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THE SPECTRAL CATEGORY AND THE CONNES INVARIANT Γ

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INTRODUCTION

In this paper we describe a new approach to the harmonic analysis of the action of a locally compact group G on a von Neumann algebra M. In particular, one would like to understand how the action of G on the linear space M is related to the algebraic structure of M. In the context of von Neumann algebras, the natural step is to compare the action of G on M with the continuous unitary representations of G and to relate the product and adjoint in M to the tensor product and conjugation of such representations.

When G is abelian, the theory of spectral subspaces is a useful tool for analyzing the action. This theory was first formalized by Arveson [1], although most of the basic techniques had long been used in theoretical physics. For non-abelian groups, no such formalism exists as yet, although there is again a long practical tradition in theoretical physics of treating particularly the action of compact non-abelian groups in terms of multiplets of operators transforming according to a given irreducible representation σ of G. The corresponding spectral subspace M_σ appears naturally as a subspace of $H_\sigma^*\otimes M$ rather than of M itself. Here H_σ is the Hilbert space of the representation σ and H_σ^* its dual space. Spectral subspaces of this type were introduced in [16] in the case of a compact group and will be employed here in preference to the subspaces of M proposed by Evans and Sund [6] as they have the advantage of exhibiting explicitly the transformation law under G.

For non-abelian groups, the spectral subspaces on their own are not much use but must be seen as part of a wider structure termed the spectral category which includes the algebras of spherical functions of Landstad [8]. These algebras arise naturally in studying spectral subspaces because, whereas the left support of M_σ is in the fixed-point algebra, its right support is a projection in the corresponding algebra of spherical functions.

Spectral subspaces for reducible representations appear naturally when relating the product in M to the tensor product of representations, since the irreducible representations of non-abelian groups are not closed under tensor products.

Their inclusion in the formalism is vital because, not only does the cross product then appear as part of the spectral category, but there is also a natural action of the representation theory on the spectral category. Both of these features will be exploited in an essential way.

On the other hand, although the spectral subspaces M_σ for reducible representations generalize the notion of point spectrum, they are no substitute for a notion of continuous spectrum. Thus our dual object for a locally compact group G is the representation theory of G treated in a purely algebraic fashion without notions of topology or Borel structure. For this reason, we can at most hope that our methods will prove adequate to handle integrable actions of G (cf. [3; III]).

The first part of this paper introduces the spectral category, compares it with other categories associated with the action of a group on a von Neumann algebra and uses it to define various invariants for the action. No attempt is made to exploit these invariants in a systematic way. Instead, in the second part of this paper, we concentrate on a single invariant, the Connes invariant Γ, generalized to the case of integrable actions of non-abelian locally compact groups. We are able to obtain satisfactory analogous of most known results for abelian groups. This serves to establish the utility of the spectral category as a device for harmonic analysis.

1. THE SPECTRAL CATEGORY

Let G be a locally compact group and let $\mathscr{H}(G)$ denote the W^*-category [7] of Hilbert spaces carrying continuous unitary representations of G, i.e. an object of $\mathscr{H}(G)$ is a Hilbert space H_σ where $\sigma : G \to \mathscr{B}(H_\sigma)$ is a continuous unitary representation of G on H_σ. An arrow from H_σ to H_τ is any bounded linear map from H_σ to H_τ. As an abstract W^*-category $\mathscr{H}(G)$ is independent of G up to equivalence. In particular $\mathscr{H}(G)$ is equivalent to the W^*-category of Hilbert spaces $\mathscr{H}$. Like $\mathscr{H}$, $\mathscr{H}(G)$ carries a natural monoidal structure (tensor product) and to economize on notations we shall suppose that both $\mathscr{H}(G)$ and $\mathscr{H}$ have a strictly associative tensor product, i.e. that they are strict monoidal W^*-categories[1]. $\mathscr{H}(G)$ differs from $\mathscr{H}$ in that it carries a natural action of G, defined on the objects by $gH_\sigma = H_\sigma$ and on the arrows by

$$x \in (H_\sigma, H_\tau) \to gx = \tau(g)x\sigma(g)^*.$$

Furthermore

$$g(x \otimes y) = gx \otimes gy.$$

[1] A convenient way of actually realizing this would be to use a full monoidal subcategory of Example 3.4 of [14], where the von Neumann algebra M would properly have to be taken from some large universe.

We denote the monoidal W^*-category of fixed points of $\mathscr{H}(G)$ under this action by Rep(G); the object corresponding to H_σ will be denoted simply by σ. Thus $t \in (\sigma, \tau)$ if $t \in (H_\sigma, H_\tau)$ and $\tau(g)t = t\sigma(g)$. In other words, Rep(G) is just the monoidal W^*-category of continuous unitary representations of G.

Now Rep(G) also acts in a natural way on $\mathscr{H}(G)$: to each object σ of Rep(G) corresponds a functor $H_\rho \to H_\sigma \otimes H_\rho$; $x \to 1_\sigma \otimes x$ and to each $t \in (\sigma, \tau)$ corresponds a natural transformation

$$H_\rho \to t \otimes 1_\rho \in (H_\sigma \otimes H_\rho,\ H_\tau \otimes H_\rho).$$

The actions of G and Rep(G) on $\mathscr{H}(G)$ commute.

Now let α denote a σ-continuous action of G on a von Neumann algebra M. Then we can form[1)] the W^*-category $\mathscr{H}(G) \otimes M$; it also carries an action of G, the tensor product of the actions on $\mathscr{H}(G)$ and M, so if $x \in (H_\sigma, H_\tau)$, $y \in M$, then

$$g(x \otimes y) = gx \otimes \alpha_g(y) = \tau(g) x \sigma(g)^* \otimes \alpha_g(y), \quad g \in G.$$

The fixed points under this action will be a W^*-category denoted by Sp(M, α) and called the *spectral category* of the action. The objects of this category will be denoted $\sigma \otimes \alpha$, $\tau \otimes \alpha$, ..., so that $t \in (\sigma \otimes \alpha, \tau \otimes \alpha)$ if and only if

$$t \in (H_\sigma, H_\tau) \otimes M$$

and

$$i \otimes \alpha_g(t) = (\tau(g)^* \otimes 1) t (\sigma(g) \otimes 1) \quad g \in G.$$

Since the action of Rep(G) on $\mathscr{H}(G)$ commutes with the action of G, Sp(M, α) carries an induced action of Rep(G). Thus an object σ of Rep(G) gives a normal $*$-functor F_σ:

$$F_\sigma(\tau \otimes \alpha) = \sigma \otimes \tau \otimes \alpha$$

$$F_\sigma(x) = 1_\sigma \otimes x$$

and each $t \in (\sigma, \sigma')$ gives a bounded natural transformation $F(t) \in (F_\sigma, F_{\sigma'})$ defined by

$$F(t)_{\tau \otimes \alpha} = t \otimes 1_\tau \otimes 1.$$

The spectral category is intimately related to various W^*-categories of modules which can be used to study the W^*-system $\{M, \alpha\}$. We have relegated this discussion to an appendix so that the main results can be reached with a minimum of formalism.

