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SPECTRAL PROPERTIES OF GENERALIZED MULTIPLIERS

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In the present paper we shall be concerned with an extension of the commutator and the multiplier of two operators. Namely, given S, T acting on Banach spaces X, Y and θ an analytic complex valued function, defined in a neighbourhood of the Cartesian product $\sigma(S)\times\sigma(T)$, we define an operator $\theta(S, T): L(Y, X)\mapsto L(Y, X)$ so that for simple functions such as $\theta(z, w) = z - w$ or $\theta(z, w) = zw$ we obtain the commutator and the multiplier of S, T (i.e. the operators $C(S, T)V = SV - VT$ and $M(S, T)V = SVT$, respectively).

We shall prove directly that the mapping $\theta\mapsto\theta(S, T)$ from the algebra of germs of analytic functions in neighbourhoods of $\sigma(S)\times\sigma(T)$ into $L(L(Y, X))$ is a morphism of algebras, and using this result we shall deduce an evaluation for the spectrum of $\theta(S, T)$ (see also [3], [5]).

Following the ideas from [4], we shall obtain some spectral properties of $\theta(S, T)$; the main result is Theorem 10, which is an extension of the results obtained in [4].

For definitions and main techniques used in this paper we refer to [8].

1. DEFINITIONS AND GENERAL RESULTS

Let X, Y be two Banach spaces, and let $S\in L(X)$, $T\in L(Y)$ be two bounded operators. Let D_S, D_T be two open sets containing $\sigma(S)$ and $\sigma(T)$, respectively. Let $\theta: D_S\times D_T\mapsto \mathbf{C}$ be an analytic function. If Γ_S, Γ_T are two regular contours contained in D_S and D_T, surrounding $\sigma(S)$ and $\sigma(T)$ respectively, we define:

$$\theta(z, T) = (2\pi i)^{-1}\int_{\Gamma_T}\theta(z, w)(w - T)^{-1}\,dw.$$

For V belonging to $L(Y, X)$, we define:

$$\theta(S, T)V = (2\pi i)^{-1}\int_{\Gamma_S}(z - S)^{-1}V\theta(z, T)\,dz.$$

It is obvious that $\theta(S, T)$ is a bounded operator from $L(Y, X)$ into $L(Y, X)$ which does not depend on the particular choice of Γ_S and of Γ_T. If θ has the particular form

$$\theta(z, w) = \sum f_j(z) g_j(w),$$

where f_j, g_j are analytic functions on D_S and D_T respectively, then using the analytic functional calculus, we obtain:

$$\theta(S, T)V = \sum f_j(S) V g_j(T).$$

By putting $\theta(z, w) = z - w$ or $\theta(z, w) = zw$, we obtain the commutator and the multiplier.

LEMMA 1. *Let θ': $D'_S \times D'_T \mapsto \mathbf{C}$ be another analytic function and suppose that $\sigma(S) \subseteq D'_S$, $\sigma(T) \subseteq D'_T$. In this case we have:*

$$(\theta\theta')(S, T) = \theta(S, T)\theta'(S, T).$$

Proof. We may suppose that we have an open set U with regular boundary Γ'_S such that $\sigma(S) \subseteq U \subseteq \overline{U} \subseteq D_S \cap D'_S$, and that Γ_S surrounds $\overline{U}$. If V belongs to $L(Y, X)$ then we have the equalities:

$$\theta(S, T)\theta'(S, T)V = (2\pi \mathrm{i})^{-1} \int_{\Gamma_S} (z - S)^{-1}(\theta'(S, T)V)\theta(z, T)\, \mathrm{d}z =$$

$$= (2\pi \mathrm{i})^{-2} \int_{\Gamma_S} \int_{\Gamma'_S} (z - S)^{-1}(z' - S)^{-1} V \theta'(z', T)\theta(z, T)\, \mathrm{d}z\, \mathrm{d}z' =$$

$$= (2\pi \mathrm{i})^{-2} \int_{\Gamma'_S} (z' - S)^{-1} \int_{\Gamma_S} (z - z')^{-1} V \theta'(z', T)\theta(z, T)\, \mathrm{d}z'\, \mathrm{d}z -$$

$$-(2\pi \mathrm{i})^{-2} \int_{\Gamma_S} (z - S)^{-1} \int_{\Gamma'_S} (z - z')^{-1} V \theta'(z', T)\theta(z, T)\, \mathrm{d}z\, \mathrm{d}z'.$$

Using the Cauchy integral formula the second integral is null while the first is equal to:

$$(2\pi \mathrm{i})^{-1} \int_{\Gamma'_S} (z' - S)^{-1} V \theta'(z', T)\theta(z', T)\, \mathrm{d}z' =$$

$$= (2\pi \mathrm{i})^{-1} \int_{\Gamma'_S} (z' - S)^{-1} V (\theta\theta')(z', T)\, \mathrm{d}z' = (\theta\theta')(S, T)V,$$

where we used the multiplicativity of the analytic functional calculus.

