J. OPERATOR THEORY
14(1985), 215–238

QUASISIMILARITY AND CLOSURES OF SIMILARITY ORBITS OF OPERATORS

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1. INTRODUCTION

In the theory of Hilbert space operators there are several relations which play roles analogous to that played by similarity in the theory of operators on finite dimensional spaces. In the finite dimensional case the Jordan form models each operator up to similarity, and this model preserves spectral and invariant subspace structures. In the infinite dimensional case such relations as quasisimilarity and asymptotic similarity have been successfully used to model special classes of operators, but the general properties of these relations have not been fully described. In the present note we examine connections between quasisimilarity and asymptotic similarity as a step towards a better understanding of these relations.

Let $\mathcal{H}$ denote a separable complex infinite dimensional Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. Operators T and S in $\mathcal{L}(\mathcal{H})$ are *similar* if there exists an invertible operator $X \in \mathcal{L}(\mathcal{H})$ such that $TX = XS$, i.e., $S = X^{-1}TX$. For T in $\mathcal{L}(\mathcal{H})$, let $\mathcal{S}(T)$ denote the similarity orbit of T, i.e., $\mathcal{S}(T) = \{S \in \mathcal{L}(\mathcal{H}) : S$ is similar to $T\}$, and let $\mathcal{S}(T)^-$ denote the norm closure of $\mathcal{S}(T)$ in $\mathcal{L}(\mathcal{H})$. Operators T and S are *asymptotically similar* if $\mathcal{S}(T)^- = \mathcal{S}(S)^-$ (equivalently, $T \in \mathcal{S}(S)^-$ and $S \in \mathcal{S}(T)^-$). In [6] C. Apostol, D. Herrero, and D. Voiculescu described the closures of the similarity orbits of operators and they subsequently provided canonical models relative to asymptotic similarity (see [5, Corollary 9.30]).

Operators T and S are *quasisimilar* ($T \underset{\mathrm{qs}}{\sim} S$) if there exist operators X and Y, each injective and with dense range, such that $TX = XS$ and $YT = SY$; for $T \in \mathcal{L}(\mathcal{H})$, let $(T)_{\mathrm{qs}} = \{S \in \mathcal{L}(\mathcal{H}) : T \underset{\mathrm{qs}}{\sim} S\}$, the quasisimilarity orbit of T. Quasisimilarity was introduced by Sz.-Nagy and C. Foiaş in [35] where it was shown to be an effective tool in the model theory of contractions; for example, quasisimilarity models were given for C_{11} contractions [35, Proposition 5.3] and $C_0(N)$ contractions [35, page 370]. Subsequently, quasisimilarity models for other classes of operators were obtained. It was shown in [3] and [37] that each algebraic operator is quasisimilar to an essentially unique Jordan model. In [7] W. S. Clary characterized the (cyclic) subnormal operators quasisimilar to the unilateral shift, and in [18] W. Hastings extended these results to subnormal operators quasisimilar to multicyclic isometries. More recently, P. Y. Wu [38] characterized the contractions with finite defect indices that are quasisimilar to multicyclic unilateral shifts.

Certain general properties of quasisimilarity are also known. Suppose $T \underset{\text{qs}}{\sim} S$. If T has a proper hyperinvariant subspace, then so does S [27], and a certain sublattice of the lattice of hyperinvariant subspaces is preserved [12]. (However, the full lattice of hyperinvariant subspaces is, in general, not preserved [21].) Each component of the spectrum of T, $\sigma(T)$, intersects $\sigma(S)$ [22]; moreover, the essential spectrum of T, $\sigma_e(T)$, intersects $\sigma_e(S)$ [11], [36].

It is an interesting open problem whether, in this case, each *component* of $\sigma_e(T)$ intersects $\sigma_e(S)$ [14], [34]. Moreover, it is not fully understood how the approximate point spectrum of T, $\sigma_\pi(T)$, is related to $\sigma_\pi(S)$, or how the semi-Fredholm domain of T, $\rho_{\text{SF}}(T)$, is related to $\rho_{\text{SF}}(S)$. It is known that, in general, quasisimilarity preserves neither compactness [27] nor quasitriangularity [16].

There is no general connection between asymptotic similarity and quasisimilarity. For example, asymptotic similarity preserves spectra, approximate point spectra, and essential spectra, while quasisimilarity does not, and quasisimilarity preserves point spectra while asymptotic similarity does not. Nevertheless, in the sequel we will show that in certain cases in which the aforementioned difficulties concerning semi-Fredholm behavior can be resolved, quasisimilarity and asymptotic similarity are closely related. In this connection, we study the following *quasisimilarity orbit inclusion property* of an operator T,

$$(T)_{\text{qs}} \subset \mathscr{S}(T)^-,$$

and we pose the following problem:

PROBLEM 1.1. *Which operators satisfy the quasisimilarity orbit inclusion property?*

For $A, T \in \mathscr{L}(\mathscr{H})$, one necessary condition that A belong to $\mathscr{S}(T)^-$ is that $\sigma(T) \subset \sigma(A)$, so in studying Problem 1.1 we are led to consider operators satisfying the following *quasisimilarity spectral inclusion property:*

(Σ) $\sigma(T) \subset \sigma(A)$ for every operator A quasisimilar to T.

Hyponormal, compact, and spectral operators satisfy (Σ) [8], [11]. Since similarity preserves the spectrum, each operator T satisfying $(T)_{\text{qs}} = \mathscr{S}(T)$ clearly satisfies (Σ). D. Herrero [25] proved the remarkable result that the set $\{T \in \mathscr{L}(\mathscr{H}): (T)_{\text{qs}} = \mathscr{S}(T)\}$ is *norm dense* in $\mathscr{L}(\mathscr{H})$. In particular, each normal operator N with finite spectrum satisfies $(N)_{\text{qs}} = \mathscr{S}(N)$ [25, p. 104], and an arbitrary normal operator M may be approximated in norm by normal operators having finite spectra;

these facts do not, however, seem to aid in the description or approximation of the operators in $(M)_{qs}$. The orbit inclusion property for an operator T is clearly equivalent to the identity $(T)_{qs}^{-} = \mathscr{S}(T)^{-}$; in this case, up to approximation, quasisimilarity reduces to similarity.

In Theorem 2.1 we provide a sufficient condition for the orbit inclusion property. As an application, in Corollary 2.2 we prove that a normal operator has the orbit inclusion property if and only if each isolated point of the essential spectrum is isolated in the spectrum. Among the non-normal operators with the orbit inclusion property, we exhibit the universal quasinilpotent operators (Corollary 2.6) and the cyclic hyponormal operators T having no normal eigenvalues and for which $\sigma_e(T) = \mathrm{bdry}(\sigma(T))$ (Corollary 2.13). In particular, operators with the orbit inclusion property include each cyclic hyponormal unilateral weighted shift (Corollary 2.14) and each cyclic subnormal operator quasisimilar to the unilateral shift (Corollary 2.15). In [29] M. Raphael proved that quasisimilar cyclic subnormal operators have equal approximate point spectra and equal essential spectra. In Proposition 2.16 we use these results to obtain the stronger conclusion that quasisimilar cyclic subnormal operators are asymptotically similar.

In [12, Theorem 4.8] it was shown that quasisimilar (injective) bilateral weighted shifts have equal spectra. In Section 3 we extend this result to approximate point spectra and essential spectra. We prove in Proposition 3.9 that (excluding one exceptional case) quasisimilar bilateral weighted shifts are asymptotically similar.

We conclude this section with some additional notation. For T in $\mathscr{L}(\mathscr{H})$, let $\sigma_p(T)$ denote the point spectrum of T, and let $\sigma_l(T)$ and $\sigma_r(T)$ denote, respectively, the left and right spectra of T; thus $\sigma_l(T) = \sigma_\pi(T)$. Let $\ker(T)$ and $\mathrm{ran}(T)$ denote the kernel and range of T. For a linear subspace $\mathscr{M} \subseteq \mathscr{H}$, let $\dim \mathscr{M}$ denote the algebraic dimension of $\mathscr{M}$. For an operator T, let $\mathrm{nul}(T) = \dim \ker(T)$ and $\mathrm{rank}(T) = \dim \mathrm{ran}(T)$. We denote similarity of operators by $\sim$ and unitary equivalence by $\approx$.

Let $\mathscr{K}(\mathscr{H})$ denote the ideal of all compact operators in $\mathscr{L}(\mathscr{H})$, and for T in $\mathscr{L}(\mathscr{H})$, let $\tilde{T}$ denote the image of T in the Calkin algebra $Q(\mathscr{H}) = \mathscr{L}(\mathscr{H})/\mathscr{K}(\mathscr{H})$ under the canonical projection. Thus $\sigma_e(T)$ is the spectrum of $\tilde{T}$ and T is semi-Fredholm if and only if $\tilde{T}$ is left or right invertible in $\mathscr{A}(\mathscr{H})$; for other properties of semi-Fredholm operators and essential spectra see [15], [26], [28].

For $T \in \mathscr{L}(\mathscr{H})$, a closed subspace $\mathscr{M}$ of $\mathscr{H}$ is *invariant* for T (or *T-invariant*) if $T\mathscr{M} \subset \mathscr{M}$.

A finite or denumerable sequence of closed subspaces of $\mathscr{H}$, $\{\mathscr{M}_n\}_{n=1}^{m}$, is a *basic sequence* if

i) for each k, $\mathscr{M}_k$ and $\bigvee_{n\neq k}\mathscr{M}_n$ (closed span) are complementary in $\mathscr{H}$, and

ii) if $m = \infty$, then $\bigcap_{k=1}^{\infty}(\bigvee_{n\geq k+1}\mathscr{M}_n) = \{0\}$.

If $T \in \mathscr{L}(\mathscr{H})$ and the $\mathscr{M}_n$'s are invariant for T, the sequence is said to be a *basic sequence for T*. The notion of a basic sequence is due to C. Apostol [1], who used this concept to characterize the operators quasisimilar to normal operators (see also [12], [14], [25]).

Finally, we denote by q_k the Jordan nilpotent k-cell acting on $\mathbf{C}^k$ $(1 \leqslant k < \infty)$.

Acknowledgment. The author thanks the refere for many helpful suggestions.

2. QUASISIMILARITY ORBITS CONTAINED IN CLOSURES OF SIMILARITY ORBITS

In this section we present conditions on an operator T that are sufficient to guarantee that $(T)_{qs} \subset \mathscr{S}(T)^-$. Since each operator $A \in \mathscr{S}(T)^-$ satisfies $\sigma(T) \subset \subset \sigma(A)$, we will begin by formulating a property that implies the quasisimilarity spectral inclusion property. To this end we require some notation and terminology.

For a subset $\Lambda \subset \mathbf{C}$, let $\Lambda^* = \{\bar{\lambda} : \lambda \in \Lambda\}$ and let $[\Lambda]_{isol}$ denote the set of isolated points of Λ, i.e., $[\Lambda]_{isol} = \{\lambda \in \Lambda : \lambda \notin [\Lambda \setminus \{\lambda\}]^-\}$. For T in $\mathscr{L}(\mathscr{H})$, let $\sigma_{qn}(T) = \{\lambda \in \mathbf{C} : \|(T-\lambda)^n x\|^{1/n} \to 0 \ (n \to \infty)$ for some nonzero x in $\mathscr{H}\}$, the "quasinilpotent" points of the spectrum of T. Thus $[\sigma(T)]_{isol} \subset \sigma_{qn}(T)$ [32, p. 424], and clearly $\sigma_p(T) \subset \sigma_{qn}(T) \subset \sigma_l(T)$, $\sigma_p(T^*)^* \subset \sigma_{qn}(T^*)^* \subset \sigma_l(T^*)^* = \sigma_r(T)$.

