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FACTORS OF TYPE III_1, PROPERTY L'_λ AND CLOSURE OF INNER AUTOMORPHISMS

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INTRODUCTION

Uffe Haagerup has succeeded to prove that any hyperfinite factor of type III_1 has a trivial bicentralizer [11, 12]. This result together with an unpublished result of the author, announced in [3], solves the longstanding problem of classification of hyperfinite factors of type III_1. It proves that they are all isomorphic to the Araki-Woods factor R_∞ (cf. [1]) and completes the classification of hyperfinite factors (cf. [3] for a review). Our aim in this paper is to give the details of the proof of the implication

M hyperfinite III_1 with trivial bicentralizer $\Rightarrow$ M isomorphic to R_∞.

Uffe Haagerup has found another proof of this implication and his proof is more direct than ours. Thus the only excuse for presenting our proof is that it gives a new characterization of property L'_λ (cf. Part II) and of the closure of $\mathrm{Int}\, M$ for M a factor of type III (cf. Part III). Also it is nice to know that the "automorphism" approach can be made to work in all cases.

The idea of our proof is the following. By previous results ([4]) one knows that any hyperfinite factor N of type III_λ, $\lambda \neq 1$ is isomorphic to the Araki-Woods factor R_λ first analyzed by R. Powers ([13], [1]). Let then $T = \dfrac{2\pi}{\log \lambda}$ and $\sigma = \sigma^\varphi_T$ be a modular automorphism of a given hyperfinite factor M of type III_1, it follows easily that $N \approx R_\lambda$ where $N = M \rtimes_\sigma \mathbf{Z}$ is the crossed product of M by $\sigma = \sigma^\varphi_T$. Let θ be the dual action of S^1 on N; one has (cf. [14])

$$M \approx N \rtimes_\theta S^1 = R_\lambda \rtimes_\theta S^1 .$$

This reduces the original problem to the classification of certain actions of S^1 on R_λ, in fact of those actions for which $\mathrm{mod}(\theta_t) = t \ \forall\, t \in S^1$. Here $\mathrm{mod}(\alpha)$, for $\alpha \in \mathrm{Aut}\, N$, is the action of α on the flow of weights of N. Since N is of type III_λ the automorphisms of its flow of weights are parametrized by S^1.

Now given two actions θ and θ' as above, one can form $\theta\otimes\tilde{\theta}'$ where $\tilde{\theta}'_t = \theta'_{-t}$ $\forall t \in S^1$. This tensor product $\alpha = \theta\otimes\tilde{\theta}'$ is an action of S^1 on $R_\lambda\otimes R_\lambda \approx R_\lambda$ such that,

1) $(R_\lambda\otimes R_\lambda)\times|_\alpha S^1$ is a factor of type III_λ;

2) $\mathrm{mod}(\alpha) = 1$.

From 1) it follows that the dual action of $\mathbf{Z}$ satisfies the two properties $\hat{\alpha}_n \in \overline{\mathrm{Int}}$, $\hat{\alpha}_n \notin \mathrm{Ct}$ $\forall n \in \mathbf{Z}$, $n \neq 0$, which allow to classify completely this action by the results of [5]. Thus one has shown that $\theta\otimes\tilde{\theta}'$ is outer conjugate to the following action of S^1 on $R_\lambda\otimes R$, where R is the hyperfinite factor of type II_1

$$\alpha_t^0 = \mathrm{id}_{R_\lambda}\otimes\beta_t$$

where β_t is an infinite tensor product action of S^1 on R.

To conclude that θ is outer conjugate to θ' one has to prove two properties:

α) $\theta\otimes\mathrm{id}_{R_\lambda} \approx \theta$;

β) $\theta\otimes\beta \approx \theta$.

One can translate what α) and β) mean on the original factor M. One finds the following two properties:

a) $M\otimes R_\lambda \approx M$;

b) $\sigma_{T_0} \in \overline{\mathrm{Int}}\,M$.

We shall explain all this in details in Part I. In Part II we shall analyze Condition a) which by [2] is equivalent to property L'_α, $\dfrac{\alpha}{1+\alpha} = \lambda$ and prove a general result which shows that b) $\Rightarrow$ a) if one knows that M is hyperfinite of type III_1.

Finally in Parts III and IV we shall prove that if the bicentralizer of M is trivial and M is hyperfinite of type III_1, then b) holds thus completing the proof.

I. UNIMODULAR ACTIONS OF S^1 ON R

We let $\lambda \in]0,1[$ and $T_0 = \dfrac{-2\pi}{\log\lambda}$. For any factor M the class of $\sigma^\varphi_{T_0}$ in $\mathrm{Out}\,M = \mathrm{Aut}\,M/\mathrm{Int}\,M$ is independent of the choice of faithful normal weight φ (cf. [6]) and we call it σ_{T_0}.

LEMMA 1. a) *Let M be a factor of type* III_1. *Then the crossed product* $N = M\times|_{\sigma_{T_0}}\mathbf{Z}$ *is a factor of type* III_λ.

b) *Let M_1, M_2 be as in* a), $N_j = M_j\times|_{\sigma_{T_0}}\mathbf{Z}$ *and θ_j the dual actions of* $\hat{\mathbf{Z}} = S^1$ (cf. [14]). *Let α_t be the action of S^1 on $N_1\otimes N_2$ given by*

$$\alpha_t = \theta_{1,t}\otimes\theta_{2,-t} \quad \forall t \in S^1.$$

Then $(N_1\otimes N_2)\times|_\alpha S^1$ *is a factor of type* III_λ.

Proof. a) Since $(\sigma_{T_0})^n$ is outer for $n \neq 0$, N is a factor (cf. [13]). The flow of weights of N is given by the dual action of $\hat{\mathbf{R}} = \mathbf{R}_+^*$ on the center of $N \times\!|_\sigma \mathbf{R} = (M \times\!|_{\sigma_{T_0}} \mathbf{Z}) \times\!|_\sigma \mathbf{R} = (M \times\!|_\sigma \mathbf{R}) \times\!|_{\sigma_{T_0}} \mathbf{Z}$, thus it is the periodic action of $\hat{\mathbf{R}}$ on a circle.

b) The tensor product $M_1 \otimes M_2 = M$ is a factor of type III_1 with modular automorphism $\sigma_{T_0} = \sigma^1_{T_0} \otimes \sigma^2_{T_0}$. By the biduality theorem ([13]) the automorphism $\sigma^j_{T_0}$, $j = 1,2$ is dual to the action of S^1 in $M_j \approx N_j \times\!|_{\theta_j} S^1$. Hence the crossed product of $M_1 \otimes M_2$ by $\sigma_{T_0} = \sigma^1_{T_0} \otimes \sigma^2_{T_0}$ is isomorphic to the crossed product of $N_1 \otimes N_2$ by the orthogonal in $S^1 \times S^1$ of the diagonal in $\mathbf{Z} \times \mathbf{Z}$ and hence to $(N_1 \otimes N_2) \times\!|_\alpha S^1$.

LEMMA 2. a) *Let R_λ be the Powers factor of type III_λ and $s \in \mathrm{Aut}\, R_\lambda$ an automorphism such that $R_\lambda \times\!|_s \mathbf{Z}$ is a factor of type III_λ then s is outer conjugate to $\mathrm{id}_{R_\lambda} \otimes s_0$ where s_0 is the aperiodic outer conjugacy class in the hyperfinite factor of type II_1 ([3]).*

b) *Let α be an action of S^1 on R_λ such that $R_\lambda \times\!|_\alpha S^1$ is a factor of type III_λ then α is outer conjugate to $\mathrm{id}_{R_\lambda} \otimes \beta$ where β is an infinite product action of S^1 on R.*

Proof. a) By [3] it is enough to show that $s \in \overline{\mathrm{Int}}\, R_\lambda$ and that $s^n \notin \mathrm{Ct}\, R_\lambda$ for any $n \neq 0$. By [3] one has $\overline{\mathrm{Int}}\, R_\lambda = \{\rho \in \mathrm{Aut}\, R_\lambda,\ \mathrm{mod}(\rho) = 1\}$. Since $R_\lambda \times\!|_s \mathbf{Z}$ is of type III_λ one has $\mathrm{mod}(s) = 1$. In general if $\mathrm{mod}(s) = x \in \mathbf{R}_+^*/\lambda^{\mathbf{Z}}$, then the invariant S of $R_\lambda \times\!|_s \mathbf{Z}$ is the closure of the subgroup generated by $\lambda^{\mathbf{Z}}$ and x. Next (cf. [3]) $\mathrm{Ct}\, R_\lambda$ is the range of the modular homomorphism $\delta : \mathbf{R}/T_0\mathbf{Z} \to \mathrm{Out}\, R_\lambda$. If one had $s^n \in \delta(\mathbf{R})$ for some $n \neq 0$ then, with $s^n = \delta(t)$, one would get

$$t \in T(R_\lambda \times\!|_s \mathbf{Z}).$$

As $R_\lambda \times\!|_s \mathbf{Z}$ is of type III_λ we thus have $t \in T_0\mathbf{Z}$ and s^n inner which is impossible since $R_\lambda \times\!|_s \mathbf{Z}$ is a factor. Thus we have shown that $s \in \overline{\mathrm{Int}}$ and $s^n \notin \mathrm{Ct}$ for $n \neq 0$.

b) Follows from a) by duality ([14]). ▨

PROPOSITION 3. *Let M_1 and M_2 be hyperfinite factors of type III_1, (R_λ, θ_j) be the dynamical systems dual to (M_j, σ_{T_0}). Then the action of S^1 on $R_\lambda \otimes R_\lambda$ given by*

$$\alpha_t = \theta_{1,t} \otimes \theta_{2,-t}$$

is outer conjugate to $\alpha^0_t = \mathrm{id}_{R_\lambda} \otimes \beta_t$, $\beta_t \in \mathrm{Aut}\, R$.

Proof. By Lemma 1 the action α_t satisfies the hypothesis of Lemma 2b.