REMARK. We may consider the operators L_S and R_T on $L(Y, X)$ given by $L_S(V) = SV$ and $R_T(V) = VT$. We note that (L_S, R_T) is a commuting pair and that the joint spectrum of this pair (in the sense of J. L. Taylor [7]) is contained in $\sigma(L_S)\times\sigma(R_T) = \sigma(S)\times\sigma(T)$. Moreover, by Corollary III.8.17 from [8],

$$\theta(L_S, R_T)V = (2\pi i)^{-2}\int_{\Gamma_S}\int_{\Gamma_T}\theta(z, w)(z - L_S)^{-1}(w - R_T)^{-1}V\,dz\,dw =$$

$$= (2\pi i)^{-2}\int_{\Gamma_S}\int_{\Gamma_T}\theta(z, w)(z - S)^{-1}V(w - T)^{-1}\,dz\,dw = \theta(S, T)V.$$

This shows in particular that the mapping $\theta \mapsto \theta(S, T)$ is an algebra homeomorphism, by Theorem 4.3 from [6]. Nevertheless, our Lemma 1 is a direct argument.

COROLLARY 2. *We have the following inclusion:*

$$\sigma(\theta(S, T)) \subseteq \{\theta(z, w) : z \in \sigma(S),\ w \in \sigma(T)\}.$$

Proof. Take a point λ which does not belong to the set on the right side and consider the function $\mu_\lambda(z, w) = (\lambda - \theta(z, w))^{-1}$ which is analytic and well defined in a neighbourhood of $\sigma(S)\times\sigma(T)$. By virtue of Lemma 1, $\lambda - \theta(S, T)$ is invertible and $(\lambda - \theta(S, T))^{-1} = \mu_\lambda(S, T)$. This ends the proof.

If Z is a Banach space and W an operator acting on Z, and if z is an element of Z, we denote by $\gamma_W(z)$ the local (analytic) spectrum of z with respect to W; if in addition, W has the single valued extension property, then we denote by $z_W(\cdot)$ the analytic Z-valued function defined on $\mathbf{C}\setminus\gamma_W(z)$ which satisfies

$$(\xi - W)z_W(\xi) = z \quad \text{for every } \xi \text{ in } \mathbf{C}\setminus\gamma_W(z).$$

We denote by $Z_W(F)$ the spectral maximal subspaces associated with W (i.e. for a closed subset F of $\mathbf{C}$ we have:

$$Z_W(F) = \{z \in Z : \gamma_W(z) \subseteq F\}).$$

If g is an analytic $L(Z)$-valued or complex-valued function defined in a neighbourhood of $\gamma_W(z)$, we put:

$$[g(W)]z = (2\pi i)^{-1}\int_\Gamma g(\xi)z_W(\xi)\,d\xi,$$

where W is supposed to have the single valued extension property, and Γ is an arbitrary contour contained in the domain of g which surrounds $\gamma_W(z)$. (The brackets indicate that in this case $g(W)$ is not completely meaningful; it would be meaningful if g were analytic in a neighbourhood of $\sigma(W)$; in this case we would have

$$g(W)z = [g(W)]z.)$$

The following lemma is a slight improvement of the multiplicativity property of the local analytic functional calculus (see [1]).

LEMMA 3. *If g is an analytic $L(Z)$-valued (or complex-valued) function defined in a neighbourhood of $\gamma_W(z)$, $z \in Z$, and if f is analytic complex-valued in a neighbourhood of $\sigma(W)$, then we have:*

$$[(fg)(W)]z = [g(W)](f(W)z).$$

Proof. We have to prove that

$$\int_\Gamma g(\xi)(f(\xi) - f(W))z_W(\xi)\, \mathrm{d}\xi = 0,$$

where Γ is a regular contour contained in the domain of f and g, which surrounds $\gamma_W(z)$. But this follows from the obvious fact that we can write $f(\xi) - f(W) = h_\xi(W)(\xi - W)$, where the mapping $\xi \mapsto h_\xi(W)$ is analytic operator-valued. This ends the proof.