[1)] This is a special case of the tensor product $\mathfrak{A} \otimes \mathfrak{B}$ of two W^*-categories. The objects of $\mathfrak{A} \otimes \mathfrak{B}$ are denoted $A \otimes B$ where A and B are objects of $\mathfrak{A}$ and $\mathfrak{B}$ respectively. $(A \otimes B, A' \otimes B') = (A, A') \otimes (B, B')$ where the tensor product is defined analogously to the tensor product of von Neumann algebras.

2. SPECTRAL SUBSPACES AND SPECTRAL INVARIANTS

If σ is a continuous unitary representation of the locally compact group G then we write

$$M_\sigma = (\sigma \otimes \alpha, i \otimes \alpha)$$

where i denotes the trivial representation of G on the Hilbert space $\mathbf{C}$. Although the linear spaces M_σ, introduced in [15, 16], are not subspaces of M, they are a useful generalization of the notion of spectral subspace to non-abelian groups. They have the advantage that their elements are determined by a simple explicit transformation law under G. Thus $x \in M_\sigma$ if and only if $x \in (H_\sigma, \mathbf{C}) \otimes M$ and

$$1 \otimes \alpha_g(x) = x\sigma(g) \otimes 1 \quad g \in G.$$

An element of M_σ thus corresponds to the physicists' notion of a multiplet of operators transforming under G according to the representation σ.

We will now discuss the relation of M_σ to other useful generalizations of the notion of spectral subspace, restricting our attention to irreducible representations of a compact group. If $x \in M_\sigma$ and $t \in (\mathbf{C}, H_\sigma)$, then $xt \otimes 1$ is an element of M and may be termed an irreducible tensor of type σ in M. The linear space of such tensors is the spectral subspace $M(\sigma)$ of M introduced by Evans and Sund [6] in the more general context of actions on Banach space. $M(\sigma)$ is the image of M under a projection p_σ onto the subspace of those elements of M which transform under G according to some multiple of the representation σ. Specifically, we have

$$p_\sigma(y) = d(\sigma) \int \operatorname{tr}(\sigma(g)^*) \alpha_g(y) \, d\mu(g), \qquad y \in M$$

where $d(\sigma)$ is the dimension of σ and μ is the normalized Haar measure.

The relation between M_σ and $M(\sigma)$ can be made more precise: both M_σ and $M(\sigma)$ have an inner product with values in the fixed-point algebra $M^\alpha =$ $= (i \otimes \alpha, i \otimes \alpha)$:

$$\langle x', x \rangle = x' x^* \quad x, x' \in M_\sigma$$

$$\langle y', y \rangle = \int \alpha_g(y' y^*) \, d\mu(g) \quad y, y' \in M(\sigma).$$

Pick $e_i \in (\mathbf{C}, H_\sigma)$, $i = 1, 2, \ldots, d(\sigma)$ with $e_i^* e_j = \delta_{ij}$; then the maps $w_i : M_\sigma \to M(\sigma)$ defined by

$$w_i(x) = \sqrt{d(\sigma)} \, x e_i \otimes 1$$

have adjoints $w_i^* : M(\sigma) \to M_\sigma$ relative to the inner products given by

$$w_i^*(y) = \sqrt{d(\sigma)} \int \alpha_g(y) e_i^* \sigma(g)^* \otimes 1 \, d\mu(g).$$

It may be verified by direct computation that the w_i are isometries expressing $M(\sigma)$ as a direct sum of $d(\sigma)$ copies of M_σ. The right supports of M_σ and $M(\sigma)$ are the same projection of M^α. However the left support of M_σ, a projection of $(\sigma\otimes\alpha, \sigma\otimes\alpha)$, is equally important, although not in evidence if one just looks at $M(\sigma) \subset M$. Perhaps the most important advantage of $M(\sigma)$ is expressed in the simple equation

$$M(\sigma)^* = M(\bar{\sigma})$$

where $\bar{\sigma}$ denotes the conjugate representation. This shows, in particular, that the left support of $M(\sigma)$ is just equal to the right supports of $M(\bar{\sigma})$ and $M_{\bar{\sigma}}$.

There is a third variant on the notion of spectral subspace combining, as far as possible, the separate advantages of $M(\sigma)$ and M_σ. Let $d(\sigma)i$ denote the trivial representation of G on the representation space H_σ of σ, and set

$$M^\sigma = (\sigma\otimes\alpha, d(\sigma)i\otimes\alpha).$$

As a linear space this is again a direct sum of $d(\sigma)$ copies of M_σ and is therefore isomorphic to $M(\sigma)$. In fact the map $p_\sigma : M \to M(\sigma)$ factorizes through M^σ: if $y \in M$, set

$$p^\sigma(y) = \int \sigma(g)^* \otimes \alpha_g(y) \, d\mu(g) \in M^\sigma;$$

then

$$p_\sigma(y) = d(\sigma)\mathrm{Tr}(p^\sigma(y))$$

where $\mathrm{Tr} : \mathscr{B}(H_\sigma)\otimes M \to M$ is the conditional expectation determined by the trace. If $x \in M^\sigma$, then

$$d(\sigma)\mathrm{Tr}(x) = \sum_i (e_i^*\otimes 1)x(e_i\otimes 1) = \sqrt{d(\sigma)} \sum_i w_i(e_i^*\otimes 1x)$$

showing that $x \to \mathrm{Tr}(x)$ is an isomorphism of M^σ and $M(\sigma)$. A simple computation shows that the inner product on $M(\sigma)$ is related to the $(d(\sigma)i, d(\sigma)i)$-valued inner product on M^σ by

$$d(\sigma)\langle \mathrm{Tr}(x'), \mathrm{Tr}(x)^* \rangle = \mathrm{Tr}\, x'x^*.$$

Note too, that if we let G act on M^σ by $x \to \sigma(g)\otimes 1x$,

$$\mathrm{Tr}(\sigma(g)\otimes 1x) = \mathrm{Tr}(x\sigma(g)\otimes 1) = \mathrm{Tr}(i\otimes\alpha_g(x)) = \alpha_g\mathrm{Tr}(x)$$

o that this action corresponds to the original action α on $M(\sigma)$.

We can describe the antilinear isomorphism from M^σ to $M^{\bar\sigma}$ directly without passing through $M(\sigma)$. Let $J_\sigma : H_\sigma \to H_{\bar\sigma}$ be an antiunitary map with $J_\sigma \sigma(g) = \bar\sigma(g) J_\sigma$, $g \in G$. Let $\bar e_i = J_\sigma e_i$ and set

$$\bar x = \sum_{i,j} (\bar e_i e_j^* \otimes 1) x (e_i e_j^* \otimes 1).$$

The map $x \to \bar x$ is the required antilinear isomorphism from M^σ to $M^{\bar\sigma}$. If x is regarded as a matrix with values in M, then $\bar x$ corresponds to the "complex-conjugate matrix". Unfortunately, this operation is a priori only well defined when $d(\sigma) < +\infty$.

