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A COHOMOLOGICAL CHARACTERIZATION OF FINITE-DIMENSIONAL C^*-ALGEBRAS

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1. INTRODUCTION

Let A be a C^*-algebra, B a C^*-algebra containing A, and let $\delta\colon A \to B$ be a derivation, i.e., a linear map for which $\delta(ab) = a\delta(b) + \delta(a)b$, for all $a, b \in A$. In [12] and [13], the authors have investigated C^*-algebras B with the property that for every C^*-subalgebra A of B each such derivation δ of A into B extends to an inner derivation of B, i.e., there exists $b \in B$ with $\delta(a) = ba - ab$, $a \in A$. Such C^*-algebras are interesting for a number of reasons, some of which are documented in the introduction to [12]. A question which naturally arises in this situation occurs when one reverses the quantifiers: what C^*-algebras A have the property that for every C^*-algebra B which contains A, each derivation of A into B is inner in B? It is easy to see that all finite-dimensional C^*-algebras A have this latter property, and in Problem 17.1 of [8], George Elliott, motivated by questions about derivations of matroid C^*-algebras, asked if the converse holds (this is not completely accurate; see the remarks at the end of next section). We show in this paper that the converse does indeed hold.

In order to state our results precisely and concisely, we introduce some concepts and notation from the Hochschild cohomology of algebras [11]. Let A be a linear, associative algebra over the complex numbers $\mathbf{C}$, M a two-sided A-module. We denote by $Z(A, M)$ the set of all derivations of A into M, i.e., linear maps of A into M which satisfy $\delta(ab) = a\delta(b) + \delta(a)b$, for all $a, b \in A$, and we denote by $B(A, M)$ the set of all inner derivations of A into M, i.e., all maps of the form $a \to ma - am$, $a \in A$, m ranging over the elements of M. $Z(A, M)$ becomes an abelian group under vector-space addition, $B(A, M)$ is a subgroup of $Z(A, M)$, and $\mathrm{H}^1(A, M)$, the first (Hochschild) cohomology group of A with coefficients in M, is defined to be the quotient group $Z(A, M)/B(A, M)$. Thus $\mathrm{H}^1(A, M) = \{0\}$ if and only if every derivation of A into M is inner in M. In most of what follows, we will be concerned with the case when A is a C^*-algebra and M is a C^*-algebra containing A. If B is a C^*-algebra and $x \in B$, we will let ad_x denote the mapping $b \to xb - bx$, $b \in B$.

The rest of our notation and terminology is standard and should cause no confusion. In particular, if H is a Hilbert space, $B(H)$ will denote the algebra of all bounded linear operators on H, $K(H)$ will denote the algebra of all compact operators in $B(H)$, and if A is a C^*-algebra with $S \subseteq A$, $C^*(S)$ will denote the C^*-subalgebra of A generated by S.

2. SOLUTION OF ELLIOTT'S PROBLEM

We consider C^*-algebras A with the following property:

(*) $\quad \mathrm{H}^1(A, B) = \{0\}$ for all C^*-algebras B which contain A.

More precisely, this means that if B is a C^*-algebra and ν is a $*$-isomorphism of A into B, then each derivation of $\nu(A)$ into B is inner in B.

2.1. THEOREM. *A C^*-algebra has Property (*) if and only if it is finite-dimensional.*

The goal of this section is to establish this theorem. For its proof, we need four lemmas.

2.2. LEMMA. *Every $*$-homomorphic image of a C^*-algebra A with Property (*) has Property (*).*

Proof. Let $\pi : A \to B$ be a surjective $*$-homomorphism, let C be a C^*-algebra containing B, and let $\delta : B \to C$ be a derivation. Define an isomorphism ν of A into $A \oplus C$ by $\nu(a) = a \oplus \pi(a)$, $a \in A$. On $\nu(A)$, define $\Delta(\nu(a)) = 0 \oplus \delta(\pi(a))$, $a \in A$. Then Δ is a derivation of $\nu(A)$ into $A \oplus C$, and $\mathrm{H}^1(\nu(A), A \oplus C) = \{0\}$ since A has Property (*). Thus there exists $x \oplus c \in A \oplus C$ with

$$\Delta(\nu(a)) = \mathrm{ad}_{x\oplus c}(\nu(a)), \quad a \in A,$$

and it follows that $\delta(\pi(a)) = \mathrm{ad}_c(\pi(a))$, $a \in A$. Since π is surjective, δ is inner in C, and $\mathrm{H}^1(B, C) = \{0\}$. Q.E.D.

