J. OPERATOR THEORY
14(1985), 311–346

SEMIGROUPS OF ISOMETRIES WITH COMMUTING RANGE PROJECTIONS

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1. INTRODUCTION

Let H denote an infinite dimensional complex Hilbert space and let $\mathscr{L}(H)$ denote the algebra of all bounded linear operators on H. Let $K(H)$ denote the algebra of compact operators on H. An isometry U on H is a linear operator satisfying $\|U(x)\| = \|x\|$ for x in H. Let m be a natural number and let $\{V(n): n \in \mathbf{Z}_+^m\}$ be a semigroup of isometries acting on H, i.e., assume that $\{V(n)\}$ satisfies $V(n+l) = V(n)V(l)$ for all $n, l \in \mathbf{Z}_+^m$. We will restrict our attention to those semigroups of isometries that have commuting range projections, i.e., $\{V(n)V^*(n): n \in \mathbf{Z}_+^m\}$ is a commuting family. In this paper we will investigate some of the properties of the C^*-algebra $C^*(V(n))$ generated by $\{V(n): n \in \mathbf{Z}_+^m\}$.

Examples of isometries with commuting range projections are the following. Let E be a subset of $\mathbf{Z}^m$ that satisfies $E + \mathbf{Z}_+^m \subset E$ (we call such a set a module over $\mathbf{Z}_+^m$). Let $\{e_p: p \in E\}$ be an orthonormal basis for $\ell^2(E)$, where e_p is defined by $e_p(q) = \delta_{p,q}$ for $q \in E$. Let $\{U(n): n \in \mathbf{Z}_+^m\}$ acting on $\ell^2(E)$ be defined by the formula $U(n)(e_p) = e_{p+n}$, for $p \in E$ and $n \in \mathbf{Z}_+^m$. Then $\{U(n): n \in \mathbf{Z}_+^m\}$ is a semigroup of isometries with commuting range projections. Let $C^*(E)$ be the C^*-algebra generated by $\{U(n): n \in \mathbf{Z}_+^m\}$. Our basic structure theorem, Theorem 4.4, asserts that from the point of view of C^*-algebras, these are the only examples. Specifically, we shall prove that if $\{V(n): n \in \mathbf{Z}_+^m\}$ is an irreducible semigroup of isometries, on the Hilbert space H, with commuting range projections $\{V(n)V^*(n): n \in \mathbf{Z}_+^m\}$, and if $V^*(n)V(l)$ is different from a multiple of the identity for all $n, l \in \mathbf{Z}_+^m$ with $|n - l| = |n| + |l| > 0$, then there is a $\mathbf{Z}_+^m$-module E such that $C^*(V(n))$ is isomorphic to $C^*(E)$. An essential ingredient in the proof of this theorem is the theory of groupoid C^*-algebras. The condition on the commutativity of the range projections is provided precisely to make use of this theory.

The idea of using groupoids to analyse the structure of concretely given C^*-algebras is not new. In [7], Cuntz studied C^*-algebras generated by isometries $T_1, \ldots, T_n$, $n \geqslant 2$, with the property $I = \sum_{i=1}^{n} T_iT_i^*$, by representing them as a crossed product. (The isometries $T_1, T_2, \ldots, T_n$ do not commute since their ranges are pairwise orthogonal.) In [20], Renault realized the Cuntz algebras as groupoid C^*-algebras. Muhly and Renault used the groupoid theory in their study of C^*-algebras of generalized Weiner-Hopf operators [17]. The same tool is used by Curto and Muhly in their analysis of C^*-algebras of commutative weighted shifts [8]. A forerunner of this circle of ideas is O'Donovan's use of covariance algebras in his study of C^*-algebras of weighted shifts [18].

Another large class of abelian semigroups of isometries with commuting range projections is given by subsemigroups of the non-negative additive reals (see Douglas [11]). The intersection of this class with ours is non-empty, in fact its members are the finitely generated one-parameter semigroups of isometries.

We now describe the contents of the sections ahead. In the next section we investigate the relation between the boundary of a module E, ∂E, and the algebraic properties of $C^*(E)$. Lemma 2.17 asserts that the algebra of compact operators, $K(\ell^2(E))$, is contained in $C^*(E)$ if and only if there is a "segment" of ∂E which is unique, i.e., it is not a translate of any other segment of ∂E. We answer the question of when, for given modules E and F, $C^*(E)$ and $C^*(F)$ are algebraically equivalent, i.e., the map $U(n) \to U(n)$ ($U(n)$ acting on $\ell^2(E)$ and $\ell^2(F)$, respectively, $n \in \mathbf{Z}_+^m$) can be extended to an isomorphism of the algebras $C^*(E)$ and $C^*(F)$. The answer is, again, phrased in terms of the geometry of ∂E and ∂F. We say that F is locally representable in E if, roughly, each segment of F is a translate of a segment of E. Theorem 2.20 asserts that $C^*(E)$ and $C^*(F)$ are algebraically equivalent if and only if E is locally representable in F and viceversa. We say that a module E_∞ is a universal module if each module E is locally representable in E_∞. Our last result shows that there exist universal modules and that every $C^*(E)$ is a quotient of $C^*(E_\infty)$.

In Section 3 we realize, for a given module E, $C^*(E)$ as a groupoid C^*-algebra. The examples there show how the structure of $C^*(E)$ is put in evidence through the groupoid approach. Section 4 is devoted primarily to the proof of the structure theorem mentioned before. Section 5 is specialized to $\mathbf{Z}_+^2$-modules. We determine when $C^*(E)$ is G.C.R.. (A C^*-algebra is G.C.R., also called type I, if all of its irreducible representations contain the underlying compact operators.) One ingredient is a corollary of the structure theorem that allows us to restrict our attention to the modules that are locally representable in E. Then we associate to E a family of weighted shifts that act on a subspace of $\ell^2(\partial E)$. It turns out that $C^*(E)$ is G.C.R. if and only if each weighted shift generates a G.C.R. C^*-algebra. We then appeal to O'Donovan's study [18] in which G.C.R. weighted shifts are characterized.

The last section is very brief. We present some possible generalizations of our results and point out some problems that require further research. In the appendix we give our proofs of the theorems of O'Donovan that we need. We also extend these results to a family of (noncommutative) weighted shifts acting on $\ell^2(E)$.

This paper is based on the author's doctoral dissertation written at the University of Iowa. I am deeply indebted to my former advisor, Professor Paul S. Muhly, for his help and encouragement. I also wish to thank Professors Raul E. Curto, Domingo A. Herrero, Marta B. Pecuch and John Quigg.

2. MODULES

The $\mathbf{Z}_+^2$-modules E and the C^*-algebras $C^*(E)$ are fairly representative of the general case, (i.e., $\mathbf{Z}_+^m$-modules). They motivate more of the subsequent concepts and results, and provide a good source of examples. For the last reason and with an eye toward Section 5 we intersperse the case $m = 2$ along the way. In particular, for a $\mathbf{Z}_+^2$-module E, we shall use a sequence to describe ∂E. For a $\mathbf{Z}_+^m$-module F it is possible to describe ∂F in a similar way. We will not do it, however, since it is too cumbersome and does not seem worthwhile.

DEFINITIONS 2.1. a) Given a $\mathbf{Z}_+^m$-module E and $p \in E$, the *diamond centered at p of radius n*, $D_n^E(p)$, is defined by the equation $D_n^E(p) = \{l \in E : |l - p| \leqslant n\}$.

b) For F a $\mathbf{Z}_+^m$-module and $q \in F$, the diamonds $D_n^E(p)$ and $D_n^F(q)$ are *equivalent*, denoted by $D_n^E(p) \sim D_n^F(q)$, if $D_n^E(p) = p - q \dotplus D_n^F(q)$.

REMARKS 2.2. 1) For a given $n \in \mathbf{N}$, there are only a finite number of classes of equivalent diamonds (because the set $\{l \in \mathbf{Z}_+^m : |l| \leqslant n\}$ is finite). Each equivalence class will be called an *abstract n-diamond* and it will be identified with the diamond in the equivalence class that is centered at the origin.

2) Assume that p is a polynomial of degree n in z_i, $\overline{z}_i$, $i \in \{1, \ldots, m\}$. We do not assume that these indeterminates commute. Let $\varepsilon_i \in \mathbf{Z}_+^m$ have the j-th coordinate equal to 1 for $j = i$ and 0 for $j \neq i$. For a given $\mathbf{Z}_+^m$-module E the operator obtained from evaluating p in $U(\varepsilon_i)$, $U^*(\varepsilon_i)$, via the map $z_i \to U(\varepsilon_i)$, $\overline{z}_i \to U^*(\varepsilon_i)$, is denoted by P. If F is another module the polynomial p evaluated in the same way, but now with $U(\varepsilon_i) \in C^*(F)$, is also denoted by P. There is no risk of confusion as long as we specify the C^*-algebra to which P belongs. The following observations are some of the reasons why diamonds play a key role in our work. If $q \in E$ then $P(e_q) = \sum_{r \in D_n^E(q)} \lambda_r e_r$. Thus $P(e_q)$ is supported in $D_n^E(q)$, and P may be viewed as smoothing or spreading out e_q. Moreover, if F is a module (F may be E) and $s \in F$ is such that $D_n^E(q) \sim D_n^F(s)$, then the operator P, $P \in C^*(F)$, is such that $P(e_s) = \sum_{r \in D_n^E(q)} \lambda_r e_{r+s-q}$. In other words the action of a polynomial P, of degree n, on e_q depends only on the equivalence class of $D_n^E(q)$.

3) Let S be a monomial in the $2m$ generators of $C^*(E)$. We may view S as induced by a partial transformation φ on E. Let $k \in \mathbf{Z}$ and let T be an operator. The symbol $T^{[k]}$ means T^k if $k \in \mathbf{Z}_+$, and $T^{*|k|}$ if $-k \in \mathbf{Z}_+$. Without loss of gene-

rality we may assume that

$$S = V(\varepsilon_m)^{[k_{mi}]} \dots V(\varepsilon_1)^{[k_{(m-1)(i-1)}]} \dots V(\varepsilon_m)^{[k_m]} \dots V(\varepsilon_1)^{[k_1]},$$

where $\sum_{s=1}^{mi} k_s = n$ is the degree of S. Let F be

$$E \cap (E - k_1\varepsilon_1) \cap (E - (k_1\varepsilon_1 + k_2\varepsilon_2)) \cap \dots \cap \left(E - \left(\sum_{t=1}^{m} \left(\sum_{s=0}^{i-1} k_{t+sm}\right)\varepsilon_t\right)\right)$$

and let φ be the partial transformation on E, with domain F, such that $\varphi(l) = l + \sum_{t=1}^{m} \left(\sum_{s=0}^{i-1} k_{t+sm}\right)\varepsilon_t$. Then S is induced by φ since $S(e_l) = e_{\varphi(l)}$ if $l \in F$ and $S(e_l) = 0$ if $l \in E \setminus F$.

Let E be a $\mathbf{Z}_+^m$-module and let Y be a subset of E. Let 1_Y denote the projection in $\ell^2(E)$ defined by the formula

$$1_Y(e_p) = \begin{cases} e_p & \text{if } p \in Y, \\ 0 & \text{if } p \in E \setminus Y. \end{cases}$$

PROPOSITION 2.3. *Let E be a module and let $P \in C^*(E)$ be a compact operator. If P is a polynomial in the generators of $C^*(E)$, then $P = 1_Y P 1_Y$ for some finite subset Y of E.*

Proof. Let n be the degree of P. The following facts are easily checked (see Remark 2.2.2).

1) If $q \in E$, then $P(e_q) \in \ell^2(D_n^E(q))$.

2) If $D_n^E(q) \cap D_n^E(s) = \emptyset$, then $P(e_q)$ is orthogonal to $P(e_s)$.

3) If $D_n^E(q) \sim D_n^E(s)$, then $\|P(e_q)\| = \|P(e_s)\|$.

Let $Q = \{q \in E : P(e_q) \neq 0\}$. Then Q is a finite set because if it were infinite, we could find a subset L, also infinite, with the property that if $q, s \in L$ and $q \neq s$, then $D_n^E(q) \sim D_n^E(s)$ and $D_n^E(q) \cap D_n^E(s) = \emptyset$. But the existence of this L, together with the facts established above, contradict the compactness of P. To conclude the proof, choose $Y = \cup D_n^E(q)$, $q \in Q$. ▨

DEFINITION 2.4. Let E be a $\mathbf{Z}_+^m$-module. The *boundary* of E, ∂E, consists of those points p of E with $p - \sum_{i=1}^{m} \varepsilon_i$ not in E.

If E is a $\mathbf{Z}_+^2$-module with $p = (p_1, p_2)$, $q = (q_1, q_2) \in \partial E$ such that $p_1 \leqslant q_1$ and $q_2 \geqslant p_2$, then the *segment* $[p, q]$ is $\{(h_1, h_2) \in \partial E : p_1 \leqslant h_1 \leqslant q_1$ and $p_2 \geqslant h_2 \geqslant q_2\}$.

For a family of $\mathbf{Z}_+^m$-modules E_j, $j \in J$, we denote by $C^*(\bigoplus_{j\in J} E_j)$ the C^*-algebra generated by $\{U(n) : n \in \mathbf{Z}_+^m\}$ acting on $\bigoplus_{j\in J} E_j$. The commutative subalgebra generated by S^*S, where S is a monomial in $U(\varepsilon_1), \ldots, U(\varepsilon_m), U^*(\varepsilon_1), \ldots, U^*(\varepsilon_m)$ is called the *diagonal subalgebra* of $C^*(\bigoplus_{j\in J} E_j)$.

Lemma 2.6, although very simple, will be fundamental in what follows. The example below illustrates particular cases of it, namely, when the index J consists of only one element and the abstract diamond has radius one. Observe that the way the polynomials are defined is independent of E.

EXAMPLE 2.5. Let E be a $\mathbf{Z}_+^2$-module.

1) The projection $(I - U(\varepsilon_1)U^*(\varepsilon_1))(I - U(\varepsilon_2)U^*(\varepsilon_2)) = 1_V$ with $V = \{q \in E : D_1^E(q) = q + \{0, \varepsilon_1, \varepsilon_2\}\}$.

2) The projection $(I - U(\varepsilon_1)U^*(\varepsilon_1))U(\varepsilon_2)U^*(\varepsilon_2) = 1_W$ with $W = \{q \in E : D_1^E(q) = q + \{0, \varepsilon_1, \varepsilon_2, -\varepsilon_2\}\}$.

3) The projection $(I - U(\varepsilon_2)U^*(\varepsilon_2))U(\varepsilon_1)U^*(\varepsilon_1) = 1_S$ with $S = \{q \in E : D_1^E(q) = q + \{0, \varepsilon_1, \varepsilon_2, -\varepsilon_1\}$.

The set $V \cup W \cup S \subset \partial E$ is the exposed boundary. A point in V is a *vertex*, a point of S is *south-exposed* while a point in W is *west-exposed*. A point in $\partial E \setminus (V \cup S \cup N)$ is a *joint*. Observe that any of the four disjoint sets that form ∂E could be empty.

LEMMA 2.6. *Let $D_n \subset \mathbf{Z}^m$ be an abstract n-diamond. There exists a polynomial $p(z_1, \ldots, z_m, \bar{z}_1, \ldots, \bar{z}_m)$, depending only on D_n, such that given a family of $\mathbf{Z}_+^m$-modules E_j, $j \in J$, the operator P (the evaluation of p in the generators) is equal to $\bigoplus_{j\in J} 1_{F_j}$, where $F_j = \{q \in E_j : D_n^{E_j}(q) = q + D_n\}$. Moreover P is in the diagonal subalgebra of $C^*(\bigoplus_{j\in J} E^j)$.*

Proof. For $l \in \mathbf{Z}^m$ write $l = l^+ - l^-$ where $l^+, l^- \in \mathbf{Z}_+^m$ and $|l| = |l^+| + |l^-|$. For $|l| \leqslant n$, define polynomials P_l according to the formulas, $P_l = U(l^+)^*U(l^-)\cdot U(l^-)^*U(l^+)$ if $l \in D_n$ and $P_l = I - U(l^+)^*U(l^-)U(l^-)^*U(l^+)$ if $l \notin D_n$. It is easy to verify the $P = \prod\{P_l : |l| \leqslant n\}$ has the required properties. ▨

NOTE. The order in which the P_l's are multiplied is immaterial. In particular, we have shown that $1_{(E+k)\cap E}$ is in $C^*(E)$, for every $k \in \mathbf{Z}^m$.