The pivotal notion of this paper is the spectral category rather than the spectral subspaces. It not only contains the spectral subspaces M_σ and M^σ but also the von Neumann algebras $(\sigma\otimes\alpha, \sigma\otimes\alpha)$. If $\sigma = i$, this is the fixed-point algebra M^α, if σ is irreducible $(\sigma\otimes\alpha, \sigma\otimes\alpha)$ is the algebra $\mathscr{B}(\bar\sigma)$ of spherical functions associated with $\bar\sigma$ in the sense of Landstad [8]. Whilst if ρ denotes the right regular representation of G then [4]

$$(\rho\otimes\alpha, \rho\otimes\alpha) = W^*(M, \alpha)$$

is the cross product of M by the action α of G. The spectral category thus unites these apparently disparate elements into an algebraic structure carrying a natural action of Rep(G) and, for this reason, is a natural tool when investigating the action α of G on M.

The spectral category also leads us to some natural spectral invariants for the action. We shall define spectral invariants so that they are subsets of objects of Rep(G) saturated under unitary equivalence. It then becomes meaningful to ask whether these invariants are closed under tensor products, conjugate and subrepresentations. In this context, it is useful to note than an invariant closed under tensor products, subrepresentations and conjugates of one-dimensional representations is automatically closed under conjugates of finite-dimensional representations. For, if σ is a d-dimensional representation, its conjugate $\bar\sigma$ may be realized by setting

$$\bar\sigma(g) = \det \sigma(g)^{-1} \sigma'(g), \quad g \in G$$

where σ' is the d-dimensional representation induced by σ on the space of totally antisymmetric tensors of rank $d - 1$ and $g \to \det \sigma(g)$ is nothing but the 1-dimensional representation on the space of totally antisymmetric tensors of rank d.

The natural definition of the spectrum of α in this framework is

$$\mathrm{Sp}(\alpha) = \{\sigma \in \mathrm{Rep}(G) \mid M_\sigma \neq 0\}.$$

Obviously, if $\sigma \in \mathrm{Sp}(\alpha)$, $\sigma \leqslant \tau$, then $\tau \in \mathrm{Sp}(\alpha)$ and the computations in this section show that $\mathrm{Sp}(\alpha)$ is closed under conjugates of finite-dimensional representations.

Of course even if M is a factor, knowing that $\sigma \in \mathrm{Sp}(\alpha)$ does not tell one much about the action, since if $e \in M^\alpha$, $e \neq 0, 1$, we do not a priori know whether this information bears on eMe, $(1-e)Me$, $eM(1-e)$ or $(1-e)M(1-e)$. Our next invariant sheds some light on this question. We define

$$\mathrm{QSp}(\alpha) = \{\sigma \in \mathrm{Rep}(G) \mid c(\sigma \otimes \alpha) = c(i \otimes \alpha)\}$$

where $c(\sigma\otimes\alpha)$, in conformity with the notation of [7], refers to the central support of $\sigma\otimes\alpha$ in the W^*-category $\mathrm{Sp}(M,\alpha)$. Clearly $\sigma, \sigma' \in \mathrm{QSp}(\alpha)$ implies $\sigma\oplus\sigma' \in \mathrm{QSp}(\alpha)$. Also, since tensoring by σ, being normal $*$-functor, preserves quasi-equivalence [7; Corollary 5.6], $\mathrm{QSp}(\alpha)$ is closed under tensor products and conjugates of 1-dimensional representations. If $\sigma \in \mathrm{QSp}(\alpha)$, then $(\sigma\otimes\alpha, \sigma\otimes\alpha)$ and M^α are Morita equivalent [7; Remark 7.9].

An action will be said to be *quasi-dominant* if the regular representation $\rho \in \mathrm{QSp}(\alpha)$. Tensoring the equality $c(\rho\otimes\alpha) = c(i\otimes\alpha)$ by σ, it now follows that every object of $\mathrm{Rep}(G)$ is in $\mathrm{QSp}(\alpha)$. Thus all objects of $\mathrm{Sp}(M, \alpha)$ are quasi-equivalent and hence have central support 1, i.e. are generators of $\mathrm{Sp}(M, \alpha)$.

The term dominant was introduced by Connes and Takesaki [3] for the action of separable locally compact groups on σ-finite von Neumann algebras. These separability conditions allow one to restrict attention to separable $\mathscr{H}(G)$ and make $\mathrm{Sp}(M, \alpha)$ into a σ-finite W^*-category. An action is then dominant if M^α is properly infinite, i.e. if $i\otimes\alpha$ has infinite multiplicity, and if $i\otimes\alpha$ and $\rho\otimes\alpha$ are equivalent as objects of $\mathrm{Sp}(M, \alpha)$. Hence, in their context, a quasi-dominant action is dominant if and only if M^α is properly infinite (cf. [7; Proposition 7.12]). When working up to multiplicity, the more economical notion is quasi-dominant.

For the remainder of this paper, the important class of actions will be the *integrable* actions, also introduced in [3]. For our purposes, it is convenient to define an action α to be integrable if $\rho\otimes\alpha$ is a generator in $\mathrm{Sp}(M, \alpha)$, or equivalent if $c(i\otimes\alpha) \leqslant c(\rho\otimes\alpha)$. It will be shown in Appendix B that this terminology is consistent with the usage of [3]. An integrable action is quasi-dominant if and only if $i\otimes\alpha$ is a generator in $\mathrm{Sp}(M, \alpha)$. In this case the center of the fixed-point algebra, the center of the cross product and the center of the spectral category are all isomorphic. In fact, the action is integrable if and only if the map $z \to z_{\rho\otimes\alpha}$ is an isomorphism from $Z\,\mathrm{Sp}(M, \alpha)$ to $Z\,W^*(M, \alpha)$ and it is then quasi-dominant if and only if $z \to z_{i\otimes\alpha}$ is an isomorphism from $Z\,\mathrm{Sp}(M, \alpha)$ to ZM^α.

The final invariant to be discussed in this section is the *monoidal spectrum:*

$$\mathrm{MSp}(\alpha) = \{\sigma \in \mathrm{Rep}(G) \mid \sigma\otimes\alpha \sim d(\sigma) i\otimes\alpha\}.$$

Here $\sim$ denotes the equivalence of objects in a W^*-category. This definition improves on [14; Definition 6.3] in that it can be employed even if M^α is not properly infinite (cf. [14; Proposition 3.6]). $\mathrm{MSp}(\alpha)$ is closed under tensor products, direct sums and conjugates of 1-dimensional representations.

To illustrate the role of this invariant, let G act on $L^\infty(G)$ by right translations. Let w_σ be the unitary operator on $\mathscr{B}(L^2(G, H_\sigma))$ defined by

$$(w_\sigma\xi)(g) = \sigma(g)\xi(g), \quad \xi \in L^2(G, H_\sigma).$$

w_σ intertwines $\sigma\otimes\rho$ and $d(\sigma)i\otimes\rho$ and is an element of $(H_\sigma, H_\sigma)\otimes L^\infty(G)$; hence every object of $\mathrm{Rep}(G)$ is in the monoidal spectrum of the action of G on $L^\infty(G)$. The same is now true of any action which is the dual of a coaction as follows at once from [9; Theorem II.2.2ii)] and the functoriality of the spectral category.