COROLLARY 4. *Let M be a hyperfinite factor of type III_1 and (R_λ, θ) the dual system to (M, σ_{T_0}). Assume that:*

α) $\theta \otimes \mathrm{id}_{R_\lambda} \approx \theta$;

β) $\theta \otimes \beta \approx \theta$.

Then M is isomorphic to the Araki-Woods factor R_∞.

Proof. Let (R_λ, θ^0) be dual to (R_∞, σ_{T_0}). It is obvious that $R_\infty \otimes R_\lambda \approx R_\infty$ and that $\sigma_{T_0} \in \overline{\mathrm{Int}}\, R_\infty$, so that α) and β) are satisfied by θ^0 (using Lemma 5 below). Thus:

$$\theta^0 \otimes (\mathrm{id}_{R_\lambda} \otimes \beta) \approx \theta^0.$$

$$\theta \otimes (\mathrm{id}_{R_\lambda} \otimes \beta) \approx \theta.$$

Hence,

$$\theta \approx \theta \otimes (\mathrm{id}_{R_\lambda} \otimes \beta) \approx \theta \otimes (\tilde{\theta} \otimes \theta^0) = (\theta \otimes \tilde{\theta}) \otimes \theta^0 = (\mathrm{id}_{R_\lambda} \otimes \beta) \otimes \theta^0 \approx \theta^0. \qquad \blacksquare$$

In the proof we used the following lemma:

LEMMA 5. *Let M be a hyperfinite factor of type* III_1 *such that*

a) $M \otimes R_\lambda \approx M$;

b) $\sigma_{T_0} \in \overline{\mathrm{Int}}\, M \quad \left(T_0 = \dfrac{-2\pi}{\log \lambda} \right)$.

Then the dual system (R_λ, θ) satisfies 4α) *and* 4β).

Proof. First $\sigma_{T_0}^{R_\lambda} = 1$ so that if $M \otimes R_\lambda \approx M$ then $\sigma_{T_0} \approx \sigma_{T_0} \otimes \mathrm{id}_{R_\lambda}$ and the dual action verifies 4α).

Next since a) holds one gets that $M \otimes R \approx M$, i.e. that M has the property $L'_{1/2}$ ([9], [2]):

There exists a sequence (u_n) of partial isometries of M such that,

1) $u_n^2 = 0, \quad u_n u_n^* + u_n^* u_n = 1 \quad \forall n \in \mathbf{N}$;

2) $\|[u_n, \Psi]\| \to 0 \quad \forall \Psi \in M_*$ (predual of M).

Now let φ be a fixed faithful normal state on M, from 2) it follows that $\sigma_{T_0}^\varphi(u_n) - u_n \to 0$ *strongly. Thus if we look at u_n as a sequence of elements of the crossed product $N = M \times\!|_{\sigma_{T_0}^\varphi} \mathbf{Z}$ we easily get:

1′) $u_n^2 = 0, \quad u_n u_n^* + u_n^* u_n = 1$;

2′) $\|[u_n, \Psi]\| \to 0 \quad \forall \Psi \in N_*$;

3) $\theta_t(u_n) = u_n \quad \forall t \in S^1$.

This of course yields factorizations $N \approx N \otimes R$ which we know already, since $N \approx R_\lambda$. In order to obtain β) it is enough using standard techniques ([9], [2]) to get a sequence (v_n) of partial isometries of N satisfying 1′), 2′) and:

3′) $\theta_t(v_n) = e^{it} v_n \quad \forall t \in S^1$.

Then in the corresponding factorization of N, $N \approx N \otimes R$ one will get $\theta \approx \theta \otimes \beta$ and thus 4β).

Now to get 3′) we note that $\sigma_{T_0}^\varphi = \lim w_n w_n^*$ by hypothesis, where $\|[w_n, \varphi]\| \to 0$ and hence $\sigma_{T_0}^\varphi(w_n) - w_n \to 0$ *strongly. Let U be the canonical unitary, implementing $\sigma_{T_0}^\varphi$, in $N = M \times\!|_{\sigma_{T_0}} \mathbf{Z}$. We have shown that $w_n^* U$ is a centralizing sequence in N, i.e.

$$\|[w_n^* U, \Psi]\| \to 0 \quad \forall \Psi \in N_*.$$

Moreover $w_n^* U$ verifies 3′) by construction. With (u_n) satisfying 1′) 2′) 3) as above one obtains a sequence v_n satisfying 1′) 2′) 3′) by setting $v_n = u_{q(n)} w_n^* U$ where $q(n)$ is so large that $u_{q(n)}$ almost commutes with $w_n^* U$. ▧

We can now conclude this section by

THEOREM 6. *Let M be a hyperfinite factor of type* III_1 *such that*

a) $M \otimes R_\lambda \approx M$,

b) $\sigma_{T_0} \in \overline{\text{Int}}\, M$,

then M is isomorphic to the Araki-Woods factor R_∞.

Proof. The dual system (R_λ, θ) satisfies α) and β) so that Corollary 4 applies. ▨

II. PROPERTY L'_λ

Let $\lambda \in]0,1[$. In this section we shall prove that for a factor M the property L'_α, $\alpha = 1/(1+\lambda)$, i.e. the existence of an isomorphism of M with $M \otimes R_\lambda$ is equivalent to the following "local" condition:

CONDITION 1. *For any $n \in \mathbf{N}$, $\varepsilon > 0$, and faithful normal states $\varphi_1, \ldots, \varphi_n \in M_*^+$ there exists $x \in M, x \neq 0$ such that:*

$$\|(\Delta_{\varphi_j}^{1/2} - \lambda^{1/2}) x \xi_{\varphi_j}\|^2 \leqslant \varepsilon \sum \varphi_j(x^* x) \quad \forall j = 1, 2, \ldots, n.$$

We have used the standard notations of the modular theory (cf. [12]). Now our goal is to prove:

THEOREM 2. *Let M be a factor of type* III, *then:*

$$M \approx M \otimes R_\lambda \Leftrightarrow M \text{ satisfies Condition 1.}$$

Before we proceed to prove this result we point out the following corollary:

COROLLARY 3. *Let M be a factor of type* III *and $\theta \in \overline{\text{Int}}\, M$, with θ^n outer for all $n \neq 0$. Assume that $N = M \times|_\theta \mathbf{Z}$ verifies $N \otimes R_\lambda \approx N$, then $M \otimes R_\lambda \approx M$.*

Proof. We shall show that M satisfies Condition 1. Let α_t, $t \in S^1$, be the dual action of $S^1 = \hat{\mathbf{Z}}$ on N. For each $\varphi \in M_*^+$, let $\hat{\varphi}$ be the dual,

$$\hat{\varphi}(\sum a_n U^n) = \varphi(a_0) \quad \forall a = \sum a_n U^n \in N = M \times|_\theta \mathbf{Z}.$$

For any $x \in N$ we let $\hat{x}_q = \dfrac{1}{2\pi} \displaystyle\int_{S^1} e^{-iqt} \alpha_t(x)\, dt$. Also we define

$$N_q = M U^q = \{y \in N, \alpha_t(y) = e^{iqt} y \quad \forall t \in S^1\}.$$

One has $\hat{x}_q \in N_q\ \forall x \in N$, and $N_{q_1} N_{q_2} = N_{q_1+q_2}\ \forall q_1, q_2 \in \mathbf{Z}$. For any faithful $\varphi \in M_*^+$, the dual $\hat{\varphi}$ is faithful; N_{q_1} is orthogonal to N_{q_2} with respect to the inner product $\langle x, y\rangle = \hat{\varphi}(y^* x)$, provided $q_1 \neq q_2$. By construction $\hat{\varphi} \circ \alpha_t = \hat{\varphi}\ \forall t \in S^1$ thus $\sigma_s^{\hat{\varphi}}$ commutes with α_t and $\sigma_s^{\hat{\varphi}}(N_q) \subset N_q\ \forall s \in \mathbf{R},\ q \in \mathbf{Z}$. Moreover the restriction of $\hat{\varphi}$ to N_0 is equal to φ and moreover $\sigma_s^{\hat{\varphi}} | N_0 = \sigma_s^{\varphi}$. Thus for any $x \in N$ one has,

$$\|(\Delta_{\hat{\varphi}}^{1/2} - \lambda^{1/2}) x \xi_{\hat{\varphi}}\|^2 = \sum_q \|(\Delta_{\hat{\varphi}}^{1/2} - \lambda^{1/2}) \hat{x}_q \xi_{\hat{\varphi}}\|^2$$

$$\hat{\varphi}(x^* x) = \sum_q \hat{\varphi}(\hat{x}_q^* \hat{x}_q).$$

Let $\varphi_1, \ldots, \varphi_n$, be faithful normal states on M and $\varepsilon > 0$. Since by hypothesis N verifies Condition 1) there exists an $x \in N$ such that,

$$x \neq 0, \quad \sum \|(\Delta_{\hat{\varphi}_j}^{1/2} - \lambda^{1/2}) x \xi_{\hat{\varphi}_j}\|^2 \leqslant \varepsilon \sum \hat{\varphi}_j(x^* x).$$

Using the above equalities one may assume that $x \in N_q$ for some q. Let now v_m be a sequence of unitaries of M with,

$$\theta^q(\Psi) = \lim_{m \to \infty} v_m \Psi v_m^* \quad \forall \Psi \in M_*^+ .$$

Then the sequence $u_m = v_m U^{-q}$ of unitaries of N_{-q} satisfies:

$$u_m \hat{\Psi} u_m^* \to \hat{\Psi} \quad \forall \Psi \in M_*^+ .$$

This implies that $\|(\Delta_{\hat{\Psi}}^{1/2} - 1) u_m \xi_{\hat{\Psi}}\| \to 0$ and hence that,

$$\|(\Delta_{\hat{\varphi}_j}^{1/2} - \lambda^{1/2}) u_m x \xi_{\hat{\varphi}_j}\| \xrightarrow[m \to \infty]{} \|(\Delta_{\hat{\varphi}_j}^{1/2} - \lambda^{1/2}) x \xi_{\hat{\varphi}_j}\|.$$

One has

$$\|(\Delta_{\hat{\varphi}_j}^{1/2} - \lambda^{1/2}) u_m x \xi_{\hat{\varphi}_j}\| = \|(J(u_m x)^* J - \lambda^{1/2} u_m x) \xi_j\| =$$

$$= \|(J x^* J J u_m^* J - \lambda^{1/2} u_m x) \xi_j\| = \|(J x^* J)(J u_m^* J - u_m) \xi_j +$$

$$+ u_m (J x^* J - \lambda^{1/2} x) \xi_j\| \leqslant \|x\|_\infty \|\Delta_{\hat{\varphi}_j}^{1/2} u_m \xi_{\hat{\varphi}_j} - u_m \xi_{\hat{\varphi}_j}\| + \|(\Delta_{\hat{\varphi}_j}^{1/2} - \lambda^{1/2}) x \xi_{\hat{\varphi}_j}\|$$

where $\xi_j \in \mathscr{P}^\natural$ is the vector corresponding to $\hat{\varphi}_j$ (cf. [7] for notations).

