lim_{|n|\to+\infty}} \operatorname{Im}\lambda_n = +\infty$ given in [1] lead to interesting examples of spectra $\sigma(m)$. In particular, it can be proved (see Theorem 3.4 (V. I. Vasjunin), part III of [1]) that any sequence of points a_n in $\mathbf{C}_+$ satisfying $\{a_n\} \in$ (CN) and $\lim_n \operatorname{Im} a_n = +\infty$ can be complemented up to such a family $\{\lambda_n\}_{n\in\mathbf{Z}}$ of points in $\mathbf{C}_\delta$ (for some $\delta > 0$) for a given $a > 0$ that $\{e^{i\lambda_n t}\}_{n\in\mathbf{Z}}$ forms an unconditional basis in $L^2(0, a)$.

In terms of problem (1) this means that such a family $\{a_n\}$ can be complemented to be the spectrum of (1). The obtained string m may be considered as a "small" perturbation of the homogeneous string in the sense that

$$0 < \inf_{x\in\mathbf{R}} |E(a_m, x)| \leqslant \sup_{x\in\mathbf{R}} |E(a_m, x)| < +\infty$$

(see [4]), which implies that $\{y_a(x, k)\}_{k\in\sigma(m)}$ forms an unconditional basis in $L^2([0, a], dm)$, $a = a_m + T(a_m)$, but at the same time

$$\sup\{\operatorname{Im} k : k \in \sigma(m)\} = +\infty.$$

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Received November 21, 1983.