COROLLARY 2.7. *Let F be a subset of a $\mathbf{Z}_+^m$-module E. Then $1_F \in C^*(E)$ if and only if F belongs to the Boolean algebra generated by $(E + p) \cap E$, $p \in \mathbf{Z}^m$. Moreover $1_F \in C^*(E)$ if and only if 1_F belongs to the diagonal subalgebra of $C^*(E)$.*

Proof. Let $1_F \in C^*(E)$, $F \subset E$. Then there exists a polynomial T in the generators of $C^*(E)$ such that $\|T - 1_F\| < 1/4$. Let n be the degree of T. As noted in Remark 2.2.2 we see that $T(e_p) = \sum \lambda_{(p,r)} e_{p+r}$, $p + r \in D_n^E(p)$, where $\lambda_{(p,r)} = \lambda_{(q,r)}$ if $D_n^E(p) \sim D_n^E(q)$. In particular for $p \in F$, $|\lambda_{(p,0)} - 1| < 1/4$ and for $p \notin F$, $|\lambda_{(p,0)}| < 1/4$. This means that if we partition E according to the different classes $D_n^E(p)$, then F is the union of some classes and $E \setminus F$ is the union of the remaining ones. Let F_i, $i \in \{1, \ldots, k\}$, be the different classes in which we have partitioned F. By Lemma 2.6 and the subsequent note, each F_i belongs to the Boolean algebra generated by $(E + p) \cap E$, $p \in \mathbf{Z}^m$, and so therefore does F. In particular, 1_F belongs to the diagonal subalgebra of $C^*(E)$. The converse is immediate because $1_{(E+p) \cap E} \in C^*(E)$ for every $p \in \mathbf{Z}^m$. ▨

DEFINITION 2.8. Let E be a $\mathbf{Z}^m_+$-module. The boundary of E, ∂E, is *periodic* if there exists $p \in \mathbf{Z}^m$, $p \neq 0$, so that $E = E + p$.

The following somewhat technical result will be used in Section 4 in the proof of Theorem 4.4. The proposition says that if, for a module E, there is a point $p \in \partial E$ such that certain small translates of sufficiently large diamonds centered at p are equivalent, then ∂E is periodic, i.e., $E = E + q$ for a non-zero q, and $|q|$ is small.

PROPOSITION 2.9. *Let k be a natural number. Let E be a module whose boundary, ∂E, is either not periodic or if $E = E + p$ with p non-zero then $|p| > 2k$. For each $p \in \partial E$ and for each $q \in D^E_k(p)$ form the diamond $D^E_n(q)$. Then there exists a j, (depending on p), such that for each $n \geqslant j$ the diamonds $D^E_n(q)$, $(q \in D^E_k(p))$, are pairwise inequivalent.*

Proof. Assume that the conclusion is false. Then there exists a sequence $\{n_i\}$ and $q, r \in D^E_k(p)$, $q \neq r$, so that $D^E_{n_i}(q) \sim D^E_{n_i}(r)$. Let $s \in E$ and let $n_i \geqslant \max\{|s - q|, |s - r|\}$. Since $s \in D^E_{n_i}(q) \cap D^E_{n_i}(r)$ we have that $s + q - r \in D^E_{n_i}(q)$ and $s + r - q \in D^E_{n_i}(r)$. Therefore $s + (r - q)$ and $s + (q - r) \in E$, i.e., $E = E + (q - r)$. This is impossible since $q - r \neq 0$ and $|q - r| \leqslant 2k$. It is clear that $D^E_j(r) \nsim D^E_j(q)$ implies that $D^E_n(r)$ and $D^E_n(q)$ are inequivalent of $n \geqslant j$. ▨

For each $q \in \partial E$ and each n, we may apply Lemma 2.6 to find a projection $Q_{q,n}$ in the diagonal subalgebra of $C^*(E)$ with range $\ell^2(\{r \in E : D^E_n(r) \sim D^E_n(q)\})$. Therefore Proposition 2.9 assures us that exists j so that if $n \geqslant j$ then the product of $Q_{q,n}$ and $Q_{r,n}$ is zero whenever $q, r \in D^E_k(p)$ and $q \neq r$.

PROPOSITION 2.10. *For a given module E, the C^*-algebra $C^*(E)$ is reducible if and only if ∂E is periodic.*

Proof. Assume that ∂E is not periodic. For each $p \in E$ we observe that $1_{\{p\}} \in C^*(E)''$ (the von Neumann algebra generated by $C^*(E)$). Indeed, $1_{\{p\}} = \inf Q_{p,n}$, $n \in \mathbf{N}$, where $Q_{p,n}$ is the projection defined above. Since $Q_{p,n} \in C^*(E)$ for all n, their infimum is in $C^*(E)''$. If $T \in C^*(E)'$ (commutant of $C^*(E)$), then, in particular, T commutes with the operators in the m.a.s.a. generated by $\{1_{\{p\}} : p \in E\}$. Therefore $T = \sum \lambda_p 1_{\{p\}}$, $p \in E$. Since T commutes with $U(\varepsilon_i)$ it follows that $\lambda_p = \lambda_{p+\varepsilon_i}$. In turn this implies that λ_p is constant, i.e., T is a multiple of the identity. Consequently $C^*(E)$ is irreducible.

Conversely, assume that $E = E - p$ for some non-zero p. The operator $U^*(p^-)U(p^+)$ is unitary and commutes with each $U(\varepsilon_i)$. Therefore $U^*(p^-)U(p^+) \in C^*(E)'$ and, since it is not a multiple of the identity, $C^*(E)$ is reducible. ▨

With arguments similar to the ones used in Corollary 2.7 we obtain the next corollary.

COROLLARY 2.11. *If E is a module with nonperiodic boundary, then the diagonal subalgebra of $C^*(E)$ is a maximal commutative subalgebra of $C^*(E)$.*

Notice that the diagonal subalgebra is separable and therefore cannot be the m.a.s.a. $\{1_{\{p\}} : p \in E\}''$, which is non-separable.

For a given $\mathbf{Z}_+^2$-module E, we will describe ∂E using a sequence. In Section 5 it will be seen that this sequence plays a role in the study of $C^*(E)$ that is similar to the role played by a weight sequence in the study of the C^*-algebra generated by a weighted shift. It is necessary first to introduce the concept of configurations. They will allow us to describe ∂E locally. Let $\overleftarrow{n}$ denote an i-tuple $(t_{-i}, \ldots, t_{-1})$ of non-negative integers, such that t_{-k} is positive for $k \neq 1$. Likewise $\overrightarrow{m}$ is a $j+1$-tuple $(t_0, \ldots, \ldots, t_j)$ of non-negative integers, with t_k positive for $k \neq 0$. In particular $\overleftarrow{0} = (t_{-1})$ with $t_{-1} = 0$ and $\overrightarrow{0} = (t_0)$ with $t_0 = 0$. The lengths of $\overleftarrow{n}$ and $\overrightarrow{m}$, namely i and $j+1$, can vary. The absolute value of $\overleftarrow{n}$, $|\overleftarrow{n}|$, is $\sum_{k=1}^{i} t_{-k}$ while the absolute value of $\overrightarrow{m}$, $|\overrightarrow{m}|$, is $\sum_{k=0}^{j} t_k$. Recall that, when $\varepsilon_1, \varepsilon_2 \in \mathbf{Z}^2$, $\varepsilon_1 = (1,0)$ and $\varepsilon_2 = (0,1)$.

DEFINITION 2.12. Given $\overleftarrow{n} = (t_{-i}, \ldots, t_{-1})$ and $\overrightarrow{m} = (t_0, \ldots, t_j)$, the configuration $C(\overleftarrow{n}, \overrightarrow{m})$ is the subset of $\mathbf{Z}^2$ which is the union of the following sets.

a) $\{l\varepsilon_1 : 0 \leqslant l \leqslant t_0\}$,

b) $\{t_0\varepsilon_1 - l\varepsilon_2 : 1 \leqslant l \leqslant t_1\}$,

c) $$\left\{\left(\sum_{s=0}^{k} t_{2s}\right)\varepsilon_1 - \left(\sum_{s=0}^{k} t_{2s+1}\right)\varepsilon_2 - l\varepsilon_1 : 0 \leqslant 2k \leqslant j-2 \text{ and } 1 \leqslant l \leqslant t_{2k+2}\right\},$$

d) $$\left\{\left(\sum_{s=0}^{k} t_{2s}\right)\varepsilon_1 - \left(\sum_{s=0}^{k-1} t_{2s+1}\right)\varepsilon_2 - l\varepsilon_2 : 1 \leqslant 2k \leqslant j-1 \text{ and } 1 \leqslant l \leqslant t_{2k+1}\right\},$$

e) $\{l\varepsilon_2 : 0 \leqslant l \leqslant t_{-1}\}$,

f) $\{t_{-1}\varepsilon_2 - l\varepsilon_1 : 1 \leqslant l \leqslant t_{-2}\}$,

g) $$\left\{\left(-\sum_{s=1}^{k} t_{-2s}\right)\varepsilon_1 + \left(\sum_{s=1}^{k} t_{-2s+1}\right)\varepsilon_2 - l\varepsilon_2 : 2 \leqslant 2k \leqslant i-1 \text{ and } 1 \leqslant l \leqslant t_{-2k-1}\right\},$$

h) $$\left\{\left(-\sum_{s=1}^{k} t_{-2s}\right)\varepsilon_1 + \left(\sum_{s=1}^{k-1} t_{-2s+1}\right)\varepsilon_2 - l\varepsilon_1 : 2 \leqslant 2k \leqslant i-2 \text{ and } 1 \leqslant l \leqslant t_{-2k-2}\right\}.$$

If $\overleftarrow{n} = \overleftarrow{0}$ then $C(\overleftarrow{0}, \vec{m})$ is the union of sets obtained from a), b), c) and d). Likewise if $\vec{m} = \vec{0}$ then $C(\overleftarrow{n}, \vec{0})$ is the union of sets obtained from e), f), g) and h).

DEFINITION 2.13. Given *a* $\mathbf{Z}_+^2$-module E, its *boundary* is described by $(\ldots, t_{-1}, \underline{p}, t_0, \ldots)$ if $\partial E = \bigcup_{i=1}^{\infty} p + C(\overleftarrow{n}_i, \vec{m}_i)$, where the configurations $C(\overleftarrow{n}_i, \vec{m}_i)$ satisfy conditions 1), 2) and 3) listed below.

1) $C(\overleftarrow{n}_i, \vec{m}_i) \subset C(\overleftarrow{n}_{i+1}, \vec{m}_{i+1})$ for all i.

2) $\overleftarrow{n}_i = (t_{-i}, \ldots, t_{-1})$ if $t_{-i} \neq \infty$ or $\overleftarrow{n}_i = (\tilde{t}_{-j}, \ldots)$, with $\tilde{t}_j = i$, if $t_{-j} = \infty$ and $j \leqslant i$. If the last alternative holds then there are only j terms to left of p.

3) Similarly $\vec{m}_i = (t_0, \ldots, t_{i-1})$ if $t_{i-1} \neq \infty$ or $\vec{m}_i = (\ldots, \tilde{t}_{j-1})$, with $\tilde{t}_{j-1} = i$, if $t_{j-1} = \infty$ and $j \leqslant i$. If the last alternative holds then there are only j terms to right of p.

Observe that ∂E is periodic if either

1) for a vertex $p \in \partial E$ the double sequence $(\ldots, t_{-1}, t_0, t_1, \ldots)$ is periodic, where ∂E is described by $(\ldots, t_{-1}, \underline{p}, t_0, t_1, \ldots)$, or

2) the module is a translate of $\mathbf{Z}_+ \times \mathbf{Z}$ or $\mathbf{Z} \times \mathbf{Z}_+$.

In either case there is a fixed configuration $C(\overleftarrow{n}, \vec{0})$ (or $C(\overleftarrow{0}, \vec{n})$) and a $k \in \mathbf{Z}^2$ so that $\partial E = \bigcup_{l \in \mathbf{Z}} p + lk + C(\overleftarrow{n}, \vec{0})$, i.e., the boundary of E is the union of translates of a fixed segment.

EXAMPLE 2.14. 1) If $E = \mathbf{Z}_+^2$, then ∂E is described by $(\infty, \underline{0}, \infty)$.

2) If $E = \mathbf{Z}_+ \times \mathbf{Z}$, then ∂E is described by $(\infty, \underline{0}, 0, \infty)$.

3) If $E = (\mathbf{Z}_+ \times \mathbf{Z}) \cup (\mathbf{Z} \times \mathbf{Z}_+)$, then ∂E is described by $(\infty, 0, \underline{0}, 0, \infty)$. Finally,

4) if $E = \bigcup_{l \in \mathbf{Z}} (\mathbf{Z}_+^2 + l(-7,7)) \cup (\mathbf{Z}_+^2 + l(-7,7) + (-2,1))$, then ∂E is described by $(\ldots 1,5,6,2,1,\underline{0},5,6,2,1,5, \ldots)$. The boundary is periodic and the period of the sequence is 4.

DEFINITION 2.15. Let E be a $\mathbf{Z}_+^m$-module and let $p \in \partial E$. The point p has *infinite multiplicity* if given a natural number n there exists $q \in \partial E$, $q \neq p$, depending upon n, such that $D_n^E(p) \sim D_n^E(q)$. The point $p \in \partial E$ has *finite multiplicity* if it does not have infinite multiplicity.

REMARKS 2.16. 1) If ∂E is periodic, then the points of ∂E have infinite multiplicity.

2) If $p \in \partial E$ has infinite multiplicity then for each n there are infinitely many points $q \in \partial E$ so that $D_n^E(p) \sim D_n^E(q)$.

3) If $p \in \partial E$ has infinite multiplicity then each $q \in \partial E$ has infinite multiplicity.

LEMMA 2.17. *Let E be a $\mathbf{Z}_+^m$-module. Then:*

1) *The intersection of the ideal of compact operators $K(\ell^2(E))$ and $C^*(E)$ is $\{0\}$ if and only if the points of ∂E have infinite multiplicity.*

2) *$K(\ell^2(E)) \subset C^*(E)$ if and only if the points of ∂E have finite multiplicity. Moreover in case 2) every non-zero ideal of $C^*(E)$ contains $K(\ell^2(E))$.*

Proof. Let $p \in \partial E$ have infinite multiplicity and let $T \in K(\ell^2(E)) \cap C^*(E)$. Fix $\varepsilon > 0$ and let $P \in C^*(E)$ be a polynomial of degree n with $\|T - P\| < \varepsilon$. Let $f \in \ell^2(D_r^E(p))$, $\|f\| = 1$, and $\|P(f)\| > \|P\| - \varepsilon$. Since T is compact, it is the norm limit of the finite rank operators $T 1_{D_s^E(p)}$, $s \in \mathbf{N}$. Choose i so that $\|T 1_{E \setminus D_i^E(p)}\| < \varepsilon$. Using Remark 2.16.2 choose $q \in E$ with $D_{r+n}^E(p) \sim D_{r+n}^E(q)$ and $D_r^E(q) \cap D_i^E(p) = \emptyset$. Let W be $U^*((q-p)^-)((q-p)^+)$. From the fact that $\|P(f)\| = \|PW(f)\| \leqslant \|(T-P)W(f)\| + \|TW(f)\|$ it follows that $\|P\| < 3\varepsilon$ and so $\|T\| < 4\varepsilon$. Since ε is arbitrary this implies that $T = 0$.

If $p \in \partial E$ has finite multiplicity then, applying Lemma 2.6, $1_{\{p\}} \in C^*(E)$. It is also clear that ∂E cannot be periodic and so $C^*(E)$ is irreducible. From [1, Corollary 2 in page 18] it follows that $K(\ell^2(E)) \subset C^*(E)$.

The remaining implications follow from those already proved and Remark 2.16.3. If $K(\ell^2(E)) \subset C^*(E)$ and I is a non-zero ideal of $C^*(E)$ then an argument in [5, I, Theorem 1] proves that $K(\ell^2(E)) \subset I$. ▨

The next examples illustrate Lemma 2.17. In each case we need only verify that p has finite multiplicity.

EXAMPLES 2.18. Let p be in the $\mathbf{Z}_+^2$-module E and let ∂E be described by $(\ldots, t_{-2}, t_{-1}, \underline{p}, t_0, t_1, \ldots)$, $t_i \neq \infty$ for all i. Then:

1) If there exists an $l \in \mathbf{Z}_+$ such that $\sum_{i=k}^{k+l} t_i \to \infty$ and $\sum_{i=-k}^{-(k+l)} t_i \to \infty$ when $k \to \infty$ then $K(\ell^2(E)) \subset C^*(E)$.

2) If just $\sum_{-k=i}^{-(k+l)} t_i \to \infty$ when $k \to \infty$, but either there is an m so that $(t_m, t_{m+1}, \ldots)$ is periodic or ∂E is described by $(\ldots, t_{-1}, p, t_0, \ldots, t_m, \infty)$, then the same conclusion holds.

DEFINITION 2.19. The $\mathbf{Z}_+^m$-module E is *locally representable* in the $\mathbf{Z}_+^m$-module F if and only if for each $p \in \partial E$ and $n \in \mathbf{N}$ there is a $q \in \partial E$ such that $D_n^E(p) \sim D_n^F(q)$. This relation is denoted by $E \leqslant F$.

Notice that $\leqslant$ is a pre-order. It is easy to see that if $E \leqslant F \leqslant E$ and $p \in \partial E$ has finite multiplicity then E is a translate of F. We recall that $C^*(E)$ is algebraically equivalent to $C^*(F)$ if the map $U(n) \to U(n)$, $n \in \mathbf{Z}_+^m$, extends to a C^*-isomorphism between the two algebras.