2.3. LEMMA. *Let A be a C^*-algebra. The following are equivalent:*

(1) *A has Property (*);*

(2) *Each faithful representation π of A on a Hilbert space H satisfies the following conditions:*

(i) $\mathrm{H}^1(\pi(A), B(H)) = \{0\}$,

(ii) *each $T \in B(H)$ is contained in $\pi(A)' + C^*(\pi(A) \cup \mathrm{ad}_T(\pi(A)))$.*

(Here and in what follows, $'$ denotes the commutant.)

Proof. (1) $\Rightarrow$ (2). It is clear that (i) holds. If $T \in B(H)$, then $\mathrm{ad}_T\, \pi(A)$ is a derivation of $\pi(A)$ into $B = C^*(\pi(A) \cup \mathrm{ad}_T(\pi(A)))$, and so there exists $x \in B$ with $\mathrm{ad}_T = \mathrm{ad}_x$ on $\pi(A)$, whence $T - x \in \pi(A)'$.

(2) $\Rightarrow$ (1). Let B be a C^*-algebra containing A, $\delta: A \to B$ a derivation. By considering a faithful representation of B, we may suppose that B acts faithfully on a Hilbert space H. By (i), there exists $T \in B(H)$ with $\delta = \mathrm{ad}_T\, A$. By (ii),

$$T \in A' + C^*(A \cup \mathrm{ad}_T(A)) = A' + C^*(A \cup \delta(A)) \subseteq A' + B,$$

and so we can find $b \in B$ with $\delta = \mathrm{ad}_T\, A = \mathrm{ad}_b\, A$. Thus $\mathrm{H}^1(A, B) = \{0\}$. Q.E.D.

2.4. LEMMA. *Let H be an infinite-dimensional Hilbert space, A a separable C^*-subalgebra of $B(H)$ which contains the identity operator I on H. Then there exists $T \in B(H)$ such that*

(i) $T \notin A + K(H)$;

(ii) $\mathrm{ad}_T(A) \subseteq K(H)$.

Proof. Suppose first that H is separable. Let $C(H) = B(H)/K(H)$ denote the Calkin algebra on H, and let $q: B(H) \to C(H)$ denote the canonical surjection. $q(A)$ is a separable C^*-subalgebra of $C(H)$ which contains $q(I)$. Since $C(H)$ is simple, we use Proposition 2.2 of [4] to find a separable, simple C^*-subalgebra B of $C(H)$ which contains $q(A)$. Since $C(H)$ is nonseparable, there exists $x \in C(H) \setminus B$. By Corollary 2 on p. 344 of [3], there is a projection $e \in C(H)$ such that

$$\|(q(I) - e)xe\| = \text{distance of } x \text{ to } B > 0, \tag{2.1}$$

$$e \text{ commutes with } B \text{ in } C(H). \tag{2.2}$$

Now $e \notin B$, for otherwise e would be a central projection in B, and since B is simple and hence by the Dauns-Hofmann theorem has trivial center, e must be 0 or $q(I)$, which is not possible by (2.1). Thus $e \notin q(A)$, and by (2.2), e commutes with $q(A)$ in $C(H)$. Any $T \in B(H)$ with $e = q(T)$ will now satisfy (i) and (ii) of the lemma.

Suppose next that H is nonseparable. Let $\{\xi_\alpha\}$ be a maximal family of unit vectors in H such that if P_α denotes the cyclic projection of A' generated by ξ_α, i.e., range of P_α = closed linear span of $\{a\xi_\alpha : a \in A\}$, then the P_α's are pairwise orthogonal. Then $I = \bigoplus_\alpha P_\alpha$ by maximality, and the range of each P_α is separable. Since H is nonseparable, there must hence be infinitely many P_α's so taking a countably infinite family $\{\alpha_n\}$ and setting $P = \bigoplus_n P_{\alpha_n}$ yields a projection $P \in A'$ whose range is separable and infinite-dimensional.