(We must remark that the local analytic spectrum of $f(W)z$ with respect to W is contained in $\gamma_W(z)$ so that the term $[g(W)]\, f(W)z$ is meaningful.)

Similary we can prove the following lemma:

LEMMA 3'. *If g is an analytic complex-valued function defined in a neighbourhood of $\gamma_W(z)$, $z \in Z$, and if f is analytic complex-valued in a neighbourhood of $\sigma(W)$, then we have:*

$$[(fg)(W)]z = f(W)[g(W)]z.$$

Now if θ_1 is an analytic complex-valued function on $D_S \times D_{T,y}$, where y is an arbitrary element of Y and $D_{T,y}$ an open neighbourhood of $\gamma_T(y)$, while V is an element of $L(Y, X)$, then we define

$$[\theta_1(S, T)V]y = (2\pi \mathrm{i})^{-1} \int_{\Gamma_S} (z - S)^{-1} V[\theta_1(z, T)]y\, \mathrm{d}z =$$

$$= (2\pi \mathrm{i})^{-2} \int_{\Gamma_S} \int_{\Gamma_{T,y}} \theta_1(z, w)(z - S)^{-1} V y_T(w)\, \mathrm{d}z\, \mathrm{d}w,$$

where $\Gamma_{T,y}$ is a contour contained in $D_{T,y}$, which surrounds $\gamma_T(y)$. The brackets indicate that in this case $\theta_1(S, T)$ is not completely meaningful but if θ_1 is analytic in a neighbourhood of $\sigma(S) \times \sigma(T)$, then

$$[\theta_1(S, T)V]y = \theta_1(S, T)Vy.$$

LEMMA 4. *On the preceding conditions and if h is analytic complex-valued on $D_{T,y}$ we have:*

$$[(\theta h)(S, T)V]y = (\theta(S, T))V[h(T)]y.$$

Proof.

$$\theta(S, T)V[h(T)]y = (2\pi i)^{-1} \int_{\Gamma_S} (z - S)^{-1} V\theta(z, T)[h(T)]y \, dz =$$

$$= (2\pi i)^{-1} \int_{\Gamma_S} (z - S)^{-1} V[(\theta h)(z, T)]y \, dz = [(h\theta)(S, T)V]y,$$

where we used Lemma 3′.

2. SPECTRAL PROPERTIES OF $\theta(S, T)$

The main result of this section is Theorem 10 which characterizes the location of the local (analytic) spectrum of $\theta(S, T)$ with respect to the properties of the spectral maximal subspaces of S and T, when both S and T are decomposable.

For every closed subsets F and K of $\mathbf{C}$ we denote by

$$\theta_F^{-1}(K) = \{z \in \sigma(S) : (\exists)\ w \in F \text{ such that } \theta(z, w) \in K\}.$$

If $\theta(z, w) = z - w$ then $\theta_F^{-1}(K) = (F + K) \cap \sigma(S)$.

Throughout this section we shall suppose that S, T are decomposable operators (see [8]).

The following proposition allows us to prove that $\theta(S, T)$ has the single valued extension property but we shall prove a more general result that is needed in the proof of Theorem 10.

PROPOSITION 5. *Let G be an open connected nonvoid subset of $\mathbf{C}$ and let $Y_0 \subseteq Y$ be a closed subspace of Y invariant under T that satisfies the following conditions:*

(a) *For every $y \in Y_0$ and $w \in \mathbf{C} \setminus \gamma_T(y)$ we have $y_T(w) \in Y_0$.*

(b) *The space generated by the set:*

$$\{y \in Y_0 : \bigcap_{\lambda \in G} \theta^{-1}_{\gamma_T(y)}(\{\lambda\}) = \emptyset\}$$

is dense in Y_0.

Let $V : G \mapsto L(Y_0, X)$ be an analytic function such that

$$(\lambda - \theta(S, T|Y_0))V(\lambda) \equiv 0.$$

In this case $V(\lambda)$ is identically null for every λ in G.

Proof. First we observe that by (a) it follows that $(w - T)^{-1}y$ belongs to Y_0 for every y in Y_0 and for every w in $\mathbf{C} \setminus \sigma(T)$ so that $\sigma(T \mid Y_0) \subseteq \sigma(T)$ and therefore $\theta(S, T|Y_0)$ is defined.