THEOREM 2.20. *Let E and F be $\mathbf{Z}_+^m$-modules, then:*

1) *There exists a representation π from $C^*(F)$ onto $C^*(E)$ such that $\pi(U(n)) = U(n)$, $n \in \mathbf{Z}_+^m$, if and only if $E \leqslant F$.*

2) *The representation π in 1) is isometric,* i.e., *$C^*(E)$ and $C^*(F)$ are algebraically equivalent, if and only if $E \leqslant F$ and $F \leqslant E$.*

Proof. 1) Assume that $E \leqslant F$. For each polynomial $p(z_1, \ldots, z_m, \bar{z}_1, \ldots, \bar{z}_m)$ let $P \in C^*(E)$ and $\tilde{P} \in C^*(F)$ be the corresponding polynomials when p is evaluated in the canonical generators of $C^*(E)$ and $C^*(F)$ respectively. We shall show that the mapping $\tilde{P} \to P$ is contractive and therefore can be extended to all of $C^*(F)$. The conclusion will follow from [1, Theorem 1.3.2], and the fact that the polynomials are dense in $C^*(E)$ and $C^*(F)$. Let P and $\tilde{P}$ be polynomials as above of degree n. Let $f = \sum \lambda_r e_r$, $r \in Q$, Q a finite subset of E, and let $p \in \partial E$ and k be large enough so that $\{r - l \in E : r \in Q$ and $|l| \leqslant n\}$ is contained in $D_k^E(p)$. By hypothesis there exists $q \in \partial F$ with $D_k^E(p) \sim D_k^F(q)$. Let $g \in l^2(F)$ be equal to $\sum \lambda_r e_{r+q-p}$, $r \in Q$, so $\|f\| = \|g\|$. By Remark 2.2.2 $\|Pf\| = \|\tilde{P}g\|$, and since these f's are dense in $l^2(E)$ it follows that $\|P\| \leqslant \|\tilde{P}\|$. For the converse, use Lemma 2.6 to produce a polynomial $p(z_1, \ldots, z_m, \bar{z}_1, \ldots, \bar{z}_m)$ such that the corresponding $P \in C^*(E)$ is non-zero while $\tilde{P} \in C^*(F)$ is zero.

2) The proof just given shows that under the additional hypothesis that $F \leqslant E$, the restriction of the mapping to polynomials is isometric and therefore the extension to $C^*(E)$ and $C^*(F)$ is also isometric. ▨

The first part of the proof of the preceding theorem suggests the following easily verifiable fact.

PROPOSITION 2.21. *Let H and R be real Hilbert spaces. Let A be a subalgebra of $\mathscr{L}(H)$ and let φ be an additive map from A into $\mathscr{L}(R)$. Let P_w, $w \in W$, W a directed set, be orthogonal finite-dimensional projections that converge strongly to the identity $I \in \mathscr{L}(R)$. Assume that for every $T \in \varphi(A)$, for every $S \in \varphi^{-1}(T)$ and for every P_w there exists an orthogonal projection $Q_w \in \mathscr{L}(H)$ such that the operators $Q_w S Q_w$ and $P_w T P_w$ are unitarily equivalent. Then φ can be extended to a linear operator of norm at most 1 from A^- (norm closure of A) into $\varphi(A)^-$.*

DEFINITION 2.22. A $\mathbf{Z}_+^m$-module E_∞ is a *universal module* if for each $\mathbf{Z}_+^m$-module E there exists a representation π_E from $C^*(E_\infty)$ onto $C^*(E)$ with $\pi_E(U(n)) = U(n)$ for $n \in \mathbf{Z}_+^m$.

Theorem 2.20 guarantees the existence of universal $\mathbf{Z}_+^m$-modules. It is enough to construct a $\mathbf{Z}_+^m$-module E so that for each $n \in \mathbf{N}$ and each abstract n-diamond D_n

there is a $p \in E$ with $D_n + p = D_n^E(p)$. For $m = 2$ there are universal modules E_∞ and F_∞ whose boundaries are described by $(\ldots, t_{-1}, \underline{p}, t_0, \ldots)$, with all $t_i \neq \infty$, and $(\infty, \underline{p}, s_0, s_1, \ldots)$ respectively. Therefore $E_\infty \leqslant F_\infty$ and $F_\infty \leqslant E_\infty$ but E_∞ is neither a rotation nor a translate of F_∞, i.e., local representability is really a local property.

3. GROUPOIDS

For the basic properties of groupoid C^*-algebras the reader is refered to [17, Section 2], from which we have adopted the terminology, or to [20]. The way in which $C^*(E)$ is realized as a groupoid C^*-algebra is almost exactly the same as the way in which the C^*-algebra of Wiener-Hopf operators is so realized in [17]. Although our modules do not satisfy 3.1.i in [17], this is balanced by the fact that $\mathbf{Z}^m$ has a nice structure.

Let E be a $\mathbf{Z}^m_+$-module and let $W(n) \in \mathscr{L}(\ell^2(\mathbf{Z}^m))$ be defined by $W(n)e_p = e_{p+n}$, for all $p, n \in \mathbf{Z}^m$. Thus $\{W(n) : n \in \mathbf{Z}^m\}$ is the minimal unitary extension of our semigroup of isometries $\{U(n) \in \mathscr{L}(\ell^2(E)) : n \in \mathbf{Z}^m_+\}$. Let $P \in \mathscr{L}(\ell^2(\mathbf{Z}^m))$ be the orthogonal projection onto $\ell^2(E)$. The C^*-algebra generated by $\{W(n)PW(-n) = P(n) : n \in \mathbf{Z}^m\}$, $C^*(P(n))$, is commutative. Let Y be its maximal ideal space. Then Y is Hausdorff and locally compact and, by Gelfand's theorem, $C_0(Y)$ (the complex-valued continuous functions that vanish at infinity) is isometrically isomorphic to $C^*(P(n))$. The group $\mathbf{Z}^m$ acts on Y naturally: for $y \in Y$, $y + n$ is defined by the formula $\hat{f}(y + n) = W(-n)fW(n)\,\hat{}\,(y)$, for all $\hat{f} \in C_0(Y)$, $n \in \mathbf{Z}^m$. (As usual $\hat{}$ denotes the Gelfand transform.) It is immediate that the action of $\mathbf{Z}^m$ on Y is continuous and so $(Y, \mathbf{Z}^m)$ is a transformation group. The set $Y \times \mathbf{Z}^m$ becomes a groupoid when we define $\{[(y, n), (x, m)] : x = y + n\}$ as the set of composable pairs, defining the product of such a pair to be $(y, n + m)$, and when we define the involution $(y, n)^{-1}$ to be $(y + n, -n)$. The maps d and r (domain and range, respectively) satisfy the equations $\mathrm{d}(x, n) = (x + n, -n)(x, n) = (x + n, 0)$ and $\mathrm{r}(x, n) = (x, n)(x + n, -n) = (x, 0)$. Thus the unit space of $Y \times \mathbf{Z}^m$, i.e., $\{\mathrm{d}(y, n) : (y, n) \in Y \times \mathbf{Z}^m\} = \{\mathrm{r}(y, n) : (y, n) \in Y \times \mathbf{Z}^m\}$, may be identified with Y.

Let X be $\{y \in Y : \hat{P}(y) = 1\}$ and observe that $C(X)$ is isomorphic to the C^*-algebra generated by $\{P(n)P|E : n \in \mathbf{Z}^m\} = \{1_{E \cap (E+n)} : n \in \mathbf{Z}^m\}$. By Corollary 2.7 $C(X)$ is isomorphic to the diagonal subalgebra of $C^*(E)$. The set X is compact because $1 = \hat{1}_E \in C(X)$. Therefore Y is σ-compact since $Y = \cup\,(X + n), n \in \mathbf{Z}^m$. The family of clopen sets $\{X + n : n \in \mathbf{Z}^m\}$ form a subbasis for the topology of Y. Thus Y is metrizable. The set $\mathbf{Z}^m$ may be seen as dense in Y. To be precise, define a map from $\mathbf{Z}^m$ into Y, $p \to y_p$, by the formula $\hat{f}(y_p) = \langle fe_p, e_p\rangle$ for all $\hat{f} \in C_0(Y)$. It is clear that $\{y_p : p \in \mathbf{Z}^m\}$ separates functions in $C_0(Y)$ and therefore is dense. Likewise $\{y_p : p \in E\}$ is dense in X. Since $y_p + n = y_{p+n}$ for all $p, n \in \mathbf{Z}^m$, the set

$\{y_p : p \in \mathbf{Z}^m\}$ is an orbit in Y. The points y_p and y_q are equal if and only if $\langle P(n)e_p, e_p\rangle = \langle P(n)e_q, e_q\rangle$ for all $n \in \mathbf{Z}^m$, in other words, if and only if $p - n$ and $q - n$ belong simultaneously to E or to $\mathbf{Z}^m \setminus E$ for all $n \in \mathbf{Z}^m$. When $p, q \in E$ this is equivalent to the condition $D_k^E(p) \sim D_k^E(q)$ for all $k \in \mathbf{N}$. Thus the map $p \to y_p$ is one-to-one if and only if ∂E is not periodic. The sequence $\{y_{p(l)} : l \in \mathbf{N}\}$ is convergent if and only if there exists an $n \in \mathbf{Z}^m$ so that the sequence is eventually in $E + n$ and, for each $k \in \mathbf{N}$, $D_k^{E+n}(p(l)) = D_k^{E+n}(p(s))$ whenever l, s are large enough. As a consequence we see that if we fix an $x \in X$, the set $\{n \in \mathbf{Z}^m : x + n \in X\}$ is a $\mathbf{Z}_+^m$-module.

The groupoid in which we are interested is the reduction (or contraction) of $Y \times \mathbf{Z}^m$ to X, i.e., $\beta = Y \times \mathbf{Z}^m | X = \{(x, n) \in Y \times \mathbf{Z}^m : x \in X \text{ and } x + n \in X\}$. The unit space of β may be identified with X. Let $C_c(\beta)$ be the space of compactly supported, continuous, complex-valued functions defined on β. Let $C_c(\beta)$ be endowed with the inductive limit topology. A Haar system $\{\lambda^u : u \in X\}$ on β is obtained if we define λ^u as counting measure on $\{(x, n) \in \beta : x = u\}$. A multiplication on $C_c(\beta)$ is defined by $h * g(x, l) = \sum_{s \in \mathbf{Z}^m} h(x, l + s)g(x + l + s, s)1_X(x + l + s)$, and an involution is defined by $h^*(x, n) = \overline{h(x + n, -n)}$, for all $h, g \in C_c(\beta)$ and $(x, n) \in \beta$. With these operations $C_c(\beta)$ becomes a topological $*$-algebra. Equipped with the norm

$$\|f\|_I = \max\left\{\sup_{u \in X} \int |f(x, n)| \, d\lambda^u(x, n), \sup_{u \in X} \int |f(x + n, -n)| \, d\lambda^u(x, n)\right\}, \quad f \in C_c(\beta),$$

$C_c(\beta)$ is a normed $*$-algebra with completion $L^1(\beta)$. The enveloping C^*-algebra of $L^1(\beta)$ is denoted by $C^*(\beta)$. Let μ be a positive Radon measure on X. Then μ induces two measures ν and ν^{-1} on according to these formulas:

$$\int f \, d\nu = \int_X \int f(x, n) \, d\lambda^u(x, n) \, d\mu(u)$$

and

$$\int f(x, n) \, d\nu^{-1}(x, n) = \int f(x + n, -n) \, d\nu(x, n),$$

for all $f \in C_c(\beta)$ [17, 2.11]. The Hilbert space $L^2(\nu^{-1})$ carries a representation of $C^*(\beta)$ which is called the representation induced off the unit space by μ and is denoted by Ind_μ. It is defined by the formula

$$\mathrm{Ind}_\mu(f)\xi(x, l) = \sum_{s \in \mathbf{Z}^m} f(x, l + s) \, g(x + l + s, -s) \, 1_X(x + l + s),$$

for $f \in C_c(\beta)$, $\xi \in L^2(\nu^{-1})$. Since $\|\mathrm{Ind}_\mu(f)\| \leq \|f\|_I$, $\mathrm{Ind}_\mu(f)$ extends to all $C^*(\beta)$ [17, 2.12].

We have paved the way then for the main result of this section.

LEMMA 3.1. *Let E be a* Z_+^m*-module. The* C^**-algebra* $C^*(E)$ *may be realized as the groupoid* C^**-algebra* $C^*(\beta)$*, where* $C^*(\beta)$ *is the enveloping* C^**-algebra of the groupoid* β *defined above.*

Proof. Let μ be a positive Radon measure on X (X the unit space of β) such that the minimal invariant (under the equivalence relation on X defined by $x \sim y$ if and only if $x+n=y$ for some $n \in \mathbf{Z}^m$) set that contains $\operatorname{supp}\mu$ is dense in X. Then, according to [17, Proposition 2.17], the induced representation Ind_μ faithfully represents $C^*(\beta)$ on $L^2(\nu^{-1})$, where ν is the measure induced by μ. Assume, without loss of generality, that $0 \in E$, and let δ_0 be the measure with mass 1 concentrated in y_0. Since $\{y_p : p \in E\}$ is a dense orbit, and hence invariant, we find that Ind_{δ_0} is faithful. The corresponding $L^2(\nu^{-1})$ is $\{\xi : \beta \to C : \sum_{n\in E} |\xi(y_n, -n)|^2 < \infty\}$. The Hilbert spaces $L^2(\nu^{-1})$ and $\ell^2(E)$ may be identified via the unitary operator V, from $L^2(\nu^{-1})$ onto $\ell^2(E)$, defined by $(V\xi)(n) = \xi(y_n, -n)$.

Let $f_{\varepsilon_i} \in C_c(\beta)$, $i \in \{1, \dots, m\}$, be defined by the formula

$$f_{\varepsilon_i}(x, n) = \begin{cases} 1 & \text{if } n = -\varepsilon_i, \\ 0 & \text{otherwise.} \end{cases}$$

It is not difficult to check that these functions generate $C_c(\beta)$ as a topological $*$-algebra. To conclude the proof we observe that $V\mathrm{Ind}_{\delta_0}(f_{\varepsilon_i})V^{-1} = U(\varepsilon_i)$, where $U(\varepsilon_i)$ are the canonical generators of $C^*(E)$. ▨

REMARKS 3.2. 1) When E is a $\mathbf{Z}_+^1$-module there are only two cases. If $E = \mathbf{Z}$ then $C^*(\mathbf{Z})$ is the C^*-algebra generated by a bilateral shift. The space Y consists of a single point. If $E = \mathbf{Z}_+$ then $C^*(\mathbf{Z}_+)$ is the C^*-algebra generated by a unilateral shift. The space Y may be identified with $\mathbf{Z} \cup \{\infty\}$, where a subbasis for the topology of Y is given by the intervals $\{(n, m)$ and $(n, \infty] : n, m \in \mathbf{Z}\}$. The action of $\mathbf{Z}$ on Y is the usual addition on $\mathbf{Z}$ and ∞ is fixed, i.e., the isotropy group of ∞ is $\mathbf{Z}$. The space X may be identified with $\mathbf{Z}_+ \cup \{\infty\}$.

2) In Lemma 3.1 the subalgebra $\{f \in C_c(\beta) : f(x, n) = 0 \text{ if } n \neq 0\}$ of $C_c(\beta)$ may be identified with $C(X)$.

3) The method used in the proof of the above lemma will be applied again in the first step of Theorem 4.4.

We keep the notation used so far.

PROPOSITION 3.3. *Let E be a* $\mathbf{Z}_+^m$*-module and let X be the maximal ideal space of the diagonal subalgebra of* $C^*(E)$*. The topological space X may be seen as the closure of* $\{y_p : p \in E\}$ *in the metric* ρ *defined by*

$$\rho(y_p, y_q) = \begin{cases} 2^{-k} & \textit{if } k \textit{ is the first natural number so that } D_k^E(p) \nsim D_k^E(q), \\ 0 & \textit{otherwise.} \end{cases}$$

Moreover, if $C^*(E)$ is irreducible then the map $p \to y_p$ is one-to-one and $\{y_p : p \in E\}$ is open in X if and only if $K(\ell^2(E)) \subset C^*(E)$.

Proof. The first part of the proposition is clear from what we already know about X.

If $K(\ell^2(E)) \subset C^*(E)$ then $1_{\{p\}} \in C^*(E)$ and therefore the singleton $\{y_p\} = \{x \in X : \hat{1}_{\{p\}}(x) = 1\}$ is an open set. If $C^*(E)$ is irreducible but $K(\ell^2(E)) \cap C^*(E) = (0)$, then we shall see that $\{y_p : p \in E\}$ is dense in itself. Thus X is a perfect metric space. Since $\{y_p : p \in E\}$ is an infinite denumerable dense subset of X, X has the power of the continuum. Each non-empty ball of X has this same cardinality, and so $\{y_p : p \in E\}$ is not open.

To see that $\{y_p : p \in E\}$ is dense in itself take $p \in E$ and $k \in \mathbf{N}$, since $1_{\{p\}} \notin C^*(E)$ the set $\{q \in E : D_k^E(p) \sim D_k^E(q)\}$ consists of more than one point. Choosing $q \neq p$ in it, we have $\rho(y_p, y_q) < 2^{-k}$. On the other hand there is a $j > k$ such that $D_j^E(p) \nsim D_j^E(q)$ and so $y_p \neq y_q$. (If $D_j^E(p) \sim D_j^E(q)$ for all j, then $E = E + p - q$ and $C^*(E)$ would be reducible by Proposition 2.10.) ▨

For a $\mathbf{Z}_+^2$-module E with $C^*(E)$ reducible we still have that $\{y_p : p \in E\}$ is open in X. In general it could be either way. As an example, let F be a $\mathbf{Z}_+^2$-module and let $\tilde{F} = \bigcup_{z \in \mathbf{Z}} F \times \{z\}$. Thus $C^*(\tilde{F})$ is reducible. The sets $\{y_q : q \in \tilde{F}\}$ and $\{y_p : p \in F\}$ may be identified via the map $y_{p \times \{z\}} \to y_p$, all $p \in F$, $z \in \mathbf{Z}$. Likewise the corresponding $\tilde{X}$ and X, Y and $\tilde{Y}$ may be identified as topological spaces. Observe, however, that if the isotropy group of y_p, $p \in \mathbf{Z}^2$, is G, then the isotropy group of $y_{p \times \{z\}}$ is $G \times \mathbf{Z}$.