As is well known, the converse is not true even if G is abelian since M can be a twisted cross product by an action of $\hat{G}$. It is precisely for this reason that the study of faithful, ergodic actions of compact abelian groups is interesting [10].

3. THE CONNES INVARIANT Γ

We come now to the problem of defining the analogue of the invariant Γ introduced by Connes [2] for the action of abelian groups. His defining formula $\Gamma(\alpha) = \bigcap_e \mathrm{Sp}(\alpha_e)$, where e runs through the non-zero projections of M^α and the spectrum is understood in the sense of Arveson, makes it clear that the reduced actions α_e on the von Neumann algebras M_e are now involved in an essential way. This formula was examined by Evans and Sund [6] for the action of compact non--abelian groups, who gave an example to show that theorems valid in the abelian case now fail. We shall define two other invariants, denoted $\Gamma_0(\alpha)$ and $\Gamma_1(\alpha)$, which can be used to give analogues of theorems valid in the abelian case and which coincide for quasi-dominant actions. For reasons already given we restrict ourselves to integrable actions.

We define $\Gamma_1(\alpha) = \bigcap_e \mathrm{QSp}(\alpha_e)$, where e runs through the (non-zero) projections of M^α. This definition involves the spectral categories $\mathrm{Sp}(M_e, \alpha_e)$ of the reduced actions. However since $(\sigma\otimes\alpha_e, \tau\otimes\alpha_e)$ can be identified with $\{t \in (\sigma\otimes\alpha, \tau\otimes\alpha) \mid t = t(1_\sigma\otimes e) = (1_\tau\otimes e)t\}$, it will not be necessary to introduce these reduced spectral categories explicitly. The first result gives alternative characterizations of $\Gamma_1(\alpha)$.

PROPOSITION 1. *The following eight equivalent conditions define* $\sigma \in \Gamma_1(\alpha)$:

a) $c(i\otimes\alpha_e) = c(\sigma\otimes\alpha_e)$ *in* $\mathrm{Sp}(M_e, \alpha_e)$, $e \in M^\alpha$.

b) *Given projections* $e \in M^\alpha$, $f \in (\sigma\otimes\alpha, \sigma\otimes\alpha)$ *with* $0 \neq f \leqslant 1_\sigma\otimes e$, *there exists* $t \in (\sigma\otimes\alpha, i\otimes\alpha)$ *with* $etf \neq 0$.

c) $c(e)_{\sigma\otimes\alpha} \geqslant 1_\sigma\otimes e$, $e \in M^\alpha$.

a′), b′) *and* c′) *obtained by restricting attention in the above to projections* $e \in ZM^\alpha$.

d) $c(e)_{\sigma\otimes\alpha} = 1_\sigma\otimes e$, $e \in ZM^\alpha$.

e) $c(i\otimes\alpha) \geqslant c(\sigma\otimes\alpha)$ *and* $et = t1_\sigma\otimes e$, $t \in (\sigma\otimes\alpha, i\otimes\alpha)$, $e \in ZM^\alpha$.

Proof. a) implies a′) implies b′) trivially. Since by [7; Proposition 5.2], $c(e)_{\sigma\otimes\alpha} = \sup_{t\in(\sigma\otimes\alpha, i\otimes\alpha)} s(et)$, b) and c) are equivalent and so are b′) and c′). Further c′) implies c) since $c(e) = c(\bar{e})$ where $\bar{e}$ is the central support of e in M^α. Now b) implies that $c(i\otimes\alpha_e) \geqslant c(\sigma\otimes\alpha_e)$, $e \in M^\alpha$. If this inequality were strict for some $e \in M^\alpha$, we could find $e' \in M^\alpha$, $0 \neq e' \leqslant e$ with $e't(1_\sigma\otimes e') = e't(1_\sigma\otimes e)(1_\sigma\otimes e') = 0$, $t \in (\sigma\otimes\alpha, i\otimes\alpha)$. Hence b) implies a) and the first six conditions are equivalent. Applying c′) to $(1 - e) \in ZM^\alpha$, we deduce

$$1 - c(e)_{\sigma\otimes\alpha} \geqslant c(1 - e)_{\sigma\otimes\alpha} \geqslant 1_\sigma\otimes(1 - e).$$

Thus $c(e)_{\sigma\otimes\alpha} \leqslant 1_\sigma\otimes e$ so c′) implies d). Now d) implies e) trivially. If $e \in ZM^\alpha$, e) implies

$$[c(e)_{i\otimes\alpha} - e]t = t[c(e)_{\sigma\otimes\alpha} - 1_\sigma\otimes e] = 0, \quad t \in (\sigma\otimes\alpha, i\otimes\alpha)$$

and hence that $c(e)_{i\otimes\alpha} = e$ since the right support of $(\sigma\otimes\alpha, i\otimes\alpha)$ is 1. Thus e) implies d) and a fortiori c′) completing the proof.

If $\sigma \in \Gamma_1(\alpha)$ and $\tau \leqslant \sigma$ then $\tau \in \Gamma_1(\alpha)$. Since, picking an isometry $v \in (\tau, \sigma)$ and using d), we have

$$c(e)_{\tau\otimes\alpha} = (v^*\otimes 1)c(e)_{\sigma\otimes\alpha}(v\otimes 1) = 1_\tau\otimes e.$$

Bearing in mind the stability properties of $\mathrm{QSp}(\alpha)$, we see that $\Gamma_1(\alpha)$ is also closed under direct sums, tensor products and conjugation of finite-dimensional representations.

Thus $\Gamma_1(\alpha)$ has the good algebraic properties we might expect of an analogue of Connes' invariant $\Gamma(\alpha)$ which is a subgroup of $\hat{G}$ in the abelian case. However $\Gamma_1(\alpha)$ is not stable under cocycle perturbations. Now, in the abelian case, $\Gamma(\alpha)$ is also the kernel of the restriction of the dual action to the center of the cross product [3; Theorem III.3.2]. When G is not abelian, the center of the cross product will no longer be preserved under the action of a dual of G. Nevertheless, the following invariant $\Gamma_0(\alpha)$ is the natural analogue of the kernel of its restriction to the center within the framework of this paper (See Proposition 2).

We define $\sigma \in \Gamma_0(\alpha)$ if for each object τ of $\mathrm{Rep}(G)$

$$z_{\sigma\otimes\tau\otimes\alpha} = 1_\sigma\otimes z_{\tau\otimes\alpha}, \quad z \in Z\,\mathrm{Sp}(M, \alpha).$$

Again $\Gamma_0(\alpha)$ is closed under subobjects, direct sums, tensor products, and conjugation of finite-dimensional representations. Furthermore, d) of Proposition 1 shows that $\Gamma_0(\alpha) \subset \Gamma_1(\alpha)$.