We recall that if $A, T, X \in \mathscr{L}(\mathscr{H})$, $AX = XT$, and X is injective, then the following properties hold:

(σ)

i) $\sigma_p(T) \subset \sigma_p(A)$;

ii) $\text{nul}(T-\lambda)^k \leqslant \text{nul}(A-\lambda)^k$, $\lambda \in \mathbf{C}$, $k = 1,2, \ldots$;

iii) $\text{rank}(T-\lambda)^k \leqslant \text{rank}(A-\lambda)^k$, $\lambda \in \mathbf{C}$, $k = 1,2, \ldots$;

iv) $\sigma_{qn}(T) \subset \sigma_{qn}(A)$;

v) If $\mathscr{M} \neq \{0\}$ is a T-invariant subspace of $\mathscr{H}$, then $\sigma(T \mid \mathscr{M}) \cap \sigma(A) \neq \emptyset$ [11];

vi) Each nonempty closed-and-open subset of $\sigma(T)$ intersects $\sigma(A)$;

vii) Each component of $\sigma(T)$ intersects $\sigma(A)$ [22];

viii) $[\sigma(T)]_{isol} \subset \sigma(A)$.

Properties i)—iii) follow immediately from the relation $(A-\lambda)^k X = X(T-\lambda)^k$, and iv) follows from the inequality $\|(A-\lambda)^k X y\|^{1/k} \leqslant \|X\|^{1/k} \|(T-\lambda)^k y\|^{1/k}$ $(y \in \mathscr{H}$, $k = 1,2, \ldots)$. Properties v) and vi) are contained in [11]. The refinement in vii) is proved in [22], while viii) follows from vii) or iv).

If the injective intertwining operator X also has dense range (i.e., X is a *quasiaffinity*), then T is a *quasiaffine transform* of A and (σ) may also be applied to the equation $T^* X^* = X^* A^*$. If A and T are quasisimilar, then the conditions described in (σ) become symmetric in A and T, e.g., $\sigma_p(T) = \sigma_p(A)$, etc., and in the sequel, references to (σ) will be to this symmetric version. Moreover, when A and T are quasisimilar, additional spectral relationships between A and T arise, such as those described by the essential spectra intersection theorems of [11], [13], [17], [34], [36].

Note that if A is quasisimilar to T, then $[\sigma_{qn}(T) \cup \sigma_{qn}(T^*)^*]^- \subset \sigma(A)$. Thus, if we are interested in the inclusion $\sigma(T) \subset \sigma(A)$, it suffices to consider the residual set $\sigma'(T) = \sigma(T) \setminus [\sigma_{qn}(T) \cup \sigma_{qn}(T^*)^*]^-$. The following property describes the condition that $\sigma'(T)$ supports sufficiently many spectral subspaces of T or T^*.

(σ') If φ is an open subset of $\mathbf{C}$ and $\varphi \cap \sigma'(T) \neq \emptyset$, then there exists a nonzero T-invariant subspace $\mathscr{M}$ such that $\sigma(T|\mathscr{M}) \subset \varphi$, or there exists a nonzero T^*-invariant subspace $\mathscr{N}$ such that $\sigma(T^*|\mathscr{N}) \subset \varphi^*$.

LEMMA 2.1. *If $T \in \mathscr{L}(\mathscr{H})$ satisfies (σ'), then $\sigma(T) \subset \sigma(A)$ for every A quasisimilar to T.*

Proof. Let $A \in (T)_{qs}$ and let X and Y be quasiaffinities such that $AX = XT$ and $YA = TY$. It suffices to prove that $\sigma'(T) \subset \sigma(A)$. Suppose $\lambda \in \sigma'(T) \setminus \sigma(A)$ and let φ be an open set such that $\lambda \in \varphi$ and $\varphi \cap \sigma(A) = \emptyset$. Condition ($\sigma'$) implies the existence of

i) a T-invariant subspace $\mathscr{M}$ with $\sigma(T|\mathscr{M}) \subset \varphi$, or

ii) a T^*-invariant subspace $\mathscr{N}$ with $\sigma(T^*|\mathscr{N}) \subset \varphi^*$.

Since $\varphi \cap \sigma(A) = \emptyset$, in case i) we have a contradiction to (σ)-(v); since $\varphi^* \cap \sigma(A^*) = \emptyset$, in case ii) we have a contradiction to (σ)-(v) applied to $A^*Y^* = Y^*T^*$.

REMARK. Each normal operator satisfies (σ'); we do not know whether every hyponormal operator satisfies (σ'). The operator $T = \sum_{k=1}^{\infty} \oplus q_k$ does not satisfy (Σ) [27] and thus does not satisfy (σ'). In this case, $\sigma(T)$ is the closed unit disk and $\sigma'(T) = \{\lambda : 0 < |\lambda| \leqslant 1\}$.

COROLLARY 2.2. *If $\sigma'(T) = \emptyset$, then T satisfies (Σ).*

Note that $\sigma'(T) = \emptyset$ is satisfied by operators T such as diagonalizable normal operators, quasinilpotent operators, the adjoint of the unilateral shift, etc.

Recall the concept of a *decomposable* operator as defined in [9]: every normal, spectral, or compact operator is decomposable [9, page 33]. It is clear from [9] that every decomposable operator satisfies (σ'); note, however, that the unilateral shift satisfies (σ') although neither it nor its adjoint is decomposable (see [9, page 10]).

COROLLARY 2.3. *Let T be in $\mathscr{L}(\mathscr{H})$. If $\mathscr{M} \neq \{0\}$ is a T-invariant subspace such that $S = T|\mathscr{M}$ is decomposable and $\sigma'(T) \subset \sigma(S)$, then T satisfies (Σ).*

Proof. For $A \in (T)_{qs}$, [11, Corollary 2.12] implies that $\sigma(S) \subset \sigma(A)$, and thus $\sigma(T) = [\sigma_{qn}(T) \cup \sigma_{qn}(T^*)^*]^- \cup \sigma'(T) \subset \sigma(A) \cup \sigma(S) \subset \sigma(A)$.

We now state the main results of this section.

THEOREM 2.4. *The following set of conditions implies that an operator $T \in \mathscr{L}(\mathscr{H})$ satisfies the quasisimilarity orbit inclusion property.*

1) $[\sigma_e(T)]_{isol} \subset [\sigma(T)]_{isol}$;

2) *T satisfies the spectral property (σ');*

3) $\mathrm{int}(\sigma_p(T)) = \mathrm{int}(\sigma_p(T^*)) = \emptyset$;

4) *For each* $\lambda \in [\sigma_e(T)]_{isol}$, let $\mathscr{H}(\lambda, T)$ *denote the Riesz subspace of* T *corresponding to the isolated subset* $\{\lambda\} \subset \sigma(T)$. *Either* $(T-\lambda) | \mathscr{H}(\lambda, T) = 0$ *or* $(T-\lambda)^k | \mathscr{H}(\lambda, T)$ *is noncompact for every* $k \geq 1$.

Condition 3) will be used to relate the semi-Fredholm behaviour of any $A \in (T)_{qs}$ to the semi-Fredholm behavior of T. Condition 2) is used only to insure that T satisfies the spectral inclusion property (Σ) (and may thus be replaced by the weaker hypothesis that T satisfies (Σ)). Hypotheses 1) and 4) are related to the known phenomenon that quasisimilarity does not preserve compactness [27]; these hypotheses will be discussed below. We will show at the conclusion of this section that none of hypotheses 1), 3), or 4) is necessary in order to conclude that $(T)_{qs} \subset \mathscr{S}(T)^-$. Indeed, for $i = 1,3,4$ there exists an operator T_i that fails to satisfy property i) but for which $(T_i)_{qs} = \mathscr{S}(T_i)$. As stated above, hypothesis 2) may be formally replaced by property (Σ); however, we know of no operator satisfying the orbit inclusion property that does not also satisfy (σ'). In any case, hypotheses 1)—4) yield positive results for both normal and certain non-normal operators T in cases when $(T)_{qs} \not\subset \mathscr{S}(T)$. To clarify these hypotheses it is helpful to consider special cases.

Clearly, every normal operator $N \in \mathscr{L}(\mathscr{H})$ satisfies 2) and 3); moreover, if λ is an isolated point of $\sigma(N)$, then $(N-\lambda) | \mathscr{H}(\lambda, N) = 0$. Indeed, we will show that for normal operators, Theorem 2.4 is best possible:

COROLLARY 2.5. *A normal operator* $N \in \mathscr{L}(\mathscr{H})$ *satisfies* $(N)_{qs} \subset \mathscr{S}(N)^-$ *if and only if* $[\sigma_e(N)]_{isol} \subset [\sigma(N)]_{isol}$.

Each quasinilpotent operator in $\mathscr{L}(\mathscr{H})$ satisfies 1), 2), and 3) trivially. A quasinilpotent Q is *universal* if $\mathscr{S}(Q)^-$ contains every quasinilpotent operator in $\mathscr{L}(\mathscr{H})$ [23], [26]. In [2], [23] it is proved that a quasinilpotent Q is universal if and only if Q^k is noncompact for every $k \geq 1$; therefore, a universal quasinilpotent operator satisfies 4).

COROLLARY 2.6. *If* $Q \in \mathscr{L}(\mathscr{H})$ *is a universal quasinilpotent, then* $(Q)_{qs} \subset \mathscr{S}(Q)^-$.

We note that a direct proof of Corollary 2.6 can be obtained without recourse to Theorem 2.4. Indeed, if Q is quasinilpotent and A is quasisimilar to Q, then A is biquasitriangular, $\sigma(A) = \sigma_e(A)$, and $\sigma(A)$ is a connected set containing 0 [1, Theorem 3.1]. It follows from [4] that A is a norm limit of quasinilpotents (indeed, a limit of nilpotents); thus, if Q is universal, then $A \in \mathscr{S}(Q)^-$. We do not know exactly which quasinilpotent operators satisfy the orbit inclusion property. Here we note only that there are known examples of infinite rank compact quasinilpotent operators K for which $(K)_{qs} = \mathscr{S}(K)$ or for which $(K)_{qs} \not\subset \mathscr{K}(\mathscr{H})$ (so that $(K)_{qs} \not\subset \mathscr{S}(K)^-$); these and related examples will be discussed below.

The proof of Theorem 2.4 depends on results of C. Apostol, D. Herrero, and D. Voiculescu [5], [6] which characterize the closure of the similarity orbit of an operator. Since we will refer to these results in some detail, for ease of reference we include a version of them sufficient to our purposes. Before doing so, we require some additional terminology.

For T in $\mathcal{L}(\mathcal{H})$, let $\rho_{SF}(T) = \{\lambda \in \mathbf{C} : T - \lambda \text{ is semi-Fredholm}\}$, the semi-Fredholm domain of T; for $\lambda \in \rho_{SF}(T)$, let $\operatorname{ind}(T-\lambda) = \dim\ker(T-\lambda) - \dim\ker(T-\lambda)^*$ and let $\text{min.ind.}(T-\lambda)^k = \min(\dim\ker(T-\lambda)^k, \dim\ker(T-\lambda)^{*k})$ $(k \geqslant 1)$. Let $\sigma_{le}(T)$ and $\sigma_{re}(T)$ denote, respectively, the left and right essential spectra of T [15]; thus $\sigma_{lre}(T) = \mathbf{C} \setminus \rho_{SF}(T) = \sigma_{le}(T) \cap \sigma_{re}(T)$.