Now $u_m x \in N_{-q}N_q = N_0$ and for m large enough one has

$$\sum \|(\Delta_{\varphi_j}^{1/2} - \lambda^{1/2})u_m x\xi_{\varphi_j}\|^2 \leqslant 2\varepsilon \sum \varphi_j(x^* u_m^* u_m x)$$

so that $N_0 = M$ satisfies Condition 1. ▨

Combining Corollary 3 with Theorem 1.6 we get:

THEOREM 4. *Let M be a hyperfinite factor of type* III_1 *and assume that* $\sigma_{T_0} \in \overline{\mathrm{Int}}\, M$ *for some* $T_0 \neq 0$. *Then M is isomorphic to the Araki-Woods factor* R_∞.

Proof. By Theorem 1.6 we just have to prove that $M \otimes R_\lambda \approx M$. For this we apply Corollary 3 with $\theta = \sigma_{T_0}$, as $M \times\!|_\theta \mathbf{Z} = R_\lambda$ satisfies the hypothesis, we get $M \approx M \otimes R_\lambda$. ▨

Let us now proceed and prove Theorem 2. By [2] a factor M verifies $M \otimes R_\lambda \approx M$ iff it satisfies property L'_α, $\alpha = 1/(1+\lambda)$, i.e.,

CONDITION 2. *For any* $n \in \mathbf{N}$, $\varepsilon > 0$, *and* $\varphi_1, \ldots, \varphi_n \in M_*^+$, *there exists a partial isometry* u, $u^2 = 0$, $u^*u + uu^* = 1$, *such that*

$$\|u\varphi_j - \lambda\varphi_j u\| < \varepsilon \quad \forall j = 1, 2, \ldots, n.$$

It is obvious that Condition 2 implies Condition 1 above. The main step in the proof of the implication $1 \Rightarrow 2$ is the next lemma.

LEMMA 5. *Let M be a factor verifying Condition* 1. *Then given* $\varphi_j \in M_*^+$, *faithful, and* $\varepsilon > 0$, *there exists a partial isometry* $u \in M$ *such that*

a) $u^2 = 0$, $u \neq 0$;

b) $\|(\Delta_{\varphi_j}^{1/2} - \lambda^{1/2})u\xi_{\varphi_j}\|^2 \leqslant \varepsilon \sum \varphi_j(u^*u)$.

Proof. Let ρ_λ be the state on $M_2(\mathbf{C})$ given by,

$$\rho_\lambda\left(\sum a_{ij}e_{ij}\right) = \frac{1}{1+\lambda}\, a_{11} + \frac{\lambda}{1+\lambda}\, a_{22} \quad \forall a_{ij} \in \mathbf{C}$$

where (e_{ij}) is the canonical system of matrix units in $M_2(\mathbf{C})$. One has $\rho_\lambda(ae_{12}) = \lambda\rho_\lambda(e_{12}a)$ $\forall a \in M_2(\mathbf{C})$, thus

$$\Delta_{\rho_\lambda}^{1/2} e_{12} = \lambda^{-1/2} e_{12}\,.$$

Thus, with the notation of [7] Proposition 1, we get

$$I(\varphi\otimes\rho_\lambda, x\otimes e_{12}) = \frac{1}{2}\,\|(\Delta^{1/2}_{\varphi\otimes\rho_\lambda} - 1)(x\otimes e_{12})\xi_{\varphi\otimes\rho_\lambda}\|^2 =$$

$$= \frac{1}{2}\,\|\Delta^{1/2}_{\varphi} x\xi_\varphi \otimes \Delta^{1/2}_{\rho_\lambda} e_{12}\xi_{\rho_\lambda} - x\xi_\varphi\otimes e_{12}\xi_{\rho_\lambda}\|^2 =$$

$$= \frac{1}{2}\,\|\Delta^{1/2}_\varphi x\xi_\varphi - \lambda^{1/2}x\xi_\varphi\|^2\lambda^{-1}\|e_{12}\xi_{\rho_\lambda}\|^2 = \frac{1}{2}\,\frac{1}{1+\lambda}\,\|(\Delta^{1/2}_\varphi - \lambda^{1/2})x\xi_\varphi\|^2.$$

Hence by [7], Theorem 2 we get:

1) $$\int_0^\infty \left(\sum_1^n \varphi_j\right)(u_{a^{1/2}}(x)^* u_{a^{1/2}}(x))\,da = \sum_1^n \varphi_j(x^*x);$$

$$\int_0^\infty \frac{1}{2}\left(\frac{1}{1+\lambda}\right)\|(\Delta^{1/2}_{\varphi_j} - \lambda^{1/2})u_{a^{1/2}}(x)\xi_{\varphi_j}\|^2\,da \leqslant$$

2)

$$\leqslant 6\left(\frac{1}{2}\,\frac{1}{1+\lambda}\right)^{1/2}\|(\Delta^{1/2}_{\varphi_j} - \lambda^{1/2})x\xi_{\varphi_j}\|\;\|x\otimes e_{12}\|^{\#}_{\varphi_j\otimes\rho_\lambda}.$$

Now

$$(\|x\otimes e_{12}\|^{\#}_{\varphi_j\otimes\rho_\lambda})^2 = (\varphi_j\otimes\rho_\lambda)(x^*x\otimes e_{22} + xx^*\otimes e_{11}) \leqslant \frac{1}{1+\lambda}\,\varphi_j(x^*x + xx^*),$$

thus we get:

3) $$\int_0^\infty \|(\Delta^{1/2}_{\varphi_j} - \lambda^{1/2})u_{a^{1/2}}(x)\xi_{\varphi_j}\|^2\,da \leqslant 12\|(\Delta^{1/2}_{\varphi_j} - \lambda^{1/2})x\xi_{\varphi_j}\|\;\|x\|^{\#}_{\varphi_j}.$$

Let $x\in M$, $x\neq 0$, be such that

$$\|(\Delta^{1/2}_{\varphi_j} - \lambda^{1/2})x\xi_{\varphi_j}\| \leqslant \varepsilon(\textstyle\sum \varphi_j(x^*x))^{1/2}.$$

Let $\eta = (\sum\varphi_j(x^*x))^{1/2}$. One has $(\|x\|^{\#}_{\varphi_j})^2 = \varphi_j(x^*x + xx^*) \leqslant 2\eta^2$ if $\lambda^{1/2} + \varepsilon \leqslant 1$. (Since $\varphi_j(xx^*) = \|\Delta^{1/2}_{\varphi_j}x\xi_{\varphi_j}\|^2$.) Hence, by 3),

4) $$\int_0^\infty \|(\Delta^{1/2}_{\varphi_j} - \lambda^{1/2})u^{1/2}_a(x)\xi_{\varphi_j}\|^2\,da \leqslant 20\varepsilon\eta^2$$

while

$$\int_0^\infty \left(\sum_1^n \varphi_j\right)(u_{a^{1/2}}(x)^* u_{a^{1/2}}(x))\,\mathrm{d}a = \eta^2.$$

This shows that for any such $x \in M$ there exists $a > 0$ with $u_a(x) \neq 0$ and $\sum_1^n \|(\Delta_{\varphi_j}^{1/2} - \lambda^{1/2})u_a(x)\xi_{\varphi_j}\|^2 \leqslant 20n\varepsilon \sum_1^n \varphi_j(u_a(x)^* u_a(x))$. This yields a partial isometry u verifying $u \neq 0$ and 5 b). Now we have to improve it to get 5 a), i.e. to get $u^2 = 0$. Let then $f = uu^*$ be the final support of u and take $y = u(1-f)$. One has $y^2 = 0$, and if u verifies 5 b) one has, with the notation of [7],

$$\|(\Delta_{\varphi_j}^{1/2} - \lambda^{1/2})y\xi_{\varphi_j}\| = \|(Jy^*J - \lambda^{1/2}y)\xi_j\| = \|(J(1-f)u^*J - \lambda^{1/2}u(1-f))\xi_j\| \leqslant$$

$$\leqslant \|J(1-f)J(Ju^*J - \lambda^{1/2}u)\xi_j\| + \lambda^{1/2}\|u(J(1-f)J - (1-f))\xi_j\| \leqslant$$

$$\leqslant \|(\Delta_{\varphi_j}^{1/2} - \lambda^{1/2})u\xi_{\varphi_j}\| + \lambda^{1/2}\|(JfJ - f)\xi_j\|.$$

Then

$$\|(JfJ - f)\xi_j\| = \|(Juu^*J - uu^*)\xi_j\| \leqslant \|JuJ(Ju^*J - \lambda^{1/2}u)\xi_j\| +$$

$$+ \lambda^{1/2}\|u(JuJ - \lambda^{-1/2}u^*)\xi_j\| \leqslant 2\|(\Delta_{\varphi_j}^{1/2} - \lambda^{1/2})u\xi_{\varphi_j}\|.$$

Hence

$$\|(\Delta_{\varphi_j}^{1/2} - \lambda^{1/2})y\xi_{\varphi_j}\| \leqslant 3\|(\Delta_{\varphi_j}^{1/2} - \lambda^{1/2})u\xi_{\varphi_j}\|.$$

Since any $u_a(y)$ satisfies $u_a(y)^2 = 0$, the proof will be ended by the following inequality:

$$(*) \qquad (\textstyle\sum \varphi_j)(y^*y) \geqslant ((1 - \lambda^{1/2}) - n\varepsilon)^2 \sum \varphi_j(u^*u).$$

Then by the same argument as above a suitable $u = u_a(y)$ gives the answer.