Letting $H_P = P(H)$, we consider AP acting on H_P, and from the first part of the proof find $T_1 \in B(H_P)$ such that

$$T_1 \notin AP + K(H_P), \tag{2.3}$$

$$\mathrm{ad}_{T_1}(AP) \subseteq K(H_P). \tag{2.4}$$

Now, define $T \in B(H)$ to be T_1 on H_P, 0 on $H_P^\perp$. Suppose there exist $a \in A$, $k \in K(H)$ with $T = a - k$. Since $TP = PT = T$, we hence have $T = aP + PkP$, and so $T_1 = T|H_P \in AP + K(H_P)$, contradicting (2.3). Thus $T \notin A + K(H)$. If $a \in A$, then

$$\mathrm{ad}_T(a) = Ta - aT = TaP - aPTP = \mathrm{ad}_{T_1}(aP)P,$$

and this is an operator which is 0 on $H_P^\perp$ and agrees by (2.4) with a compact operator on H_P. Thus $\mathrm{ad}_T(A) \subseteq K(H)$. Q.E.D.

2.5. Lemma *If A is a seprable C^*-algebra with Property (*), then A is finite-dimensional.*

Proof. Since every derivation of A into itself is inner in A, by Theorem 7.3 of [1] and Corollary 3.10 of [2], we may write $A = B \oplus C$, where B is commutative and C is a finite direct sum of unital C^*-algebras which are either simple or homogeneous of finite degree. By Lemma 2.2, each direct summand of A has Property (*), so we may assume that A is either unital and simple, or homogeneous of finite degree.

Suppose first that A is unital and simple. Assume that A is infinite-dimensional. Then any irreducible representation ρ of A acts faithfully on an infinite-dimensional Hilbert space H, and so by Lemma 2.4 we can find $T \in B(H)$ with $T \notin \rho(A) + K(H)$, $\mathrm{ad}_T(\rho(A)) \subseteq K(H)$. If there exists $x \in \rho(A) + K(H)$ with $\mathrm{ad}_T = \mathrm{ad}_x$ on $\rho(A)$, then $T - x \in \rho(A)' = \mathbf{C}\cdot I$, which contradicts the fact that $T \notin \rho(A) + K(H)$. Thus $\mathrm{ad}_T|\rho(A)$ is an outer derivation of $\rho(A)$ into $\rho(A) + K(H)$, which contradicts the hypothesis that A has Property (*).

Suppose next that A is n-homogeneous, n a fixed positive integer. Assume A is infinite-dimensional. Then the spectrum X of A is a locally compact Hausdorff space with an infinite number of points. Suppose X is discrete. Fix a separable, infinite-dimensional Hilbert space H, and denote by c (resp., c_0) the algebra of all diagonal operators in $B(H)$ with convergent diagonal (resp., diagonal converging to 0). (All diagonal operators referred to here and in what follows are diagonal, relative to some fixed orthonormal basis of H.) Then A is isomorphic to $c_0 \otimes M_n$, where M_n denotes the algebra of $n \times n$ complex matrices. Now, let $\tilde{K}$ denote the C^*-algebra generated by $K(H)$ and the identity operator I on H. Denote by S a unilateral shift (of multiplicity 1) on H, and let $T = S \otimes 1 \in B(H) \otimes M_n$. Then $\delta = \mathrm{ad}_T|\tilde{K} \otimes M_n$ is a derivation of $\tilde{K} \otimes M_n$, and since $A \subseteq \tilde{K} \otimes M_n$ and A has Property (*), we find $x \in \tilde{K} \otimes M_n$ with $\delta = \mathrm{ad}_x$ on A. Thus $T - x \in$ commutant of A in $B(H) \otimes M_n = D \otimes 1$, where D is the algebra of all diagonal operators in $B(H)$. We may hence restrict to the diagonal entries in $T - x$ to obtain $d \in D$, $k \in K(H)$ with $S = d + k$, which is not possible (S is Fredholm of index -1, so d is Fredholm of index -1, contradicting the fact that a normal Fredholm operator has index 0).