If σ is a nonempty closed-and-open subset of $\sigma(T)$, let P_σ denote the Riesz idempotent for T corresponding to σ and let $\mathcal{H}(\sigma, T) = \operatorname{ran}(P_\sigma)$ denote the Riesz invariant subspace for T corresponding to σ [30, p. 31], [32, p. 421]; when $\sigma = \{\lambda\}$ $(\lambda \in \mathbf{C})$, we denote P_σ by P_λ and $\mathcal{H}(\sigma, T)$ by $\mathcal{H}(\lambda, T)$. Let $\sigma_0(T) = [\sigma(T)]_{\text{isol}} \cap \rho_{SF}(T)$, the set of normal eigenvalues of T [26, p. 5]; thus $\sigma_0(T) = \{\lambda \in [\sigma(T)]_{\text{isol}} : \dim \mathcal{H}(\lambda, T) < \infty\}$.

If $\lambda \in [\sigma_e(T)]_{\text{isol}}$ and $\psi : Q(\mathcal{H}) \to \mathcal{L}(\mathcal{H}_\psi)$ is a faithful $*$-representation of the Calkin algebra, then the Riesz Decomposition Theorem [32, p. 421], [30] implies that $\psi(\tilde{T})$ is similar to $(\lambda + Q_\lambda) \oplus R_\lambda$, where Q_λ is quasinilpotent and $R_\lambda - \lambda$ is invertible. Following [5, p. 4] we define a function $k^+(\lambda, \tilde{T})$ $(\lambda \in \mathbf{C})$ by

$$k^+(\lambda, \tilde{T}) = \begin{cases} 0 & \text{if } \lambda \notin [\sigma_e(T)]_{\text{isol}}. \\ 1 & \text{if } \lambda \in [\sigma_e(T)]_{\text{isol}} \text{ and } Q_\lambda = 0. \\ n & \text{if } \lambda \in [\sigma_e(T)]_{\text{isol}},\ Q_\lambda \text{ is nilpotent of order } n \geqslant 2, \text{ and } Q_\lambda^{n-1} + Q_\lambda^* \\ & \text{is not invertible.} \\ n + 1/2 & \text{if } \lambda \in [\sigma_e(T)]_{\text{isol}},\ Q_\lambda \text{ is nilpotent of order } n \geqslant 2, \text{ and} \\ & Q_\lambda^{n-1} + Q_\lambda^* \text{ is invertible.} \\ +\infty & \text{if } \lambda \in [\sigma_e(T)]_{\text{isol}} \text{ and } Q_\lambda \text{ is not nilpotent.} \end{cases}$$

As in [5, p. 4] we further define $\sigma_{ne}(T) = \{\lambda \in [\sigma_e(T)]_{\text{isol}} : 1 \leqslant k^+(\lambda, \tilde{T}) < +\infty\}$ (the "essentially nilpotent" points of the spectrum of T) and $\sigma_{me}(T) = \{\lambda \in \sigma_{ne}(T) : k^+(\lambda, \tilde{T}) = n + 1/2 \text{ for some } n,\ 1 \leqslant n < +\infty\}$.

Hypothesis 4) of Theorem 2.4 implies that $k^+(\cdot, \tilde{T})$ is valued in $\{0, 1, +\infty\}$, so in particular, $\sigma_{me}(T) = \emptyset$ and $\sigma_{ne}(T) = \{\lambda \in [\sigma_e(T)]_{\text{isol}} : (T - \lambda) | \mathcal{H}(\lambda, T) = 0\}$. (In the last identity we are also assuming hypothesis 1).) Hypotheses 1) and 4) are used to exclude an obstruction to orbit inclusion similar to that which occurs when a compact operator is quasisimilar to a noncompact operator.

We require the following theorem of C. Apostol, D. Herrero, and D. Voiculescu [5], [6].

THEOREM 2.7. [5, Theorem 9.1] *Let $T \in \mathcal{L}(\mathcal{H})$ satisfy $\sigma_{me}(T) \subset \operatorname{int}(\sigma(T))$. An operator $A \in \mathcal{L}(\mathcal{H})$ belongs to $\mathcal{S}(T)^-$ if and only if A satisfies the following properties:*

(S) (spectral conditions) *$\sigma_0(A) \subset \sigma_0(T)$ and each component of $\sigma_{lre}(A) \setminus \sigma_{ne}(A)$ intersects $\sigma_e(T) \setminus \sigma_{ne}(T)$.*

(F) (Fredholm conditions) $\rho_{SF}(A) \subset \rho_{SF}(T)$, $\text{ind}(\lambda - A) = \text{ind}(\lambda - T)$ *for each* $\lambda \in \rho_{SF}(A)$, *and* $\text{min. ind.}(\lambda - A)^k \geqslant \text{min. ind.}(\lambda - T)^k$ *for each* $\lambda \in \rho_{SF}(A)$, $k = 1, 2, \ldots$.

(A) (algebraic properties) $\dim \mathscr{H}(\lambda, A) = \dim \mathscr{H}(\lambda, T)$ *for each* $\lambda \in \sigma_0(A)$, *and* $k^+(\lambda, \tilde{A}) \leqslant k^+(\lambda, \tilde{T})$ *for each* $\lambda \in \mathbf{C}$. *If* $\lambda \in \sigma_{ne}(A)$ *is isolated in* $\sigma(A)$, *then* λ *is isolated in* $\sigma(T)$ *and*

$$\text{rank}((\lambda - A)^k \mid \mathscr{H}(\lambda, A)) \leqslant \text{rank}((\lambda - T)^k \mid \mathscr{H}(\lambda, T)) \quad \textit{for all } k \geqslant k^+(\lambda, \tilde{A}).$$

As noted in [5, p. 5], properties (S) and (F) also imply that $\sigma(T) \subset \sigma(A)$ and each component of $\sigma(A)$ intersects $\sigma(T)$, and that $\sigma_e(T) \subset \sigma_e(A)$ and each component of $\sigma_e(A)$ intersects $\sigma_e(T)$.

Proof of Theorem 2.4. We will show that if T satisfies properties 1)—4) and A is quasisimilar to T, then A satisfies (S), (F), and (A) of Theorem 2.7. Well-known properties of semi-Fredholm operators and the hypothesis that $\text{int}(\sigma_p(T)) = \text{int}(\sigma_p(T^*)) = \emptyset$ imply that $\rho_{SF}(T) \cap \sigma(T) = \sigma_0(T)$; since $A \in (T)_{qs}$, (σ)-(i) implies that $\rho_{SF}(A) \cap \sigma(A) = \sigma_0(A)$. Thus we have

$$\sigma(T) = \sigma_0(T) \,\dot{\cup}\, \sigma_{lre}(T), \quad \sigma(A) = \sigma_0(A) \,\dot{\cup}\, \sigma_{lre}(A). \tag{2.1}$$

We will first show that A satisfies the Fredholm property (F). Since T satisfies (σ'), Lemma 2.1 implies that

$$\sigma(T) \subset \sigma(A). \tag{2.2}$$

If α is an isolated point of $\sigma(A)$, then $\alpha \in \sigma(T)$ ((σ)-(viii)), and (2.2) shows that α is isolated in $\sigma(T)$; thus we have

$$[\sigma(A)]_{isol} \subset [\sigma(T)]_{isol}. \tag{2.3}$$

It now follows from [34, Lemma 5] that

$$\sigma_0(A) \subset \sigma_0(T). \tag{2.4}$$

Moreover, from (2.1), (2.2), and (2.4) we have $\rho_{SF}(A) = \rho(A) \cup \sigma_0(A) \subset \rho(T) \cup \sigma_0(T) = \rho_{SF}(T)$, i.e.,

$$\rho_{SF}(A) \subset \rho_{SF}(T). \tag{2.5}$$

We thus conclude from (σ)-(ii) and (2.5) that A satisfies property (F).

We next show that A satisfies property (A). Note first that if σ is a nonempty closed-and-open subset of both $\sigma(A)$ and $\sigma(T)$, then

$$\dim \mathscr{H}(\sigma, A) = \dim \mathscr{H}(\sigma, T). \tag{2.6}$$

Indeed, if X is an injective operator such that $AX = XT$, then $P_\sigma(A)X = XP_\sigma(T)$ [11, Lemma 2.1], so

$$\dim \mathscr{H}(\sigma, A) = \dim P_\sigma(A)\mathscr{H} \geqslant \dim P_\sigma(A)X\mathscr{H} = \dim XP_\sigma(T)\mathscr{H} =$$

$$= \dim P_\sigma(T)\mathscr{H} = \dim \mathscr{H}(\sigma, T);$$

now (2.6) follows by symmetry in A and T. The relations (2.6) and (2.3) immediately show that

$$\text{(2.7)} \qquad \dim \mathscr{H}(\lambda, A) = \dim \mathscr{H}(\lambda, T) \quad \text{for each } \lambda \in \sigma_0(A).$$

Observe now that if $\lambda \in \sigma_{ne}(A)$ is isolated in $\sigma(A)$, then λ is isolated in $\sigma(T)$ and

$$\text{(2.8)} \quad \operatorname{rank}[(\lambda - A)^k \mid \mathscr{H}(\lambda, A)] \leqslant \operatorname{rank}[(\lambda - T)^k \mid \mathscr{H}(\lambda, T)] \ \text{ for } k = 1,2, \ldots .$$

That λ is isolated in $\sigma(T)$ follows from (2.3) (and (2.6) implies that $\lambda \in \sigma_e(T)$) If Y is an injective operator such that $YA = TY$, then $Y(\lambda - A)^k P_\lambda(A) = (\lambda - T)^k P_\lambda(T) Y$ and thus

$$\operatorname{rank}[(\lambda - T)^k \mid \mathscr{H}(\lambda, T)] = \operatorname{rank}[(\lambda - T)^k \mid P_\lambda(T)\mathscr{H}] \geqslant \operatorname{rank}[(\lambda - T)^k \mid P_\lambda(T)Y\mathscr{H}] =$$

$$= \operatorname{rank}(\lambda - T)^k P_\lambda(T) Y = \operatorname{rank}[Y(\lambda - A)^k P_\lambda(A)] = \operatorname{rank}[(\lambda - A)^k P_\lambda(A)] =$$

$$= \operatorname{rank}[(\lambda - A)^k \mid \mathscr{H}(\lambda, A)].$$

We now verify the following inclusions:

$$\text{(2.9)} \qquad [\sigma_e(A)]_{\text{isol}} \subset [\sigma(A)]_{\text{isol}}\,; \quad [\sigma_e(A)]_{\text{isol}} \subset [\sigma_e(T)]_{\text{isol}}$$

Let α be an isolated point of $\sigma_e(A)$, but suppose that α is not isolated in $\sigma(A)$. Thus, from (2.1), there exists a sequence of distinct points $\{\alpha_n\} \subset \sigma(A) \setminus \sigma_e(A) = \sigma_0(A)$ such that $\alpha_n \to \alpha$ $(n \to \infty)$; (2.4) shows that $\alpha_n \in \sigma_0(T)$ and thus $\alpha \in \sigma_e(T)$ [15]. Since α is not isolated in $\sigma(T)$, hypothesis (I) of Theorem 2.4 implies that there is a sequence of distinct points $\{\beta_n\} \subset \sigma_e(T)$ convergent to α. Since $\sigma(T) \subset \sigma(A)$, $\beta_n \in \sigma(A)$, and since α is isolated in $\sigma_e(A)$, then for large n, $\beta_n \in \sigma(A) \setminus \sigma_e(A) = \sigma_0(A) \subset \sigma_0(T)$; this contradicts the fact $\beta_n \in \sigma_e(T)$. Thus α is isolated in $\sigma(A)$. Moreover, (2.3) shows that α is isolated in $\sigma(T)$. Since $\alpha \in \sigma_e(A)$, it follows from [34, Lemma 5] that $\alpha \in \sigma_e(T)$, i.e., $\alpha \in [\sigma(A)]_{\text{isol}}$ and $\alpha \in [\sigma_e(T)]_{\text{isol}}$.