Let E be a module. Let the groupoid β, with unit space X and Haar system $\{\lambda^u : u \in X\}$, be as before. In the sequel we shall use [20, Chapter II, Proposition 4.4] combined with [17, 2.15]. That is, the map $V \to \bar{I}_V$ is a one-to-one order preserving map from the lattice of open invariant subsets of X into the two-sided ideals in $C^*(\beta)$. For each such set V, $\bar{I}_V$ is canonically isomorphic to $C^*(\beta/V)$ and the quotient $C^*(\beta)|I_V$ is canonically isomorphic to $C^*(\beta/F)$ where $F = X \setminus V$. (The Haar systems on β/V and β/F are $1_{\beta/V}\,\lambda$ and $1_{\beta/F}\,\lambda$ respectively. Recall that $C^*(\beta)$ is isomorphic to $C^*(E)$.) We shall identify some subsets of $\mathbf{Z}^m$ with open sets of X.

EXAMPLES 3.4. 1) Let E be $\mathbf{Z}_+ \times \mathbf{Z}$. The unit space X is $\mathbf{Z}_+ \cup \{\infty\}$. The open invariant set $V = Z_+$ corresponds to the ideal $\bar{I}_V$ in $C^*(E)$ generated by $I - U(\varepsilon_1)U^*(\varepsilon_1)$. The quotient $C^*(E)|\bar{I}_V$ is isomorphic to $C(\pi \times \pi)$ (the continuous functions on the torus). Thus the length of the canonical composition series [1, Theorem 1.5.5] is 2.

2) Let E be $\mathbf{Z}_+ \times \mathbf{Z}_+$. To the open invariant set $\mathbf{Z}_+ \times \mathbf{Z}_+ = V$ of X it corresponds the ideal $\bar{I}_V = K(\ell^2(E))$. The quotient $C^*(E)|K(\ell^2(E))$ is isomorphic to

$C^*((\mathbf{Z}_+ \times \mathbf{Z}) \oplus (\mathbf{Z} \times \mathbf{Z}_+))$. The C^*-algebra $C^*(E)$ was first analyzed by Douglas and Howe in [12]; and from the groupoid point of view by Muhly and Renault in [17].

3) Likewise for $F = (\mathbf{Z}_+ \times \mathbf{Z}) \cup (\mathbf{Z} \times \mathbf{Z}_+)$, the quotient $C^*(F)/K(\ell^2(F))$ is isomorphic to $C^*((\mathbf{Z}_+ \times \mathbf{Z}) \oplus (\mathbf{Z} \times \mathbf{Z}_+))$. Thus the canonical composition series of $C^*(F)$ (and $C^*(\mathbf{Z}_+ \times \mathbf{Z}_+)$) has length 3. In fact, for every module F with ∂F not periodic and with a finite number of vertices the length of the canonical composition series is 3.

4) Let θ be a $\mathbf{Z}_+^2$-module whose boundary is described by a sequence $(\ldots, t_{-1}, \underline{p}, t_0, t_1, \ldots)$ with $t_i \neq \infty$ and $\lim_{|i| \to \infty} t_i = \infty$. In Example 2.18.1 we saw that $K(\ell^2(\theta)) \subset C^*(\theta)$. The quotient $C^*(\theta)/K(\ell^2(\theta))$ is isomorphic to $C^*(E \oplus F)$, where E and F are the modules in 2) and 3). The canonical composition series of $C^*(\theta)$ has length 4.

A complete description of the unit spaces corresponding to several $\mathbf{Z}_+^2$-modules is done in [21]. Among them is the module $E_\alpha = \{(m, n) \in \mathbf{Z}^2 : m + n\alpha \geqslant 0\}$, α irrational. The semigroup $\Gamma_\alpha^+ = \{m + n\alpha \geqslant 0\}$ determines a C^*-algebra isomorphic to $C^*(E_\alpha)$ (see Douglas [11, page 148]). The groupoid approach is an observation of Muhly and Renault.

PROPOSITION 3.5. *Let j be a natural number. There exists a $\mathbf{Z}_+^2$-module E_j such that $C^*(E_j)$ has its canonical composition series of length j, so, in particular, $C^*(E_j)$ is* G.C.R. . *Moreover, if $j \geqslant 5$ then $C^*(E_j)/K(\ell^2(E_j))$ is isomorphic to $C^*(E_{j-1})$.*

Proof. For $j = 1$ let $E_1 = \mathbf{Z}^2$. Examples 3.3 provide us with modules E_2, E_3, and E_4. Let $j \geqslant 5$ and assume that we have E_{j-1} with the required properties. Assume also that ∂E_{j-1} is described by a sequence $(\ldots, t_{-1}, \underline{p}, t_0, \ldots)$ with t_0 and t_{-1} nonzero and $t_i \neq \infty$ for all $i \in \mathbf{Z}$. Since $E_4 \leqslant E_{j-1}$, ∂E_{j-1} has horizontal and vertical segments of large length. Moreover, there are increasing sequences, $\{s_i : i \in \mathbf{N}\}$ and $\{r_i : i \in \mathbf{N}\}$, of odd and even numbers respectively, such that $\lim_{i\to\infty} t_{-r_i} = \lim_{i\to\infty} t_{s_i} = \infty$. We shall construct E_j with the property that E_j be open in the corresponding unit space X_j; and $X \setminus E_j$ may be topologically and algebraically identified with the unit space X_{j-1}. This will be obtained if $E_{j-1} \leqslant E_j$ and, if $F \leqslant E_j$ but $E_j \nleqslant F$ then $F \leqslant E_{j-1}$. We ask ∂E_j to satisfy that:

1) ∂E_j has a segment that is not a translate of any segment of ∂E_{j-1}, say $C(\vec{n}, \vec{m})$ with $\vec{n} = (h_{-l}, \ldots, h_{-1})$ and $\vec{m} = (h_0, \ldots, h_{l-1})$.

2) ∂E_j is the union of this segment and translate of infinite nested segments of ∂E_{j-1} whose union is ∂E_{j-1}. The segments that glue together the translates of two segments of ∂E_{j-1} are segments of $\mathbf{Z}_+ \times \mathbf{Z}_+$. (So no new segments are added, since $\mathbf{Z}_+ \times \mathbf{Z}_+ \leqslant E_{j-1}$.)

Let l in 1) be even and define $l_1 = l$, while $l_i = \left(\sum_{k=1}^{i-1} r_k - s_k\right) + l + i - 1$ if $i > 1$. Let $\{h_z : z \in \mathbf{Z}\}$ be a sequence such that:

i) h_z is as in 1) for $-l \leqslant z < l$,

ii) $h_{l_i+d} = t_{-r_i+d}$ for $0 \leqslant d \leqslant r_i + s_i$ and $i \geqslant 1$,

iii) $h_{-l_{i-1}-d} = t_{-r_i-d}$ for $0 \leqslant d \leqslant r_i + s_i$ and $i \geqslant 1$.

Define E_j with ∂E_j described by $(\ldots, h_{-1}, \underline{0}, h_0, h_1, \ldots)$. Applying [20, Chapter II, Proposition 4.4] we complete the proof. ▨

If $m > 2$, then there are $\mathbf{Z}_+^m$-modules $\tilde{E}_j$ whose canonical composition series has length j. Simply define $\tilde{E}_j = E_j \times \mathbf{Z}^{m-2}$, where E_j are the modules of the above proposition. Then $C^*(\tilde{E}_j)$ is isomorphic to $C^*(E_j) \otimes C^*(\mathbf{Z}^{m-2})$.

4. THE MAIN THEOREM

Let $\{V(n) : n \in \mathbf{Z}_+^m\}$ be a semigroup of isometries acting on the Hilbert space H. Let $\{W(n) : n \in \mathbf{Z}^m\}$ be its minimal unitary extension acting on the Hilbert space K. That is, for each $n \in \mathbf{Z}_+^m$, H is invariant under $W(n)$ and there is no proper subspace of K that reduces $\{W(n) : n \in \mathbf{Z}_+^m\}$. The existence of the minimal unitary extension is a classical result due to Îto and Brehmer (see [10] for references and additional material). Let $P \in \mathscr{L}(K)$ be the orthogonal projection onto H and let $\{P(n) = W(n)PW(-n): n \in \mathbf{Z}^m\}$. If we know that $\{P(n): n \in \mathbf{Z}^m\}$ is a commuting family of projections, then we could proceed to build a groupoid as in the discussion previous to Lemma 3.1. For this reason we prove:

PROPOSITION 4.1. *The family of projections $\{P(n): n \in \mathbf{Z}^m\}$ is commutative if and only if the ranges of the semigroup of isometries $\{V(n) : n \in \mathbf{Z}_+^m\}$, $\{V(n)V^*(n) : n \in \mathbf{Z}_+^m\}$, form a commutative family.*

Proof. Since, for $n \in \mathbf{Z}_+^m$, $V(n) = PW(n)P|H$ it follows $V(n)V^*(n) = P\,P(n)P|H$. Moreover, $P(n) \leqslant P$ because $W(-n)H^\perp \subset H^\perp$ and therefore $V(n)V^*(n) = P(n)|H$. Thus $\{V(n)V^*(n) : n \in \mathbf{Z}_+^m\}$ is a commuting family if and only if $\{P(n) : n \in \mathbf{Z}_+^m\}$ is commutative.

From the definition $P(k) = W(-k^-)P(k^+)W(k^-)$. Computing $P(k)P(h)$ and assuming that $\{P(n) : n \in \mathbf{Z}_+^m\}$ is commutative we obtain:

$$P(k)P(h) = W(-k^-)P(k^+)W(k^-)W(-h^-)P(h^+)W(h^-) =$$
$$= W(-k^-)W(-h^-)P(k^+ + h^-)P(h^+ + k^-)W(h^-)W(k^-) =$$
$$= W(-k^-)W(-h^-)P(h^+ + k^-)P(k^+ + h^-)W(h^-)W(k^-) = P(h)P(k).$$ ▨

DEFINITION 4.2. Let $\{V(n): n \in \mathbf{Z}_+^m\}$ be a semigroup of isometries on H, with commuting range projections. The C^*-algebra generated by

$$\{P(n)P|H: n \in \mathbf{Z}^m\} = \{V^*(n^-)V(n^+)V^*(n^+)V(n^-): n \in \mathbf{Z}^m\},$$

$C^*(P(n)P|H)$ is called the *diagonal subalgebra* of $C^*(V(n))$.

REMARK 4.3. Notice that the diagonal subalgebra contains all the monomials $V^*(n_{2k})V(n_{2k-1}) \dots V^*(n_2)V(n_1)$, $n_j \in \mathbf{Z}_+^m$ for all $j \in \{1, \dots, 2k\}$, such that $\sum_{i=1}^{k} n_{2i-1} - \sum_{i=1}^{k} n_{2i} = 0$. In fact

$$V^*(n_{2k})V(n_{2k-1}) \dots V^*(n_2)V(n_1) =$$

$$= PW(-n_{2k})\, PW(n_{2k-1})P \dots PW(n_1)P|H =$$

$$= PW(-n_{2k})PW(n_{2k})W(n_{2k-1} - n_{2k})P \dots PW(n_1)P|H =$$

$$= P\, P(-n_{2k})P(n_{2k-1} - n_{2k}) \dots P\left(\sum_{i=0}^{k-2} [n_{2(k-i)-1} - n_{2(k-i)}] - n_2\right) \cdot$$

$$\cdot V^*\left(\sum_{i=1}^{k} n_{2i}\right)V\left(\sum_{i=1}^{k} n_{2i-1}\right).$$

$\left(\text{So, if } \sum_{i=1}^{k} n_{2i} = \sum_{i=1}^{k} n_{2i-1}, \text{ then } V^*\left(\sum_{i=1}^{k} n_{2i}\right)V\left(\sum_{i=1}^{k} n_{2i}\right) = I.\right)$ Thus each monomial is a partial isometry.

We are prepared to prove our basic structure theorem. The proof is divided in four steps. The first step is, using a technique similar to the one used in Lemma 3.1, to associate to $C^*(V(n))$ a groupoid β. The enveloping C^*-algebra $C^*(\beta)$ turns out to be isomorphic, for suitable $\mathbf{Z}_+^m$-modules E_i, $i \in J$, to $C^*(\bigoplus_{i \in J} E_i)$. The second step is to show that the map $U(n) \to V(n)$, $n \in \mathbf{Z}_+^m$, extends to a representation π from $C^*(\bigoplus_{i \in J} E_i)$ onto $C^*(V(n))$. In the third step the full hypotheses are used to prove that π is faithful. Finally, in the last step, we construct a module E with the properties:

i) $E_j \leqslant E$ for all $j \in J$ and

(ii) for each $p \in E$ and $k \in \mathbf{N}$ there is a module E_j and a point $q \in E_j$ so that $D_k^E(p) \sim D_k^{E_j}(q)$.

THEOREM 4.4. *Let* $\{V(n): n \in \mathbf{Z}_+^m\}$ *be a semigroup of isometries acting on* H *and assume that* $\{V(n)V^*(n): n \in \mathbf{Z}_+^m\}$ *is a commuting family. Then:*

1) *There exists a countable collection of* $\mathbf{Z}_+^m$*-modules* $\{E_j: j \in J\}$, *and a representation* π *from the* C^**-algebra* $C^*(\bigoplus_{j\in J} E_j)$, *acting on* $\bigoplus_{j\in J} l^2(E_j)$, *onto the* C^**-algebra generated by* $\{V(n): n \in \mathbf{Z}_+^m\}$, $C^*(V(n))$.

2) *If* $C^*(V(n))$ *is irreducible and if* $V^*(k^-)V(k^+)$ *is not a multiple of the identity for all non-zero* $k \in \mathbf{Z}^m$, *then the representation* π *is faithful. Moreover, in this case it is possible to choose just one* $\mathbf{Z}_+^m$*-module* E *such that* $C^*(E)$ *is algebraically equivalent to* $C^*(V(n))$.

2′) *When* $m = 2$, *the conditions in* 2) *could be replaced by the equivalent conditions:* $C^*(V(n))$ *is irreducible and either* $I - V(\varepsilon_1)V^*(\varepsilon_1)$ *or* $I - V(\varepsilon_2)V^*(\varepsilon_2)$ *has infinite rank.*

Proof. Step 1. Let $\{W(n): n \in \mathbf{Z}^m\}$ be the minimal unitary extension of $\{V(n): n \in \mathbf{Z}_+^m\}$ acting on a Hilbert space K. Let P denote the orthogonal projection of K onto H. By Proposition 4.1, the family $\{P(n) = W(n)PW(-n): n \in \mathbf{Z}^m\}$ is commutative. Let $C^*(P(n))$ be the C^*-algebra generated by $\{P(n): n \in \mathbf{Z}^m\}$ and let Y be the maximal ideal space of $C^*(P(n))$. The group $\mathbf{Z}^m$ acts continuously on Y if we define $y + n$, $y \in Y$ and $n \in \mathbf{Z}^m$, by the formula $\hat{f}(y + n) = W(-n)fW(n)(y)$, for all $\hat{f} \in C_0(Y)$ where ^ is, as usual, the Gelfand transform. If $X = \{y \in Y: \hat{P}(y) = 1\}$, then X is a compact set and $C(X)$ is isomorphic to the C^*-algebra generated by $\{P(n)P|H: n \in \mathbf{Z}^m\}$, i.e., the diagonal subalgebra of $C^*(V(n))$. Form the groupoid $\beta = Y \times \mathbf{Z}^m|X = \{(x, n) \in Y \times \mathbf{Z}^m: x \in X, x + n \in X\}$. The set of composable pairs is $\{[(x, n), (y, l)]: x + n = y\}$ and a composition of such a pair is $(x, n + l)$, the involution is given by $(x, n)^{-1} = (x + n, -n)$. The unit space of β is identified with X. A Haar system on β is $\{\lambda^u: u \in X\}$ where λ^u is counting measure on $\{(x, n): x = u\}$. The space $C_c(\beta)$, equipped with the inductive limit topology, is a topological $*$-algebra when a multiplication is defined by

$$h * g(x, n) = \sum_{s\in\mathbf{Z}^m} h(x, n + s)g(x + n + s, -s)1_X(x+n+s)$$

and an involution is defined by $h^*(x, n) = \overline{h(x+n, n)}$. Thus the subalgebra of $C_c(\beta)$, $\{f: f(x, n) = 0 \text{ if } n \neq 0\}$ is commutative and may be identified with $C(X)$. The functions on $C_c(\beta)$ are finite sum of functions b_k defined by: $b_k(x, n) = 0$ if $n \neq -k$ and $b_k(x, -k) = b(x)$, where b is some function of $C(X)$. (We may assume that b is zero outside $X \cap (X + k)$.) Let $f_k \in C_c(\beta)$ be defined by

$$f_k(x, n) = \begin{cases} 1 & \text{if } n = -k, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Let

$$\tilde{b}(x, n) = \begin{cases} b(x) & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

It is easy to verify that:

i) $f_k * f_l^* = f_{k-l}$ if $l \in \mathbf{Z}_-^m$ and $k \in \mathbf{Z}^m$,

ii) $b_k = \tilde{b} * f_k$, where b_k, b and $\tilde{b}$ are as above. As a consequence it follows that $f_{\varepsilon_1}, \ldots, f_{\varepsilon_m}$ generates $C_c(\beta)$.

