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NONNUCLEAR SUBALGEBRAS OF C^*-ALGEBRAS

BRUCE BLACKADAR

1. INTRODUCTION

There has been much work in the last decade or so attempting to shed light on the internal structure of non-type-I C^*-algebras. Much of this work has concerned the structure of nuclear C^*-algebras. Nuclear C^*-algebras are characterized by a large number of equivalent structure properties which are of great technical use in applications and which make the algebras relatively tractable objects for study; at the same time the class of nuclear C^*-algebras appears to be broad enough to include almost all C^*-algebras which arise "naturally". See [4] for a survey of the structure of nuclear C^*-algebras.

The class of nuclear C^*-algebras is closed under many standard operations such as quotients, extensions, inductive limits, tensor products, and crossed products by amenable groups. It was long an open question, however, whether a C^*-subalgebra of a nuclear C^*-algebra is necessarily nuclear. Some experts believed this to be the case until Choi [1] produced a counterexample: the nonnuclear "Choi algebra" C can be embedded in the nuclear "Cuntz algebra" O_2.

The main result of this note is:

THEOREM 1. *Let A be a C^*-algebra which is not type* I. *Then A contains a nonnuclear C^*-subalgebra.*

(Of course, a type I C^*-algebra cannot contain a nonnuclear subalgebra, since any subalgebra of a type I C^*-algebra is type I.)

Theorem 1 is a consequence of Theorem 2, which is of independent interest.

THEOREM 2. *Let A be a non-type-*I *C^*-algebra. Then A contains a C^*-subalgebra which has O_2 as a quotient.*

Proof of Theorem 1 from Theorem 2. Let B be a subalgebra of A and $\pi : B \to O_2$ a surjective homomorphism. Regard the Choi algebra C as a subalgebra of O_2. Then $\pi^{-1}(C)$ is a nonnuclear C^*-subalgebra of A. ▨

To prove Theorem 2. it suffices to find an example of such a subalgebra for A a UHF algebra, since every non-type-I C^*-algebra has a C^*-subalgebra with a UHF quotient [5, 6.7.4]. In fact, it suffices to find an example in the case that A is the crossed product of a UHF algebra by some automorphism, since by a result of Voiculescu [7] any such crossed product can be embedded in a larger UHF algebra.

Most of the work of this paper was done during the AMS Summer Research Institute on Operator Algebras and Applications, Kingston, Ont., summer 1980. The question was raised whether a *finite* nuclear C^*-algebra could have a nonnuclear subalgebra. George Elliott and I were able to construct such an example, and the construction of this paper was an outgrowth of that example. I am grateful to the organizers of the Institute, and to the National Science Foundation for a support grant.

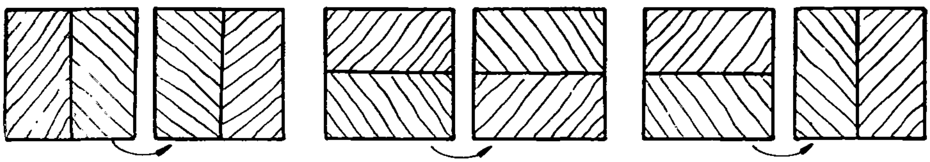
The construction done in Kingston actually showed that the crossed product of the CAR algebra by the tensor product shift automorphism contains a nonnuclear subalgebra [2, 6.2.1]. The missing step in finishing Theorem 2 was the problem of embedding this crossed product into a UHF algebra. This problem remained open (with some experts conjecturing a negative solution) until the beautiful recent result of Voiculescu.

2. CONSTRUCTION OF THE EXAMPLE

It will be convenient to describe our C^*-algebras concretely as algebras of operators on $L^2(X)$, where X is the unit square $[0, 1]^2$ with Lebesgue measure. A typical point of X will be denoted by its base 2 expansions $(.a_0a_1a_2\ldots, .b_1b_2\ldots)$, where $a_i, b_j \in \{0,1\}$. If $a \in \{0,1\}$, write $\bar{a}$ for $1 - a$.