Let us prove $(*)$. Let $\eta^2 = \sum \varphi_j(u^*u) = \sum \varphi_j(e)$. One has $\|f\xi_j\| = \|u^*\xi_j\| \leqslant \leqslant \lambda^{1/2}\|u\xi_j\| + \varepsilon\eta$ since $\|Ju^*\xi_j - \lambda^{1/2}u\xi_j\| \leqslant \varepsilon\eta$ for all j. Thus $|\langle fe\xi_j, \xi_j\rangle| \leqslant \|f\xi_j\|\,\|e\xi_j\| \leqslant \leqslant (\lambda^{1/2}\|e\xi_j\| + \varepsilon\eta)(\|e\xi_j\|) \leqslant \lambda^{1/2}\|e\xi_j\|^2 + \varepsilon\eta^2$. Then, $|\sum \langle fe\xi_j, \xi_j\rangle \leqslant \lambda^{1/2}\eta^2 + n\varepsilon\eta^2$. Let $\Psi(x) = \sum \langle x\xi_j, \xi_j\rangle \ \forall x \in M$. One has $\eta^2 = \Psi(e)$, $|\Psi(fe)| \leqslant (\lambda^{1/2} + n\varepsilon)\Psi(e)$. Assume that $\lambda^{1/2} + n\varepsilon < 1$, then

$$|\Psi(e - fe)| \geqslant (1 - (\lambda^{1/2} + n\varepsilon))\Psi(e)$$

$$\|e(1-f)\xi_\Psi\| \geqslant (1 - (\lambda^{1/2} + n\varepsilon))\|e\xi_\Psi\|$$

since $|\Psi(e - fe)| = |\langle e(1-f)\xi_\Psi, e\xi_\Psi\rangle| \leqslant \|e\xi_\Psi\|\,\|e(1-f)\xi_\Psi\|$.

Proof of Theorem 2. We let $\varphi_1^0, \ldots, \varphi_n^0$ be faithful normal states on M and $\delta > 0$. We want to prove Condition 2. We let $\xi_j^0 \in \mathscr{P}^\natural = \mathscr{H}^+$ be the vectors associated to the φ_j^0 with the notations of [7]. The argument we shall use is an exhaustion argument similar to the one used in [7].

Thus we let $\mathscr{R}$ be the set of $(n+1)$ uples $(u, \alpha_1, \ldots, \alpha_n)$ where:

a) u is a partial isometry $u \in M$, $u^2 = 0$,

b) $\alpha_j \in \mathscr{H}$, one has $(u^*u + uu^*)\alpha_j = \alpha_j$, $\xi_j = \xi_j^0 - \alpha_j - J\alpha_j \in \mathscr{P}^\natural$ and ξ_j commutes with $u^*u + uu^*$,

c) One has $\|\xi_j u - \lambda^{1/2} u \xi_j\|^2 \leqslant \delta \sum \varphi_j^0(u^*u + uu^*) \quad \forall j$,

d) $\|\alpha_j\|^2 \leqslant \delta \sum \varphi_j^0(u^*u + uu^*) \quad \forall j$,

We define a partial ordering $r \leqslant r'$ by:

1) u' is an extension of u, i.e. $u = eu' = u'e$ where $e = u^*u + uu^*$,

2) $e(\alpha_j' - \alpha_j) = 0 \quad \forall j$,

3) $\|\alpha_j' - \alpha_j\|^2 \leqslant \delta \sum \varphi_j^0(e' - e) \quad \forall j$.

One proves exactly as in [7] that $\leqslant$ is a partial order on $\mathscr{R}$, that any totally ordered subset is countable and majorized by some element of $\mathscr{R}$. Let then $r = (u, \alpha_j)$ be a maximal element of $\mathscr{R}$. Let $e = u^*u + uu^*$ and $f = 1 - e$. We shall assume that $f \neq 0$ and contradict the maximality of r.

Let $M_f = fMf$ act standardly in $\mathscr{H}_f = f\mathscr{H}f$. By b) one has $e\alpha_j = \alpha_j$ hence $f\alpha_j = 0$, $Jf\alpha_j = (J\alpha_j)f = 0$. Thus $f\xi_j f = f\xi_j^0 f$. Let $\eta_j = f\xi_j^0 f$.

By Lemma 5 there exists a non-zero partial isometry $v \in M_f$, $v^2 = 0$ such that $\|(Jv^*J - \lambda^{1/2}v)\eta_j\|^2 \leqslant \dfrac{\lambda\delta}{16} \sum \langle v^*v\eta_j, \eta_j\rangle$. Let $e_1 = v^*v + vv^*$ then $\|(Je_1J - e_1)\eta_j\| \leqslant$ $\leqslant 2(1 + \lambda^{-1/2})\|(Jv^*J - \lambda^{1/2}v)\eta_j\|$.

Thus $\|[e_1, \eta_j]\|^2 \leqslant \delta \sum \langle e_1\eta_j, \eta_j\rangle \leqslant \delta \sum \varphi_j^0(e_1)$ since $\|e_1\eta_j\| = \|e_1 f\xi_j^0 f\| \leqslant \|e_1\xi_j^0\|$ for any j. We let $u' = u + v$ and $\alpha_j' = \alpha_j + e_1\xi_j(1 - e_1)$. Since ξ_j commutes with f one has $e_1\xi_j(1 - e_1) = e_1\xi_j(f - e_1) \in f\mathscr{H}f$. One has $\alpha_j' - \alpha_j \in e_1\mathscr{H}$, this vector is orthogonal to $e\mathscr{H}$ thus 2) holds. Also $\|\alpha_j' - \alpha_j\|^2 = \|e_1\eta_j(f - e_1)\|^2 = \|[e_1, \eta_j]\|^2 \leqslant$ $\leqslant \delta \sum \varphi_j^0(e_1)$ thus 3) holds, while 1) is obvious by construction. Let us check that $r' \in \mathscr{R}$, i.e. that r' satisfies a), b), c), d). Condition a) is clear. One has $e_1(\alpha_j' - \alpha_j) =$ $= \alpha_j' - \alpha_j$ hence $(e + e_1)\alpha_j' = \alpha_j'$ so that b) follows from the equality

$$\xi_j' = \xi_j^0 - \alpha_j' - J\alpha_j' = \xi_j - (\alpha_j' - \alpha_j) - J(\alpha_j' - \alpha_j) =$$

$$= \xi_j - e_1\xi_j(1 - e_1) - (1 - e_1)\xi_j e_1 = (1 - e_1)\xi_j(1 - e_1) + e_1\xi_j e_1 \in \mathscr{P}^\natural.$$

Condition d) follows from the equality $\|\alpha_j'\|^2 + \|\alpha_j\|^2 = \|\alpha_j' - \alpha_j\|^2 \ \forall j$, and 3). Let us check c). One has:

$$(Ju'^*J - \lambda^{1/2}u')\xi_j' = (Ju^*J - \lambda^{1/2}u)\xi_j' + (Jv^*J - \lambda^{1/2}v)\xi_j'$$

$$(Ju^*J - \lambda^{1/2}u)\xi_j' = (Ju^*J - \lambda^{1/2}u)(1 - e_1)\xi_j(1 - e_1) = (1 - e_1)(\xi_j u - \lambda^{1/2}u\xi_j)(1 - e_1).$$

Thus $\|(Ju^*J - \lambda^{1/2}u)\xi'_j\|^2 \leqslant \delta \sum \varphi^0_j(u^*u + uu^*)$. Also, $(Jv^*J - \lambda^{1/2}v)\xi'_j = (Jv^*J - \lambda^{1/2}v)(e_1\xi_j e_1) = e_1(\eta_j v - \lambda^{1/2}v\eta_j)e_1$, thus $\|(Jv^*J - \lambda^{1/2}v)\xi'_j\|^2 \leqslant \delta \sum \varphi^0_j(v^*v + vv^*)$.

Hence c) follows, since $\xi'_j u - \lambda^{1/2}u\xi'_j$ is orthogonal to $e_1(\eta_j v - \lambda^{1/2}v\eta_j)e_1$. This proves that any maximal element of $\mathscr{R}$ yields a partial isometry u such that:

$$u^2 = 0, \quad uu^* + u^*u = 1$$

$$\|\lambda^{1/2}u\xi^0_j - \xi^0_j u\| \leqslant 5n^{1/2}\delta^{1/2}.$$

(Since $\|\alpha_j\| \leqslant n^{1/2}\delta^{1/2}$ and $\|\xi_j - \xi^0_j\| \leqslant 2\|\alpha_j\|$.) Thus M satisfies Condition 2 and by [2], $M \approx M \otimes R_\lambda$. ▨

III. $\overline{\mathrm{Int}}\, M$ FOR TYPE III FACTORS

In this section we shall prove a characterization of approximately inner automorphisms of type III factors, which will be crucial in the proof (cf. Section IV) of $\sigma_{T_0} \in \overline{\mathrm{Int}}\, M$ for M a hyperfinite factor of type III_1 with trivial bicentralizer.

THEOREM 1. *Let M be a type* III *factor, and $\theta \in \mathrm{Aut}\, M$. Then $\theta \in \overline{\mathrm{Int}}\, M$ iff for any $\xi_1, \ldots, \xi_n \in \mathscr{P}^\natural$ and $\varepsilon > 0$ there exists a non zero $x \in M$ such that:*

$$\|x\xi_j - \theta(\xi_j)x\|^2 \leqslant \varepsilon^2 \sum \|x\xi_j\|^2 \quad \forall j = 1, \ldots, n.$$

The condition is clearly necessary. To prove that it is sufficient we shall use the same technique as in Section II. We first need a lemma which allows to take the supremum $e \vee f$ of two projections in such a way that $e \vee f$ commutes approximately with ξ_j's if e and f do. The usual supremum is too discontinuous to work.