To complete the proof of (A) we will show that

$$\text{(2.10)} \qquad k^+(\lambda, \tilde{A}) \leqslant k^+(\lambda, \tilde{T}) \quad (\lambda \in \mathbf{C}).$$

We may assume that $\lambda \in [\sigma_e(A)]_{isol}$, and thus, from (2.9), $\lambda \in [\sigma(A)]_{isol}$ and $\lambda \in [\sigma_e(T)]_{isol} \subset [\sigma(T)]_{isol}$. Thus $k^+(\lambda, \tilde{T}) > 0$, and it suffices to consider the case when $1 \leqslant k^+(\lambda, \tilde{T}) < +\infty$. In this case, our hypothesis on T shows that $(T - \lambda) \mathscr{H}(\lambda, T) = 0$, and since $(A - \lambda)P_\lambda(A)X = X(T - \lambda)P_\lambda(T) = 0$ for some quasi-affinity X, it follows that $(A - \lambda)P_\lambda(A) = 0$, whence $k^+(\lambda, \tilde{A}) = 1 = k^+(\lambda, \tilde{T})$. From (2.7), (2.8), and (2.10) we see that A satisfies property (A).

Since $\sigma_0(A) \subset \sigma_0(T)$, to prove that A satisfies (S) it suffices to show that each component of $\sigma_{lre}(A) \setminus \sigma_{ne}(A)$ intersects $\sigma_e(T) \setminus \sigma_{ne}(T)$. To this end, we first establish that

$$(2.11) \qquad [\sigma_{lre}(A) \setminus \sigma_{ne}(A)]_{isol} \subset \sigma_e(T) \setminus \sigma_{ne}(T).$$

Let α be an isolated point of $\sigma_{lre}(A) \setminus \sigma_{ne}(A)$. We consider first the case when α is not an isolated point of $\sigma_e(A)$. In this case, let $\{\alpha_n\} \subset \sigma_e(A) = \sigma_{lre}(A)$ be a sequence of distinct points convergent to α. Since α is isolated in $\sigma_{lre}(A) \setminus \sigma_{ne}(A)$, then $\alpha_n \in \sigma_{ne}(A) \subset [\sigma_e(A)]_{isol} \subset [\sigma_e(T)]_{isol}$ ((2.9)). Thus $\alpha \in \sigma_e(T) \setminus [\sigma_e(T)]_{isol} \subset \sigma_e(T) \setminus \sigma_{ne}(T)$. In the case when α is isolated in $\sigma_e(A)$, then (2.9) implies that $\alpha \in [\sigma_e(T)]_{isol} \subset [\sigma(T)]_{isol}$. If $\alpha \in \sigma_{ne}(T)$, then hypothesis 4) of Theorem 2.4 implies that $k^+(\lambda, \tilde{T}) = 1$; since $\alpha \in [\sigma_e(A)]_{isol}$, (2.10) implies that $1 \leqslant k^+(\alpha, \tilde{A}) \leqslant k^+(\alpha, \tilde{T}) = 1$, contradicting the fact that $\alpha \notin \sigma_{ne}(A)$. Thus $\alpha \notin \sigma_{ne}(T)$ and (2.11) is established.

It is easy to see that $\sigma_{lre}(A) \setminus \sigma_{ne}(A)$ and $\sigma_e(T) \setminus \sigma_{ne}(T)$ are compact; to show that each component σ of $\sigma_{lre}(A) \setminus \sigma_{ne}(A)$ intersects $\sigma_e(T) \setminus \sigma_{ne}(T)$, it thus suffices to consider the case when σ is a closed-and-open subset of $\sigma_{lre}(A) \setminus \sigma_{ne}(A)$ (see [22, p. 48]); in view of (2.11), we may also assume that σ is an infinite set.

Suppose that σ is an infinite closed-and-open subset of $\sigma_{lre}(A) \setminus \sigma_{ne}(A)$ and suppose that φ is an open set such that $\sigma \subset \varphi$,

$$(2.12) \qquad [\sigma_{lre}(A) \setminus \sigma_{ne}(A)] \cap \varphi^- = \sigma, \quad \text{and} \quad [\sigma_e(T) \setminus \sigma_{ne}(T)] \cap \varphi^- = \emptyset.$$

We assert that $\Delta \equiv \sigma_{ne}(A) \cap \varphi$ is finite (or empty); indeed, if $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence of distinct points of Δ, with $\alpha_n \to \alpha$ $(n \to \infty)$, then (2.12) shows that $\alpha \in \sigma$, and $\alpha_n \in \sigma_{ne}(A) \subset [\sigma_e(A)]_{isol} \subset \sigma_e(T)$. Thus $\alpha \in \sigma_e(T)$, and since α is not isolated in $\sigma_e(T)$, then $\alpha \in [\sigma_e(T) \setminus \sigma_{ne}(T)] \cap \sigma$, contradicting (2.12). Now we conclude that Δ is finite, so by taking φ smaller, we may assume that $\sigma_{ne}(A) \cap \varphi^- = \emptyset$ and

$$(2.13) \qquad \sigma_{lre}(A) \cap \varphi^- = \sigma_e(A) \cap \varphi^- = (\sigma_e(A) \setminus \sigma_{ne}(A)) \cap \varphi^- = \sigma.$$

We next claim that $\Lambda \equiv \sigma_0(A) \cap \varphi$ is finite or empty. Clearly, all limit points of Λ belong to $\sigma_e(A) \cap \varphi^- = \sigma$. If Λ is infinite, let $\{\alpha_n\} \subset \Lambda$ be a convergent sequence of distinct points of Λ and let $\alpha = \lim \alpha_n$. Since $\alpha_n \in \sigma_0(A) \cap \varphi \subset \sigma_0(T) \cap \varphi \subset \partial\sigma(T) \cap \varphi$ (the first inclusion uses (2.4)), then $\alpha \in \sigma_e(T)$ [15], and since α

is not isolated in $\sigma(T)$, then α is not isolated in $\sigma_e(T)$ (hypothesis 1). Thus $\alpha \in [\sigma_e(T) \setminus \sigma_{ne}(T)] \cap \sigma$, a contradiction to (2.12). Since A is finite, by further shrinking φ we may assume that $\sigma_0(A) \cap \varphi^- = \emptyset$, and thus

$$\sigma(A) \cap \varphi^- = [\sigma_0(A) \cup \sigma_{lre}(A)] \cap \varphi^- = \sigma_{lre}(A) \cap \varphi^- = \sigma. \tag{2.14}$$

The last relation shows that σ is a closed-and-open subset of $\sigma(A)$. Let $\tau = \sigma(T) \cap \varphi$. Property ($\sigma$)-(vi) and (2.2) imply that

$$\tau = \sigma(A) \cap \tau = \sigma(A) \cap \varphi \cap \sigma(T) = \sigma(A) \cap \varphi^- \cap \sigma(T) = \sigma \cap \sigma(T),$$

and thus τ is nonempty and compact. If τ is infinite, let α be a limit point of τ. Since $\sigma(T) = \sigma_0(T) \cup \sigma_{lre}(T)$, then $\alpha \in \sigma_e(T) \setminus [\sigma(T)]_{isol} \subset \sigma_e(T) \setminus [\sigma_e(T)]_{isol}$; hence $\alpha \in [\sigma_e(T) \setminus \sigma_{ne}(T)] \cap \sigma$, a contradiction to (2.12).

Thus τ is finite; let $\tau = \{\alpha_1, \ldots, \alpha_n\} = \sigma(T) \cap \varphi \subset \sigma$. Since $[\sigma_e(T) \setminus \sigma_{ne}(T)] \cap \sigma = \emptyset$, for each i, either $\alpha_i \in \sigma_0(T)$, or $\alpha_i \in \sigma_{ne}(T) \cap [\sigma(T)]_{isol}$ and $(T - \alpha_i) \mid \mathscr{H}(\alpha_i, T) = 0$ (hypotheses 1) and 4)). Thus $TP_\tau(T)$ is algebraic (with minimal polynomial p).

Let f denote the characteristic function of $\sigma \cup \tau$. Since σ and τ are isolated subsets of $\sigma(A)$ and $\sigma(T)$ respectively, f is analytic in a neighborhood of $\sigma(A) \cup \sigma(T)$ and thus $f(A)X = Xf(T)$ [11, Lemma 2.1]. Since $f(A) = P_\sigma(A)$ and $f(T) = P_\tau(T)$, then $AP_\sigma(A)X = AXP_\tau(T) = XTP_\tau(T)$, and so $f(AP_\sigma(A))X = Xf(TP_\tau(T)) = 0$. Since $\partial\sigma = \partial\sigma(A \mid P_\sigma(A)\mathscr{H}) \subset \sigma(AP_\sigma(A))$, then $\partial\sigma$, and hence σ, is finite. This final contradiction implies that σ intersects $\sigma_e(T) \setminus \sigma_{ne}(T)$. Thus A satisfies property (S) and the proof of Theorem 2.4 is complete.

We next consider some examples concerning the hypotheses of Theorem 2.4. In [25, p. 107] D. Herrero discusses compact operators K with the properties that $\sigma(K)$ is denumerable, K admits no denumerable basic sequence of invariant subspaces, and $(K)_{qs} = \mathscr{S}(K)$. Clearly $0 \in [\sigma_e(K)]_{isol} \setminus [\sigma(K)]_{isol}$, so this example shows that hypothesis 1) is not necessary in order to obtain the conclusion of Theorem 2.4. Nevertheless, a stronger variant of hypothesis 1) is necessary for orbit inclusion, as the following result shows.

PROPOSITION 2.8. *Let $T \in \mathscr{L}(\mathscr{H})$, let $\lambda \in [\sigma_e(T)]_{isol}$, and suppose there exists a sequence of distinct points $\{\lambda_n\}_{n=1}^\infty \subset \sigma_0(T)$ such that $\lim \lambda_n = \lambda$. If there exists a T-invariant closed subspace $\mathscr{N}$ such that the family $\mathscr{N}$, $\mathscr{H}(\lambda_1, T)$, $\mathscr{H}(\lambda_2, T)$, ... is a basic sequence for T, then for each k, $0 \leqslant k \leqslant \infty$, there exists $A_k \in (T)_{qs}$ such that $k^+(\lambda, \tilde{A}_k) \geqslant k$. In particular, if in this case $k^+(\lambda, \tilde{T}) < \infty$, then T does not satisfy the quasisimilarity orbit inclusion property.*

REMARK. If $\{\mathscr{H}(\lambda_i, T)\}_{i=1}^\infty$ is a basic sequence for T, the hypothesis will be satisfied with $\mathscr{N} = \{0\}$.