Let $C^*(\beta)$ be the enveloping C^*-algebra of β (with respect to the Haar system $\{\lambda^u : u \in X\}$). In order to find a faithful induced representation of $C^*(\beta)$, we observe that X is a Hausdorff compact, second countable space and thus is metrizable and second countable. Let $\{y_k : k \in I\}$ be dense in X (I a countable set) and from each orbit $O(y_k) = \{y \in X : y = y_k - n$ for some $n \in Z^m\}$ choose a point x_j. Therefore $\{x_j : j \in J\}$ satisfies:

i) x_k and x_l are not in the same orbit if $k \neq l$,

ii) $\bigcup_{j \in J} O(x_j)$ is dense in X.

Let μ be the positive Radon measure on X that is concentrated in $\bigcup_{k \in J} \{x_k\}$ and $\mu(\{x_k\}) = k^{-2}$. The measure ν induced by μ satisfies

$$\int_\beta h \mathrm{d}\nu = \int_X \int_\beta h(x, n)\, \lambda^u(x, n) \mathrm{d}\mu(u) =$$

$$= \int_X \sum_{\{n:(u,n) \in \beta\}} h(u, n)\, \mathrm{d}\mu(u) = \sum_{k \in J} \sum_{\{n:(x_k, n) \in \beta\}} h(x_k, n),$$

for all $h \in C_c(\beta)$. The measure ν^{-1} is defined by

$$\int_\beta h(x + n, -n) \mathrm{d}\, \nu(x, n) = \sum_{k \in J} \sum_{\{n:(x_k, n) \in \beta\}} h(x_k + n, -n) k^{-2}.$$

(See [17, Section 2].) Then the representation induced off μ, Ind_μ, is faithful on $L^2(\nu^{-1})$ [17, Proposition 2.15 and 2.16]. For each $x \in X$ the subset of $\mathbf{Z}^m$, $\{n : x + n \in X\}$, is a module. (Since $P \leqslant P(-l)$ for $l \in Z_+^m$ it follows that $\hat{P} \leqslant \widehat{P(-l)}$ and therefore for $y \in X$, $\hat{P}(y + l) = \hat{P}(-l)(y) = 1$, i.e., $y + l \in X$.) Let E_k denote the module $\{n : x_k + n \in X\}$, $k \in J$. We can write $L^2(\nu^{-1}) = \{\xi : \beta \to C : \sum_{k \in J} \sum_{n \in E_k} |\xi(x_k + n, -n)|^2 < \infty\}$. For each $\xi \in L^2(\nu^{-1})$ let $V_k \xi$ be defined by $V_k \xi(n) = k^{-1} \xi(x_k + n, -n)$ for $n \in E_k$. Let V be a unitary operator from $L^2(\nu^{-1})$ onto $\bigoplus_{k \in J} \ell^2(E_k)$ defined by $V\xi = \bigoplus_{k \in J} V_k \xi$, $\xi \in L^2(\nu^{-1})$. A computation shows that $V \mathrm{Ind}_\mu(f_{\varepsilon_i}) V^{-1} = U(\varepsilon_i)$ for f_{ε_i} defined as in (1) and $i \in \{1, \ldots, m\}$. Thus $V \mathrm{Ind}_\mu V^{-1}$ is a faithful representation of $C^*(\beta)$ onto $C^*(\bigoplus_{k \in J} E_k)$. (Recall that $U(\varepsilon_i)$ are the canonical generators of $C^*(\bigoplus_{k \in J} E_k)$ and $\{f_{\varepsilon_i} : i \in \{1, \ldots, m\}\}$ generates $C_c(\beta)$.)

Step 2. We show that if we define $\pi(f_{\varepsilon_i}) = V(\varepsilon_i)$, $i \in \{1, \ldots, m\}$, then π extends to a representation of $C^*(\beta)$ onto $C^*(V(n))$. Observe that, by construction, π extends to a faithful representation of $\{f \in C_c(\beta) : f(x, n) = 0 \text{ if } n \neq 0\}$ onto the diagonal subalgebra of $C^*(V(n))$.

Since $\mathbf{Z}^m$ is discrete, X is an open set of β (to be precise $X \times \{0\}$ is open in β). Thus β is an r-discrete groupoid in the sense of [20], so, by [20, Chapter I, Example 3.28 b; and Chapter II, Theorem 1.21 and Corollary 1.22] we need only verify that π restricted to $C_c(\beta)$ is a $*$-representation. We recall that π is a representation in this sense if π is $*$-homomorphism of the involutive algebra $C_c(\beta)$ and it is continuous when $C_c(\beta)$ has the inductive limit topology while $C^*(V(n))$ has the weak operator topology.

First we show that the map π is a homomorphism from the involutive algebra {polynomials in $f_{\varepsilon_i}, f^*_{\varepsilon_i}$, $i \in \{1, \ldots, m\}$} onto the involutive algebra {polynomials in $V(\varepsilon_i)$, $V^*(\varepsilon_i)$, $i \in \{1, \ldots, m\}$} (no topology yet). It suffices to verify that a polynomial evaluated in $V(\varepsilon_i)$, $V^*(\varepsilon_i)$ is zero whenever it yields zero when evaluated in $f_{\varepsilon_i}, f^*_{\varepsilon_i}$.

Using Remark 4.3 for monomials in $C^*(V(n))$ and $C^*(\bigoplus_{j \in J} E_j)$ and that $V\mathrm{Ind}_\mu V^{-1}$ is faithful we obtain that, for $\{n_i : i \in \{1, \ldots, 2k\}\} \subset \mathbf{Z}^m_+, f^*_{n_{2k}} * \ldots \ldots * f_{n_1} = b_n * f_n$, where $n = \sum_{i=1}^{k} n_{2i-1} - n_{2i}$ and $b_n \in \{h \in C_c(\beta) : h(x, k) = 0 \text{ if } k \neq 0\}$, and that $\pi(b_n * f_n) = \pi(b_n) V^*(n^-) V(n^+)$. Let $1_{X \cap (X+n)}$ be defined by

$$1_{X \cap (X+n)}(x, k) = \begin{cases} 1 & \text{if } x \in X \cap (X+n) \text{ and } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\pi(1_{X \cap (X+n)}) = V^*(n^-)V(n^+)V^*(n^+)V(n^-)$ and so $\pi(1_{X \cap (X+n)})V^*(n^-)V(n^+) = V^*(n^-)V(n^+)$. If a function $\sum_{|n| \leqslant k} b_n * f_n$ is zero, where each b_n is a polynomial in $\{1_{X \cap (X+s)} : s \in \mathbf{Z}^m\}$, then $b_n * 1_{X \cap (X+n)} = 0$, and so $b_n = b_n * (1_X - 1_{X \cap (X+n)})$. Thus $\pi(\sum_{|n| \leqslant k} b_n * f_n) = \sum_{|n| \leqslant k} \pi(b_n) [I - \pi(1_{X \cap (X+n)})] \pi(f_n) = 0$.

We now prove simultaneously that π can be extended to the whole algebra $C_c(\beta)$ and that it is continuous. For $g \in C_c(\beta)$, $g = \sum_{|n| \leqslant k} d_n * f_n$, where $d_n \in \{d \in C_c(\beta) : \operatorname{supp} d \subset X \times \{0\}\}$, we have that

$$\|g\| = \max_{|n| \leqslant k} \|d_n * 1_{X \cap (X+n)}\|.$$

In fact we may write $g = \sum_{|n| \leqslant k} d_n * 1_{X \cap (X+n)} * f_n$. Assume that $\{g^l : l \in \mathbf{N}\}$, contained in the $*$-algebra generated by f_{ε_i}, $i \in \{1, \ldots, m\}$, goes to zero in the inductive limit topology. That is, there exists k_0 such that $\bigcup_{l\in N}\operatorname{supp}\{g^l\} \subset \{(x, n) \in \beta : |n| \leqslant k_0\}$ and $\|g^l\| \to 0$ where $l \to \infty$. If $g^l = \sum_{|n|\leqslant k_0} d_n^l * f_n$, then $\|d_n^l * 1_{X\cap X+n}\| \to 0$ when $l \to \infty$. Since $\|d_n^l * 1_{X\cap(X+n)}\| = \|\pi(d_n^l * 1_{X\cap X+n})\| = \|\pi(d_n^l * 1_{X\cap X+n} * f_n)\|$, we see that $\pi(g^l)$ goes to zero not only in the weak operator topology but also in the norm topology of $\mathscr{L}(H)$. By Step 1 we may think of π as a representation of $C^*(\bigoplus_{j\in J} E_j)$ onto $C^*(V(n))$, with $\pi(U(n)) = V(n)$ for all $n \in \mathbf{Z}_+^m$.

Step 3. In this step we will see that π is faithful whenever $C^*(V(n))$ is irreducible and $V^*(k^-)V(k^+)$ is not a multiple of the identity for all $k \in \mathbf{Z}^m\setminus\{0\}$.

It is enough to prove that $\|P\| = \|\pi(P)\|$ whenever P is a polynomial. Given such a P there exists $i \in J$ and $f \in \ell^2(E^i)$ satisfying $\|f\| = 1$, $f = \sum_{q\in F} \lambda_q \varepsilon_q$ with F a finite subset of E^i, such that $\|Pf\|$ almost attains the norm of $\|P\|$. Let l be the degree of P. Let $r \in E_i$, and let k be large enough so that $\{q - n : q \in F, q + n \in E_i$ and $|n| \leqslant l + 1\} \subset D_k^{E_i}(r)$. What we are going to do, roughly, is to find a finite-dimensional subspace, H_0, of H with the property that the action of P on $\ell^2(F)(\ell^2(F) \subset \ell^2(D_k^{E_i}(r)))$ is exactly the same that the action of $\pi(P)$ on H_0. There are two cases to consider, in the first we use Proposition 2.9.

Case 1. There exists j such that E_j contains a point p with $D_k^{E_j}(p) \sim D_k^{E_i}(r)$ and such that either ∂E_j is not periodic or ∂E_j is periodic but if $n \in \mathbf{Z}^m\setminus\{0\}$ and $E_j + n = E_j$ then $|n| > 2k$.

By Proposition 2.9 there exists h, $h > 2k$, so that $s \neq s'$, $s, s' \in D_k^{E_j}(p)$ implies that $D_h^{E_j}(s) \nsim D_h^{E_j}(s')$. Let $Q \in C^*(\bigoplus_{j\in J} E_j)$ be the projection (given by Lemma 2.6) equal to $\bigoplus_{t\in J} 1_{F_t}$, where $F_t = \{q \in E_t : D_{h+k}^{E_t}(q) \sim D_{h+k}^{E_j}(p)\}$. For n such that $p + n \in D_k^{E_j}(p)$ define $Q(n) = U^*(n^-)U(n^+)QU^*(n^+)U(n^-)$. (Notice that $Q(0) = Q$ and $Q(n)$ would be zero if $|n| \leqslant k$ and $p + n \notin D_k^{E_j}(p)$.) Since $Q(n) \leqslant \bigoplus_{t\in J} 1_{L_t(n)}$, where $L_t(n) = \{q \in E_t : D_h^{E_t}(q) \sim D_h^{E_j}(p + n)\}$, we have that $Q(n)Q(n') = 0$ whenever $n \neq n'$.

Let x be a unit vector in $\pi(Q)H$ ($\pi(Q)$ is not zero since $Q \neq 0$ and π is faithful restricted to the diagonal subalgebra of $C^*(\bigoplus_{t\in J} E_t)$). For n such that $p + n \in D_k^{E_j}(p)$ let $x_n = V^*(n^-)V(n^+)x$.

Assertion. a) $x_n \in \pi(Q(n))H$ and $\|x_n\| = 1$; and so $\{x_n\}$ forms an orthonormal set.

b) If $|n| \leqslant k - 1$ then $V(\varepsilon_i)x_n = x_{n+\varepsilon_i}$.

c) If $|n| \leqslant k-1$ then $V^*(\varepsilon_i)x_n = x_{n-\varepsilon_i}$ when $p+n-\varepsilon_i \in D_k^{E_j}(p)$ and $V^*(\varepsilon_i)x_n = 0$ when $p + n - \varepsilon_i \notin D_k^{E_j}(p)$.

From $U^*(n^+)U(n^-)U^*(n^-)U^+(n)Q = Q$ follows a). From $U(\varepsilon_i)U^*(n^-)U(n^-)Q = U^*(n^-)U(n^+ - \varepsilon_i)Q$ follows b). Finally, if $p - n - \varepsilon_i \notin D_k^{E_j}(p)$ then $U^*(n^- - \varepsilon_i)U(n^-)Q = 0$, this implies c).

To complete the proof of Case 1 define $f' = \sum \lambda_{r+n}x_n$ where $r + n \in F$ and where λ_{r+n} is the coefficient of e_{r+n} in the definition of the function f. Then f' has been constructed in such a way that $\|f'\| = 1$ and $\|\pi(P)f'\| = \|Pf\|$. This concludes Case 1.

Case 2. Let $L = \{j \in J : E_j$ contains a point p_j with $D_k^{E_j}(p_j) \sim D_k^{E_i}(r)\}$. If $j \in L$, then ∂E_j is periodic and there is an $s_j \in \mathbf{Z}^m \setminus \{0\}$, $|s_j| \leqslant 2k$, with $E_j + s_j = E_j$.

We would like to find a $j \in L$ that satisfies:

(†) There is an $h, h > k$ and possibly very large, such that if $D_h^{E_j}(p_j) \sim D_h^{E_t}(n)$ then $E_t + s_j = E_t$. Assume that we have obtained such a j. Let $Q = \bigoplus_{t \in J} 1_{A_t} \in C^*(\bigoplus_{t \in J} E_t)$, where $A_t = \{n \in E_t : D_h^{E_t}(n) \sim D_h^{E_j}(p_j)\}$. Since π is faithful restricted to the diagonal subalgebra $\pi(Q) \neq 0$. Let $x \in \pi(Q)H$ and $x \neq 0$. The span of $\{Mx$: where M is a monomial in $V(\varepsilon_i)$, $V^*(\varepsilon_i)$, $i \in \{1, \ldots, m\}\}$ is H because $C^*(V(n))$ is irreducible. Let $I_{s_j} = \{t \in J : E_t = E_t + s_j\}$, the operator $U^*(s_j^-)U(s_j^+) \in C^*(\bigoplus_{t \in I_{s_j}} E_t)$ is unitary and commutes with $U(\varepsilon_i)$, $i \in \{1, \ldots, m\}$, which implies $V^*(s_j^-)V(s_j^+) \in C^*(V(n))'$. By hypothesis $V^*(s_j^-)V(s_j^+)$ is not a multiple of the identity, but this contradicts the irreducibility of $C^*(V(n))$. Thus only Case 1 can occur.

We then need to consider only the following possibility. (By our assumption $L \cap I_{s_i}$ is non-empty, p_i may be chosen as r.) For each $j \in L \cap I_{s_i}$, $h \in \mathbf{N}$, and $D_h^{E_j}(p)$ there is a $j' \in L \setminus I_{s_i}$ and a $q \in E_{j'}$ so that $D_h^{E_{j'}}(q) \sim D_h^{E_j}(p)$. Arguing as in Theorem 2.20 it follows that $C^*(\bigoplus_{j \in J} E_j)$ is algebraically equivalent to $C^*(\bigoplus_{j \in J \setminus (I_{s_i} \cap L)} E_j)$. Since $\{s_j : j \in L\}$ is a finite set we can obtain a $j \in L$ that satisfies (†).

We now prove statement 2'). Let $m = 2$, we will see that Case 2 cannot occur. Without loss of generality we may assume that if $j, j' \in J$, $j \neq j'$, then either $E_j \not\leqslant E_{j'}$ or $E_{j'} \not\leqslant E_j$. Let r and E_i be as in Case 2. Let $Q \in C^*(\bigoplus_{j \in J} E_j)$ be such that $Q|\ell^2(E_i) = 1_{\{q : D_{|s_i|}^{E_i}(q) \sim D_{|s_i|}^{E_i}(r)\}}$ and $Q|\ell^2(E_j) = 0$ if $j \neq i$. For each $n \in \mathbf{Z}^2$ let $Q(n) = U^*(n^+)U(n^-)QU^*(n^-)U(n^+)$ and notice that for $n \neq n'$ either $Q(n) = Q(n')$ or $Q(n)Q(n') = 0$. (We discard the n's with $Q(n) = 0$.) There exist n_b, $b \in \{1, \ldots, d\}$ and $d \leqslant |s_i|$ that satisfy:

I) $Q(n_b) \neq Q(n_{b'})$ if $b \neq b'$,

II) $\sum_{b=1}^{d} Q(n_b)MQ = (I - U(\varepsilon_1)U^*(\varepsilon_1)U(\varepsilon_2)U^*(\varepsilon_2))MQ$ for each monomial in $C^*(\bigoplus_{j \in J} E_j)$.