We shall first prove the following inclusion:

$$(1)\qquad \gamma_S(V(\lambda)y) \subseteq \theta^{-1}_{\gamma_T(y)}(\{\lambda\}),\quad (\forall)\ \lambda \in G,\ (\forall)\ y \in Y_0.$$

To prove this, suppose that y and λ are fixed ; passing to the complement we have to prove that:

$$B = \{\eta \in D_S : (\forall)\ w \in \gamma_T(y),\ \theta(\eta, w) \neq \lambda\} \subseteq \mathbf{C}\setminus\gamma_S(V(\lambda)y).$$

Let η_0 be fixed in B and let V_0 be a relatively compact open neighbourhood of η_0 so that $\bar{V}_0 \subseteq B$ (it is obvious that B is an open set).

By an easy compactness argument, it follows that there is an open neighbourhood $D_{T,y}$ of $\gamma_T(y)$ such that for every point w in $D_{T,y}$ and η in V_0 we have $\theta(\eta, w) \neq \lambda$ so that the map $r_\eta(w) = (\lambda - \theta(\eta, w))^{-1}$ is analytic and well defined in $D_{T,y}$. We define

$$h_\eta(z, w) = (\eta - z)^{-1}(\theta(z, w) - \theta(\eta, w))r_\eta(w),$$

$h_\eta : D_S \times D_{T,y} \mapsto \mathbf{C}$, and we observe that

$$(2)\qquad (\eta - z)h_\eta(z, w) = 1 - (\lambda - \theta(z, w))r_\eta(w).$$

For η in V_0 define

$$R(\eta) = [h_\eta(S, T|Y_0)V(\lambda)]y$$

and note that the map $\eta \mapsto R(\eta) \in X$ is analytic in V_0 (remark that by (a) $\gamma_{T|Y_0}(y) \subseteq \gamma_T(y)$, $(\forall)\ y \in Y_0$ so that the term $[h(S, T \mid Y_0)V(\lambda)]y$ is meaningful). Using (2) and Lemma 4 we obtain:

$$(\eta - S)R(\eta) = V(\lambda)y - (\lambda - \theta(S, T|Y_0))V(\lambda)[r_\eta(T|Y_0)]y = V(\lambda)y,$$

since $(\lambda - \theta(S, T|Y_0))V(\lambda) \equiv 0$ by hypothesis. Hence

$$V_0 \subseteq \mathbf{C}\setminus\gamma_S(V(\lambda)y)$$

so that we have proved (1).

For any fixed λ in G we consider $D_{\lambda,r}$ to be the open disk with center at λ and radius r (r being so small that $\bar{D}_{\lambda,r}$ is contained in G). Then from (1) we have that:

$$\gamma_S(V(\lambda')y) \subseteq \theta^{-1}_{\gamma_T(y)}(\bar{D}_{\lambda,r}),\quad (\forall)\ \lambda' \in D_{\lambda,r},$$

whence

$$V(\lambda')y \in X_S(\theta^{-1}_{\gamma_T(y)}(\bar{D}_{\lambda,r})),\ (\forall)\ \lambda' \in D_{\lambda,r}$$

and hence for every $\lambda' \in G$. Since r was arbitrary small we obtain

$$V(\lambda')y \in X_S(\theta^{-1}_{\gamma_T(y)}(\{\lambda\})).$$

Since this holds for every λ' in G, we obtain

$$\gamma_S(V(\lambda')y) \subseteq \bigcap_{\lambda \in G} \{\theta^{-1}_{\gamma_T(y)}(\{\lambda\})\}, \quad (\forall)\ \lambda, \lambda' \in G$$

whence $V(\lambda')y = 0$, $(\forall)\ \lambda' \in G$, for y in dense subspace of Y_0 (by (b) and since S has the single valued extension property). This ends the proof.

Corollary 6. *$\theta(S, T)$ has the single valued extension property.*

Proof. Since T is decomposable, every point in Y can be written as a sum of elements z such that the diameter of $\gamma_T(z)$ is smaller than a fixed $\delta > 0$. But for δ small enough, we have that the intersection $\bigcap_{\lambda \in G} \theta_F^{-1}(\{\lambda\})$ is void if the diameter of F is smaller than δ (by the uniform continuity of θ), where F is an arbitrary closed subset of $\mathbf{C}$ contained in D_S. Therefore Y satisfies conditions (a), (b) in the preceding proposition and hence $\theta(S, T)$ has the single valued extension property.

From now on we shall consider a fixed V in $L(Y, X)$ and for this V we consider the analytic function $V(\lambda)$ defined on $\mathbf{C}\setminus\gamma_{\mathfrak{M}}(V)$ such that $(\lambda - \mathfrak{M})V(\lambda) = V$ for every λ in $\mathbf{C}\setminus\gamma_{\mathfrak{M}}(V)$, which exists because $\mathfrak{M}$ has the single valued extension property (by $\mathfrak{M}$ we denote the operator $\theta(S, T)$).