Proof. Assume first that the existence of the A_k's has been established. If $k^+(\lambda, \tilde{T}) < \infty$, let k denote an integer larger than $k^+(\lambda, \tilde{T})$. Now $k^+(\lambda, \tilde{T}) < k \leqslant$ $\leqslant k^+(\lambda, \tilde{A}_k)$ and thus, from [5, page 7], $A_k \in (T)_{qs} \setminus \mathscr{S}(T)^-$. We will show how to construct A_∞; the other A_k's may be produced by suitable (minor) modifications of the following method.

For $n \geqslant 1$, let $\mathscr{H}_n = \mathscr{H}(\lambda_n, T)$. For $n \geqslant 0$, let $\alpha(n) = n(n+1)/2$ and let $\mathscr{M}_n = \mathscr{H}_{\alpha(n)+1} \dotplus \ldots \dotplus \mathscr{H}_{\alpha(n+1)}$. Since $\mathscr{N}, \mathscr{H}_1, \mathscr{H}_2, \ldots$ is a basic sequence for T, it follows readily that $\mathscr{N}, \mathscr{M}_0, \mathscr{M}_1, \mathscr{M}_2, \ldots$ is also a basic sequence for T. Let $\mathscr{N}'$, $\mathscr{M}_0', \mathscr{M}_1', \ldots$ denote isomorphic copies of $\mathscr{N}, \mathscr{M}_0, \mathscr{M}_1, \ldots$, and let $\mathscr{H}' = \mathscr{N}' \oplus$ $\oplus \sum_{i=0}^{\infty} \oplus \mathscr{M}_i'$ (an external orthogonal direct sum). Using the technique of C. Apostol [1] (see also [12], [25]), we see that there exist operators $R \in \mathscr{L}(\mathscr{N}')$, $R_i \in \mathscr{L}(\mathscr{M}_i')$ $(i \geqslant 0)$ such that $R \approx T|\mathscr{N}$, $R_i \approx T|\mathscr{M}_i$, and $R' \equiv R \oplus \sum_{i=0}^{\infty} \oplus R_i$ is quasisimilar to T. Moreover, since $\mathscr{N}$ is a complemented invariant subspace for T, it follows hat $\sigma(T|\mathscr{N}) \subset \sigma(T)$, $\sigma_e(T|\mathscr{N}) \subset \sigma_e(T)$, and thus λ cannot be a limit point of $\sigma_e(T|\mathscr{N})$. Hence if $\lambda \in \sigma_e(R)$, then $\lambda \in [\sigma_e(R)]_{\text{isol}}$.

The Riesz Decomposition Theorem [32, p. 421] implies that $\sigma(R_n) = \sigma(T|\mathscr{M}_n) =$ $= \{\lambda_{\alpha(n)+1}, \ldots, \lambda_{\alpha(n+1)}\}$, so there exists an orthogonal decomposition

$$(*) \qquad \mathscr{M}_n' = \mathscr{L}_1^{(n)} \oplus \ldots \oplus \mathscr{L}_{n+1}^{(n)}$$

such that R_n is similar to an operator of the form

$$S_n = (\lambda_1^{(n)} + N_1^{(n)}) \oplus \ldots \oplus (\lambda_{n+1}^{(n)} + N_{n+1}^{(n)}),$$

where $\lambda_i^{(n)} = \lambda_{\alpha(n)+i}$ and $N_i^{(n)}$ is nilpotent. Thus, up to an additional similarity, we may assume that $\|N_i^{(n)}\| < 1/n$ for each i, $1 \leqslant i \leqslant n+1$.

Let $e_i^{(n)}$ denote a unit vector in $\mathscr{L}_i^{(n)}$ and define the operator $W_i^{(n)}: \mathscr{L}_{i+1}^{(n)} \to \mathscr{L}_i^{(n)}$ by $W_i^{(n)}(v) = (1/i)(v, e_{i+1}^{(n)})e_i^{(n)}$ $(1 \leqslant i \leqslant n)$. Since the $\lambda_i^{(n)}$'s are distinct, the operators $\lambda_i^{(n)} + N_i^{(n)}$ have mutually disjoint spectra, so Rosenblum's Corollary [30, Corollary 0.15, p. 9] implies that S_n is similar to an operator T_n in $\mathscr{L}(\mathscr{M}_n')$ whose operator matrix relative to $(*)$ is of the form

$$T_n = \begin{pmatrix} \lambda_1^{(n)} + N_1^{(n)} & W_1^{(n)} & & & \\ & \lambda_2^{(n)} + N_2^{(n)} & W_2^{(n)} & & \\ & & \ddots & \ddots & \\ & & & \ddots & W_n^{(n)} \\ & & & & \lambda_{n+1}^{(n)} + N_{n+1}^{(n)} \end{pmatrix}.$$

Thus [27, Theorem 2.5] implies that T is quasisimilar to $A \equiv R \oplus \sum_{n=0}^{\infty} \oplus T_n$, so it suffices to prove that $k^+(\lambda, \tilde{A}) = +\infty$.

Let $V_n = T_n - S_n$ $(n \geq 0)$, and let $V = \sum_{n=0}^{\infty} \oplus V_n$. For each $k \geq 1$, either $V_n^k = 0$ (if $k > n$), or $\|V_n^k\| = (1/k!)$ (if $n \geq k$), so clearly V is quasinilpotent, and the same reasoning shows that V^k is not compact for every $k \geq 1$. Let $K = 0_{\mathcal{N}'} \oplus \oplus \sum_{n=0}^{\infty} \oplus (S_n - \lambda 1_{\mathcal{M}_n'}) = 0_{\mathcal{N}'} \oplus \sum_{n=0}^{\infty} \oplus \left(\sum_{i=1}^{n-1} \oplus (\lambda_i^{(n)} - \lambda) - N_i^{(n)} \right)$. Since $\lim \lambda_n = \lambda$ and $N_i^{(n)}$ is a finite rank operator with $\|N_i^{(n)}\| < 1/n$, then K is compact, and $A - K = R \oplus (V + \lambda)$. Since λ is not a limit point of $\sigma_e(R)$, then λ is an isolated point of $\sigma_e(A)$, and since V^k is noncompact for every $k \geq 1$, it follows that $k^+(\lambda, \tilde{A}) = +\infty$ (cf. [26, p. 231]).

Proof of Corollary 2.5. As noted earlier, each normal operator $N \in \mathcal{L}(\mathcal{H})$ satisfies 2) and 3) of Theorem 2.4. Moreover, if N satisfies 1), i.e., if $[\sigma_e(N)]_{\mathrm{isol}} \subset \subset [\sigma(N)]_{\mathrm{isol}}$, then N also satisfies 4), and thus $(N)_{\mathrm{qs}} \subset \mathcal{S}(N)^-$.

For the converse, suppose λ is an isolated point of $\sigma_e(N)$ but λ is not isolated in $\sigma(N)$. Thus there exists a sequence of distinct points $\{\lambda_n\}_{n=1}^{\infty} \subset \sigma_0(N)$ such that $\lambda_n \to \lambda$ $(n \to \infty)$. Let $\mathcal{H}_n = \ker(N - \lambda_n) = \mathcal{H}(\lambda_n, N)$ and let $\mathcal{N} = \mathcal{H} \ominus \bigvee_{n=1}^{\infty} \mathcal{H}_n$. The normality of N readily implies that $\mathcal{N}, \mathcal{H}_1, \mathcal{H}_2, \ldots,$ is a basic sequence for N. Since $k^+(\lambda, \tilde{N}) = 0$, it follows from Proposition 2.8 that $(N)_{\mathrm{qs}} \not\subset \mathcal{S}(N)^-$.

In contrast to Proposition 2.8, the next results (which we state without proof) show that no *finite* subset of $\sigma_0(T)$ can prevent an operator from satisfying the orbit inclusion property.

PROPOSITION 2.9. *Let $T \in \mathcal{L}(\mathcal{H})$ and $\alpha \in \sigma_0(T)$. If A is quasisimilar to T, then there exist complementary A-invariant subspaces $\mathcal{K}_1, \mathcal{K}_2$, such that $A|\mathcal{K}_1 \sim T|\mathcal{H}(\alpha, T)$ and $A|\mathcal{K}_2 \underset{\mathrm{qs}}{\sim} T|\mathcal{H}(\sigma(T)\setminus\{\alpha\}, T)$.*

REMARK. In general, α will not be an isolated point of $\sigma(A)$ (see [34, p. 112]).

COROLLARY 2.10. *Let $T \in \mathcal{L}(\mathcal{H})$, let σ denote a finite subset of $\sigma_0(T)$, and let $\sigma' = \sigma(T)\setminus\sigma$. Then $(T)_{\mathrm{qs}} = \{A \in \mathcal{L}(\mathcal{H})$: there exist complementary A-invariant subspaces $\mathcal{K}_1(A), \mathcal{K}_2(A)$ such that $A|\mathcal{K}_1(A) \sim T|\mathcal{H}(\sigma, T)$ and $A|\mathcal{K}_2(A) \underset{\mathrm{qs}}{\sim} T|\mathcal{H}(\sigma', T)\}$. Thus, if $(T|\mathcal{H}(\sigma', T))_{\mathrm{qs}} \subset \mathcal{S}(T|\mathcal{H}(\sigma', T))^-$, then T satisfies the quasisimilarity orbit inclusion property.*

The preceding results suggest the following question that appears to be open.

QUESTION 2.11. *Let $T \in \mathcal{L}(\mathcal{H})$, let σ denote a nonempty closed-and-open subset of $\sigma(T)$, and let $\sigma' = \sigma(T)\setminus\sigma$. If A is quasisimilar to T, do there exist complementary A-invariant subspaces $\mathcal{M}$, $\mathcal{N}$ such that $A|\mathcal{M} \underset{\mathrm{qs}}{\sim} T|\mathcal{H}(\sigma, T)$ and $A|\mathcal{N} \underset{\mathrm{qs}}{\sim} T|\mathcal{H}(\sigma', T)$?*

An affirmative answer to this question would permit spectral decomposition techniques to be employed in studying the orbit inclusion property (as was done in a limited way in Corollary 2.10).

Hypothesis 3) of Theorem 2.4 is not a necessary condition for the orbit inclusion property. In [25, Theorem 3], D. Herrero exhibited operators T such that $(T)_{qs} = \mathcal{S}(T)$ and $\sigma(T) \neq \sigma_{lre}(T)$, so that either $\operatorname{int}(\sigma_p(T)) \neq \emptyset$ or $\operatorname{int}(\sigma_p(T^*)) \neq \emptyset$. In the case of the unilateral shift U, $\operatorname{int}(\sigma_p(U^*)) \neq \emptyset$ and $(U)_{qs} \neq \mathcal{S}(U)$ [7], [18], but U does satisfy the orbit inclusion property. We will prove the following more general result.