As before if we let $x \in \pi(Q)H$ and $x \neq 0$, then the span of $\{\pi(M)x : M$ monomial$\}$ is H. Therefore $\sum_{b=1}^{d} \pi(Q_{n_b}) = I - V(\varepsilon_1)V^*(\varepsilon_1)V(\varepsilon_2)V^*(\varepsilon_2)$ and so there is a $\pi Q(n_c)$ that has infinite rank. For each $Q(n)$ and for each monomial M we have that $MQ(n)$ is either equal to $(U^*(s_i^-)U(s_i^+))^{[t]}Q(n)$ for some $t \in \mathbf{Z}$ or $Q(n)MQ(n) = 0$. If $y \in \pi(Q(n_c))H$, $y \neq 0$, then $\{[V^*(s_i^-)V(s_i^-)]^{[t]}y : t \in \mathbf{Z}\}$ spans $\pi(Q(n_c))H$. Thus $V^*(s_i^-)V(s_i^+)$ is not a multiple of the identity. As before $V^*(s_i^-)V(s_i^+) \in C^*(V(n))'$ (and is a unitary operator). This contradicts the irreducibility of $C^*(V(n))$ and so we conclude that π is faithful.

Step 4. Assume that $C^*(V(n))$ is irreducible and that π is a faithful representation of $C^*(\bigoplus_{j \in J} E_j)$ onto $C^*(V(n))$, where $\pi(U(n)) = V(n)$ for $n \in \mathbf{Z}_+^m$.

Let $J = \{1,2,\ldots,n\}$ if J is finite and $J = \mathbf{N}$ if J infinity. The cardinality of J is 1 in case one of the points chosen from X in Step 1 has a dense orbit. Let $\pi^{-1} = \rho = \bigoplus_{k \in J} \rho_k$ and observe that $\rho_k(C^*(V(n)))$ acts on $\ell^2(E_k)$. It is convenient to consider two cases.

Case 1. $\rho = \bigoplus_{k=1}^{n} \rho_k$, i.e., J is a finite set. The faithfulness of ρ implies that $(0) = \operatorname{Ker}\rho = \bigcap_{k=1}^{n} \operatorname{Ker}\rho_k$ and by [9, Lemma 2.11.4], there is one k such that $\operatorname{Ker}\rho_k = (0)$. (To apply this lemma, we notice that the zero ideal of $C^*(V(n))$ is primitive because the identity representation of $C^*(V(n))$ is irreducible.)

Case 2. $J = \mathbf{N}$. For each $I \subset \mathbf{N}$ we write $\rho = \rho^I \oplus \rho^{I^c}$, where $\rho^I = \bigoplus_{k \in I} \rho^k$. As before we see that either $\operatorname{Ker}(\rho^I) = (0)$ or $\operatorname{Ker}(\rho^{I^c}) = (0)$. The following assertion says that for each pair of diamonds that appear in $\{E_k : k \in \mathbf{N}\}$ there is a particular E_k such that both diamonds appear in E_k. This will allow us to construct an E such that the diamonds in E are exactly the diamonds that appear in $\{E_k : k \in \mathbf{N}\}$.

Assertion. For each pair of points p_i, p_j, with $p_i \in E_i$, $p_j \in E_j$, and for each $n \in \mathbf{N}$ there exists k and $\tilde{p}_i$, $\tilde{p}_j \in E_k$ such that $D_n^{E_i}(p_i) \sim D_n^{E_k}(\tilde{p}_i)$ and $D_n^{E_j}(p_j) \sim D_n^{E_k}(\tilde{p}_j)$.

To prove the assertion, assume it is not true, i.e., assume that there exists a pair i, j, $p_i \in E_i$, $p_j \in E_j$, and an $n \in \mathbf{N}$ such that no E_k exists so that $D_n^{E_i}(p_i) \sim D_n^{E_k}(\tilde{p}_i)$ and $D_n^{E_j}(p_j) \sim D_n^{E_k}(\tilde{p}_j)$. Let I be the set $\{l :$ there exists $q_l \in E_l$ with $D_n^{E_l}(q_l) \sim D_n^{E_i}(p_i)\}$. Notice that $i \in I$ and $j \in I^c$. Let p and q be the polynomials given by Lemma 2.6 such that when evaluated in the generators of $C^*(U(n))$ acting on $\ell^2(E)$, p and q are $1_{\{r : D_n^E(r) \sim D_n^{E_i}(p_i)\}}$ and $1_{\{r : D_n^E(r) \sim D_n^{E_j}(p_j)\}}$. Call P and Q the operators $C^*(V(n))$ that are the polynomials p and q evaluated in $V(\varepsilon_s)$, $V^*(\varepsilon_s)$,

$s \in \{1, \ldots, m\}$. Then $\rho^I(P) \neq 0$ but $\rho^{I^c}(P) = 0$, and $\rho^I(Q) = 0$ but $\rho^{I^c}(Q) \neq 0$. Consequently, neither ρ^I nor ρ^{I^c} is faithful, contradicting the observation made just before the assertion.

Now we construct an E such that $E_i \leqslant E$ for all $i \in \mathbf{N}$ and such that for any given $p \in E$ and $n \in \mathbf{N}$ there exists $i \in \mathbf{N}$ and $q \in E_i$ such that $D_n^E(p) \sim D_n^{E_i}(q)$. This implies that $C^*(E)$ is algebraically equivalent to $C^*(\bigoplus_{i \in \mathbf{N}} E_i)$.

Let $p_j \in E_j$ be fixed for all $j \in J$ and let $f\colon \mathbf{N} \to \mathbf{N}$ be a surjective function such that $f^{-1}(n)$ is an infinite set for all $n \in \mathbf{N}$. We shall write $E = \bigcup_{i \in \mathbf{N}} B_i$, where the sets B_i's satisfy the following properties.

I) $B_i = D_{l_i}^{E_{j_i}}(h_i) - h_i$.

II) $B_i \cap \{k \in \mathbf{Z}^m : |k| \leqslant l_{i-1}\} = B_{i-1}$ for $i \geqslant 2$.

III) For $i \leqslant n$, $\{k \in \mathbf{Z}^m : |k| \leqslant i$ and $r_i + k \in B_n\} = D_i^{E_{f(i)}}(p_{f(i)}) - p_{f(i)}$.

The points r_i, h_i will be chosen inductively, as well as the radii l_i's and the indexes j_i's. We shall ask also for $\{l_i\}$ to be increasing and $l_n \geqslant |r_i| + n$ for all pairs i, n with $i \leqslant n$.

Let $B_1 = D_1^{E_{f(1)}}(p_{f(1)}) - p_{f(1)}$, i.e., $j_1 = f(1)$, $p_{f(1)} = h_1 - r_1$ and $l_1 = 1$. Assume that we have, for $i \leqslant n$, B_i's satisfying I, II and III. By the assertion there is a module $E_{j_{n+1}}$, containing points h_{n+1} and g_{n+1}, such that $D_{l_n}^{E_{j_n}}(h_n) \sim$ $\sim D_{l_n}^{E_{j_{n+1}}}(h_{n+1})$, $D_{l_n}^{E_{f(n+1)}}(p_{f(n+1)}) \sim D_{l_n}^{E_{j_{n+1}}}(g_{n+1})$. Choose $l_{n+1} = 2(|h_{n+1} - g_{n+1}| + l_n)$. If we set $B_{n+1} = D_{l_{n+1}}^{E_{j_{n+1}}}(h_{n+1}) - h_{n+1}$ and $r_{n+1} = g_{n+1} - h_{n+1}$, it is clear that I, II and III are satisfied for $i \leqslant n + 1$.

The set $E = \bigcup_{i \in \mathbf{N}} B_i$ is a module that satisfies all the required properties. This completes the proof of the last step of the theorem. ▨

REMARK 4.5. Let A be a C^*-algebra and let I be a primitive ideal. If the lemma "if $I \supset \bigcap_{n \in J} I_n$ then $I \supset I_n$ for some n" extends to the case when J is countable, then it would not be necessary to analyse two cases in Step 4 of Theorem 4.4. The next example shows that the lemma is not true when $J = \mathbf{N}$.

Let $E_i \leqslant E_{i+1}$ be an strictly increasing sequence of $\mathbf{Z}_+^2$-modules. Let E be the least upper bound of $\{E_i : i \in \mathbf{N}\}$ for the preorder $\leqslant$. Since ∂E is not periodic, $C^*(E)$ is irreducible. The representation $\pi = \bigoplus_{i \in J} \pi_i$ from $C^*(E)$ onto $C^*(\bigoplus_{i \in J} E_i)$ is faithful, i.e., $\operatorname{Ker} \pi = \bigcap_{i \in J} \operatorname{Ker} \pi_i = (0)$, but $\{\operatorname{Ker} \pi_i\}$ is a strictly decreasing chain of ideals.

The following corollary will be fundamental for the problem of deciding when $C^*(E)$ is G.C.R. for a given $\mathbf{Z}_+^2$-module.

COROLLARY 4.5. *Let E be a $\mathbf{Z}_+^2$-module. Then the $C^*(E)$ is* G.C.R. *if and only if every module F, $F \leqslant E$, has periodic boundary or $K(\ell^2(F)) \subset C^*(F)$.*

Proof. Assume that $C^*(E)$ is G.C.R., that $F \leqslant E$ and ∂F is not periodic. By Proposition 2.10, $C^*(F)$ is irreducible and, by Theorem 2.20, $C^*(F)$ is a quotient of $C^*(E)$. Thus $C^*(F)$ is G.C.R. and $K(\ell^2(F)) \subset C^*(F)$. To prove the converse assume that for each module $F \leqslant E$, ∂F is periodic or $K(\ell^2(F)) \subset C^*(F)$, but $C^*(E)$ is not G.C.R. . Then there exists a representation π of $C^*(E)$, on some Hilbert space H, such that $\pi(C^*(E))$ is irreducible but $\pi(C^*(E)) \cap K(H) = (0)$. Therefore $I - \pi(U(\varepsilon_i)U^*(\varepsilon_i))$, $i \in \{1,2\}$, are infinite-dimensional projections. By Theorem 4.4 $\pi(C^*(E))$ is algebraically equivalent to $C^*(F)$, for some $\mathbf{Z}^2$-module F. By Theorem 2.20 $F \leqslant E$. Since $K(\ell^2(F)) \cap C^*(F) = (0)$, it follows, by hypothesis, that ∂F is periodic. This implies that $\pi(C^*(E))$ is reducible. This contradiction completes the proof of the proposition. ▨

5. WEIGHTED SHIFTS

In this section we focus on $\mathbf{Z}_+^2$-modules. Our main objective is to determine when, for a module E, $C^*(E)$ is G.C.R. . It turns out that each $C^*(E)$ has a naturally associated family of weighted shifts that act on a subspace of $\ell^2(E)$. Theorem 5.8 asserts that $C^*(E)$ is G.C.R. if and only if each of its associated weighted shifts is G.C.R. . Since, as we have already observed in the introduction, O'Donovan determined when a weighted shift is G.C.R. [18], our theorem completely determines when $C^*(E)$ is G.C.R. . Along the way we present several results that show that the relation between weighted shifts and the algebras $C^*(E)$ is very closed indeed. In general, similar concepts can be found in both settings. For instance we shall show that there are also universal shifts. Another example is that both G.C.R. bilateral weighted shifts, with weight sequences bounded away from zero, and G.C.R. $C^*(E)$ have representations that are periodic. This means, roughly, that the weight sequence of a G.C.R. bilateral weighted shift is "made up" of periodic sequences. Likewise if $C^*(E)$ is G.C.R., then ∂E is locally periodic.

DEFINITION 5.1. 1) Let $I_z = [0,1]$ and let $\Omega = \underset{z \in \mathbf{Z}}{\times} I_z$ be endowed with the product topology. A *word* $(a_i, \ldots, a_{i+k})$ of (a_i), $(a_i) \in \Omega$, is a finite sequence of consecutive numbers in (a_i).

2) For two sequences (a_i) and (b_i) we write $(a_i) \leqslant (b_i)$ if and only if for every word $(a_i, \ldots, a_{i+k})$ of (a_i) and for every $\varepsilon > 0$ there exists a word of (b_i), $(b_j, \ldots, b_{j+k})$, with the same length, such that $|a_{i+s} - b_{j+s}| < \varepsilon$ for $s \in \{0,1, \ldots, k\}$. (Observe that $\leqslant$ is a pre-order.)

3) The sequence $a = (a_i)$ of Ω contains a unique word $(a_i, \ldots, a_{i+k})$ if there exists $\varepsilon > 0$ such that for every j, $j \neq i$, we have $\sum_{s=0}^{k} |a_{i+s} - a_{j+s}| \geqslant \varepsilon$.

4) As usual (a_i) is *periodic* if there is an integer n such that $a_i = a_{i+n}$ for all i.

The definition of diagonal spectrum for weighted shifts is in [18]. Here we express this concept in terms of the elements in Ω.

DEFINITION 5.2. Let ψ be the shift transformation acting on Ω, i.e., $\psi(a_i) = (a_{i-1}), (a_i) = a \in \Omega$. The *diagonal spectrum* of a, Ω_a, is defined to be the minimal closed set invariant under ψ and ψ^{-1} containing the element a.

REMARK 5.3. An equivalent definition is $\Omega_a = \{b \in \Omega : b \leqslant a\}$. Also we find that $b \leqslant a$ if and only if $\Omega_b \subset \Omega_a$. Since Ω_a is closed and Ω is compact, the set Ω_a is also compact.

LEMMA 5.4. *Let e be an element of Ω with the property that if $b \leqslant e$ then either b contains a unique word or b is periodic. Then there exists $c \leqslant e$ such that c is periodic.*

Proof. Let F be the family $\{\Omega_b : b \leqslant e\}$, ordered by inclusion. Then F is non-empty because $\Omega_e \in F$: also, each set in F is non-empty. Applying Hausdorff's Maximality theorem, we find a maximal chain. This chain satisfies the finite intersection property and so the intersection, call it C, is not empty. Let $c \in C$ and so $\Omega_c \subset C$. By maximality of the chain we see that $\Omega_c = C$. Assume c is not periodic. Then c contains a unique word $(c_i, \ldots, c_{i+k})$. Let ψ be the shift transformation acting on Ω and consider the sequence $\{\psi^n(c) : n \in \mathbf{N}\}$. By compactness of Ω_c some subsequence $\psi^{n_j}(c)$ converges to $d \in \Omega_c$. Then $c \notin \Omega_d$ because the word $(c_i, \ldots, c_{i+k})$ cannot be approximated by a word in d (the unique word is shifted to infinity). But $\Omega_d \in F$, contradicting the fact that the chain is maximal. Therefore we find that c is periodic and the cardinality of Ω_c is the period of c. ▨

Let (X, d) be a compact metric space, and let M be the set of closed subsets of X. We recall that the Hausdorff metric $\hat{d}$ is defined on M by the formula

$$\hat{d}(A, B) = \inf\{\varepsilon : B \subset A_\varepsilon \text{ and } A \subset B_\varepsilon\},$$

where for a subset $Y \subset X$, $Y_\varepsilon = \{x : \text{there exists } y \in Y \text{ with } d(x, y) \leqslant \varepsilon_i\}$.

We regard Ω as a metric space (Ω, d) with metric $d(a, b) = \sum_{i \in \mathbf{Z}} |a_i - b_i|/2^{|i|}$. It is known that the metric topology is equivalent to the product topology.

PROPOSITION 5.5. *Let M be the set of closed subsets of Ω and let $\hat{d}$ be the Hausdorff metric. Then the map f from (Ω, d) into $(M, \hat{d})$ given by $f(b) = \Omega_b$ is not continuous, but for every $b \in \Omega$ and for every $\varepsilon > 0$ there exists a δ, depending on b and ε; such, that if $d(b, c) < \delta$ then $\Omega_b \subset (\Omega_c)_\varepsilon$;* i.e., *$f$ is lower semicontinuous.*

Proof. To show that f is not continuous it suffices to consider $a \in \Omega$ such that every word with rational numbers is in a. It is clear that $\Omega_{\psi^n_{(a)}} = \Omega$ for all $n \in \mathbf{Z}$; and that there exists a subsequence $\{n_j\} \subset \mathbf{N}$ such that $\psi^{n_j}_{(a)} \to (0)$ or $\psi^{-n_j}_{(a)} = (0)$.

To prove the second part suppose that b and $\varepsilon > 0$ are fixed. Let $n \in \mathbf{N}$ be such that $\sum_{i>n} \frac{1}{2^i} < \frac{\varepsilon}{3}$. Since the set $\{(b_k, \ldots, b_{k+2n}) : k \in \mathbf{Z}\}$ is dense in the compact set $\{(a_{-n}, \ldots, a_n) : a \leqslant b\}$ there is an $m \in \mathbf{N}$ such that $\{(b_i, \ldots, b_{i+2n}) : -m \leqslant \leqslant i \leqslant m - 2n\}_{\varepsilon/3} \supset \{(a_{-n}, \ldots, a_n) : a \leqslant b\}$. It is immediate to verify that if we set $\delta = \varepsilon/(2^{2n+m} \cdot 3)$, then $(\Omega_c)_\varepsilon \supset \Omega_b$ whenever $d(b, c) < \delta$. ▨

The transformation ψ is ergodic if Ω is endowed with the product measure whose factors are Lebesgue measure. By a result of Halmos [14, page 26], for almost every $a \in \Omega$, $\Omega_a = \Omega$. A simple argument shows that $\{a \in \Omega : \Omega = \Omega_a\}$ is a dense G_δ subset of Ω.