We conclude that X is not discrete; since A is separable, X contains a convergent sequence $\{x_k\}$ of distinct points. Using the structure theorem for n-homogeneous algebras ([9] or [17]) we may suppose that the closure F of $\{x_k\}$ in X occupies an open subset U of X over which the structure bundle of A is bundle isomorphic to the trivial bundle $U \times M_n$. Let I denote the norm-closed, two-sided ideal of all elements $a \in A$ with $a(x) = 0$ for all $x \in F$. By Lemma 2.2, A/I has Property (*), and by the choice of U, A/I is n-homogeneous, has spectrum F, and has trivial structure bundle over F. Thus A/I is isomorphic to $c \otimes M_n$, and $c \otimes M_n$ hence has Property (*). The argument given in the discrete case above now applies to show that this is impossible, and so A is finite-dimensional. Q.E.D.

Proof of Theorem 2.1. The fact that a finite-dimensional C^*-algebra has Property (*) follows straigtforwardly from the proof of Lemma 1 of [8]. Suppose therefore that A is a C^*-algebra with Property (*). We assert first that A is liminal: in fact, every irreducible representation of A acts on a finite-dimensional space.

Suppose not, and consider an irreducible representation ρ of A which acts on an infinite-dimensional Hilbert space H. Let $\mathcal{S}$ denote the set of all separable C^*-subalgebras of $\rho(A)$. For each $S \in \mathcal{S}$, use Lemma 2.4 to choose $T_S \in B(H)$ such that $\|T_S\| = 1$ and, with I denoting the identity operator on H,

$$T_S \notin \mathbf{C} \cdot I + S + K(H), \tag{2.5}$$

$$\mathrm{ad}_{T_S}(S) \subseteq K(H). \tag{2.6}$$

New, let $C = \ell_\infty(\mathcal{S}, B(H))$, the C^*-algebra of all norm-bounded functions of $\mathcal{S}$ into $B(H)$ with pointwise operations and supremum norm. For an $x \in C$, we denote the value of x at $S \in \mathcal{S}$ by a subscripted x_S. Define an isomorphism $\nu: \rho(A) \to$ $\to C$ by

$$\nu(\rho(a))_S = \rho(a), \quad a \in A, \ S \in \mathcal{S}.$$

Set

$$J = \{x = (x_S) \in C: \text{there exists } S_x \in \mathcal{S} \text{ with } x_S \in K(H) \text{ for all } S \in \mathcal{S} \text{ which contain } S_x\}.$$

Claim 1. J is a norm-closed, two-sided ideal of C.

It is clear that J is a two-sided ideal of C. To see that J is norm-closed, let $\{x_n\}$ be a sequence in J with $\|x_n - x\| \to 0$. For each n, there exists $S_n \in \mathcal{S}$ with $(x_n)_S \in K(H)$, for all $S \in \mathcal{S}$ with $S_n \subseteq S$. Let $S_x = C^*(\bigcup_n S_n)$; $S_x \in \mathcal{S}$. For each $S \in \mathcal{S}$,

$$\|(x_n)_S - x_S\| \to 0. \tag{2.7}$$

If $S \in \mathcal{S}$ with $S_x \subseteq S$, then $S_n \subseteq S$ for each n, and so $(x_n)_S \in K(H)$, for all n, whence by (2.7), $x_S \in K(H)$. This shows that $x \in J$, and Claim 1 holds.

Set $b = (T_S) \in C$.

Claim 2. $\mathrm{ad}_b(\nu(\rho(A))) \subseteq J$.

Let $a \in A$, and set $S_0 = C^*(\rho(a))$. Then $S_0 \in \mathcal{S}$, and for all $S \in \mathcal{S}$ with

$S_0 \subseteq S$,

$$\mathrm{ad}_b(\nu(\rho(a)))_S = \mathrm{ad}_{T_S}(\rho(a)) \in K(H),$$

by (2.6), since $\rho(a) \in S$. This verifies Claim 2.