PROPOSITION 2.12. *An operator* $T \in \mathcal{L}(\mathcal{H})$ *satisfies* $(T)_{qs} \subset \mathcal{S}(T)^-$ *if it has the following properties:*

1) $[\sigma_e(T)]_{isol} \cap [\sigma_0(T)]^- = \emptyset$;
2) T *satisfies* (Σ);
3) T *is cyclic,* $\operatorname{int}(\sigma_p(T)) = \emptyset$, *and* $\operatorname{int}(\sigma(T)) \subset \rho_{SF}(T)$;
4) $k^+(\cdot, T)$ *is valued in* $\{0, 1, +\infty\}$.

Proof. Property 4) implies that $\sigma_{me}(T) = \emptyset$. We will show that each operator quasisimilar to T satisfies (F), (A), and (S) of Theorem 2.7. Much of the proof is very close to the proof of Theorem 2.4, so we give a sketch and supply details only when necessary.

Since T is cyclic, then $\operatorname{nul}(T - \lambda)^* \leqslant 1$ $(\lambda \in \mathbf{C})$ [24, Proposition 1(i)]. Thus, since $\operatorname{int}(\sigma_p(T)) = \emptyset$, it follows that

$$\rho_{SF}(T) \cap \operatorname{int}(\sigma(T)) = \rho_{SF}^{(-1)}(T) \equiv \{\lambda \in \rho_{SF}(T) : \operatorname{ind}(T - \lambda) = -1\} \tag{2.15}$$

and $\operatorname{nul}(T - \lambda) = 0$ for all $\lambda \in \rho_{SF}^{(-1)}(T)$. It follows that

$$\sigma_e(T) = \sigma_{lre}(T) \quad \text{and} \quad \rho_{SF}(T) \cap \sigma(T) = \rho_{SF}^{(-1)}(T) \cup \sigma_0(T). \tag{2.16}$$

A result of D. Herrero [24, Theorem 1] implies that each component of $\rho_{SF}^{(-1)}(T)$ is simply connected, and thus

$$[\rho_{SF}^{(-1)}(T)]^- \cap [\sigma_e(T)]_{isol} = \emptyset \tag{2.17}$$

(note that $\operatorname{bdry}(\rho_{SF}^{(-1)}(T)) \subset \sigma_{lre}(T) \setminus [\sigma_e(T)]_{isol}$). Now hypothesis 1), (2.16), and (2.17) imply that

$$[\sigma_e(T)]_{isol} \subset [\sigma(T)]_{isol}. \tag{2.18}$$

Let A be an operator quasisimilar to T. Since quasisimilarity preserves cyclicity [24, Proposition 1(vii)] and $\operatorname{int}(\sigma_p(A)) = \operatorname{int}(\sigma_p(T)) = \emptyset$, then (as above)

$$\begin{gathered} \rho_{SF}(A) \cap \sigma(A) = \rho_{SF}^{(-1)}(A) \cup \sigma_0(A), \\ \operatorname{nul}(A - \lambda) = 0 \quad \text{for all } \lambda \text{ in } \rho_{SF}^{(-1)}(A), \text{ and} \\ \sigma_e(A) = \sigma_{lre}(A). \end{gathered} \tag{2.19}$$

Since T satisfies (Σ) it follows as in the proof of Theorem 2.4 that $\sigma_0(A) \subset \sigma_0(T)$; moreover, $\rho_{SF}^{(-1)}(A) = \operatorname{int}(\rho_{SF}^{(-1)}(A)) \subset \operatorname{int}(\sigma(T)) \subset \rho_{SF}(T)$ (hypothesis 3). Since also

$\rho(A) \subset \rho(T)$, we conclude from (2.19) that $\rho_{SF}(A) \subset \rho_{SF}(T)$, and thus property (F) holds.

To prove property (A), it follows as in Theorem 2.4 that $\dim \mathscr{H}(\lambda, A) = \dim \mathscr{H}(\lambda, T)$ $(\lambda \in \sigma_0(A))$, and that if $\lambda \in \sigma_{ne}(A) \cap [\sigma(A)]_{isol}$, then $\lambda \in [\sigma(T)]_{isol}$ and $\operatorname{rank}(A-\lambda)^k \mathscr{H}(\lambda, A) \leqslant \operatorname{rank}(T-\lambda)^k \mathscr{H}(\lambda, T)$ $(k \geqslant 1)$. It also follows as before that if $\lambda \in [\sigma_e(A)]_{isol}$, then $\lambda \notin [\sigma_0(A)]^-$. Since A is cyclic, $[\sigma_e(A)]_{isol} \cap [\rho_{SF}^{(-1)}(A)]^- = \emptyset$, and thus (as in Theorem 2.4),

$$[\sigma_e(A)]_{isol} \subset [\sigma(A)]_{isol} \quad \text{and} \quad [\sigma_e(A)]_{isol} \subset [\sigma_e(T)]_{isol}. \tag{2.20}$$

Using (2.20), hypothesis 4) shows (as before) that $k^+(\lambda, \hat{A}) \leqslant k^+(\lambda, \tilde{T})$ $(\lambda \in \mathbf{C})$, so property (A) holds.

It remains to prove property (S). Exactly as in Theorem 2.4 we may prove that $[\sigma_{lre}(A) \setminus \sigma_{ne}(A)]_{isol} \subset \sigma_e(T) \setminus \sigma_{ne}(T)$. It thus suffices to show that each infinite open-and-closed subset σ of $\sigma_{lre}(A) \setminus \sigma_{ne}(A)$ intersects $\sigma_e(T) \setminus \sigma_{ne}(T)$. Assuming the contrary, let φ denote a bounded open set such that $\sigma \subset \varphi$,

$$[\sigma_{lre}(A) \setminus \sigma_{ne}(A)] \cap \varphi^- = \sigma, \quad \text{and} \quad [\sigma_e(T) \setminus \sigma_{ne}(T)] \cap \varphi^- = \emptyset. \tag{2.21}$$

Exactly as before, we may assume

$$\sigma_{ne}(A) \cap \varphi = \emptyset, \quad \sigma_0(A) \cap \varphi = \emptyset, \quad \text{and} \quad \sigma \doteq \sigma_{lre}(A) \cap \varphi^-. \tag{2.22}$$

We assert that $\rho_{SF}^{(-1)}(A) \cap \varphi = \emptyset$. Let H be a component of $\rho_{SF}^{(-1)}(A)$. Now (2.21) and (2.22) imply that $\operatorname{bdry}(H) \cap \operatorname{bdry}(\varphi) \subset \sigma_{lre}(A) \cap \operatorname{bdry}(\varphi) = \emptyset$. Since H is simply connected [24] and $\sigma_{lre}(A) \cap \varphi = \sigma \neq \emptyset$, we conclude that either $H^- \subset \varphi$ or $H^- \cap \varphi = \emptyset$. Suppose there exists a component H_1 of $\rho_{SF}^{(-1)}(A)$ such that $H_1^- \subset \varphi$. Since $H_1 \subset \operatorname{int}(\sigma_p(A^*)^*) \subset \operatorname{int}(\sigma(T)) \subset \rho_{SF}^{(-1)}(T)$, there exists a component K of $\rho_{SF}^{(-1)}(T)$ such that $H_1 \subset K$. Since $\operatorname{bdry}(K) \subset \sigma_e(T) \setminus \sigma_{ne}(T)$, (2.21) implies that $\operatorname{bdry}(K) \cap \varphi^- = \emptyset$. Let $\mathcal{O}$ denote the component of φ that contains H_1^-. Since $\operatorname{bdry}(K) \cap \mathcal{O}^- = \emptyset$, then $H_1^- \subset \mathcal{O} \subset \mathcal{O}^- \subset K$. Since $\mathcal{O}$ is a component of φ, then $\operatorname{bdry}(\mathcal{O}) \subset \operatorname{bdry}(\varphi)$; thus if $x \in \operatorname{bdry}(\mathcal{O}) \subset K \subset \rho_{SF}^{(-1)}(T)$, then $x \in \operatorname{bdry}(\varphi) \cap \operatorname{int}(\sigma(A)) \subset \sigma(A) \setminus \sigma_{lre}(A)$ (use (2.22)), and so $x \in \rho_{SF}^{(-1)}(A)$. Since $x \in \operatorname{bdry}(\varphi)$, the component of x in $\rho_{SF}^{(-1)}(A)$ is neither disjoint from φ nor is it contained in φ. From this contradiction we conclude that $\rho_{SF}(A) \cap \varphi = \emptyset$, and by also using (2.21) and (2.22) we see that σ is a closed-and-open subset of $\sigma(A)$, i.e., $\sigma(A) \cap \varphi^- = \sigma(A) \cap \varphi = \sigma$.

Let

$$\tau = \sigma(T) \cap \varphi \quad (\subset \sigma). \tag{2.23}$$

If K is a component of $\rho_{SF}^{(-1)}(T)$ $(\subset \sigma(A))$ and $K \cap \varphi \neq \emptyset$, then (2.23) implies that $K^- \subset \sigma$, so $\operatorname{bdry}(K) \subset \sigma \cap [\sigma_e(T) \setminus \sigma_{ne}(T)]$, a contradiction to (2.21). Thus $\rho_{SF}^{(-1)}(T) \cap \varphi = \emptyset$, and it follows as in the proof of Theorem 2.4 that τ is finite. Exactly as in Theorem 2.4, we obtain a contradiction to the fact that σ is infinite, so the proof is complete.

Recall that if S is a hyponormal operator and $\lambda \in [\sigma(S)]_{\mathrm{isol}}$, then $\mathcal{H}(\lambda, S) = \ker(S - \lambda)$ (see [13]). If $\lambda \in \sigma_p(S)$, then $\ker(S - \lambda)$ reduces S, so $S \mid \bigvee_{\lambda \in \sigma_p(S)} \ker(S - \lambda)$ is normal; thus if S is separably acting, then $\operatorname{int}(\sigma_p(S)) = \emptyset$. Let T be a hyponormal operator in $\mathcal{L}(\mathcal{H})$ and let ψ be as in the definition of $k^+(\lambda, \tilde{T})$. If $\lambda \in [\sigma_e(T)]_{\mathrm{isol}}$, then since $S = \psi(\tilde{T})$ is hyponormal and $\lambda \in [\sigma(S)]_{\mathrm{isol}}$, it follows that $(S - \lambda)\mathcal{H}(\lambda, S) = 0$, i.e., $k^+(\lambda, \tilde{T}) = 1$.

COROLLARY 2.13. *If $T \in \mathcal{L}(\mathcal{H})$ is a cyclic hyponormal operator, $\sigma_0(T) = \emptyset$, and $\sigma_e(T) = \operatorname{bdry}(\sigma(T))$, then $(T)_{qs} \subset \mathcal{S}(T)^-$.*

Proof. Hypothesis 1) of Proposition 2.12 is satisfied trivially, and property 2) follows from [8]. The preceding remarks show that $k^+(\cdot, \tilde{T})$ is valued in $\{0,1\}$ and that $\operatorname{int}(\sigma_p(T)) = \emptyset$. Since $\sigma_e(T) = \operatorname{bdry}(\sigma(T))$, then $\operatorname{int}(\sigma(T)) \subset \rho_{SF}(T)$. Thus properties 3) and 4) hold and the result follows from Proposition 2.12.