Lemma 7. *For every y in Y we have*

$$\gamma_S(Vy) \subseteq \theta^{-1}_{\gamma_T(y)}(\gamma_{\mathfrak{M}}(V)).$$

Proof. Fix y in Y. We have to prove that:

$$B_1 = \{\eta \in D_S : (\forall)\ w \in \gamma_T(y),\ \theta(\eta, w) \notin \gamma_{\mathfrak{M}}(V)\} \subseteq \mathbf{C}\setminus\gamma_S(Vy).$$

Taking a point $\eta_0 \in B_1$, for every relatively compact open neighbourhood E of η_0 such that $\bar{E} \subseteq B_1$, there is a neighbourhood $D_{T,y}$ of $\gamma_T(y)$ such that $\theta(\eta, w) \notin \gamma_{\mathfrak{M}}(V)$, $(\forall)\ \eta \in E$ and $(\forall)\ w \in D_{T,y}$. We consider the analytic X-valued function $R : E \mapsto X$ defined by

$$R(\eta) = c_2 \int_{\Gamma_S} \int_{\Gamma_{T,y}} h_\eta(z, w)(z - S)^{-1} V(\theta(\eta, w)) y_T(w)\, dz\, dw$$

for η in E, $\Gamma_{T,y}$ a contour contained in $D_{T,y}$ surrounding $\gamma_T(y)$, $c_k = (2\pi i)^{-k}$ $k = 1, 2$, and $h_\eta : D_S \times D_T \mapsto \mathbf{C}$ the analytic function defined by

$$h_\eta(z, w) = (\eta - z)^{-1}(\theta(\eta, w) - \theta(z, w)), \quad w \in D_T,\ z \in D_S\setminus\{\eta\}$$

and if $\eta \in D_S$ then $h_\eta(\eta, w)$ is the partial derivative with respect to z evaluated at (η, w) of the function $(z, w) \mapsto -(\theta(\eta, w) - \theta(z, w))$. It is obvious that

$$(\eta - z)h_\eta(z, w) = \theta(\eta, w) - \theta(z, w).$$

We shall prove that $(\eta - S)R(\eta) = Vy$ and the proposition will follow. We have:

$$(\eta - S)R(\eta) = c_2 \int_{\Gamma_S} \int_{\Gamma_{T,y}} (z - S)^{-1}\theta(\eta, w)V(\theta(\eta, w))y_T(w)\,dz\,dw -$$

$$- c_2 \int_{\Gamma_S} \int_{\Gamma_{T,y}} (z - S)^{-1}V(\theta(\eta, w))\theta(z, T)y_T(w)\,dz\,dw =$$

$$= c_1 \int_{\Gamma_{T,y}} (\theta(\eta, w) - \theta(S, T))V(\theta(\eta, w))y_T(w)\,dw =$$

$$= c_1 \int_{\Gamma_{T,y}} Vy_T(w)\,dw = Vy,$$

by Lemma 3 and because of the equality

$$(\theta(\eta, w) - \theta(S, T))V(\theta(\eta, w)) = V,$$

since $\theta(\eta, w) \in \mathbf{C}\setminus\gamma_{\mathfrak{M}}(V)$, $(\forall)\ \eta \in E$, $(\forall)\ w \in D_{T,y}$.

LEMMA 8. *Let W be an open nonvoid connected subset of $\mathbf{C}$ and let δ be a positive number. Then there is an open covering $\{G_j\}_{j=1}^n$ of $\sigma(T)$, G_j with diameter less than δ, such that for every family of closed sets $\{F_j'\}_{j=1}^n$, with*

(c) $F_j' \subseteq G_j \cap \sigma(T)$, $\quad j = 1, \ldots, n$,

(d) $F_j' \subseteq \bigcup_{i \neq j} F_i'$, $\quad j = 1, \ldots, n$,

we have $\bigcap_{\lambda \in W} \theta_F^{-1}(\{\lambda\}) = \emptyset$, *where* $F = \cup F_j'$.