Let E_n be the multiplication operator corresponding to the characteristic function of $\{(.a_0a_1\ldots, .b_1b_2\ldots): a_n = 0\}$ if $n \geqslant 0$, $\{(.a_0a_1\ldots, .b_1b_2\ldots): b_{-n} = 0\}$ if $n < 0$. Let T_n be the operator defined by $T_nf(.a_0a_1\ldots, .b_0b_1\ldots) = f(.a_0a_1\ldots \ldots\bar{a}_n\ldots, .b_0b_1\ldots)$ if $n \geqslant 0$, and $T_nf(.a_0a_1\ldots, .b_0b_1\ldots) = f(.a_0a_1\ldots, .b_0b_1\ldots \ldots\bar{b}_n\ldots)$ if $n < 0$. Then E_n and T_n generate a 2×2 matrix algebra for each n, and the different matrix algebras commute. The C^*-algebra M generated by all the E_n and T_n is thus a UHF algebra of type 2^∞. Let D be the C^*-subalgebra of M generated by all the E_n and 1. Then D is the commutative C^*-algebra generated by all multiplication operators by characteristic functions of subrectangles of X with sides parallel to the sides of X and dyadic rational corners, and is a standard diagonal in M. Let U be the operator defined by $Uf(.a_0a_1\ldots, .b_1b_2\ldots) = f(.a_1a_2\ldots, .a_0b_1b_2\ldots)$. Then U is unitary, $UE_nU^* = E_{n+1}$, $UT_nU^* = T_{n+1}$, so ad U leaves M invariant and the induced automorphism of M is the shift on tensor product factors. Let A be the C^*-algebra generated by M and U. Then A is isomorphic to the crossed product of M by ad U (this crossed product is simple, so the representation on $L^2(X)$ is faithful).

The operators E_0, E_{-1}, T_0, T_{-1}, and U can be pictured geometrically: E_0 is multiplication by the characteristic function of the left of the square, E_{-1} by that of the bottom half of the square. $1-E_0$ corresponds to the right half of the square, $1-E_{-1}$ the top half. T_0 interchanges the left and right halves; T_{-1} interchanges the top and bottom halves. U maps the bottom half to the left half and the top half to the right half (the standard "cutting and stacking" transformation of ergodic theory).



Let $S_1 = U(1 - E_{-1})$, $S_2 = T_0S_1$. S_1 may be thought of as mapping the top half to the right half and vanishing on the bottom half; similarly S_2 maps the top half to the left half and vanishes on the bottom half. S_1 and S_2 are partial isometries; $S_1^*S_1 = S_2^*S_2 = 1 - E_{-1}$; $S_1S_1^* = 1 - E_0$, $S_2S_2^* = E_0$.

Let B be the C^*-subalgebra of A generated by S_1, S_2, and all of the elements of D which "vanish along the top of the square", i.e., the ideal K of D generated by the characteristic functions of dyadic rectangles whose top edge lies strictly below the top of the square. Let J be the (closed two-sided) ideal of B generated by K. It is easy to see that J is a proper ideal of B. Let $\pi: B \to B/J$ be the quotient map, and $s_i = \pi(S_i)$. Then B/J is generated by s_1 and s_2. We have $s_1s_1^* + s_2s_2^* = 1$ since $S_1S_1^* + S_2S_2^* = 1$; and $s_i^*s_i = 1$ $(i = 1,2)$ since $S_i^*S_i = 1 - E_{-1}$ and $E_{-1} \in J$. Thus B/J is isomorphic to O_2.

REMARK 3. Theorem 2 shows that even a UHF algebra is "residually infinite", i.e., contains a subalgebra which has a quotient containing proper isometries. Thus there is no intrinsic characterization of AF algebras or sub-AF algebras (those which can be embedded in AF algebras) by any type of residual finiteness. Voiculescu's result and Theorem 2 (along with earlier results of Pimsner [6]) show that a surprising number of C^*-algebras can be embedded in AF algebras.

In the same vein, it follows from Theorem 1 that no subclass of the class of nuclear C^*-algebras (except the type I C^*-algebras) can have its members characterized among all C^*-algebras by a condition like "A contains no subset (or subalgebra) of the form (or with the property) ..." (cf. [3, § 10].) There is, however, some hope that the subnuclear C^*-algebras (ones which can be embedded in a nuclear C^*-algebra) could be characterized by such a condition. It may well be that subnuclear C^*-algebras are also characterized by an intrinsic condition such as C^*-exactness.

QUESTION 4. If A is a subnuclear C^*-algebra, is there a sub-AF algebra which has A as a quotient?

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BRUCE BLACKADAR
Department of Mathematics,
University of Nevada,
Reno, Nevada 89557,
U.S.A.

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