By Lemma 2.2, $\nu(\rho(A))$ has Property (*), and so by Claims 1 and 2, there exists $x \in \nu(\rho(A)) + J$ with

$$\mathrm{ad}_b = \mathrm{ad}_x \quad \text{on } \nu(\rho(A)). \tag{2.8}$$

Choose $a_0 \in A$, $j \in J$ with $x = \nu(\rho(a_0)) + j$. Then there exists $S_1 \in \mathscr{S}$ such that

$$x_S - \rho(a_0) \in K(H), \quad \text{for all } S \in \mathscr{S} \text{ with } S_1 \subseteq S. \tag{2.9}$$

By (2.8),

$$T_S - x_S \in \rho(A)' = \mathbf{C}\cdot I, \quad \text{for all } S \in \mathscr{S}. \tag{2.10}$$

Let $S_2 = C^*(\{\rho(a_0)\} \cup S_1) \in \mathscr{S}$. Then we conclude by (2.9) and (2.10) that

$$T_{S_2} \in \rho(a_0) + \mathbf{C}\cdot I + K(H) \subseteq \mathbf{C}\cdot I + S_2 + K(H),$$

which contradicts (2.5). It follows that A is liminal.

Claim 3. Each separable C^*-subalgebra of A is contained in a separable C^*-subalgebra of A with Property (*).

If this is so, then by Lemma 2.5 each separable subalgebra of A is finite-dimensional, and this implies that A is finite-dimensional, since otherwise A would contain by [14] an element with infinite spectrum, and hence an infinite-dimensional separable subalgebra.

Suppose Claim 3 fails. Then A has a separable C^*-subalgebra B such that, with $\mathscr{C}$ denoting the set of all separable C^*-subalgebras of A which contain B, each element of $\mathscr{C}$ does not have Property (*). Since A is liminal, each $C \in \mathscr{C}$ is liminal ([7], Proposition 4.2.4), and so every representation of each $C \in \mathscr{C}$ is type I ([7], Theorem 5.5.2). Since type I von Neumann algebras are injective ([16], Corollary 7.2.1), we conclude from Corollary 3.3 of [5] and Lemma 2.3 that each $C \in \mathscr{C}$ has a faithful representation π_C on a Hilbert space H_C with an operator $T_C \in B(H_C)$ such that $\|T_C\| = 1$ and

$$T_C \notin \pi_C(C)' + C^*(\pi_C(C) \cup \mathrm{ad}_{T_C}(\pi_C(C))). \tag{2.11}$$

By Proposition 2.10.2 of [7] there exists for each $C \in \mathscr{C}$ a Hilbert space K_C containing H_C and a representation $\tilde{\pi}_C$ of A on K_C with

$$\tilde{\pi}_C(b)|H_C = \pi_C(b), \quad \text{for all } b \in C.$$

We note that if $P_C =$ projection of K_C onto H_C, then $P_C \in \tilde{\pi}_C(C)'$.

Set $\pi = \oplus\{\tilde{\pi}_C : C \in \mathscr{C}\}$, $H = \oplus\{K_C : C \in \mathscr{C}\}$. Since $A = \cup\{C \in \mathscr{C}\}$ and each π_C is faithful, π is faithful. Define $T \in B(H)$ as follows: for $\xi = (\xi_C) \in H$, set

$$(T\xi)_C = T_C P_C \xi_C, \quad C \in \mathscr{C}.$$

Claim 4. For each separable C^*-subalgebra D of $\pi(A)$ which contains $\pi(B)$,

$$T \notin D' + C^*(D \cup \mathrm{ad}_T(D)).$$

Suppose this is false for some such D. Let $C_0 = \pi^{-1}(D) \in \mathscr{C}$. Let P_0 be the projection in $B(H)$ defined as follows: for $\xi = (\xi_C) \in H$,

$$(P_0\xi)_C = \begin{cases} 0, & C \neq C_0 \\ P_{C_0}\xi_{C_0}, & C = C_0. \end{cases}$$