LEMMA 2. *There exists a constant $c < \infty$ such that for any $e, f \in \mathrm{Proj}\, M$ and any $\xi_1, \ldots, \xi_n \in \mathscr{P}^\natural$ there exists $E \in \mathrm{Proj}\, M$ such that:*

a) $E \leqslant c(e + f)$ *(in M^+),*

b) $e(1 - E)e \leqslant \frac{1}{9}e, \quad f(1 - E)f \leqslant \frac{1}{9}f,$

c) $\sum \|[E, \xi_j]\|^2 \leqslant c \sum \|[(e + f), \xi_j]\| \|(e + f)\xi_j\|.$

Proof. Let $e(a) = E_{a^{1/2}}(e + f)$ be the spectral projection of $e + f$ corresponding to $[a^{1/2}, +\infty[$. By construction one has

$$(*) \qquad a^{1/2}e(a) \leqslant e + f, \quad e(a) \leqslant a^{-1/2}(e + f).$$

Next we shall estimate $\|e(e(a) - 1)e\|$ and prove:

$$(**) \qquad \|e(1 - e(a))e\| \leqslant \sin^2\frac{\theta}{2} \quad \text{for } a^{1/2} = 1 - \cos\theta,\ \theta \in \left[0, \frac{\pi}{2}\right].$$

To prove this is quite simple since (cf. [5]) one can assume that $M = M_2(\mathbf{C})$, that $e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $f = \begin{bmatrix} \cos^2\alpha & -\sin\alpha\cos\alpha \\ -\sin\alpha\cos\alpha & \sin^2\alpha \end{bmatrix}$ for some $\alpha \in [0, \pi/2]$. Then a simple computation shows that the spectrum of $e+f$ consists of $1 \pm \cos\alpha$. Clearly $(**)$ is trivially true $(e(a) = 1)$ if $\alpha \geqslant \theta$. If $\alpha \leqslant \theta$ one gets $e(a) = \begin{bmatrix} \cos^2\frac{\alpha}{2} & -\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} \\ -\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} & \sin^2\frac{\alpha}{2} \end{bmatrix}$ for simple geometric reasons of symmetry. Then a direct estimate gives:

$$\|e(1-e(a))e\| \leqslant \operatorname{Trace}(e(1-e(a))e) = \left(\cos^2\frac{\alpha}{2} - 1\right)^2 + \sin^2\frac{\alpha}{2}\cos^2\frac{\alpha}{2} =$$

$$= \sin^2\frac{\alpha}{2} \leqslant \sin^2\frac{\theta}{2}\cdot$$

Thus $(**)$ is proven. This shows that provided $c \geqslant 9$, any $a \in J = [1/81, 4/81]$ will work, since, as $a \geqslant 9^{-2}$ one has $a^{-1/2} \leqslant 9$ and as $a \leqslant 4/81$ one has $\sin\frac{\theta}{2} \leqslant 1/3$ if $1 - \cos\theta = a^{1/2} \leqslant 2/9$. But now by [7] one has:

$$\int \sum_j \|[e(a), \xi_j]\|^2 \, da \leqslant 12 \sum_j \|[(e+f), \xi_j]\| \, \|(e+f)\xi_j\|$$

so that in the interval I there exists an a such that

$$\sum_j \|[e(a), \xi_j]\|^2 \leqslant (12 \times 27) \sum_j \|[(e+f), \xi_j]\| \, \|(e+f)\xi_j\|.$$ ▨

We are now ready to prove the crucial lemma in the proof of Theorem 1.

LEMMA 3. *Let M, θ, ξ_j and ε be as in Theorem* 1 *and c be as in Lemma* 2. *There exists a projection $E \in M$ and an $x \in M$ such that,*

1) $\|x\|_\infty \leqslant 1, \quad x = \theta(E)xE,$
2) $\sum \|x\xi_j\|^2 \geqslant 2^{-6}c^{-1}\sum \|E\xi_j\|^2,$
3) $\|[E, \xi_j]\|^2 \leqslant \varepsilon^2 \sum \|E\xi_j\|^2 \quad \forall j = 1, \ldots, n,$
4) $\|x\xi_j - \theta(\xi_j)x\|^2 \leqslant \varepsilon^2 \sum \|x\xi_j\|^2 \quad \forall j = 1, \ldots, n.$

Proof. First, exactly as in Section 2, we can use the technique of [7] to find a partial isometry $u \in M$, $u \neq 0$ such that:

$$\|u\xi_j - \theta(\xi_j)u\| \leqslant \varepsilon\eta, \quad \text{where } \eta^2 = \sum \|u\xi_j\|^2.$$

Let then $e = u^*u$ and $f = \theta^{-1}(uu^*)$. One has

$$\|[e, \zeta_j]\| \leqslant 2\varepsilon\eta, \quad \|[f, \zeta_j]\| \leqslant 2\varepsilon\eta.$$

Also $\|\theta(\zeta_j)u\| \leqslant \|u\xi_j\| + \varepsilon\eta$ hence:

$$\|f\zeta_j\| = \|\theta^{-1}(u^*)\zeta_j\| \leqslant \|u\zeta_j\| + \varepsilon\eta = \|e\xi_j\| + \varepsilon\eta.$$

Thus by Lemma 2 there exists $E \in \operatorname{Proj} M$ such that: a), b) and c') hold where:

c') $\|[E, \zeta_j]\|^2 \leqslant cn4\varepsilon\eta(2\eta + \varepsilon\eta) \leqslant 10nc\varepsilon\eta^2$.

We let $x = \theta(E)uE$. By construction 1) holds. Also 3) and 4) are clear for ε small enough, provided we prove 2).

One has $\|(eE - e)\xi_j - \zeta_j(eE - e)\| \leqslant 2\varepsilon\eta + 2\varepsilon\eta + (10nc\varepsilon)^{1/2}\eta$, $\|\zeta_j(eE - e)\| = \|(E - 1)e\zeta_j\| \leqslant (1/3)\|e\zeta_j\|$, thus given δ one has $\|(eE - e)\zeta_j\| \leqslant (1/3)\|e\xi_j\| + \delta\eta$ for ε small enough. Then $\|\theta(E)uE\xi_j - \theta(E)u\xi_j\| = \|\theta(E)u(eE - e)\zeta_j\| \leqslant (1/3)\|e\xi_j\| + \delta\eta$, and,

$$\|\theta(E)uE\xi_j\| \geqslant \|\theta(E)u\zeta_j\| - \frac{1}{3}\|e\zeta_j\| - \delta\eta,$$

$$\|\theta(E)u\xi_j\| \geqslant \|\theta(E)\theta(\zeta_j)u\| - \varepsilon\eta = \|E\zeta_j f\| - \varepsilon\eta \geqslant$$

$$\geqslant \|Ef\zeta_j\| - 3\varepsilon\eta \geqslant \frac{2}{3}\|f\zeta_j\| - 3\varepsilon\eta.$$

As $\|f\zeta_j\| = \|\theta^{-1}(u^*)\zeta_j\| = \|\theta(\zeta_j)u\| \geqslant \|e\zeta_j\| - \varepsilon\eta$, we get:

$$\|\theta(E)uE\xi_j\| \geqslant \frac{2}{3}\|e\xi_j\| - \left(\frac{2}{3} + 3\right)\varepsilon\eta - \frac{1}{3}\|e\xi_j\| - \delta\eta.$$

Thus with ε small enough we get:

$$\|x\xi_j\| \geqslant \frac{1}{3}\|e\xi_j\| - \delta\eta, \quad \eta^2 = \sum\|e\xi_j\|^2.$$

Thus for δ small enough one gets

$$\sum\|x\xi_j\|^2 \geqslant 2^{-6}\sum(\|e\xi_j\|^2 + \|f\xi_j\|^2),$$

$$\sum\|x\xi_j\|^2 \geqslant \sum\left(\frac{1}{8}\right)^2\langle(e + f)\xi_j, \xi_j\rangle \geqslant \sum\left(\frac{1}{8}\right)^2\frac{1}{c}\langle E\xi_j, \xi_j\rangle. \quad ▨$$

LEMMA 4. *Let M and θ be as in Theorem 1. There exists a bounded sequence (y_n) of elements of M such that y_n does not tend to 0 strongly and :*

$$\|y_n\varphi - \theta(\varphi)y_n\| \to 0 \qquad \varphi \in M_* .$$

Proof. Using Lemma 3 and the exhaustion argument of Section II one constructs for any $\xi_1, \ldots, \xi_n \in \mathscr{P}^\natural$ and $\varepsilon > 0$ an element x of M such that:

1) $\|x\|_\infty \leqslant 1$,
2) $\sum\|x\xi_j\|^2 \geqslant 2^{-6}c^{-1}$,
3) $\|x\xi_j - \theta(\xi_j)x\| \leqslant \varepsilon$.

With this one easily gets the desired sequence since M is finitely generated. ▨

Proof of Theorem 1. We consider the set X of elements of the predual of $M_2(\mathbf{C})\otimes M = N$ of the form: $\begin{bmatrix}\varphi & 0\\ 0 & \theta(\varphi)\end{bmatrix}$, $\varphi \in M_*^+$. We fix a faithful normal state φ_0 on M and let $\Psi_0 = \begin{bmatrix}\varphi_0 & 0\\ 0 & \theta(\varphi_0)\end{bmatrix}$.