Since α is integrable $Z\mathrm{Sp}(M, \alpha)$ and $ZW^*(M, \alpha)$ are isomorphic and we can expect that $\Gamma_0(\alpha)$ can be defined using the cross product. Let w_σ denote the unitary operator on $H_\sigma \otimes L^2(G) \simeq L^2(G, H_\sigma)$ defined by:

$$(w_\sigma \xi)(g) = \sigma(g)\xi(g), \quad \xi \in L^2(G, H_\sigma)$$

then

$$w_\sigma \in (\sigma \otimes \rho,\ d(\sigma) i \otimes \rho).$$

PROPOSITION 2. *$\sigma \in \Gamma_0(\alpha)$ if and only if $w_\sigma \otimes 1$ commutes with $1_\sigma \otimes z$ for each $z \in Z W^*(M, \alpha)$.*

However, since $\rho \otimes \alpha$ is a generator in $\mathrm{Sp}(M, \alpha)$, we can find partial isometries $v_k \in (\tau \otimes \alpha, \rho \otimes \alpha)$ with $\sum_k v_k^* v_k = 1_{\tau\otimes\alpha}$ [7; Proposition 7.3]. Hence

$$1_\sigma \otimes z_{\tau\otimes\alpha} = \sum_k 1_\sigma \otimes v_k^* z_{\sigma\otimes\rho\otimes\alpha} 1_\sigma \otimes v_k = z_{\sigma\otimes\tau\otimes\alpha}$$

and $\sigma \in \Gamma_0(\alpha)$.

Turning to the stability properties of Γ_0 under a change in the action, we first note that if we let G act trivially on $\mathscr{B}(H)$ then $\mathrm{Sp}(M \otimes \mathscr{B}(H), \alpha \otimes 1) = \mathrm{Sp}(M, \alpha) \otimes \mathscr{B}(H)$ so that $z \to z \otimes 1$ is an isomorphism from $Z\,\mathrm{Sp}(M, \alpha)$ to $Z\,\mathrm{Sp}(M, \alpha \otimes 1)$ hence $\Gamma_0(\alpha \otimes 1) = \Gamma_0(\alpha)$ and $\Gamma_1(\alpha \otimes 1) = \Gamma_1(\alpha)$.

Now let $g \to a(g)$ be a continuous unitary cocycle for the action α, then the perturbed action ${}_a\alpha$ is defined by

$${}_a\alpha_g(x) = a(g)\alpha_g(x)a(g)^*.$$

a is said to be square integrable if ${}_a\alpha$ is integrable [3; Chapter III]. If we regard M as acting on a Hilbert space H and define, cf. [18], $A \in \mathscr{B}(L^2(G, H))$ by

$$(A\xi)(g) = a(g^{-1})\xi(g), \quad \xi \in L^2(G, H),$$

then A is a unitary operator implementing an isomorphism of $W^*(M, \alpha)$ and $W^*(M, {}_a\alpha)$. Since $w_\sigma \otimes 1$ commutes with $1_\sigma \otimes A$, we deduce from Proposition 2,

COROLLARY 3. *If a is a square integrable cocycle for the action α, then $\Gamma_0(\alpha) = \Gamma_0({}_a\alpha)$.*

Thus, if $[\alpha]$ denotes the set of all actions which can be obtained from α by tensoring with a trivial action on some $\mathscr{B}(H)$ and then perturbing by a square integrable cocycle, we have $\Gamma_0(\alpha) = \Gamma_0(\beta)\ \forall\, \beta \in [\alpha]$. We can now draw two simple conclusions on the relation between Γ_0 and Γ_1.

PROPOSITION 4. a) *If α is quasi-dominant, $\Gamma_1(\alpha) = \Gamma_0(\alpha)$.*
b) $\Gamma_0(\alpha) = \bigcap_{\beta \in [\alpha]} \Gamma_1(\beta)$.

Proof. If $\sigma \in \Gamma_1(\alpha)$ then by Proposition 1d)

$$z_{\sigma\otimes\alpha} = 1_\sigma \otimes z_{i\otimes\alpha}, \quad z \in Z\,\mathrm{Sp}(M, \alpha).$$

If α is quasi-dominant, $i\otimes\alpha$ is a generator of $\mathrm{Sp}(M, \alpha)$ and, arguing as in the latter half of the proof of Proposition 2, we see that $\sigma \in \Gamma_0(\alpha)$ proving a). To prove b), we simply consider the W^*-system $\{M\otimes\mathscr{B}(H_\rho), \alpha\otimes\mathrm{ad}\,\rho\}$. $\alpha\otimes\mathrm{ad}\,\rho$ is a quasi--dominant element of $[\alpha]$. Hence $\Gamma_1(\alpha\otimes\mathrm{ad}\,\rho) = \Gamma_0(\alpha\otimes\mathrm{ad}\,\rho) = \Gamma_0(\alpha)$ proving b).

We close with some elementary remarks.

Let $H = \{h \in G \mid \sigma(h) = 1_\sigma, \sigma \in \Gamma_0(\alpha)\}$, then H is a closed normal subgroup of G. In the case of compact groups since $\Gamma_0(\alpha)$ is closed under tensor products, direct sums, subrepresentations and conjugations, $\Gamma_0(\alpha)$ corresponds to the set of representations of G/H. The same remark applies to $\Gamma_1(\alpha)$.

Any action of the form $\{\mathscr{B}(H_\sigma), \mathrm{ad}\,\sigma\}$ is a cocycle perturbation of the trivial action so that $\Gamma_0(\mathrm{ad}\,\sigma)$ contains only trivial representations. Even for such simple actions Γ_1 can contain nontrivial representations. If σ is the irreducible two-dimensional representation of $\mathbf{P}_3$, for example, Γ_1 corresponds to the representations of $\mathbf{P}_3/\mathbf{Z}_3 \simeq \mathbf{P}_2$.

If α is an integrable action of a locally compact abelian group then using [3; Theorem III.3.2, Lemma III.3.3] or [11; Theorem 3.1], we see that Γ_0 and Γ_1 coincide with Connes' invariant Γ in restriction to irreducible representations.

We have restricted ourselves to integrable actions. For general actions of locally compact groups what is missing is not so much a suitable definition of Γ_0 which might be defined using Proposition 2 or set equal to $\Gamma_0({}_a\alpha)$ for some square--integrable cocycle a but an invariant $\Gamma_1(\alpha)$ which can be computed using spectral properties of the action α.

4. APPLICATIONS TO A CLASS OF ACTIONS

We clarify the relations between ZM^α, ZM, $ZW^*(M, \alpha)$ for integrable actions whose Connes invariant is maximal. Let π_α denote the embedding of M into $W^*(M, \alpha)$ defined by:

$$(\pi_\alpha(x)\xi)(g) = \alpha_g^{-1}(x)\xi(g), \quad \xi \in L^2(G, H).$$

Note that if $x \in M^\alpha$ then $\pi_\alpha(x) = 1_\rho \otimes x$.

For any action α we have an injection $\Phi : Z(M) \cap M^\alpha \to Z\,\mathrm{Sp}(M, \alpha)$ defined by $\Phi(z)_\sigma = 1_\sigma \otimes z$.