Proof. Let us denote by θ_z the function defined on D_T by $\theta_z(w) = \theta(z, w)$ $z \in D_S$. We have to find a covering $\{G_j\}_{j=1}^n$ of $\sigma(T)$ such that $\bigcap_{\lambda \in W} \theta_F^{-1}(\{\lambda\}) = \emptyset$, for every closed subsets F_j', $j = 1, 2, \ldots, n$, which satisfies (c), (d). This is equivalent to find a covering $\{G_j\}_{j=1}^n$ such that W is not contained in $\theta_z(\cup F_j')$ for every closed subsets F_j', $j = 1, 2, \ldots, n$, which satisfies (c), (d) and for every z in $\sigma(S)$.

If this is not true, then for every open covering $\{G_j\}_{j=1}^n$, of $\sigma(T)$ with open sets of diameter less than δ there exist some closed sets $\{F_j'\}_{j=1}^n$, for which (c), (d) are fulfilled and for which there exists a z in $\sigma(S)$ such that $W \subseteq \theta_z(\cup F_j')$.

We consider a lattice consisting of squares with sides of length less than $\delta/2\sqrt{2}$ that covers $\mathbf{C}$. By the compactness of $\sigma(T)$ we can choose a finite number $P_1, \ldots, P_k$ of them which also cover $\sigma(T)$. For $\varepsilon > 0$, $\varepsilon \leqslant \delta/4\sqrt{2}$ consider $P_i^\varepsilon = \{z \in \mathbf{C}:$

$: d(z, P_i) < \varepsilon\}$. Clearly the diameter of P_i^ε is less than δ. We denote the boundary of P_i by ∂P_i. Plainly $\{P_i^\varepsilon\}_{i=1}^k$ is also a covering of $\sigma(T)$. Set $Q_i^\varepsilon = \{z \in \mathbf{C} : d(z, \partial P_i) < \varepsilon\}$.

From our hypothesis, it follows that there are some closed sets $F_j' \subseteq P_j^\varepsilon \cap \sigma(T)$, which satisfy (d), and there is a z_ε in $\sigma(S)$ such that $W \subseteq \theta_{z_\varepsilon}(\cup F_j')$. By (d), we obtain

$$F_i' \subseteq \bigcup_{j \neq i} P_j^\varepsilon \cap P_i^\varepsilon \subseteq Q_i^\varepsilon, \quad \text{for } i = 1, \ldots, n.$$

Hence $F_i' \subseteq Q_i^\varepsilon \cap \sigma(T)$.

Thus for every $\varepsilon > 0$ there exists an z_ε in $\sigma(S)$ such that

$$W \subseteq \theta_{z_\varepsilon}(\sigma(T) \cap \{z \in \mathbf{C} : d(z, \cup \partial P_i) < \varepsilon\}).$$

Letting $\varepsilon \to 0$ and using an easy compactness argument we obtain that there is a z_0 in $\sigma(S)$ such that:

$$W \subseteq \theta_{z_0}(\sigma(T) \cap (\cup \partial P_i)).$$

Since W is open and nonvoid, and since θ_z is analytic we obtain a contradiction by the following remark.

REMARK. Let D be an open subset of $\mathbf{C}$, let f be an analytic function $f : D \mapsto \mathbf{C}$, and K a compact subset of D with void interior. Then $f(K)$ is a subset of $\mathbf{C}$ of the first Baire category, and hence cannot contain a nonvoid open subset of $\mathbf{C}$.

Proof. We write $D = \bigcup_n D_n$, where D_n are disjoint open connected nonvoid subsets of $\mathbf{C}$. Because K is compact, there is a p such that $K \subseteq \bigcup_{n=1}^{p} D_n$, and we can write $K = \bigcup_{n=1}^{p} K_n$, where each set $K_n = D_n \cap K$ is compact.

For every $n = 1, 2, \ldots, p$ we have that either f is constant on D_n or f' is not identical null on D_n.

Let $n \in \{1, 2, \ldots, p\}$ be such that f' is not identical null on D_n. Since f' is also an analytic function and K_n is compact there exist $w_1, \ldots, w_q \in K_n$ such that $f'(z) \neq 0$ for every $z \in K_n \setminus \{w_1, w_2, \ldots, w_q\}$ and hence by the inverse mapping theorem for analytic functions there are two open sets V_z, W_z with the property $z \in V_z$, $f(z) \in W_z$ and f is a homeomorphism from V_z onto W_z. We select an open neighbourhood V_z' of z such that $\overline{V_z'} \subseteq V_z$. Since

$$K_n \setminus \{w_1, \ldots, w_q\} \subseteq \bigcup_z V_z',$$

and since the usual topology on $\mathbf{C}$ has a countable basis, we can find a countable subset $\{z_r\}_{r \in \mathbf{N}}$ of $K_n \setminus \{w_1, \ldots, w_q\}$ such that

$$K_n \setminus \{w_1, \ldots, w_q\} \subseteq \bigcup_r V_{z_r}',$$

and hence

$$K_n \setminus \{w_1, \ldots, w_q\} \subseteq \bigcup_r V'_{z_r} \cap K_n \subseteq \bigcup_r \overline{V}'_{z_r} \cap K_n .$$