Assume that $\{e_n : n \in \mathbf{Z}\}$ is a fixed orthonormal basis of a Hilbert space H. Consider the set B consisting of the bilateral weighted shifts with weights in [0,1] and with polar decomposition $U\mathrm{Diag}(a_i)$, where $Ue_n = e_{n+1}$ and $\mathrm{Diag}(a_i)e_n = a_n e_n$, and define a map φ from B onto Ω by $\varphi(U\mathrm{Diag}(a_i)) = (a_i)$. We define a measure ν on B induced by μ (the product of Lebesgue measure on each factor of Ω) and the map φ, by the formula $\nu(\varphi^{-1}(R)) = \mu(R)$ for all measurable subset R of Ω. Then we may say that almost every (with respect to ν) bilateral weighted shift $U\mathrm{Diag}(a_i)$ has the property that $\Omega_{(a_i)} = \Omega$. Likewise, for each finite set $\{t_1, \ldots, t_n\} \subset [0,1]$, consider the set $B(t_1, \ldots, t_n) = \{U\mathrm{Diag}(a_i) \in B : a_i \in \{t_1, \ldots, t_n\}$ for $i \in \mathbf{N}$ and $\Omega_a = \times_{-\infty}^{\infty} \{t_1, \ldots, t_n\}\}$. Give $\times_{-\infty}^{\infty} \{t_1, \ldots, t_n\} = \beta$ the measure γ which is the product of counting measure on each factor. Let η be the measure on $B(t_1, \ldots, t_n)$ induced by γ and the map φ, defined by the formula $\eta(\varphi^{-1}(R)) = \gamma(R)$ for all measurable subset R of β. Then we may say that the bilateral weighted shifts V with weights in $\{t_1, \ldots, t_n\}$ have the property that almost all (with respect to η) satisfy $\Omega_{\varphi(t)} = \beta$. (This is [18, Corollary of 2.4.1].)

Let $S = U\mathrm{Diag}(a_i)$ and $T = U\mathrm{Diag}(b_i)$ be in B. In [18] is implicit the following observation (a direct proof of it could be given using arguments similar to the ones used in Theorem 2.20). There exists a representation π from $C^*(T)$ onto $C^*(S)$ with $\pi(T) = S$ if $(a_i) \leqslant (b_i)$.

The above comment implies the amusing result that there exists a universal bilateral weighted shift $T \in B$. (In the sense that for every bilateral weighted shift $R \in B$ there exists a representation π from $C^*(T)$ onto $C^*(R)$ with $\pi(T) = R$.) Moreover, almost all bilateral weighted shifts are universal.

PROPOSITION 5.6. 1) *Let T be a* G.C.R. *weighted shift with positive weights bounded away from zero. Then there exists a representation π of $C^*(T)$ such that* $|\pi(T)| = (\pi(T^*T))^{1/2} = \mathrm{Diag}(c_i)$ *with (c_i) periodic.*

2) *If E is a module such that $C^*(E)$ is* G.C.R., *then there exists a module F, with $F \leqslant E$, and ∂F periodic.*

Proof. 1) Let $U\mathrm{Diag}(a_i)$ be the polar decomposition of S. It is known that $C^*(S)$ is reducible if either some $a_i = 0$ or (a_i) is periodic. Let $T = U\mathrm{Diag}(b_i)$ and let $(a_i) \leqslant (d_i)$. Since $C^*(T)$ is G.C.R. and $C^*(S)$ is a representation of $C^*(T)$ either $C^*(S)$ is reducible (i.e., (a_i) periodic) or $K(H) \subset C^*(S)$. In the last case, [18, Theorem 2.5.1] asserts that (a_i) has a unique word. But these are precisely the alternatives in the hypothesis of Lemma 5.4. Therefore there exists a periodic sequence (c_i), $(c_i) \leqslant (b_i)$. Thus $C^*(U\mathrm{Diag}(c_i))$ is the required representation of $C^*(T)$.

2) It suffices to analyse the case when ∂E is described by $(\ldots, t_{-1}, \underline{p}, t_0, t_1, \ldots)$ with $\sup\{t_n : n \in \mathbf{Z}\} < \infty$. Otherwise we can choose either $\mathbf{Z}_+ \times \mathbf{Z}$ or $\mathbf{Z} \times \mathbf{Z}_+$ as F. We may assume that t_0 and t_{-1} are non-zero, i.e., that p is a vertex. Let Q be the family $\{F : p \in \partial F$ and ∂F is described by $\{\ldots, s_{-1}, \underline{p}, s_0, \ldots\}$ with s_0 and s_{-1} non-zero and $\sup\{s_n : n \in \mathbf{Z}\} \leqslant \sup\{t_n : n \in \mathbf{Z}\}\}$. Define a one-to-one function f from Q into Ω by $f(F) = 1/(2^{s_{2i}} 3^{s_{2i-1}})$ where ∂F is described by $(\ldots, s_{-1}, \underline{p}, s_0, \ldots)$. The following facts are easily verified.

i) From Proposition 2.10, it follows that $C^*(F)$ is reducible if and only if (F) is periodic.

ii) From Lemma 2.17, it follows that $K(l^2(F)) \subset C^*(F)$ if and only if $f(F)$ contains a unique word.

iii) From Theorem 2.20, it follows that if F and $G \in Q$, then $G \leqslant F$ if and only if $f(G) \leqslant f(F)$.

iv) $\Omega_{f(F)} = \{f(G) : G \leqslant F\}$.

Let $d \in \Omega$ be $(1/(2^{t_{2i}} 3^{t_{2i+1}}))$, then since $C^*(E)$ is G.C.R., we see that for every $(a_i) \in \Omega_d$, either (a_i) is periodic or (a_i) contains a unique word. Lemma 5.4 tells us that there is a periodic $(c_i) \in \Omega_d$. So $f^{-1}((c_i)) \leqslant E$ and $\partial f^{-1}((c_i))$ is periodic. ▨

Our next result shows that the converse of Proposition 5.6 is not true. Proposition 5.7 answers negatively Question 6 in [21].

PROPOSITION 5.7. *Let F be a finite subset of $(0,1]$. There is a sequence $d \in \Omega$, with $d_n \in F$ for $n \in \mathbf{Z}$, which does not have a unique word but every $b \in \Omega_d$ admits a $c \in \Omega_b$ such that c is periodic.*

Proof. It suffices to consider that F has only two elements, s and t. We will construct d in an inductive way.

Step 1 is to define $d_0 = s$. Assume that after the $(2n-1)$-th step we have defined a word l (centered at the 0-slot) of length 3^{2n-2}. The $2n$-th step is to form the word lll, i.e., we add the same word at both sides of l. Step $2n+1$ is to form the word $allla$, where $a = (t_1, \ldots, t_{3^{2n-1}})$ and $t = t_i$ for $i \in \{1, \ldots, 3^{2n-1}\}$.

With the even steps we guarantee that d does not contain a unique word. We will show that $c = (c_n)$ with $c_n = t$ for all n satisfies the relation $c \leqslant b$ for every $b \leqslant d$.

Assume not, i.e., there exists $b \leqslant d$ and $k \in \mathbf{N}$ such that the word with length k, $(t, \ldots, t)$, does not appear in b. Choose $p \in \mathbf{N}$ with $k < (2p+1)/27$ and consider the word of b equal to $(b_{-p}, \ldots, b_0, \ldots, b_p)$. Assume that the word constructed in the i-th step is the first one such that $(b_{-p}, \ldots, b_p)$ is a subword of it. If $i = 2n$ then the word obtained in the i-th is lll (where l has been obtained in the $2n-1$-th step).

Since the word lll has length 3^{2n-1} we have that $2p + 1 \leqslant 3^{2n-1}$. Thus $k < 3^{2n-4}$. The word l has a subword $(t, \ldots, t)$ of length 3^{2n-3}, and so l cannot be a subword of $(b_{-p}, \ldots, b_p)$. Therefore the word $(b_{-p}, \ldots, b_p)$ must start in the first l and finish in the second l or start in the second l and finish in the third one. (Here the order is from left to right and the "second" l is in the center.) Since there is an $i \in \{-p, \ldots, p\}$ so that $b_i = s$ it follows that $(b_{-p}, \ldots, b_p)$ contains a word $(t, \ldots, t)$ of length 3^{2n-3}. This contradiction shows that i must be an odd number $2n + 1$.

The word $(b_{-p}, \ldots, b_p)$ must contain a word $(t, \ldots, t)$ of length 3^{2n-3}. (Otherwise $(b_{-p}, \ldots, b_p) = (t, t, \ldots, t)$.) Since $2p + 1 \leqslant 3^{2n}$ it follows that $k \leqslant 3^{2n-3}$. This contradiction concludes the proof of the proposition. ▨

We now associate to each module E with infinitely many vertices a family of weighted shifts, bilateral or unilateral. These operators act on the Hilbert space $\ell^2(V)$, where V is the set of vertices in E and the orthonormal basis associated with each shift is $\{e_q : q \in V\}$. Using Lemma 2.6 we can construct projections $Q(n, m)$ $C^*(E)$, $(n, m) \in \mathbf{N}^2$, such that $Q(n, m)\ell^2(E)$ is the span of $\{e_p$: the segment $[p, p + (n, -m)]$ contains only the vertices p and $p + (n, -m)\}$. Then $U^*(m\varepsilon_2)U(n\varepsilon_1) \cdot Q(n, m) | \ell^2(V)$ is a weighted shift with weights 0 or 1. To see this, let $\{e_{q_i}\} = V$ be an enumeration of the vertices when ∂E is traversed in the positive sense. We have that

$$U^*(m\varepsilon_2)U(n\varepsilon_1)Q(n, m)e_{q_i} = \begin{cases} e_{q_{i+1}} & \text{if } Q(n, m)e_{q_i} = e_{q_i}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $p \in \partial E$ be a vertex and let ∂E be described by $(\ldots, t_{-1}, \underline{p}, t_0, t_1, \ldots)$. Then $Q(n, m)$ is different from zero if and only if $(n, m) = (t_{2k}, t_{2k+1})$ for some $k \in \mathbf{Z}$. Let $\{a_{(n,m)} : (n, m) \in \mathbf{N}^2\}$ be a double sequence of positive numbers such that $a_{(n,m)} \neq a_{(r,s)}$ if $(n, m) \neq (r, s)$, and $\lim\limits_{n+m \to \infty} a_{(n,m)} = 0$. It is easy to verify that $T = \sum\limits_{(n,m) \in \mathbf{N}^2} a_{(n,m)} U^*(m\varepsilon_2)U(n\varepsilon_1)Q(n, m)$ belongs to $C^*(E)$. The operator $T | \ell^2(V)$ is a weighted shift. (If the sequence that describes ∂E contains ∞ on the left, then $T | \ell^2(V)$ is a forward shift. If ∞ appears on the right, then $T | \ell^2(V)$ is a backward shift. Otherwise $T | \ell^2(V)$ is a bilateral weighted shift.) Since we can choose the sequence $\{a_{(n,m)} : (n, m) \in \mathbf{N}^2\}$ in many different ways we talk about the family of weighted shifts associated to the module E.

THEOREM 5.8. *Let E be a module with infinitely many vertices. Let R be a weighted shift associated to E. Then:*

1) $C^*(E) \cap K(\ell^2(E)) = (0)$ *if and only if* $C^*(R) \cap K(\ell^2(V)) = (0)$.

2) $C^*(E)$ *is* G.C.R. *if and only if* $C^*(R)$ *is* G.C.R. .

Proof. Let $R = \sum_{(n,m)\in\mathbf{N}^2} a_{(m,n)} U^*(m\varepsilon_2)U(n\varepsilon_1)Q(n,m)\,\ell^2(V)$ for some double sequence $\{a_{(n,m)}: (n,m) \in \mathbf{N}^2\}$.

1) Assume first that R is a bilateral weighted shift. Let $\{q_k : k \in \mathbf{Z}\}$ be an enumeration of the vertices of E and let ∂E be described by $(\ldots, t_{-1}, \underline{q_0}, t_0, \ldots)$. Let $U\mathrm{Diag}(d_i)$ be the polar decomposition of R (where $U\mathrm{Diag}(d_i)e_{q_k} = d_k e_{q_{k+1}}$ and $d_k = a_{(t_{2k}, t_{2k+1})}$). The following facts are equivalent:

i) $C^*(E) \cap K(\ell^2(E)) = (0)$.

ii) q_0 has infinite multiplicity.

iii) There exists a sequence $\{n_i : i \in \mathbf{N}\}$ such that the segments of ∂E, $[q_{-i}, q_i]$ and $[q_{n_i - i}, q_{n_i + i}]$, are a translate of each other for all $i \in \mathbf{N}$. ($\lim n_i$ may be either $+\infty$ or $-\infty$.)

iv) $\lim_{i\to\infty} d_{k+n_i} = d_k$; in fact $d_{k+n_i} = d_k$ if $i \geqslant |k| + 1$.

v) $C^*(R) \cap K(\ell^2(V)) = (0)$.

The equivalence of i) and ii) follows from Lemma 2.17. The equivalence of ii) and iii) follows from the definition of infinite multiplicity. The equivalence of iii) and iv) follows from inspection of the weight sequence. The equivalence of iv) and v) is [18, Theorem 2.5.1].

If R is a unilateral weighted shift, then ∂E is described by $(\infty, \underline{q_0}, t_0, t_1, \ldots)$ (where the set of vertices is $\{q_i : i \in \mathbf{Z}_+\}$). We still can have the five equivalences if we change iii) to iii′) and iv) to iv′) as follows.

iii′) There exists a sequence $\{n_i : i \in \mathbf{N}\}$ with $\lim_{i\to\infty} n_i = +\infty$, such that the segments of ∂E, $[q_0 + i\varepsilon_2, q_i]$ and $[q_{n_i} + i\varepsilon_2, q_{n_i+i}]$, are a translate of each other for all $i \in \mathbf{N}$.

iv′)
$$\lim d_{k+n_i} = \begin{cases} d_k & \text{if } k \geqslant 0, \\ 0 & \text{if } k = -1. \end{cases}$$

In fact $d_{k+n_i} = d_k$ if $i \geqslant k+1$ and $k \geqslant 0$.

The equivalence of iv′) and v) is [18, Theorem 3.2.1]. (See the correct statement in Appendix.) If R is the adjoint of a unilateral shift we argue in a similar way.

2) If $C^*(E)$ is G.C.R., then so is $C^*(R)$, since subalgebras of G.C.R. C^*-algebras are G.C.R. [9]. To prove the converse, assume that $C^*(E)$ is not G.C.R. . By Corollary 4.5 there is a module $F \leqslant E$, such that $C^*(F)$ is irreducible and $C^*(F) \cap$

$\cap$ $K(\ell^2(F)) = (0)$. The module F has infinitely many vertices (see Example 3.4.3). Using the sequence $\{a_{(n, m)}: (n, m) \in \mathbf{N}^2\}$ which defines R, form the weighted shift $\tilde{R}$ associated to F. Then $C^*(\tilde{R})$ is irreducible and does not contain the underlying compact operators. Thus $C^*(\tilde{R})$ is not G.C.R.; and so, since $C^*(\tilde{R})$ is a representation of $C^*(R)$, $C^*(R)$ is not G.C.R. . ▨

COROLLARY 5.9. *Let the boundary of a module E be described by $(\ldots, t_{-1}, p, t_0, \ldots)$ and let the boundary of a module E' be described by $(\ldots, (t_{-1}), p, f(t_0), \ldots)$ where f is a one to one function from $\mathbf{N} \cup \{\infty\}$ into $\mathbf{N} \cup \{\infty\}$ with $f(\infty)=\infty$. Then $C^*(E) \cap K(\ell^2(E)) = (0)$ if and only if $C^*(E') \cap K(\ell^2(E')) = (0)$; and $C^*(E)$ is* G.C.R. *if and only if $C^*(E')$ is* G.C.R. .

Proof. If E and E' have a finite number of vertices there is nothing to prove. Otherwise we shall construct unitarily equivalent weighted shifts R and R', associated to E and E' respectively. Use a suitable sequence $\{a_{(n, m)}: (n, m) \in \mathbf{N}^2\}$ to define R. Let $\{a'_{(i, l)}: (i, l) \in \mathbf{N}^2\}$ be another suitable sequence such that $a'_{(f(n), f(m))} = a_{(n, m)}$ for all $(n, m) \in \mathbf{N}^2$. (Observe that there could be $(i, l) \notin f(N) \times f(N)$.) Let R' be the weighted shift associated to E' defined using $\{a'_{(i, l)}: (i, l) \in \mathbf{N} \times \mathbf{N}\}$. If (d_k) and (d'_k) are the weight sequences of R and R' respectively then $d_k = a_{(t_{2k}, t_{2k+1})} = a'_{(f(t_{2k}), f(t_{2k+1}))} = d'_k$. By using Theorem 5.8 we complete the proof. ▨

DEFINITION 5.10. The *boundaries* of the modules E and E' are a finite perturbation of each other if there exist $p, q \in \partial E$ and $p', q' \in \partial E'$ such that $p + C(\overleftarrow{n}_i, \vec{0}) \subset \partial E$, $q + C(\overleftarrow{0}, \vec{m}_i) \subset \partial E$, $p' + C(\overleftarrow{n}_i, \vec{0}) \subset \partial E'$ and $q' + C(\overleftarrow{0}, \vec{m}_i) \subset \partial E$, where $\lim_{i\to\infty} |\overleftarrow{n}_i| = \lim_{i\to\infty} |\vec{m}_i| = \infty$.