PROPOSITION 5. *Let α be an integrable action of G on M, then the following conditions are equivalent:*

a) *Φ is an isomorphism.*

b) *$ZM^\alpha = Z(M) \cap M^\alpha$ and α is quasi-dominant.*

c) $ZW^*(M, \alpha) \subset \pi_\alpha(ZM)$.
d) $ZW^*(M, \alpha) = \pi_\alpha(ZM^\alpha)$.
e) $\Gamma_0(\alpha)$ *contains every object* σ *of* Rep(G).
e′) $\Gamma_1(\alpha)$ *contains every object* σ *of* Rep(G).
f) $\rho \in \Gamma_0(\alpha)$.
f′) $\rho \in \Gamma_1(\alpha)$.

Proof. a) and b) are obviously equivalent. c) could have been written $ZW^*(M,\alpha) = \pi_\alpha(Z(M) \cap M^\alpha)$ and is therefore again equivalent to a). Since $ZW^*(M, \alpha)$ commutes with $\pi_\alpha(M)$, d) is apparently stronger than c) but a glance at b) shows they are again equivalent. a) implies e) trivially. Now by Proposition 1e) and Proposition 4 a), f′) implies f). On the other hand, if $\rho \in \Gamma_0(\alpha)$ then $z_{\rho\otimes\alpha} = 1_\rho \otimes z_{\iota\otimes\alpha}$, $\forall z \in Z\mathrm{Sp}(M, \alpha)$. Thus $x \to 1_\rho \otimes x = \pi_\alpha(x)$ is an isomorphism of ZM^α and $ZW^*(M, \alpha)$ so f) implies d), completing the proof.

For compact groups, it was implicitly shown in [15; cf. (3.6)] using condition b) of Proposition 1 that b) follows from e′).

The original motivation for Proposition 5 comes of course from [2; Theorem 2.4.1] and [3; Corollary III.3.4]. In this sense we have:

COROLLARY 6. *Let α be an integrable action of G on M; then $W^*(M, \alpha)$ is a factor if and only if α acts ergodically on the centre of M and one of the equivalent conditions of Proposition 5 hold.*

Motivated by the discussion in [9; Chapter VI] and [12] we close by noting conditions on the relative commutant which imply the equivalent conditions in Proposition 5.

PROPOSITION 7. *Let α be an integrable action of G on M. The following properties imply the equivalent conditions of Proposition 5:*
1) $\pi_\alpha(ZM)' \cap W^*(M, \alpha) = \pi_\alpha(M)$.
2) $\pi_\alpha(M)' \cap W^*(M, \alpha) \subset \pi_\alpha(M)$.
3) $\pi_\alpha(M)' \cap W^*(M, \alpha) = \pi_\alpha(ZM)$.
4) $(M^\alpha)' \cap M = ZM$ *and* α *is quasi-dominant.*
Furthermore 1) $\Rightarrow$ 2) $\Leftrightarrow$ 3) $\Rightarrow$ 4).

Proof. The equivalence of 2) and 3) is evident. Intersecting 1) with $\pi_\alpha(M)'$, we see that 1) implies 3). Since $ZW^*(M, \alpha) \subset \pi_\alpha(M)' \cap W^*(M, \alpha)$, 3) implies condition c) of Proposition 5, so that in particular α is quasi-dominant. But now 3) implies 4) by [9; Proposition VI.1.5], [12]. 4) obviously implies condition b) of Proposition 5, completing the proof.

Note that 2) holds whenever $\pi_\alpha(M)$ is maximal abelian in $W^*(M, \alpha)$.

APPENDIX A: CATEGORIES OF MODULES

In this appendix, we relate the spectral category to categories of modules associated with the W^*-system $\{M, \alpha\}$.

We regard $H_\sigma \otimes M$ as a right M-module in the obvious way and give it a M-valued inner product

$$\langle h \otimes m, h' \otimes m' \rangle = (h, h') m^* m', \quad h, h' \in H_\sigma, \; m, m' \in M,$$

and a norm

$$\|x\|^2 = \|\langle x, x \rangle\|, \quad x \in H_\sigma \otimes M.$$

$H_\sigma \otimes M$ is just a Hilbert direct sum of $\dim H_\sigma$ copies of M considered as a right M-module and is an example of a self-dual right Hilbert module in the sense of Paschke [13]. The bounded M-module homomorphisms from $H_\sigma \otimes M$ to $H_\tau \otimes M$ can be identified naturally with $(H_\sigma, H_\tau) \otimes M$, the set of arrows from $H_\sigma \otimes M$ to $H_\tau \otimes M$ in the W^*-category $\mathscr{H}(G) \otimes M$. A formal proof of these simple statements can be got by noting that $H_\iota \otimes M$ is a generator of $\mathscr{H}(G) \otimes M$ and then invoking [17; Theorem 2.3]. Now $H_\sigma \otimes M$ also carries a natural action of G:

$$g(h \otimes m) = \sigma(g) h \otimes \alpha_g(m), \quad h \in H_\sigma, \; m \in M, \; g \in G.$$

This action makes $H_\sigma \otimes M$ into a self-dual Hermitian $\{M, \alpha\}$-module in the sense of [17] and identifies $\mathscr{H}(G) \otimes M$ as a full subcategory of the W^*-G-category $\mathrm{Hmod}\{M, \alpha\}$ introduced there.

Now, as discussed in Section 1, $\mathscr{H}(G)$ and hence $\mathscr{H}(G) \otimes M$ carry a natural action of $\mathrm{Rep}(G)$ commuting with the action of G. The action of $\mathrm{Rep}(G)$ extends in an obvious way to an action of $\mathrm{Hmod}\{M, \alpha\}$ commuting with the action of G. It follows that $\Gamma\mathrm{Hmod}\{M, \alpha\}$, the fixed-point W^*-category of $\mathrm{Hmod}\{M, \alpha\}$ under G, carries an action of $\mathrm{Rep}(G)$ and contains $\mathrm{Sp}(M, \alpha)$ as a full subcategory stable under the action of $\mathrm{Rep}(G)$.

Unlike $\mathrm{Sp}(M, \alpha)$, $\Gamma\mathrm{Hmod}\{M, \alpha\}$ is closed under direct sums and has sufficient subobjects and one might be tempted to use it in place of $\mathrm{Sp}(M, \alpha)$ to study the invariant Γ. However, even if α is integrable, not every object of $\mathrm{Hmod}\{M, \alpha\}$ need be square integrable. Instead, one should consider the full subcategory $\mathrm{Hmod}^2\{M, \alpha\}$ of square integrable modules and the corresponding fixed-point category $\Gamma\mathrm{Hmod}^2\{M, \alpha\}$, which still contains $\mathrm{Sp}(M, \alpha)$ provided α is integrable.

APPENDIX B

B.1. For an action α of a locally compact abelian group G on a von Neumann algebra M, Connes and Takesaky introduced in [3] the concept of integrability: α is said to be *integrable* when the set of $x \in M$ such that $\int_G \alpha_g(x^* x)\, \mathrm{d}\mu_l(g) < +\infty$

is σ-weakly dense in M. $d\mu_l$ denotes left invariant Haar measure on G and the integral is defined as limit of the increasing net $\int_K \alpha_g(x^*x)\, d\mu_l(g)$ indexed by compact sets K of G. Also we shall consider right invariant Haar measure $d\mu_r$ as specified by the condition

$$\int f(g)\, d\mu_r(g) = \int f(g)\Delta(g^{-1})\, d\mu_l(g)$$

for any continuous function f on G with compact support.