We have $P_0 \in D'$, and the construction of π implies that $DP_0 = \pi_{C_0}(C_0)P_{C_0}$. Since $TP_0 = P_0 T = T_{C_0}P_{C_0}$, it follows that

$$\begin{aligned} P_0 C^*(D \cup \mathrm{ad}_T(D))P_0 &\subseteq C^*(DP_0 \cup \mathrm{ad}_{TP_0}(DP_0)) \subseteq \\ &\subseteq C^*(\pi_{C_0}(C_0)P_{C_0} \cup \mathrm{ad}_{T_{C_0}P_{C_0}}(\pi_{C_0}(C_0)P_{C_0})). \end{aligned} \tag{2.12}$$

Since $P_0 D' P_0 \subseteq (\pi_{C_0}(C_0)P_{C_0})'$ and $T \in D' + C^*(D \cup \mathrm{ad}_T(D))$, we conclude

$$T_{C_0}P_{C_0} \in (\pi_{C_0}(C_0)P_{C_0})' + C^*(\pi_{C_0}(C_0)P_{C_0} \cup \mathrm{ad}_{T_{C_0}P_{C_0}}(\pi_{C_0}(C_0)P_{C_0})),$$

and compressing to H_{C_0}, we obtain

$$T_{C_0} \in \pi_{C_0}(C_0)' + C^*(\pi_{C_0}(C_0) \cup \mathrm{ad}_{T_{C_0}}(\pi_{C_0}(C_0))),$$

which contradicts (2.11). This verifies Claim 4.

Since A has Property (*), we conclude by Lemma 2.3 that

$$T \in \pi(A)' + C^*(\pi(A) \cup \mathrm{ad}_T(\pi(A))).$$

But we now notice that $C^*(\pi(A) \cup \mathrm{ad}_T(\pi(A)))$ is the union of all C^*-subalgebras of the form $C^*(D \cup \mathrm{ad}_T(D))$, D ranging over the separable C^*-subalgebras of $\pi(A)$ which contain $\pi(B)$. We may hence find $x \in \pi(A)'$ and such a D for which

$$T - x \in C^*(D \cup \mathrm{ad}_T(D)),$$

and since $\pi(A)' \subseteq D'$, we conclude that

$$T \in D' + C^*(D \cup \mathrm{ad}_T(D)),$$

which contradicts Claim 4. This final contradiction proves Claim 3, and hence also Theorem 2.1. Q.E.D.

The following corollary also answers a question from [8].

2.6. COROLLARY. *Let A be a simple C^*-algebra. Then A has Property* (*) *if and only if there is a positive integer n for which A is isomorphic to the algebra of $n \times n$ matrices over* $\mathbf{C}$.

REMARKS. (1). In [8], Elliott actually asked the following question: if A is an algebra over $\mathbf{C}$ with the property that $\mathrm{H}^1(A, B) = 0$ for all algebras B over $\mathbf{C}$ which contain A, is A a finite direct sum of finite-dimensional matrix algebras over $\mathbf{C}$ (it is *not* assumed here that A and the B's are C^*-algebras). We will show that the answer to this question is affirmative; the proof is very short and follows from classical results in the Hochschild cohomology of algebras (in a private communication, Professor Elliott informed us that he knew this answer, and that the original question should have been stated in the C^*-category).

Suppose then that A is an algebra over $\mathbf{C}$ with the above property. We will show that if M is a two-sided A-module, then $\mathrm{H}^1(A, M) = \{0\}$. If this is so, then by Theorem 2.3 of [10] and Theorem 1 of [15], A must be semisimple and finite-dimensional, and so A is a direct sum of the desired form by Wedderburn's theorem.

Let M be a two-sided A-module. Let $B = A \times M$ (Cartesian product) with the vector-space operations defined coordinate-wise. We define a multiplication on B as follows: for (a_1, m_1) and (a_2, m_2) in B, set

$$(a_1, m_1)(a_2, m_2) = (a_1a_2, a_1m_2 + m_1a_2).$$

One easily checks that B becomes an algebra over $\mathbf{C}$ with these operations, and the map $\nu: a \to (a, 0)$, $a \in A$, defines an algebra isomorphism of A into B.