Let ω be a free ultrafilter on $\mathbf{N}$, and let $N^\omega_{\Psi_0}$ be the asymptotic centralizer of Ψ_0. This is a von Neumann algebra with a faithful trace τ which is obtained as the quotient by an ideal J of the C^*-algebra of all bounded sequences (y_n), $y_n \in N$, such that,

$$\|[y_n, \Psi_0]\| \to 0.$$

The ideal J is the kernel of the trace τ, defined by:

$$\tau((y_n)) = \lim_{n\to\omega} \Psi_0(y_n)$$

$$J = \{((y_n)),\ \lim_{n\to\omega} \tau((y_n^* y_n)) = 0\}.$$

We let P be the von Neumann subalgebra of $N^\omega_{\Psi_0}$ obtained as the intersection of the centralizer of the Ψ_ω, $\Psi \in X$, where:

$$\Psi_\omega((y_n)) = \lim_{n\to\omega} \Psi(y_n).$$

(Note that if $y_n \in J$ then $\|y_n\xi_{\Psi_0}\| \xrightarrow[n\to\omega]{} 0$ and hence $\|y_n\xi_\Psi\| \xrightarrow[n\to\omega]{} 0$ for any Ψ.)

For any $x \in N$ let $\tilde{x}$ be the constant sequence: $\tilde{x}_n = x\ \forall n$. One has $\tilde{e}_{11} \in P$, $\tilde{e}_{22} \in P$ where $e_{11} = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$, $e_{22} = \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$. To prove Theorem 1 we just have to prove that $\tilde{e}_{11}$ and $\tilde{e}_{22}$ are equivalent projections in P. Then this yields a sequence of unitaries u_n such that for any $\varphi \in M_*$ $\left\| \left[\begin{bmatrix} \varphi & 0 \\ 0 & \theta(\varphi) \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ u_n & 0 \end{bmatrix} \right] \right\| \to 0$, i.e. such that $\theta = \lim u_n u_n^*$.

To prove that $\tilde{e}_{11}$ is equivalent to $\tilde{e}_{22}$ it is enough to show that any element z of the center Z of P is of the form (y_n), $y_n = \begin{bmatrix} a_n & 0 \\ 0 & \theta(a_n) \end{bmatrix}$, since then $\tau(z\tilde{e}_{11}) = \tau(z\tilde{e}_{22})$ $\forall\, z \in Z$. Any $y = (y_n)$ in Z is of the form $y_n = \begin{bmatrix} a_n & 0 \\ 0 & b_n \end{bmatrix}$ because $\tilde{e}_{jj} \in P$. Moreover, $\|[a_n, \varphi]\| \xrightarrow[n\to\omega]{} 0$, $\|[b_n, \varphi]\| \xrightarrow[n\to\omega]{} 0$ and hence (a) and (b) are centralizing sequences in M. They commute, since $\left(\begin{bmatrix} b_n & 0 \\ 0 & 0 \end{bmatrix}\right) \in P$. Thus it is enough to show that for any centralizing sequence (e_n) of idempotents, $(e_n) \nsim 0$, there exists sequence (y_n) of elements of M such that:

1) $\|y_n\|$ is bounded,

2) $\|y_n\varphi - \theta(\varphi)y_n\| \to 0 \quad \forall\, \varphi \in M_*^+$,

3) $\theta(e_n)y_n e_n \nrightarrow 0$ strongly when $n \to \omega$.

Then $\begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix}$ cannot be centrally disjoint from $\begin{bmatrix} 0 & 0 \\ 0 & \theta(e) \end{bmatrix}$ in the von Neumann algebra P, and it follows that any element of Z has the desired form.

Now we already have a sequence (y_n) satisfying 1), 2) by Lemma 4. To get the required sequence, it is enough to show that for some $\alpha > 0$,

$$\lim_{q\to\omega} \|\theta(e_n)y_q e_n \xi_{\varphi_0}\| \geqslant \alpha \|e_n \xi_{\varphi_0}\| \quad \forall\, n \in \mathbf{N}.$$

One has, from 2), that $\|\theta(e_n)y_q e_n\xi_{\varphi_0} - y_q e_n^2 \xi_{\varphi_0}\| \xrightarrow[q\to\infty]{} 0$, moreover, since $(y_q^* y_q)$ is centralizing and does not tend weakly to 0 the weak limit: $\lim_{q\to\omega} y_q^* y_q$ is a non zero scalar: α^2. Then $\lim_{q\to\omega} \|y_q e_n^2 \xi_{\varphi_0}\|^2 = \alpha^2 \varphi_0(e_n)$, which ends the proof. ▨

IV. $\overline{\mathrm{Int}}\, M$, $C^*(M, M')$ AND THE MODULAR OPERATOR

In [4] we have shown that if M is a factor of type II_1 then the closure of inner automorphisms, $\overline{\mathrm{Int}}\, M$, is characterized by,

$$\theta \in \overline{\mathrm{Int}}\, M \Leftrightarrow \|\textstyle\sum \theta(x_i)y_i\| = \|\textstyle\sum x_i y_i\| \quad \forall\, x_i \in M,\ y_i \in M'.$$

This result fails for trivial reasons when M is of type II_∞, since $\theta \in \mathrm{Aut}\, M$ may have a non trivial module,

$$\mathrm{mod}_M(\theta) \in \mathbf{R}_+^*$$

while the norm equality above is automatic if M is hyperfinite. In general, for arbitrary factors, it is natural to expect that,

$$(*)\qquad \theta \in \overline{\operatorname{Int}} M \Leftrightarrow \begin{cases} \text{a) } \|\sum \theta(x_i)y_i\| = \|\sum x_i y_i\| & \forall x_i \in M,\ y_i \in M'; \\ \text{b) } \operatorname{mod}_M(\theta) = 1. \end{cases}$$

For factors of type III_1 the second condition b) is automatic. For hyperfinite factors the first condition a) is automatic. Thus (*) would imply that $\operatorname{Aut} M = \overline{\operatorname{Int}} M$ for M hyperfinite of type III_1. This, together with Theorem 2.4 would settle the uniqueness problem. It is in trying to prove (*) that we have met the bicentralizer problem. In this section we shall show that if we assume that M is hyperfinite of type III_1 with trivial bicentralizer then $\sigma_{T_0} \in \overline{\operatorname{Int}} M$.

THEOREM 1. *Let M be a hyperfinite factor of type* III_1 *with trivial bicentralizer, then* $\sigma_{T_0} \in \overline{\operatorname{Int}} M$, $\forall\, T_0 \in \mathbf{R}$, *and M is isomorphic to* R_∞.

With a little more work one can prove that (*) holds for any factor of type III_1 with trivial bicentralizer. The general conjecture remains however open.

Let us now assume that M is hyperfinite of type III_1. Let A be the C^*-algebra $C^*(M, M')$ generated by M and M'. Since M is semidiscrete ([4], [10]) one has $A = M \otimes_{\min} M'$ and for any $\theta \in \operatorname{Aut} M$ there exists an automorphism $\tilde{\theta}$ of A such that

$$\tilde{\theta}(\textstyle\sum x_i y_i) = \sum \theta(x_i) y_i \qquad \forall\, x_i \in M,\ y_i \in M'.$$

LEMMA 2. *With the above notations, assume that for any faithful $\varphi \in M_*^+$ one has,*

$$\|(\sigma_{T_0}^{\varphi})^{\sim}(X) + f(\Delta_\varphi)\| = \|X + f(\Delta_\varphi)\|$$

for any $X \in A$ and $f \in C_0(\mathbf{R}_+^*)$. Then $\sigma_{T_0} \in \overline{\operatorname{Int}} M$.

Proof. We fix a faithful $\Psi \in M_*^+$ and let $\theta = \sigma_{T_0}^{\Psi}$. We shall prove that θ satisfies the hypothesis of Theorem 1 of Section III. We let $\xi_1, \ldots, \xi_n \in \mathscr{P}^\natural$ be given, and $\varepsilon > 0$. Let,

$$\varphi(x) = \sum \langle x\xi_j, \xi_j\rangle \qquad \forall\, x \in M.$$

We may assume (using an auxiliary ξ_{n+1}) that φ is faithful. Let $\xi \in \mathscr{P}^\natural$ be the unique vector such that,

$$\varphi(x) = \langle x\xi, \xi\rangle \qquad \forall\, x \in M.$$

As $\langle x\xi_j, \xi_j\rangle \leqslant \langle x\xi, \xi\rangle$ $\forall\, x \in M_+$, there exists an element T_j of M' such that $T_j\xi = \xi_j$, and $\|T_j\| \leqslant 1$. Thus there exists $b_j, b_j \in M$, $\|b_j\| \leqslant 1$ with

$$\xi_j = \xi b_j = b_j^* \xi \qquad \forall\, j = 1, 2, \ldots, n.$$

To prove that θ satisfies the hypothesis of Theorem 1 of Section III it is enough to find $x \in M$, $x \neq 0$, with

$$\|x\xi_j - \sigma^{\varphi}_{T_0}(\xi_j)x\| \leqslant \varepsilon\|x\xi\|.$$

(Since $\sigma^{\Psi}_{T_0}(x) = u\sigma^{\varphi}_{T_0}(x)u^* \quad \forall x \in M$, with $u = (D\Psi : D\varphi)_{T_0}$.) Let $\alpha = \sigma^{\varphi}_{T_0}$. Put $X = \sum_{j=1}^{n} |Jb_j^*J - b_j^*|^2 \in A$. One has

$$\tilde{\alpha}(X) = \sum_{j=1}^{n} |Jb_j^*J - \alpha(b_j^*)|^2.$$

Let $f \in C_c^\infty(\mathbf{R}_+^*)$ be such that

a) $f(\lambda) \in [0, 1]$, $\forall \lambda \in \mathbf{R}_+^*$, $f(1) = 1$,

b) $|(\lambda^{1/2} - 1)(f(\lambda))| \leqslant \varepsilon \quad \forall \lambda \in \mathbf{R}_+^*$.