In [9] such actions are characterized by semi-finiteness of an operator-valued weight ξ_α from M to M^α.

In Section 2 we have proposed, for non-abelian G, to say that α is an integrable action when the object $\rho\otimes\alpha$ is a generator in the category $\mathrm{Sp}(M, \alpha)$. Our aim in the present appendix is to show the equivalence of the two definitions.

B.2. In Section 1, $\mathrm{Sp}(M, \alpha)$ was defined as fixed points of the action $\tilde{\alpha}$: $t \to \tilde{\alpha}_g(t) = (\tau(g)\otimes 1)(1\otimes\alpha_g(t))(\sigma(g)^*\otimes 1)$, $t \in (H_\sigma\otimes M, H_\tau\otimes M)$, of G on the category $\mathscr{H}(G)\otimes M$. We want to derive from this action an operator-valued weight of $\mathscr{H}(G)\otimes M$ onto $\mathrm{Sp}(M, \alpha)$ whose semi-finiteness will characterize integrable α.

Let α be an action of a locally compact non-abelian group G on a W^*-category $\mathfrak{A}$ for which objects are invariant, that is $\alpha_g(t) \in (A, B)$ $\forall g \in G$ whenever $t \in (A, B)$ in $\mathfrak{A}$.

Let $Q_\alpha = \left\{ t \in \mathfrak{A} \,\middle|\, \int \alpha_g(t^*t)\, d\mu_l(g) < +\infty \right\}$. Essentially for the same reasons as for weights defined on a W^*-category [7], we can see that Q_α is a left ideal in $\mathfrak{A}$. Let P_α the closed linear span of $\{s^*t \mid s \in Q_\alpha,\ t \in Q_\alpha\}$ and let $P_\alpha^+ = P_\alpha \cap \mathfrak{A}^+$ where $\mathfrak{A}^+$ denotes the positive part of $\mathfrak{A}$, i.e. elements t^*t, $t \in \mathfrak{A}$.

For $t \in P_\alpha^+$, let ξ_α defined by $\xi_\alpha(t) = \int \alpha_g(t)\, d\mu_l(g)$. Clearly ξ_α extends linearly to P_α giving a normal $\mathfrak{A}^\alpha$-valued weight on $\mathfrak{A}$. Moreover, if $t \in P_\alpha$ and $a \in \mathfrak{A}^\alpha$, $b \in \mathfrak{A}^\alpha$, with atb defined in $\mathfrak{A}$, then

$$atb \in P_\alpha \quad \text{and} \quad \xi_\alpha(atb) = a\xi_\alpha(t)b.$$

B.3. Following [3], we say that an object A of $\mathfrak{A}$ is square-integrable if $Q_\alpha \cap (A, A)$ is σ-weakly dense in (A, A).

LEMMA. *Let e_A be the right support of $Q_\alpha \cap (A, A)$ in (A, A). Then A is square-integrable if and only if $e_A = 1_A$. Moreover $e_A \in \mathfrak{A}^\alpha$ and $ae_A = e_Ba$ $\forall a \in (A, B) \cap \mathfrak{A}^\alpha$, hence the mapping $A \to e_A$ belongs to the center of $\mathfrak{A}^\alpha$.*

Proof. For a von Neumann algebra M with an hereditary cone P^+ in M^+, it is well known that the left support of the corresponding left ideal Q is $s(Q) = \bigvee\{e \in P^+ \mid e^*e = e\}$. Also the σ-weak closure of Q is $Ms(Q)$. Hence, if A is a square-integrable object, then $e_A = 1_A$. The converse is trivial.

Since P_α^+ is globally invariant under α, $s(Q_\alpha \cap (A, A)) = e_A$ belongs to $\mathfrak{A}^\alpha$.

If $t \in Q_\alpha \cap (B, B)$, $a \in \mathfrak{A}^\alpha \cap (A, B)$, then $ta \in Q_\alpha$ hence $ta = tae_A$. If moreover $z \in (C, B)$, then $z^*ta \in Q_\alpha$ and, for any projection e in $P^+ \cap (B, B)$ we have $ea \in Q_\alpha$, so that $ea = eae_A$. Hence $e_B a = e_B a e_A$ and the result follows.

B.4. Using the previous lemma, we can characterize a square-integrable object in the following way.

PROPOSITION. *There exists a unique element c in the center of $\mathfrak{A}^\alpha$ such that an object A is square-integrable if and only if $c(A) \leqslant c$.*

Proof. Let $c = \bigvee \{c(B) \mid B \text{ square-integrable object}\} \in \mathfrak{A}^\alpha$. For an object A of $\mathfrak{A}$ suppose $c(A) \leqslant c$. Then $c_A = 1_A$, but, for any square-integrable object B, the lemma in (B1) gives

$$ae_A = e_B a = a \qquad \forall\, a \in \mathfrak{A}^\alpha \cap (A, B)$$

hence $c(B)_A \leqslant e_A$ and so $c_A \leqslant e_A$ that is, using again the same lemma, A is square-integrable.

We immediately see that any object quasi-majorized by a square-integrable object is also square-integrable.

If we define quasi-dominant objects as objects whose central support in $\mathfrak{A}^\alpha$ is c, we obtain also that dominant objects are quasi-equivalent. Quasi-dominant objects quasi-majorize any square-integrable object.

B.5. We give now some examples of the previous notions.

B.5.1. The category Rep(G) of unitary representations of a locally compact group has been defined in Section 1 as fixed points of an action of G on $\mathscr{H}(G)$. Square-integrable objects of Rep(G) correspond to square-integrable representations σ in the sense that there exists a dense subset of vectors ξ in H_σ such that

$$\int |(\sigma(g)\xi, \xi)|^2 \, d\mu_l(g) < +\infty.$$

In that case $c = c(\lambda)$, where λ is the left regular representation of G, which is a quasi-dominant object (see [5] and [3], Example 2.8).

B.5.2. In [7], weights on a von Neumann algebra M have been described as objects of a W^*-category $\mathscr{W}(M)$ obtained as fixed points of an action of $\mathbf{R}$ on a category equivalent to M.

Square-integrable object with respect to this action correspond to integrable weights φ in the sense of [3], i.e. those for which the subset

$Q_\varphi = \left\{ x \in M_{s(\varphi)} \middle| \int_{-\infty}^{+\infty} \sigma_t^\varphi(x^* x)\, dt < +\infty \right\}$ is σ-weakly dense in $M_{s(\varphi)}$. ($s(\varphi)$ denotes the support of φ.) In that case $c = c(\varphi)$ for a dominant weight φ.

B.5.3. One-cocycles for an action of a locally compact non-abelian group G on a properly-infinite σ-finite von Neumann algebra M studied in [3] can also be considered as objects of a W^*-category $Z^1(G, \{M, \alpha\})$, as in Appendix A, which is obtained as fixed points of an action of G on a W^*-category equivalent to M. In that case, square-integrable cocycles in the sense of [3] correspond to square-integrable objects for this action and $c = c(a)$ for a dominant cocycle a.