COROLLARY 2.14. *If T is a cyclic hyponormal unilateral weighted shift, then $(T)_{qs} \subset \mathcal{S}(T)^-$.*

Proof. Since T is cyclic and hyponormal, then $T \neq 0$, $\sigma_p(T) = \emptyset$, and the spectrum of T is a disk of positive radius [33, Theorem 4, Theorem 8, Proposition 26]. The fact that $\operatorname{int}(\sigma(T)) = \rho_{SF}^{(-1)}(T)$ follows from [33, Theorem 6, Theorem 8, Proposition 26], and thus $\sigma_e(T) = \operatorname{bdry}(\sigma(T))$. The result now follows from Corollary 2.13.

COROLLARY 2.15. *If T is a cyclic subnormal operator quasisimilar to the unilateral shift, then $(T)_{qs} \subset \mathcal{S}(T)^-$.*

Proof. Since T is quasisimilar to the unilateral shift U, $\sigma_p(T) = \emptyset$. It follows from [8] that T and U have equal spectra, and it follows from [7], [29] that they have equal essential spectra. Thus $\sigma_e(T) = \sigma_e(U) = \operatorname{bdry}(\sigma(U)) = \operatorname{bdry}(\sigma(T))$, and the result follows from Corollary 2.13.

It follows from Corollary 2.5 and [24, Proposition 1 (viii)] that there exist cyclic diagonalizable normal operators that do not satisfy the orbit inclusion property. Does each *pure* cyclic subnormal operator have the orbit inclusion property? We do not know if this is the case, but an affirmative answer would be consistent with Corollary 2.15 and the following result.

PROPOSITION 2.16. *Quasisimilar cyclic subnormal operators are asymptotically similar.*

Proof. Let A and T be cyclic subnormal operators, $A \underset{qs}{\sim} T$. As noted above, A and T have equal spectra and equal essential spectra, so cyclicity implies (as above) that $\rho_{SF}(A) = \rho_{SF}(T)$, and (F) follows. As before, $\sigma_0(A) = \sigma_0(T)$ and $k^+(\lambda, \tilde{A}) = k^+(\lambda, \tilde{T})$ ($\lambda \in \mathbf{C}$). The rest of the proof of (A) follows as in Theorem 2.4. Since $\sigma_{lre}(A) = \sigma_e(A) = \sigma_e(T) = \sigma_{lre}(T)$ and $\sigma_{ne}(A) = [\sigma_e(A)]_{\mathrm{isol}} = [\sigma_e(T)]_{\mathrm{isol}} = \sigma_{ne}(T)$, property (S) holds, and since $\sigma_{me}(A) = \sigma_{me}(T) = \emptyset$, the result follows from Theorem 2.7.

To examine hypothesis 4) of Theorem 2.4 we consider quasinilpotent operators. As noted in [25, page 107], there exist infinite rank compact quasinilpotent operators K such that $(K)_{qs} = \mathcal{S}(K)$; each strictly cyclic compact injective unilateral weighted shift has this property [25], [20], [33]. This example shows that hypothesis 4) is not necessary for orbit inclusion. More simply, note that if T is a nonzero finite rank nilpotent operator, then clearly T does not satisfy hypothesis 4), but nonetheless $(T)_{qs} = \mathcal{S}(T)$. Indeed, if T is a finite rank nilpotent, then so is each $A \in (T)_{qs}$, and $\mathrm{rank}(T^j) = \mathrm{rank}(A^j)$, $\mathrm{nul}(T^j) = \mathrm{nul}(A^j)$ for all $j \geqslant 1$. It now follows from [26, Corollary 2.8] that A is similar to T.

In contrast to these results, note that no infinite rank compact nilpotent operator satisfies the orbit inclusion property. More generally, we have the following result.

PROPOSITION 2.15. *If T is a nilpotent operator and T^k is an infinite rank compact operator for some $k \geqslant 1$, then T doe snot satisfy the quasisimilarity orbit inclusion property.*

Proof. By a result of C. Apostol, R. Douglas, and C. Foiaş [3] and L. Williams [37], T is quasisimilar to a Jordan nilpotent operator J, and thus T^k is quasisimilar to J^k. Since T^k has infinite rank, so does the Jordan operator J^k, and thus J^k is not compact. Since each similarity of T^k is compact, it follows that $J^k \notin \mathcal{S}(T^k)^-$, and thus $J \notin \mathcal{S}(T)^-$.

REMARK. The first example above concerning quasinilpotent operators shows that we cannot replace "nilpotent" by "quasinilpotent" in the preceding result. On the other hand, T. Hoover's example [27] shows that not every compact quasinilpotent satisfies the orbit inclusion property.

QUESTION 2.16. *Which quasinilpotent operators and which nilpotent operators satisfy the quasisimilarity orbit inclusion property?*

3. EQUALITY OF ESSENTIAL SPECTRA OF QUASISIMILAR BILATERAL WEIGHTED SHIFTS

Let $\{e_n\}_{n=-\infty}^{\infty}$ denote an orthonormal basis for $\mathcal{H}$. For a bounded sequence $\alpha = \{\alpha_n\}_{n=-\infty}^{\infty} \subset \mathbf{C}$, we define the bilateral weighted shift $W_\alpha \in \mathcal{L}(\mathcal{H})$ by $W_\alpha e_n = \alpha_n e_{n+1}$ ($n \in \mathbf{Z}$). We consider only injective shifts (all $\alpha_n \neq 0$) and up to unitary equivalence we may assume each $\alpha_n > 0$ [33, Corollary 1, p. 52]. From [11, Theorem 4.2] we have the following criterion for quasisimilarity of (injective) bilateral weighted shifts.

LEMMA 3.1. *Bilateral weighted shifts W_α and W_β are quasisimilar if and only if the following conditions are satisfied:*

1) *There exists an integer k such that*

$$\sup_{i \geqslant \max(1,1-k)} (\alpha_0 \dots \alpha_{i-1+k})/(\beta_0 \dots \beta_{i-1}) < \infty$$

and

$$\sup_{i \geqslant \max(1,1-k)} (\beta_{-1} \dots \beta_{-(i+k)})/(\alpha_{-1} \dots \alpha_{-i}) < \infty.$$

2) *There exists an integer m such that*

$$\sup_{i \geqslant \max(1,1-m)} (\beta_0 \dots \beta_{i-1+m})/(\alpha_0 \dots \alpha_{i-1}) < \infty$$

and

$$\sup_{i \geqslant \max(1,1-m)} (\alpha_{-1} \dots \alpha_{-(i+m)})/(\beta_{-1} \dots \beta_{-i}) < \infty.$$

Using this result, it is shown in [11] that quasisimilar invertible bilateral weighted shifts are similar, but that there exist quasisimilar weighted shifts that are not similar; in any case, quasisimilar shifts have equal spectra [11, Theorem 4.6]. We will show that quasisimilar shifts also have equal approximate point spectra and equal essential spectra. As an application, we will show that (excluding one exceptional case) quasisimilar shifts are asymptotically similar.

PROPOSITION 3.2. *Quasisimilar bilateral weighted shifts have equal approximate point spectra.*

We defer the proof briefly to present an application.

COROLLARY 3.3. *Quasisimilar bilateral weighted shifts have equal essential spectra.*

Proof. Let W_α and W_β denote quasisimilar weighted shifts. From [33, Theorem 9] we know that either

i) $\sigma_p(W_\alpha) = \sigma_p(W_\beta) = \emptyset$ and $\dim\ker(W_\alpha - \lambda)^* = \dim\ker(W_\beta - \lambda)^* \leqslant 1$ $(\lambda \in \mathbf{C})$ or

ii) $\sigma_p(W_\alpha^*) = \sigma_p(W_\beta^*) = \emptyset$ and $\dim\ker(W_\alpha - \lambda) = \dim\ker(W_\beta - \lambda) \leqslant 1$ $(\lambda \in \mathbf{C})$.

In case i), $W_\alpha - \lambda$ is Fredholm if and only if $W_\alpha - \lambda$ has closed range, or, equivalently, if and only if $W_\alpha - \lambda$ is bounded below. Thus $\sigma_e(W_\alpha) = \sigma_\pi(W_\alpha)$, $\sigma_e(W_\beta) = \sigma_\pi(W_\beta)$, and the desired conclusion follows from Proposition 3.2. In case ii), we apply the same argument to W_α^* and W_β^*, which are also quasisimilar bilateral weighted shifts.

In order to prove Proposition 3.2 we require additional notation. Following W. Ridge [31], for a bilateral weighted shift W_α we define

$$r(W_\alpha)^+ = \lim_n \sup_{j \geqslant 0} (\alpha_j \dots \alpha_{j+n-1})^{1/n};$$

this parameter, and several others defined below, are used in [31] to describe the approximate point spectrum of W_α. Let $\mathscr{H}_+ = \bigvee_{n\geq 0} \{e_n\}$ and let W_α^+ denote the *unilateral* weighted shift on $\mathscr{H}^+$ defined by $W_\alpha^+ e_n = \alpha_n e_{n+1}$ $(n \geq 0)$; clearly $\mathrm{r}(W_\alpha)^+ = \mathrm{r}(W_\alpha^+)$, the spectral radius of W_α^+.

LEMMA 3.4. *If* $W_\alpha \underset{qs}{\sim} W_\beta$, *then* $\mathrm{r}(W_\alpha)^+ = \mathrm{r}(W_\beta)^+$.

Proof. We may assume $\|W_\alpha\|$, $\|W_\beta\| \leq 1$ and we may thus also assume that the integers k and m of Lemma 3.1 are nonnegative. In particular, there exists a constant $M > 1$ such that

i) $(\alpha_0 \ldots \alpha_{i-1+k}) < M(\beta_0 \ldots \beta_{i-1}) \quad (i \geq 1)$

and

ii) $(\beta_0 \ldots \beta_{i-1+m}) < M(\alpha_0 \ldots \alpha_{i-1}) \quad (i \geq 1)$.

For $n > k + m$ and $j \geq 1$, we have

$$\begin{aligned}(\alpha_j \ldots \alpha_{j+n-1}) &= (\alpha_0 \ldots \alpha_{j-1}\alpha_j \ldots \alpha_{j+n-1})/(\alpha_0 \ldots \alpha_{j-1}) < \\ &< M(\beta_0 \ldots \beta_{j+n-1-k})/(\alpha_0 \ldots \alpha_{j-1}) < \\ &< M^2(\beta_0 \ldots \beta_{j+n-1-k})/(\beta_0 \ldots \beta_{j-1+m}) = \\ &= M^2(\beta_{j+m} \ldots \beta_{j-1+n-k}).\end{aligned}$$

For $j = 0$,

$$(\alpha_0 \ldots \alpha_{n-1}) < M(\beta_0 \ldots \beta_{n-1-k}) < M^2(\beta_m \ldots \beta_{n-1-k}).$$

Thus

$$\begin{aligned}\sup_{j\geq 0}(\alpha_j \ldots \alpha_{j+n-1}) &\leq M^2 \sup_{j\geq 0}(\beta_{j+m} \ldots \beta_{j-1+n-k}) \leq \\ &\leq M^2 \sup_{j\geq -m}(\beta_{j+m} \ldots \beta_{j-1+n-k}),\end{aligned}$$

and so $\|(W_\alpha^+)^n\| \leq M^2\|(W_\beta^+)^{n-(k+m)}\|$. Now [11, Lemma 4.7] implies that $\mathrm{r}(W_\alpha^+) \leq \mathrm{r}(W_\beta^+)$. By symmetry, we have $\mathrm{r}(W_\alpha)^+ = \mathrm{r}(W_\alpha^+) = \mathrm{r}(W_\beta^+) = \mathrm{r}(W_\beta)^+$.