One has

$$0 \leqslant X + |f(\Delta_\varphi) - 1|^2 \leqslant 4n + 1$$

and

$$\|(4n + 1) - X - |f(\Delta_\varphi) - 1|^2\| = 4n + 1$$

since ξ is an eigenvector for the eigenvalue $4n + 1$. Thus, by hypothesis, we get,

$$\|(4n + 1) - \tilde{\alpha}(X) - |f(\Delta_\varphi) - 1|^2\| = 4n + 1$$

which shows that there exists a unit vector η_1, such that,

α) $\|(Jb_j^*J - \alpha(b_j^*))\eta_1\| \leqslant \varepsilon \quad \forall j$,

β) $\|(f(\Delta_\varphi) - 1)\eta_1\| \leqslant \varepsilon$.

Let $\eta = f(\Delta_\varphi)\eta_1$. Then $\eta \in \operatorname{Dom} \Delta_\varphi^{1/2}$ and,

α') $\|(Jb_j^*J - \alpha(b_j^*))\eta\| \leqslant 3\varepsilon$,

β') $\|(\Delta_\varphi^{1/2} - 1)\eta\| \leqslant \varepsilon$,

γ') $\|\eta\| \geqslant 1 - \varepsilon$.

Since $M\xi$ is dense in the domain of $\Delta^{1/2}$ one can replace η by $x\xi$ for some $x \neq 0$, $x \in M$ and then one has (for ε small enough)

α'') $\|x\xi b_j - \alpha(b_j^*)x\xi\| \leqslant 4\varepsilon\|x\xi\| \quad \forall j$,

β'') $\|x\xi - \xi x\| \leqslant 2\varepsilon\|x\xi\|$.

As $\alpha(\xi) = \xi$ one has $\alpha(\xi_j) = \alpha(b_j^*)\xi$ and thus,

$$\|x\xi_j - \alpha(\xi_j)x\| \leqslant 6\varepsilon\|x\xi\|. \qquad ▨$$

The C^*-algebra $A = C^*(M, M')$ together with the unitaries Δ^{it}, $t \in \mathbf{R}$ does not quite form a covariant system because the automorphisms $X \to \Delta^{it}X\Delta^{-it}$ of A do not depend continuously on t in the pointwise norm topology. We shall now replace A by another C^*-algebra, A_Δ which will be better in this respect.

LEMMA 3. a) *Let K be a compact metric space, μ a probability measure on K and a, b be bounded *-strongly continuous maps from K to M and M' then*

$$\int_K a(t)b(t)\,d\mu(t) \in \mathscr{L}(\mathscr{H}).$$

b) *The subset $\mathscr{B}$ of $\mathscr{L}(\mathscr{H})$ of elements of the form* a) *is a *-subalgebra.*

c) *Let* $A_\Delta =$ norm-closure of $\{T \in \mathscr{B},\ t \to \Delta^{it} T \Delta^{-it}$ is norm continuous$\}$. *Then A_Δ is a C^*-algebra, $\theta_t = \Delta^{it}\cdot\Delta^{-it}$ is a norm continuous action of* $\mathbf{R}$ *on A_Δ.*

d) *For any $X \in A$ and $f \in C_c(\mathbf{R})$ one has,*

$$\int \Delta^{is} X \Delta^{-is} f(s)\,ds \in A_\Delta .$$

Proof. a) Obvious.

b) One has for instance

$$\left(\int_{K_1} a_1(t_1)b_1(t_1)\,d\mu_1(t_1)\right)\left(\int_{K_2} a_2(t_2)b_2(t_2)\,d\mu_2(t_2)\right) =$$

$$= \int_{K_1\times K_2} (a_1(t_1)a_2(t_2))(b_1(t_1)b_2(t_2))\,d\mu_1(t_1)\,d\mu_2(t_2).$$

c) By construction A_Δ is the norm closure of a *-algebra, thus it is a C^*-algebra. By construction $t \to \theta_t(Y)$ is norm continuous for any $Y \in A_\Delta$.

d) It is enough to prove d) when $X = ab$ for $a \in M, b \in M'$. One has $\int \Delta^{is} X \Delta^{-is} f(s)\,ds = \int (\Delta^{is} a \Delta^{-is})(\Delta^{is} b \Delta^{-is}) f(s)\,ds$. Thus it belongs to $\mathscr{B}$. Since it is θ-norm continuous by construction, the conclusion follows. ▨

We now consider the C^*-dynamical system (A_Δ, θ) and the C^*-crossed product $A_\Delta \times|_\theta \mathbf{R}$. The system (A_Δ, Δ^{it}) defines a covariant representation π of (A_Δ, θ) in $\mathscr{H}$ and we let $B = \pi(A_\Delta \times|_\theta \mathbf{R})$ be the corresponding C^*-subalgebra of $\mathscr{L}(\mathscr{H})$. Any element of B is a norm limit of operators of the form

$$\int Y(s)\Delta^{is}\,ds$$

where Y is a norm continuous map from a compact of $\mathbf{R}$ to A_Δ. We shall prove

LEMMA 4. a) *Let $\alpha = \sigma^\varphi_{T_0}$ as above. Under the hypothesis of Theorem* 1 *there exists an automorphism $\tilde\alpha$ of A_Δ such that*

$$\tilde\alpha\left(\int a(s)b(s)\,d\mu(s)\right) = \int \alpha(a(s))b(s)\,d\mu(s)$$

(with the notations of Lemma 3*).*

b) *There exists an automorphism* β *of* $B = \pi(A_\Delta \times\!\mid \mathbf{R})$ *such that*

$$\beta\left(\int Y(s)\Delta^{is}\,ds\right) = \int \tilde{\alpha}(Y(s))\Delta^{is}\,ds.$$

Before we prove Lemma 4, let us show that it implies that $\sigma_{T_0} \in \overline{\operatorname{Int} M}$. With the notations of Lemma 2, we have to show that

$$\|4n + 1 - \tilde{\alpha}(X) - |f(\Delta_\varphi) - 1|^2\| = 4n + 1.$$

Let $h \in C_c^\infty \mathbf{R})$, $h(t) \geqslant 0 \;\; \forall t \in \mathbf{R}$, $\int h(t)\,dt = 1$.

One has,

$$\int \Delta_\varphi^{it} X \Delta_\varphi^{-it} h(t)\,dt \in A_\Delta,$$

$$\int \Delta_\varphi^{it}(4n + 1 - |f(\Delta_\varphi) - 1|^2)\Delta_\varphi^{-it} h(t)\,dt = 4n + 1 - |f(\Delta_\varphi) - 1|^2,$$

$$Z = \left(\int \Delta_\varphi^{it}(4n + 1 - X - |f(\Delta_\varphi) - 1|^2)\Delta_\varphi^{-it} h(t)\,dt\right)\int \Delta^{it} h(t)\,dt \in B$$

and

$$\beta(Z) = \left(\int \Delta_\varphi^{it}(4n + 1 - \tilde{\alpha}(X) - |f(\Delta_\varphi) - 1|^2)\Delta_\varphi^{-it} h(t)\,dt\right) g(\Delta_\varphi)$$

where $g = \hat{h}$. As $\|g(\Delta_\varphi)\| \leqslant 1$ we see that,

$$\|\beta(Z)\| \leqslant \|(4n + 1) - \tilde{\alpha}(X) - |f(\Delta_\varphi) - 1|^2\|.$$

But by hypothesis, $\|\beta(Z)\| = \|Z\|$ and $\|Z\| \geqslant 4n + 1$ because ξ is an eigenvector for the eigenvalue $4n + 1$. Thus we get,

$$\|(4n + 1) - \tilde{\alpha}(X) - |f(\Delta_\varphi) - 1|^2\| = 4n + 1. \qquad ▨$$

Thus the proof of Theorem 1 is reduced to the proof of Lemma 4. To prove Lemma 4 we first recall the following simple fact.

LEMMA 5. *Let* $\mathscr{H}_1, \mathscr{H}_2$ *be two Hilbert spaces,* $\mathscr{A}$ *a* *-*subalgebra of* $\mathscr{L}(\mathscr{H}_1)$ *and* π *a* *-*homomorphism of* $\mathscr{A}$ *in* $\mathscr{L}(\mathscr{H}_2)$.

a) *Let* ξ_2 *be a cyclic vector for* $\pi(\mathscr{A})$. *If* $\|\xi_2\| = 1$ *and*

$$|\langle \pi(a)\xi_2, \xi_2\rangle| \leqslant \|a\| \quad \forall a \in \mathscr{A}$$

then π extends to a norm continuous representation $\bar{\pi}$ of the closure of $\mathscr{A}$ in $\mathscr{L}(\mathscr{H}_1)$ to $\mathscr{L}(\mathscr{H}_2)$.

b) *With the hypothesis of a) assume that ξ_1 is a cyclic vector for $\mathscr{A}$ such that $\|\xi_1\| = 1$ and,*

$$\|\pi(a)\| \geqslant \langle a\xi_1, \xi_1\rangle \quad a \in \mathscr{A}$$

then the representation $\bar{\pi}$ is isometric.

Proof. Cf. [8].

Applying Lemma 5, let us show that Lemma 4 holds if we assume the following two conditions:

A. There exists a sequence $(\xi_\nu)_{\nu\in\mathbf{N}}$, $\xi_\nu \in \mathscr{H}\otimes\mathscr{H}$ such that,

1) $\|(\Delta_\varphi^{it}\otimes\Delta_\varphi^{it})\xi_\nu - \xi_\nu\| \xrightarrow[\nu\to\infty]{} 0 \quad \forall t \in \mathbf{R}$,

2) $\langle(a\otimes b)\xi_\nu, \xi_\nu\rangle \to \langle ab\xi, \xi\rangle \quad \forall a \in M,\ b \in M'$.

B. There exists a sequence Ψ_ν of normal states on $\mathscr{L}(\mathscr{H})$ such that,

1) $\Psi_\nu(\Delta_\varphi^{it}) \to 1 \quad \forall t \in \mathbf{R}$,

2) $\Psi_\nu(ab) \to \langle a\xi, \xi\rangle\langle b\xi, \xi\rangle \quad \forall a \in M,\ b \in M'$.