Notice that if ∂E is a finite perturbation of $\partial E'$, there need not be an $r \in \mathbf{Z}^2$ such that $(E + r \setminus E') \cup (E' \setminus E + r)$ is a finite set. It is evident that if ∂E is periodic and $\partial E'$ is a non-trivial finite perturbation, then $K(\ell^2(E')) \subset C^*(E')$.

PROPOSITION 5.11. *Let E and E' be modules such that $\partial E'$ is a finite perturbation of ∂E. Then:*

1) *$C^*(E)$ is* G.C.R. *if and only if $C^*(E')$ is* G.C.R. . *Moreover, the length of the canonical composition series for $C^*(E)$ differs at most one from the length of the composition series for $C^*(E')$.*

2) *If E is a universal module, so is E'.*

3) *It is possible for $C^*(E)$ to contain $K(\ell^2(E))$ while $C^*(E') \cap K(\ell^2(E')) = (0)$ and $C^*(E')$ is irreducible. In particular $C^*(E)$ need not be isomorphic to $C^*(E')$.*

Proof. Assume that $C^*(E)$ is G.C.R. and that neither ∂E nor $\partial E'$ is periodic. Then by Proposition 2.10 $C^*(E)$ and $C^*(E')$ are irreducible. Therefore $K(\ell^2(E)) \subset C^*(E)$; we shall prove that $K(\ell^2(E')) \subset C^*(E')$. If not, Lemma 2.17 asserts that every $t \in \partial E'$ has infinite multiplicity. Let $p, q \in \partial E$ and $p', q' \in \partial E'$ be such that

$p + C(\overleftarrow{n}_i, \vec{0}) \subset \partial E$, $p' + C(\overleftarrow{n}_i, \vec{0}) \subset \partial E'$, $q + C(\overleftarrow{0}, \vec{m}_i) \subset \partial E$ and $q' + C(\overleftarrow{0}, m_i) \subset \partial E'$, with $\lim_{i\to\infty} \overleftarrow{n}_i = \lim_{i\to\infty} \vec{m}_i = \infty$. Since p' has infinite multiplicity, for each $n \in \mathbf{N}$ there is $p(n) \in \partial E'$ with $D_n^{E'}(p') \sim D_n^{E'}(p(n))$ and $D_n^{E'}(p(n)) \cap \partial E'$ is contained in either $p' + C(\overleftarrow{n}_i, \vec{0})$ or $q' + C(\overleftarrow{0}, \vec{m}_i)$ for some i sufficiently large. Consequently $D_n^{E'}(p(n))$ is equivalent to $D_n^{E}(\tilde{p})$ for some $\tilde{p} \in \partial E$, so $E' \leqslant E$. By Theorem 2.20 $C^*(E')$ is a representation of the G.C.R. algebra $C^*(E)$. This is a contradiction and hence $K(\ell^2(E')) \subset C^*(E')$. We will show that $C^*(E)|K(\ell^2(E))$ is isomorphic to $C^*(E')|K(\ell^2(E'))$; and so the canonical composition series of $C^*(E)$ and $C^*(E')$ have the same length. Let $\beta(\beta')$ be the groupoid corresponding to $C^*(E)(C^*(E'))$. By Lemma 3.3 $E(E')$ may be seen as an open set of the unit space $X(X')$. The groupoids $(\beta|X\setminus E, \lambda_1)$ and $(\beta'|X'\setminus E', \lambda_2)$ are topologically isomorphic in the sense of [17, 2.9], where the Haar systems λ_1 and λ_2 are $\{\lambda_1^x$ is counting measure on $\{(x, n) \in \beta|X\setminus E\} : x \in X\setminus E\}$ and $\{\lambda_2^x$ is counting measure on $\{(x, n) \in \beta'|X'\setminus E'\} : x \in X'\setminus E'\}$. First we define a homeomorphism φ from $X\setminus E$ onto $X'\setminus E'$. Let $x = \lim s_n \in X\setminus E$, with $\{s_n\} \subset E$ and set

$$\varphi(x) = \begin{cases} \lim\limits_{n\to\infty} s_n + p' - p & \text{if the first coordinate of } s_n \text{ goes to } -\infty, \\ & \text{or the second one goes to } \infty, \\ \lim\limits_{n\to\infty} s_n + q' - q & \text{if the first coordinate of } s_n \text{ goes to } \infty, \\ & \text{or the second one goes to } -\infty. \end{cases}$$

Since φ is equivariant ($\varphi(x + n) = \varphi(x) + n$ and $\varphi^{-1}(y + n) = \varphi^{-1}(y) + n$), we can extend φ to an algebraic isomorphism of $\beta|X\setminus E$ onto $\beta'|X'\setminus E'$ which is a homeomorphism carrying λ_1^x to $\lambda_2^{\varphi(x)}$. By [17, Proposition 2.15], [20, Chapter II, Proposition 4.4] and Lemma 2.17, the ideal $K(\ell^2(E))$ is isomorphic to $C^*(\beta|E)$, $K(\ell^2(E'))$ is isomorphic to $C^*(\beta'|E')$, $C^*(E)|K(\ell^2(E))$ is isomorphic to $C^*(\beta|X\setminus E)$ and $C^*(E')|K(\ell^2(E'))$ is isomorphic to $C^*(\beta'|X'\setminus E')$. By [17, Proposition 2.10] we see that $C^*(E)|K(\ell^2(E))$ and $C^*(E')|K(\ell^2(E'))$ are isomorphic. To finish Part 1) we observe that if ∂E is periodic and $\partial E'$ is a non-trivial perturbation of ∂E, then the length of the canonical composition series of $C^*(E)$ and $C^*(E')$ is 2 and 3, respectively

Parts 2) and 3) are easily verified. ▨

If ∂E and $\partial E'$ are finite perturbation of the boundary of $\mathbf{Z}_+^2$, then in [3] it was shown that $C^*(E)$ and $C^*(E')$ are spatially isomorphic.

6. CONCLUDING REMARKS

Let E be a $\mathbf{Z}_+^m$-module.

1) It would be desirable to have a version of Corollary 4.5 in the general case.

CONJECTURE. *$C^*(E)$ is* G.C.R. *if and only if for each $F \leqslant E$ either $K(\ell^2(F)) \subset C^*(F)$ or $C^*(F)$ is reducible and a* G.C.R. *algebra.*

2) What is the structure of a C^*-algebra generated by non-commutative weighted shifts $T_1, \ldots, T_m \in \mathscr{L}(l^2(E))$, where the polar decomposition of T_i is $U(\varepsilon_i)D_i$? This might be useful in an extension of Theorem 5.8. In the appendix we characterize those algebras that contain $K(l^2(E))$.

3) When $C^*(E)$ and $C^*(E')$ are spatially isomorphic? When $C^*(E)$ and $C^*(E')$ are isomorphic as C^*-algebras?

Let $C^*(E)$ and $C^*(E')$ be algebraically isomorphic. If $C^*(E)$ is G.C.R. then the canonical map implements a spatial isomorphism. This is simply because E' is a translate of E.

4) Let G be a locally compact abelian group and let G_+ be a subsemigroup of G that is the closure of its interior and satisfies $G_+ - G_+ = G$. A module over G_+ is a closed subset $E \subset G$ such that $E + G_+ \subset E$. It follows that E has positive Haar measure and so we may form $L^2(E)$. For $f \in L^1(G)$ define the operator $U(f)$ by the formula

$$U(f)\,\xi(s) = \int_G f(t)\xi(s-t)1_E(s-t)\,dt$$

for $\xi \in L^2(G)$, $s \in E$. These operators might be called generalized Wiener Hopf operators (see [17]). In a subsequent paper we plan to investigate the C^*-algebra generated by $\{U(f): f \in L^1(G)\}$.

7. APPENDIX

Here we present our proofs of the O'Donovan's theorems that determine when the ideal of compact operators are contained in the C^*-algebra generated by a weighted shift. (There is a minor error in his original statement in the case of unilateral weighted shifts.) The method used also applies in a more general context.

THEOREM. [18, Theorem 2.5.1]. *Let $V = U\mathrm{Diag}(d_k) \in \mathscr{L}(H)$ be a bilateral weighted shift with $d_k > 0$ for all $k \in \mathbf{Z}$. Then $C^*(V) \cap K(H) = (0)$ if and only if there exists $n_i \to \pm\infty$ with $d_{n_i+k} \to d_k$, for all $k \in \mathbf{Z}$.*

Proof. Assume that $n_i \to \infty$ satisfy the hypothesis. That is, for each word $(d_{-m}, \ldots, d_m)$, the $\lim_{n_i \to \infty} (d_{-m+n_i}, \ldots, d_{m+n_i})$ exists and is $(d_{-m}, \ldots, d_m)$. Let $\{e_k : k \in \mathbf{Z}\}$ be the orthonormal basis such that $\mathrm{Diag}(d_i)e_k = d_k e_k$ and let H_n be the subspace spanned by $\{e_k : |k| \leqslant n\}$. The first ingredient of the proof is that given a polynomial $p(V, V^*)$ in V and V^*, its action on each H_n can be approximated as closely as we want by $U^{*n_i} p(V, V^*) U^{n_i}$ if n_i is large enough. The second ingredient is that for a compact operator F the $\lim_{l\to\infty} \|F|H_l^\perp\|$ is zero. Assume that there is a compact F in $C^*(V)$ with $\|F\| = 1$. Let ε be positive and let $p(V, V^*)$ be a polynomial so that $\|p(V, V^*) - F\| \leqslant \varepsilon$. Let $f \in H_n$ be a unit vector with $\|p(V, V^*)f\| \geqslant \|p(V, V^*)\| - \varepsilon$. Let $l \in \mathbf{N}$ be such that $\|F H_l^{\perp}\| < \varepsilon$. Choose n_i that satisfies

$$\|p(V, V^*)U^{n_i}f\| \geqslant \|p(V, V^*)f\| - \varepsilon \quad \text{and} \quad n_i > 2\max(l, n + \text{degree } p).$$

Then

$$\varepsilon \geqslant \|(p(V, V^*) - F)U^{n_i}f\| \geqslant \|p(V, V^*)U^{n_i}f\| - \|FU^{n_i}f\| \geqslant$$

$$\geqslant \|p(V, V^*)f\| - 2\varepsilon \geqslant \|p(V, V^*)\| - 3\varepsilon \geqslant 1 - 4\varepsilon.$$

This is a contradiction since ε is arbitrary.

For the converse we use the contrapositive to obtain a finite rank operator in $C^*(V)$. Assume that the conclusion is not valid, i.e., that (d_k) contains a unique word $(d_{-m}, \ldots, d_m)$. If δ is $\inf_{l \neq -m} \max \left\{ |d_m - d_l|, \ldots, \left| \prod_{s=0}^{2m} d_{-m+s} - \prod_{s=0}^{2m} d_{l+s} \right| \right\}$, then $\delta > 0$. Consider the diagonal operators $|V^k|$ with $k \in \{1, \ldots, 2m+1\}$. Take continuous real functions $\{f_k : k \in \{1, \ldots, 2m+1\}\}$ with range $[0, 1]$, $f_k\left(\prod_{s=0}^{k-1} d_{-m+s}\right) = 1$, $\operatorname{supp} f_k \subset \left[\prod_{s=0}^{k-1} d_{-m+s} - \delta, \prod_{s=0}^{k-1} d_{-m+s} + \delta\right]$ and $f_k(0) = 0$. (This last condition is necessary if the weight sequence is not bounded away of zero.) Let $R = \sum_{k=1}^{2m+1} f_k(|V^k|)$. Then $R \in C^*(V)$ since f_k is the uniform limit of polynomials with constant terms equal to zero. Furthermore R is diagonal and $\langle Re_{-m}, e_{-m}\rangle = 2m + 1$ while $0 \leqslant \langle Re_l, e_l\rangle \leqslant 2m$ for $l \neq m$. If g is a continuous increasing positive function with $g(2m) = 0$ and $g(2m+1) = 1$ then $g(R)$ is the projection onto the span of $\{e_{-m}\}$. Since $g(0) = 0$ we obtain again that $g(R) \in C^*(V)$. This is a contradiction since we have assumed that there is a non-zero compact operator in $C^*(V)$. ▨

THEOREM. [18, Theorem 3.2.1]. *Let $V = U\mathrm{Diag}(d_k) \in \mathscr{L}(H)$ be a unilateral weighted shift with $d_k > 0$ for all $k \in \mathbf{Z}_+$. Then: $C^*(V) \cap K(H) = (0)$ if and only if there exists $n_i \to \infty$ with $d_{n_i+k} \to d_k$, for all $k \in \mathbf{Z}_+$, and $d_{n_i-1} \to 0$.*

Proof. If the condition on (d_k) is satisfied, then we can prove that there is no non-zero compact operator in $C^*(V)$ just by copying the argument in the bilateral case. The converse is also very similar, the only variant is the necessity of using $|V^*|$. ▨

With the same arguments we can prove the following two propositions. In general the weighted shifts are not commutative. The weights are always positive.

PROPOSITION 7.1. *Let E be a $\mathbf{Z}_+^m$-module and let $\{T_i \in \mathscr{L}(\ell^2(E)) : i \in \{1, \ldots, m\}\}$ be weighted shifts with polar decomposition $T_i = U(\varepsilon_i)D_i$, $D_i e_n = a_{n,i} e_n$ for all*

$n \in E$. Let $C^*(T_1, \ldots, T_m)$ be the C^*-algebra generated by $\{T_i : i \in \{1, \ldots, m\}\}$. Then: $C^*(T_1, \ldots, T_m) \cap K(\ell^2(E)) = (0)$ if and only if for each $p \in E$, $k \in \mathbf{N}$, and $\varepsilon > 0$ there exists an $l \in E$, $l \neq p$ such that

1) $|a_{p+n,i} - a_{l+n,i}| < \varepsilon$ if $p+n,\ l+n \in E$ and $|n| \leqslant k$,

2) $|a_{l+n,i}| < \varepsilon$ if $p+n \notin E$ but $l+n,\ p+n+\varepsilon_i \in E$ and $|n| \leqslant k$.

PROPOSITION 7.2. *Let $\{T_i, \tilde{T}_i \in \mathscr{L}(\ell^2(\mathbf{Z}^m)) : i \in \{1, \ldots, m\}\}$ be weighted shifts with polar decomposition $T_i = U(\varepsilon_i)D_i$ and $\tilde{T}_i = U(\varepsilon_i)\tilde{D}_i$, where $D_i e_n = a_{n,i}e_n$ and $\tilde{D}_i e_n = \tilde{a}_{n,i}e_n$ for $n \in \mathbf{Z}^m$. The map $T_i \to \tilde{T}_i$ extends to a representation from $C^*(T_1, \ldots, T_m)$ onto $C^*(\tilde{T}_1, \ldots, \tilde{T}_m)$ if and only if for each $q \in \mathbf{Z}^m$, $k \in \mathbf{N}$ and $\varepsilon > 0$ there exists a $p \in \mathbf{Z}^m$ such that $|a_{p+n,i} - \tilde{a}_{q+n,i}| \leqslant \varepsilon$ for all $|n| \leqslant k$.*

We were interested in extending [18, Theorem 2.1.1] to our setting. Thus we have the following simple and very general proposition.

PROPOSITION 7.3. *Let $A \subset \mathscr{L}(H)$ and let $\{e_i : i \in I\}$ be an orthonormal basis for H. Assume that the C^*-algebra generated by A is irreducible. If $\{f_i : i \in I\}$ is another orthonormal basis for H such that for every $T \in A$ the matrices of T with respect o $\{e_i : i \in I\}$ and $\{f_i : i \in I\}$ are equal, then $f_i = \lambda e_i$ for all $i \in I$, where λ is complex number of absolute value 1.*

Proof. It is clear that for every S in β, the $*$-algebra generated by A, the matrices of S with respect to $\{e_i\}$ and $\{f_i\}$ are the same. Let $S_\alpha \in \beta$ be a net that converges weakly to $e_n \otimes e_m$. ($(e_n \otimes e_m)x = \langle x, e_m \rangle e_n$ for all $x \in H$.) It is immediate that S_α also converges weakly to $f_n \otimes f_m$, and so the conclusion holds. ▨

COROLLARY 7.4. 1) *Let $T \in \mathscr{L}(H)$ and let $\{e_n : n \in \mathbf{N}\}$ be an orthonormal basis for H. If T is irreducible then the only bases for which T have the same matrix that T has with respect to $\{e_n : n \in \mathbf{Z}\}$ are $\{\lambda e_n : n \in \mathbf{Z}\}$, $|\lambda| = 1$. In particular, if T is an irreducible shift this was obtained in* [18, Theorem 2.1.1].

2) *Let E be a $\mathbf{Z}_+^m$-module. If ∂E is not periodic then the basis $\{e_p : p \in E\}$ is the only one for which $U(\varepsilon_i)$, $i \in \{1, \ldots, m\}$, have these shift form.*

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Received May 7, 1984.