But $\overline{V}'_{z_r} \cap K_n$ is a compact subset of V_{z_r} with void interior and f is a homeomorphism from V_{z_r} onto W_{z_r}, so that $f(\overline{V}'_{z_r} \cap K_n)$ is also a compact subset of $\mathbf{C}$ with void interior.

Finally we obtain

$$f(K_n) \subseteq \bigcup_r f(\overline{V}'_{z_r} \cap K_n) \cup \left(\bigcup_{s=1}^{q} \{f(w_q)\} \right)$$

where each set in the right side is closed with void interior.

If n is such that f is constant on D_n then the image of K_n by f is reduced to one point. We conclude that $f(K) = \bigcup_{n=1}^{p} f(K_n)$ is contained in a countable union of closed sets with void interior.

LEMMA 9. *Let $\delta > 0$, let W be an open nonvoid subset of* $\mathbf{C}$ *and let $\{G_j\}_{j=1}^{n}$ be the open covering from Lemma* 8. *Let $\{F_j\}_{j=1}^{n}$ be a family of closed sets such that $F_j \subseteq G_j \cap \sigma(T)$ and*

$$\sum Y_T(F_j) = Y.$$

Let $\tilde{Y} = Y \oplus \ldots \oplus Y$ (n times), $\tilde{T} = T \oplus \ldots \oplus T$ (n times), and let

$$\tilde{Y}_0 = \left\{ \tilde{y} = \bigoplus_{i=1}^{n} y_i : y_i \in Y_T(F_i), \sum_{i=1}^{n} y_i = 0 \right\}.$$

Then $\tilde{Y}_0$ satisfies conditions (a), (b) *from Proposition* 5.

Proof. That $\tilde{Y}_0$ satisfies condition (a) is obvious (see also Lemma IV.6.6 and the proof of Theorem IV.6.7 from [8]).

To prove (b) note that for every $\tilde{y} = \bigoplus_{i=1}^{n} y_i \in \tilde{Y}_0$, we have that $\gamma_T(y_j) \subseteq F_j \subseteq$ $\subseteq G_j \cap \sigma(T)$ and

$$\gamma_T(y_j) \subseteq \bigcup_{j \neq i} \gamma_T(y_i), \quad j = 1, \ldots, n,$$

$\left(\text{since } \sum_{i=1}^{n} y_i = 0\right)$. Therefore the sets $F'_j = \gamma_T(y_j)$, $j = 1, \ldots, n$ satisfy conditions (c), (d) in Lemma 8 and hence

$$\bigcap_{\lambda \in W} \theta_F^{-1}(\{\lambda\}) = \emptyset, \quad F = \bigcup_{i=1}^{n} \gamma_T(y_i), \quad (\forall)\ \tilde{y} = \bigoplus_{i=1}^{n} y_i \in \tilde{Y}_0 .$$

But this is exactly condition (b), since

$$\bigcup_{i=1}^{n} \gamma_T(y_i) = \gamma_{\tilde{T}}(\tilde{y}).$$

THEOREM 10. *Let K be a compact nonvoid subset of* $\mathbf{C}$. *Then the following statements are equivalent:*

(i) $\gamma_{\mathfrak{M}}(V) \subseteq K$.

(ii) $\gamma_S(Vy) \subseteq \theta^{-1}_{\gamma_T(y)}(K)$, $(\forall)\ y \in Y$.

(iii) $V(Y_T(F)) \subseteq X_S(\theta_F^{-1}(K))$ *for every closed subset F of $\sigma(T)$.*

(If $\theta(z, w) = z - w$, then $\theta_F^{-1}(K) = (F + K) \cap \sigma(S)$ and we obtain the Theorem IV.6.7 from [8].)

Proof. The implication (i) $\Rightarrow$ (ii) follows from Lemma 7 and (iii) is merely a rewriting of (ii). So the difficult part of the proof is the implication (iii) $\Rightarrow$ (i).