Let $\mathscr{A}_1$ be the *-subalgebra of $\mathscr{L}(\mathscr{H}\otimes\mathscr{H})$ formed by the operators $T = \int_K a(t)\otimes b(t)\,\mathrm{d}\mu(t)$ with the notations of Lemma 3 a). For $T \in \mathscr{A}_1$ put $\rho(T) = \int_K a(t)b(t)\,\mathrm{d}\mu(t) \in \mathscr{B} \subset \mathscr{L}(\mathscr{H})$. By A 2) one has, using the Lebesgue dominated convergence theorem,

$$\langle\rho(T)\xi, \xi\rangle = \lim_{\nu\to\infty} \langle T\xi_\nu, \xi_\nu\rangle.$$

This shows that ρ is well defined and satisfies the hypothesis of Lemma 5 a). Moreover, as $\xi\otimes\xi$ is cyclic for $\mathscr{A}_1$, Lemma 5 b) combined with B 2) shows that ρ is an isometry. Let then $X = \int a(s)b(s)\,\mathrm{d}\mu(s) \in \mathscr{B}$, one has

$$\|\tilde{\alpha}(X)\| = \left\|\int \alpha(a(s))b(s)\,\mathrm{d}\mu(s)\right\| = \left\|\int \alpha(a(s))\otimes b(s)\,\mathrm{d}\mu(s)\right\| =$$

$$= \left\|(\Delta_\varphi^{iT_0}\otimes 1)\int a(s)\otimes b(s)\,\mathrm{d}\mu(s)(\Delta_\varphi^{-iT_0}\otimes 1)\right\| =$$

$$= \left\|\int a(s)\otimes b(s)\,\mathrm{d}\mu(s)\right\| = \left\|\int a(s)b(s)\,\mathrm{d}\mu(s)\right\| = \|X\|.$$

Since $\tilde{\alpha}$ commutes with $\theta_t = \Delta_\varphi^{it}\cdot\Delta_{\bar{\varphi}}^{-it}$ we have proven Lemma 4 a). Let us prove Lemma 4 b). We have just shown that there exists an isometric homomorphism of A_Δ in

$\mathscr{L}(\mathscr{H}\otimes\mathscr{H})$ such that,

$$\eta\left(\int_K a(s)b(s)\,\mathrm{d}\mu(s)\right)=\int_K a(s)\otimes b(s)\,\mathrm{d}\mu(s).$$

Moreover one has,

$$\eta(\Delta_\varphi^{is}Y\Delta_\varphi^{-is})=(\Delta_\varphi^{is}\otimes\Delta_\varphi^{is})\eta(Y)(\Delta_\varphi^{-is}\otimes\Delta_\varphi^{-is})\qquad\forall s\in\mathbf{R},\ Y\in A_\Delta .$$

Let $\mathscr{A}_2$ be the *-algebra of elements of $\mathscr{L}(\mathscr{H})$ of the form,

$$T=\int Y(s)\Delta_\varphi^{is}\,\mathrm{d}s\qquad Y\in C_c(\mathbf{R},A_\Delta).$$

For $T\in\mathscr{A}_2$ put,

$$\eta_1(T)=\int\eta(Y(s))(\Delta_\varphi^{is}\otimes\Delta_\varphi^{is})\,\mathrm{d}s.$$

The vector $\xi\otimes\xi$ is cyclic for $\eta_1(\mathscr{A}_2)$ and by B one has, $\langle\eta_1(T)\xi\otimes\xi,\xi\otimes\xi\rangle=$ $=\lim_{\nu\to\infty}\Psi_\nu(T)$, thus Lemma 5 a) shows that $\|\eta_1(T)\|\leqslant\|T\|$ $\forall T\in\mathscr{A}_2$. As the vector ξ is cyclic for $\mathscr{A}_2$ one has using A

$$\|\langle T\xi,\xi\rangle\|\leqslant\|\eta_1(T)\|.$$

Thus

$$\|\eta_1(T)\|=\|T\|\qquad\forall T\in\mathscr{A}_2 .$$

We hence have,

$$\left\|\int\tilde{\alpha}(Y(s))\Delta^{is}\,\mathrm{d}s\right\|=\left\|\int\eta(\tilde{\alpha}(Y(s)))\Delta^{is}\otimes\Delta^{is}\,\mathrm{d}s\right\|=$$

$$=\left\|\int(\Delta^{iT_0}\otimes 1)\eta(Y(s))(\Delta^{iT_0}\otimes 1)(\Delta^{is}\otimes\Delta^{is})\mathrm{d}s\right\|=$$

$$=\left\|(\Delta^{iT_0}\otimes 1)\left(\int\eta(Y(s))(\Delta^{is}\otimes\Delta^{is})\mathrm{d}s\right)(\Delta^{iT_0}\otimes 1)\right\|=$$

$$=\left\|\int\eta(Y(s))(\Delta^{is}\otimes\Delta^{is})\mathrm{d}s\right\|=\left\|\int Y(s)\Delta^{is}\,\mathrm{d}s\right\|.$$

This proves Lemma 4 b).

Proof of A. Since M is semi-discrete there exists a (non normal) state Ψ on $\mathscr{L}(\mathscr{H}\otimes\mathscr{H})$ whose restriction to the C^* tensor product $M\otimes_{\min}M'$ is given by,

$$\Psi(x\otimes y) = \langle xy\xi, \xi\rangle \quad \forall x \in M,\ y \in M'.$$

This shows that given $\varepsilon > 0$, $x_1, \ldots, x_n \in M$, $y_1, \ldots, y_n \in M'$ there exists a *normal* state Ψ' on $\mathscr{L}(\mathscr{H}\otimes\mathscr{H})$ such that,

α) $\Psi'(x\otimes 1) = \langle x\xi, \xi\rangle \quad \forall x \in M$,

β) $\Psi'(1\otimes y) = \langle y\xi, \xi\rangle \quad \forall y \in M'$,

γ) $|\Psi'(x_i\otimes y_i) - \langle x_i y_i\xi, \xi\rangle| \leqslant \varepsilon \quad \forall i = 1, \ldots, n$.

Since the unit ball of M (and M') is separable for the strong topology it follows that there exists a sequence Ψ'_ν of normal states on $\mathscr{L}(\mathscr{H}\otimes\mathscr{H})$ such that,

a) $\Psi_\nu(x\otimes 1) = \langle x\xi, \xi\rangle \quad \forall x \in M$,

b) $\Psi_\nu(1\otimes y) = \langle y\xi, \xi\rangle \quad \forall y \in M'$,

c) $\Psi_\nu(x\otimes y) \to \langle xy\xi, \xi\rangle \quad \forall x \in M, y \in M'$.

Replacing Ψ_ν by $\Psi_\nu^{N_\nu}$,

$$\Psi_\nu^{N_\nu} = \frac{1}{2N_\nu}\int_{-N_\nu}^{N_\nu} (\Delta^{is}\otimes\Delta^{is})\Psi_\nu(\Delta^{-is}\otimes\Delta^{-is})ds$$

one can assume that $\|\Psi_\nu - (\Delta^{is}\otimes \Delta^{is})\Psi_\nu(\Delta^{-is}\otimes\Delta^{-is})\| \xrightarrow[\nu\to\infty]{} 0$ for any $s \in \mathbf{R}$. Let then ξ_ν be the canonical vector in the natural cone of $M\otimes M'$ in $\mathscr{H}\otimes\mathscr{H}$ which is such that,

d) $\Psi_\nu(x\otimes y) = \langle (x\otimes y)\xi_\nu, \xi_\nu\rangle \quad \forall x \in M,\ y \in M'$.

Since $\|\Psi_\nu - (\Delta^{is}\otimes\Delta^{is})\Psi_\nu(\Delta^{-is}\otimes\Delta^{-is})\| \to 0$ one gets

$$\|(\Delta^{is}\otimes\Delta^{is})\xi_\nu - \xi_\nu\| \to 0 \quad \forall s \in \mathbf{R}.$$

Moreover, Condition A 2) is clearly fulfilled by c) and d).

Proof of B. By [11] Proposition 1.4 b'), there exists a sequence P_n of linear maps of norm one from $\mathscr{L}(\mathscr{H})$ to $\mathscr{L}(\mathscr{H})$ such that,

α) For each n there exists unitaries u_j^n in M, with $\|[u_j^n, \varphi]\| \leqslant 1/n \quad \forall j$, and scalars $\lambda_j^n \geqslant 0$, $\sum \lambda_j^n = 1$, with

$$P_n(T) = \sum \lambda_j^n u_j^n T u_j^{n*} \quad \forall T \in \mathscr{L}(\mathscr{H}).$$

β) $P_n(x) \to \varphi(x)1$ in the *-strong topology, for any $x \in M$.

For any $\nu \in \mathbf{N}$, let Ψ_ν be the state on $\mathscr{L}(\mathscr{H})$ given by,

$$\Psi_\nu(T) = \sum \lambda_j^\nu \langle T u_j^{\nu*}\xi, u_j^{\nu*}\xi\rangle \quad \forall T \in \mathscr{L}(\mathscr{H}).$$

For any $t \in \mathbf{R}$, one has, using α),

$$\sup_j \| \Delta^{it} u_j^{\nu *} \xi - u_j^{\nu *} \xi \| \xrightarrow[\nu \to \infty]{} 0.$$

Thus, $\Psi_\nu(\Delta^{it}) \to 1 \ \forall t \in \mathbf{R}$. For any $a \in M, b \in M'$ one has,

$$\Psi_\nu(ab) = \sum \lambda_j^\nu \langle abu_j^{\nu *}\xi, u_j^{\nu *}\xi \rangle =$$

$$= \langle (\sum \lambda_j^\nu u_j^\nu a u_j^{\nu *}) b\xi, \xi \rangle = \langle P_\nu(a) b\xi, \xi \rangle.$$

Thus $\Psi_\nu(ab) \to \langle a\xi, \xi \rangle \langle b\xi, \xi \rangle$. ▨

REMARK. One can in fact show that the homomorphism $\pi : A_\Delta \times|_\theta \mathbf{R} \to B$ is an isomorphism.

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