LEMMA 3.5. *If* $W_\alpha \underset{qs}{\sim} W_\beta$, *then* W_α^+ *is bounded below if and only if* W_β^+ *is bounded below.*

Proof. Let k, m, and M be as in the proof of Lemma 3.4. If $\alpha_i \geq \delta > 0$ $(i \geq 0)$, then for $i \geq 2 + m$, inequalities i) and ii) imply that

$$\begin{aligned}\beta_{i-1} &= (\beta_0 \ldots \beta_{i-2}\beta_{i-1})/(\beta_0 \ldots \beta_{i-2}) > \\ &> (\alpha_0 \ldots \alpha_{i-1+k})/(M^2\alpha_0 \ldots \alpha_{i-2-m}) = \\ &= (1/M^2)(\alpha_{i-1-m} \ldots \alpha_{i-1+k}) > (1/M^2)\delta^{k+m+1}.\end{aligned}$$

Thus W_β^+ is bounded below if W_α^+ is, and the result follows by symmetry.

For $T \in \mathscr{L}(\mathscr{H})$, let $m(T) = \inf_{\|x\|=1} \|Tx\|$; thus for $S \in \mathscr{L}(\mathscr{H})$, $m(TS) \geqslant m(T)m(S)$ [33, page 68]. Note that $m((W_\alpha^+)^n) = \inf_{j \geqslant 0} (\alpha_j \ldots \alpha_{j+n-1})$; following [31], we define $\mathrm{i}(W_\alpha)^+ = \lim_n \inf_{j \geqslant 0} (\alpha_j \ldots \alpha_{j+n-1})^{1/n}$.

LEMMA 3.6. *W_α^+ is bounded below if and only if W_α^{+n} is bounded below for some $n \geqslant 1$.*

Proof. If W_α^{+n} is bounded below, then $m(W_\alpha^{+n}) > 0$ and thus $\delta \equiv \inf_{j \geqslant 0} (\alpha_j \ldots \ldots \alpha_{j+n-1}) > 0$. If $N > \|W_\alpha^+\|$, then for $j \geqslant 0$, $\alpha_j/N \geqslant (\alpha_j/N) \ldots (\alpha_{j+n-1}/N) \geqslant \geqslant \delta/N^n > 0$, so W_α^+ is bounded below; the converse is obvious.

LEMMA 3.7. *If $W_\alpha \underset{\text{qs}}{\sim} W_\beta$, then $\mathrm{i}(W_\alpha)^+ = \mathrm{i}(W_\beta)^+$.*

Proof. If W_α^+ is not bounded below, then Lemma 3.6 implies that for each n, $m(W_\alpha^{+n}) = 0$, whence $\mathrm{i}(W_\alpha)^+ = \lim_n m(W_\alpha^{+n})^{1/n} = 0$. In this case, Lemma 3.5 implies that $m(W_\beta^+) = 0$, so the preceding argument shows that $\mathrm{i}(W_\beta)^+ = 0$. We may thus assume that both W_α and W_β are bounded below. We retain the notation of the proof of Lemma 3.4. For $j \geqslant k+1$,

$$
\begin{aligned}
&(\alpha_j \ldots \alpha_{j+n-1}) = \\
&= (\alpha_0 \ldots \alpha_{j-1}\, \alpha_j \ldots \alpha_{j+n-1})/(\alpha_0 \ldots \alpha_{j-1}) > \\
&> (1/M^2)(\beta_0 \ldots \beta_{j+n-1+m})/(\beta_0 \ldots \beta_{j-1-k}) = \\
&= (1/M^2)(\beta_{j-k} \ldots \beta_{j+n-1+m}).
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\inf_{j \geqslant k+1} (\alpha_j \ldots \alpha_{j+n-1}) \geqslant \\
&\geqslant (1/M^2) \inf_{j \geqslant k+1} (\beta_{j-k} \ldots \beta_{j+n-1+m}) \geqslant \\
&\geqslant (1/M^2) \inf_{j \geqslant k} (\beta_{j-k} \ldots \beta_{j+n-1+m}) = \\
&= (1/M^2)\, m((W_\beta^+)^{n+m+k}) \geqslant \\
&\geqslant (1/M^2) m(W_\beta^{+n})\, m((W_\beta^+)^{m+k}).
\end{aligned}
$$

Now

$$
\begin{aligned}
m(W_\alpha^{+n}) &= \min\{(\alpha_0 \ldots \alpha_{n-1}), \ldots, (\alpha_k \ldots \alpha_{k+n-1}), \inf_{j \geqslant k+1} (\alpha_j \ldots \alpha_{j+n-1})\} \geqslant \\
&\geqslant \min\{(\alpha_0 \ldots \alpha_{n-1}), \ldots, (\alpha_k \ldots \alpha_{k+n-1}), (1/M^2) m(W_\beta^{+n})\, m((W_\beta^+)^{m+k})\}.
\end{aligned}
$$

For each j, $0 \leqslant j \leqslant k-1$,

$$(\alpha_j \ldots \alpha_{j+n-1}) \geqslant$$

$$\geqslant (1/M)(\beta_0 \ldots \beta_{j+n-1+m})/(\alpha_0 \ldots \alpha_{j-1}) \geqslant$$

$$\geqslant (1/M)(\beta_0 \ldots \beta_{j+n-1+m}) \geqslant (1/M^2) m((W_\beta^+)^{n+j+m}) \geqslant$$

$$\geqslant (1/M^2) m(W_\beta^{+n}) m((W_\beta^+)^{m+j}).$$

Thus

$$m(W_\alpha^{+n}) \geqslant (1/M^2) m(W_\beta^{+n}) \min_{0 \leqslant j \leqslant k} m((W_\beta^+)^{m+j}),$$

and so

$$\mathrm{i}(W_\alpha)^+ = \lim_n m(W_\alpha^{+n})^{1/n} \geqslant \lim_n m(W_\beta^{+n})^{1/n} = \mathrm{i}(W_\beta)^+ .$$

The result now follows by symmetry.

Following [31], we define the following additional parameters for a shift W_γ:

$$\mathrm{i}(W_\gamma)^- = \lim_n \inf_{k \leqslant 0} (\gamma_{k-1} \ldots \gamma_{k-n})^{1/n},$$

$$\mathrm{r}(W_\gamma)^- = \lim_n \sup_{k \leqslant 0} (\gamma_{k-1} \ldots \gamma_{k-n})^{1/n}.$$

LEMMA 3.8. *If* $W_\alpha \underset{\mathrm{qs}}{\sim} W_\beta$, *then* $\mathrm{i}(W_\alpha)^- = \mathrm{i}(W_\beta)^-$ *and* $\mathrm{r}(W_\alpha)^- = r(W_\beta)^-$.

Proof. Let $f_n = e_{-n}$ ($n \in \mathbf{Z}$). Relative to the orthonormal basis $\{f_n\}$, W_α^* and W_β^* are bilateral weighted shifts such that the weight sequences for $(W_\alpha^*)^+$ and $(W_\beta^*)^+$ are, respectively, $\alpha_{-1}, \alpha_{-2}, \alpha_{-3}, \ldots$ and $\beta_{-1}, \beta_{-2}, \beta_{-3}, \ldots$. Since $W_\alpha^* \underset{\mathrm{qs}}{\sim} W_\beta^*$, Lemma 3.4 implies that $\mathrm{r}(W_\alpha)^- = \mathrm{r}(W_\alpha^*)^+ = \mathrm{r}(W_\beta^*)^+ = \mathrm{r}(W_\beta)^-$ and Lemma 3.7 implies that $\mathrm{i}(W_\alpha)^- = \mathrm{i}(W_\alpha^*)^+ = \mathrm{i}(W_\beta^*)^+ = \mathrm{i}(W_\beta)^-$.

Proof of Proposition 3.2. In [31, Theorem 3] Ridge proved that for a bilateral weighted shift W_γ, if $\mathrm{r}(W_\gamma)^- < \mathrm{i}(W_\gamma)^+$, then

$$\sigma_\pi(W_\gamma) = \{\lambda : \mathrm{i}(W_\gamma)^+ \leqslant |\lambda| \leqslant \mathrm{r}(W_\gamma)^+\} \cup \{\lambda : \mathrm{i}(W_\gamma)^- \leqslant |\lambda| \leqslant \mathrm{r}(W_\gamma)^-\};$$

otherwise,

$$\sigma_\pi(W_\gamma) = \{\lambda : \min(\mathrm{i}(W_\gamma)^-, \mathrm{i}(W_\gamma)^+) \leqslant |\lambda| \leqslant \max(\mathrm{r}(W_\gamma)^-, \mathrm{r}(W_\gamma)^+)\}.$$

From this result, and Lemmas 3.4, 3.7 and 3.8, it is clear that quasisimilar bilateral weighted shifts have equal approximate point spectra.

We conclude with an application concerning asymptotic similarity.

PROPOSITION 3.9. *If* $W_\alpha \underset{\mathrm{qs}}{\sim} W_\beta$ *and* 0 *is not an isolated point of* $\sigma_e(W_\alpha)$ $(= \sigma_e(W_\beta))$, *then* W_α *and* W_β *are asymptotically similar.*

Proof. The results of [11] and the preceding results of this section imply that $\sigma(W_\alpha) = \sigma(W_\beta)$, $\sigma_e(W_\alpha) = \sigma_e(W_\beta)$, $\rho_{SF}(W_\alpha) = \rho_{SF}(W_\beta)$, $\sigma_e(W_\alpha) = \sigma_{ire}(W_\alpha)$, and $\sigma_0(W_\alpha) = \emptyset$. Since our hypothesis implies that $\sigma_{ne}(W_\alpha) = \emptyset$, Theorem 2.7 shows that $W_\alpha \in \mathscr{S}(W_\beta)^-$; the result follows by symmetry.

The following example shows that in Proposition 3.9 we cannot dispose of the hypothesis that $0 \notin [\sigma_e(W_\alpha)]_{isol}$.

EXAMPLE 3.10. Define the weighted shifts W_α and W_β as follows:

1) $\alpha_n = \beta_n = 1/n$ for $n < 0$;

2) The weight sequence $\{\alpha_n\}_{n=0}^\infty$ is given by

$1/2, 1, 1/4, 1/4, 1, 1/8, 1/8, 1/8, 1/8, 1,$ (8 terms of $1/16$), $1, \ldots$;

3) The weight sequence $\{\beta_n\}_{n=0}^\infty$ is given by

$$1/2^{1/2}, 1/2^{1/2}, 1/4, 1/2, 1/2, 1/8, 1/8, 1/8, 1/8^{1/2},$$

$$1/8^{1/2}, \text{ (7 terms of } 1/16), 1/4, 1/4, \ldots .$$

It is straightforward to verify that with $k = 0$ and $m = 1$, W_α and W_β satisfy the hypotheses of Lemma 3.1, and thus W_α and W_β are quasisimilar. Clearly W_β is compact and W_α is noncompact, so $W_\alpha \notin \mathscr{S}(W_\beta)^-$. T. Hoover's example showing that quasisimilarity does not preserve compactness involves operators with denumerable sequences of finite dimensional reducing subspaces; in contrast, neither W_α nor W_β admits a pair of complementary nontrivial invariant subspaces (see [19], [25]).

Research partially supported by a National Foundation Research Grant.

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Received January 4, 1984; revised March 19, 1985.