Let $\lambda_0 \notin K$. We shall show that $\lambda_0 \notin \gamma_{\mathfrak{M}}(V)$. Let $0 < \varepsilon < d(\lambda_0, K)/4$ and let W be the open disk with center at λ_0 and radius ε. By the uniform continuity of θ there is a $\delta > 0$ such that:

(3) $|\theta(z, w) - \theta(z, w')| < \varepsilon$, $(\forall)\ z \in \sigma(S)$, $(\forall)\ w, w' \in \sigma(T)$ with $|w - w'| < \delta$.

With δ as before, we consider the covering $\{G_j\}_{j=1}^{n}$ from Lemma 8. Since T is decomposable, we can find closed sets $F_j \subseteq G_j \cap \sigma(T)$ such that $Y = \sum_{j=1}^{n} Y_T(F_j)$. We denote

$$Y_j = Y_T(F_j), \quad X_j = X_S(\theta_{F_j}^{-1}(K)), \quad j = 1, \ldots, n$$

and from (iii) we have $V(Y_j) \subseteq X_j$ for $j = 1, \ldots, n$. We denote by V_j, $V_j : Y_j \mapsto X_j$, the restriction $V_j = V|Y_j$ and let $\mathfrak{M}_j = \theta(S|X_j, T|Y_j)$ for $j = 1, \ldots, n$.

Using Corollary 2 for $\theta(S_j, T_j)$ we obtain that

$$\sigma(\theta(S_j, T_j)) \subseteq \theta(\sigma(S_j) \times \sigma(T_j)) \subseteq \theta(\theta_{F_j}^{-1}(K) \times F_j), \quad j = 1, \ldots, n.$$

This inclusion, the fact that the diameter of F_j is less than δ, as well as (3) show that $W \subseteq \mathbf{C} \setminus \sigma(\mathfrak{M}_j)$ for $j = 1, \ldots, n$.

For λ belonging to W we consider

$$R_j(\lambda) = (\lambda - \mathfrak{M}_j)^{-1} V_j$$

and we shall prove that:

(4) $$\sum_{j=1}^{n} R_j(\lambda) y_j = 0 \quad \text{if } y_j \in Y_j,\ j = 1, \ldots, n, \text{ and } \sum_{j=1}^{n} y_j = 0.$$

If we prove this, then it follows that

$$R(\lambda)y = \sum_{j=1}^{n} R_j(\lambda)y_j, \quad y \in Y, \ y = \sum_{j=1}^{n} y_j, \ y_j \in Y_j$$

is well defined, $R(\lambda) \in L(Y, X)$ and the mapping $\lambda \mapsto R(\lambda) \in L(Y, X)$ is analytic. Since for every $y = \sum_{j=1}^{n} y_j$, $y_j \in Y_j$, we have

$$(\lambda - \mathfrak{M})R(\lambda)y = \sum_{j=1}^{n} (\lambda - \mathfrak{M}_j)R_j(\lambda)y_j = \sum_{j=1}^{n} V_j y_j = Vy,$$

it follows that $\lambda_0 \in W \subseteq \mathbf{C} \setminus \gamma_{\mathfrak{M}}(V)$.

So we have only to prove (4). We use the notations in Lemma 9. If we define

$$\tilde{R}(\lambda)\tilde{y} = \sum_{j=1}^{n} R_j(\lambda)y_j, \quad \text{for } \tilde{y} = \bigoplus_{j=1}^{n} y_j \in \tilde{Y}_0,$$

then (4) is equivalent to:

$$\tilde{R}(\lambda)\tilde{y} = 0, \quad (\forall)\ \lambda \in W, \ (\forall)\ \tilde{y} = \bigoplus_{j=1}^{n} y_j \in Y_0.$$

But this is a consequence of Lemma 9, Proposition 5, and of the fact that the map $\lambda \mapsto \tilde{R}(\lambda) \in L(\tilde{Y}_0, X)$ is analytic and

$$(\lambda - \theta(S, \tilde{T}|\tilde{Y}_0))\tilde{R}(\lambda) = V, \quad (\forall)\ \lambda \in W.$$

To prove the last equality observe that for every $\tilde{y} = \bigoplus_{j=1}^{n} y_j \in \tilde{Y}_0$ and for every λ in W,

$$(\lambda - \theta(S, \tilde{T}|\tilde{Y}_0))\tilde{R}(\lambda)\tilde{y} = \sum_{j=1}^{n} (\lambda - \theta(S_j, T_j))R_j(\lambda)y_j =$$

$$= \sum_{j=1}^{n} V_j y_j = V\left(\sum_{j=1}^{n} y_j\right) = 0.$$

This ends the proof of the theorem.

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