B.6. In the following proposition we apply these ideas to the category $\mathrm{Sp}(M, \alpha)$ considered as fixed points of the action of G described in B.2. We shall see that $H_i \otimes M$ is a square-integrable object if and only if $\rho \otimes \alpha$ is a generator in $\mathrm{Sp}(M, \alpha)$ (i denotes the trivial representation of G), hence our definition of integrable action coincides with the one in [3].

PROPOSITION. *Let α be a σ-continuous action of G on M. The following conditions are equivalent:*

1) *The object $H_i \otimes M$ is square-integrable*, i.e. *the set* $\left\{ x \in M \middle| \int \alpha_g(x^* x)\, d\mu_l(g) < +\infty \right\}$ *is σ-weakly dense in M.*

2) $c(i \otimes \alpha) \leqslant c(\rho \otimes \alpha)$ *in* $\mathrm{Sp}(M, \alpha)$.

3) $\rho \otimes \alpha$ *is a generator in* $\mathrm{Sp}(M, \alpha)$.

An action satisfying these conditions is said to be integrable.

Proof. 2) $\Rightarrow$ 1). Using Lemmas 2.11 and 2.10 of [3] we see that the object $H_\rho \otimes M$ is square-integrable for the action described in B.2. The proposition in B.4 then shows that 2) $\Rightarrow$ 1).

2) $\Leftrightarrow$ 3). 3) $\Rightarrow$ 2) is clear. Now suppose 2). Then $c(\sigma \otimes \alpha) \leqslant c(\sigma \otimes \rho \otimes \alpha)$ $\forall \sigma \in \mathscr{H}(G)$ since tensoring by σ is a functor. Moreover $c(\sigma \otimes \rho \otimes \alpha) = c(\rho \otimes \alpha)$ since $\sigma \otimes \rho$ and ρ are quasi-equivalent objects in $\mathrm{Rep}(G)$. Hence result.

1) $\Rightarrow$ 2). Let $a \in M^\alpha$. If $at = 0$ $\forall t \in (\rho \otimes \alpha, i \otimes \alpha)$ implies $a = 0$, we have $c(\rho \otimes \alpha)_{i \otimes \alpha} = 1$ and 1) $\Rightarrow$ 2) is proved. But, if we denote ad the action of G on $\mathscr{H}(G)$ described in Section 2, we have $\eta \otimes y = (y\eta) \otimes 1 \in ((H_\rho, H_i) \otimes M) \cap Q_{\tilde{\alpha}}$ $\forall \eta \in (H_\rho, H_i) \cap Q_{\mathrm{ad}}$, $\forall y \in M \cap Q_\alpha^*$, hence $t = \xi_{\tilde{\alpha}}(\eta \otimes y) \in (\rho \otimes \alpha, i \otimes \alpha)$, and, if $0 = a\xi_{\tilde{\alpha}}(\eta \otimes y) = \xi_{\tilde{\alpha}}(\eta \otimes ay)$ then $\xi_{\tilde{\alpha}}(\eta \otimes y)\xi \otimes 1 = 0$ $\forall \xi \in (H_i, H_\rho)$. Since $M \cap Q_\alpha^*$ is σ-weakly dense in M, the conclusion follows from the next lemma.

B.7. LEMMA. *For any $x \in Q_\alpha^*$, there exists a net $\{\eta_K\}$, $\eta_K \in (H_\rho, H_i) \cap Q_{\mathrm{ad}}$ and a net $\{\xi_K\}$, $\xi_K \in (H_i, H_\rho)$, such that $x = \lim_K \xi_{\tilde{\alpha}}(\eta_K \otimes x)\xi_K \otimes 1$ for the σ-weak topology.*

Proof. First of all, if $\eta \in (H_\rho, H_l)$ and η^*1 is a continuous function with compact support, then, for any function f in $L^2(G, \mathrm{d}\mu_l)$, we have:

$$\left(f, \int \mathrm{ad}_g(\eta^*\eta)\,\mathrm{d}\mu_l(g) f\right) = \int (f, \rho(g)\eta^*\eta\rho(g)^* f)\,\mathrm{d}\mu_l(g) =$$

$$= \int (\eta\rho(g)^* f, \eta\rho(g)^* f)\,\mathrm{d}\mu_l(g) < +\infty$$

since $\eta f' = (1, \eta f') = (\eta^*1, f') = \int \overline{\eta^*1}(h) f'(h)\,\mathrm{d}\mu_l(h) < +\infty \quad \forall f' \in L^2(G, \mathrm{d}\mu_l)$.

Hence $\eta \in Q_{\mathrm{ad}}$.

This shows that we can choose $\eta_K \in (H_\rho, H_l) \cap Q_{\mathrm{ad}}$ and $\xi_K \in (H_l, H_\rho)$, K compact set in G, such as η_K^*1 and $\xi_K 1$ be positive continuous functions with support in K and satisfying the normalization condition:

$$\int \eta_K \rho(g)^* \xi_K \, d\mu_l(g) = \int \overline{\eta_K^* 1}(h) \xi_K 1(hg^{-1}) \Delta(g)^{-1/2}\,\mathrm{d}\mu_l(h)\,\mathrm{d}\mu_l(g) =$$

$$= \int \overline{\eta_K^* 1}(h) \xi_K 1(hg^{-1}) \Delta(g)^{1/2}\,\mathrm{d}\mu_l(h)\,\mathrm{d}\mu_r(g) =$$

$$= \int \overline{\eta_K^* 1}(h) \xi_K 1(g^{-1}) \Delta(gh^{-1})^{1/2}\,\mathrm{d}\mu_l(h)\,\mathrm{d}\mu_r(g) =$$

$$= \int \overline{\eta_K^* 1}(h) \Delta(h)^{-1/2}\,\mathrm{d}\mu_l(h) \int \xi_K 1(g^{-1}) \Delta(g)^{-1/2}\,\mathrm{d}\mu_l(g) = 1.$$

Since $x\eta_K \otimes 1 \in (H_\rho, H_l) \otimes M \cap P_{\tilde{\alpha}}$, we deduce that, $\forall \Phi \in M_*$,

$$|\Phi(\varphi_{\tilde{\alpha}}(\eta_K \otimes x)\xi_K \otimes 1) - \Phi(x)| = |\xi_K \otimes \Phi(\varphi_{\tilde{\alpha}}(\eta_K \otimes x)) - \Phi(x)| =$$

$$= \left| \int (\eta_K \rho(g)^* \xi_K \Phi(\alpha_g(x)) - \Phi(x))\,\mathrm{d}\mu_l(g) \right| \leqslant \sup_K |\Phi(\alpha_g(x)) - \Phi(x)|$$

and the lemma is proved when we choose subsets K decreasing to $\{e_G\}$.

B.8. Another immediate consequence of the previous lemma is that for integrable action α, the collection of spectral subspaces M_σ defined in Section 2 is σ-weakly total in M. More precisely, the subspace M_ρ is σ-weakly dense in M since in that case Q_α is σ-weakly dense in M.

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