Suppose $\delta: A \to M$ is a derivation. Then the mapping $\Delta: \nu(A) \to B$ defined by $\Delta: (a, 0) \to (0, \delta(a))$, $a \in A$, is a derivation of $\nu(A)$ into B. Since $\mathrm{H}^1(\nu(A), B) = \{0\}$, there exists $(a_0, m_0) \in B$ with $\Delta = \mathrm{ad}_{(a_0, m_0)}|\,\nu(A)$, and since $\mathrm{ad}_{(a_0, m_0)}((a, 0)) = (a_0a - aa_0, m_0a - am_0)$, $a \in A$, we conclude that $\delta(a) = \mathrm{ad}_{m_0}(a)$, $a \in A$, whence $\mathrm{H}^1(A, M) = \{0\}$. Q.E.D.

(2). The analog of Theorem 2.1 in the W^*-category is far from the truth. Indeed, if R is an injective W^*-subalgebra of a W^*-algebra M, then $\mathrm{H}^1(R, M) = \{0\}$. This follows from the fact that R is amenable, i.e., $\mathrm{H}^1(R, X) = \{0\}$ for any dual, normal, Banach R-bimodule X (see [6]), and from the fact that any W^*-algebra M which contains R is such an R-bimodule. We thus close with the following question: if R is a W^*-algebra with $H^1(R, M) = \{0\}$ for all W^*-algebras M which contain R, is R injective?

REFERENCES

1. AKEMANN, C.; ELLIOTT, G.; PEDERSEN, G. K.; TOMIYAMA, J., Derivations and multipliers of C^*-algebras, *Amer. J. Math.*, **98**(1976), 679–708.
2. AKEMANN, C.; PEDERSEN, G. K., Central sequences and inner derivations of separable C^*-algebras, *Amer. J. Math.*, **101**(1979), 1047–1061.
3. ARVESON, W. B., Notes on extensions of C^*-algebras, *Duke Math. J.*, **44**(1977), 329–355.
4. BLACKADAR, B. E., Weak expectations and nuclear C^*-algebras, *Indiana Univ. Math. J.*, **27**(1978), 1021–1026.
5. CHRISTENSEN, E., Extensions of derivations. II, *Math. Scand.*, **50**(1982), 111–122.
6. CONNES, A., On the classification of von Neumann algebras and their automorphisms, in *Symposia Mathematica*, vol. 20, Academic Press, New York, 1976, pp. 435–478.
7. DIXMIER, J., *C^*-algebras*, North-Holland, Amsterdam–New York–Oxford, 1977.
8. ELLIOTT, G., Derivations of matroid C^*-algebras, *Invent. Math.*, **9** (1970), 253–269.
9. FELL, J. M. G., The structure of algebras of operator fields, *Acta. Math.*, **106**(1961), 233–280.
10. HOCHSCHILD, G., Semisimple algebras and generalized derivations, *Amer. J. Math.*, **64**(1941), 677–694.
11. HOCHSCHILD, G., On the cohomology groups of an associative algebra, *Ann. of Math.*, **46**(1945), 58–67.
12. LAZAR, A. J.; TSUI, S. -K.; WRIGHT, S., Derivations from subalgebras of C^*-algebras with continuous trace, to appear.
13. LAZAR, A. J.; TSUI, S.-K.; WRIGHT, S., Derivations from subalgebras of separable C^*-algebras, *Michigan Math. J.*, **31**(1984), 65–72.
14. OGASAWARA, T., Finite-dimensionality of certain Banach algebras, *J. Sci. Hiroshima Univ.*, **17**(1954), 359–364.
15. ROSENBERG, A.; ZELINSKY, D., Cohomology of infinite algebras, *Trans. Amer. Math. Soc.*, **82**(1956), 85–98.
16. TOMIYAMA, J., Tensor products and projections of norm one in von Neumann algebras, mimeographed lecture notes, Univ. of Copenhagen, 1970.
17. TOMIYAMA, J.; TAKESAKI, M., Applications of fibre bundles to a certain class of C^*-algebras, *Tôhoku Math. J.*, **13**(1961), 498–523.

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