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NORM LIMITS OF FINITE DIRECT SUMS OF I_∞ FACTORS

BERNDT BRENKEN

Consider unital C^*-algebras $\mathfrak{A}$ which are direct limits $\varinjlim(\mathfrak{A}_n, \varphi_n)$, where $\{\mathfrak{A}_n \mid n \in \mathbf{N}\}$ is a sequence of von Neumann algebras, each a finite direct sum of countably decomposable type I_∞ factors and where $\varphi_n : \mathfrak{A}_n \to \mathfrak{A}_{n+1}$ is an injective unital homomorphism. Call such an algebra a type I_∞ sequence algebra. Included in this class are the type I_∞ funnels of [9]. The isomorphism classes of these algebras are completely described by the isomorphism classes of monoids associated with the algebras in a manner analogous to the dimension group theory of AF algebras [2, 4, 5]. In fact, the enveloping groups of these monoids are the K_0 groups of these algebras; however, these are zero ($K_0(\mathcal{M}) = 0$ for a type I_∞ factor $\mathcal{M}$). For algebras where the maps φ_n are "finite embeddings" we conclude that the isomorphism classes are described by isomorphism classes of certain dimension groups. The monoid associated with a type I_∞ sequence algebra has a partial ordering and the ideal structure of the algebra is reflected in the (order) ideal structure of the partially ordered monoid. Simple conditions involving the semilattice consisting of all idempotents in the monoid distinguish various ideal structures.

The countable decomposability of the factors ensures that each embedding φ_n is normal ([6, 9]). All representations are on separable Hilbert spaces and all homomorphisms of C^*-algebras are $*$-homomorphisms. If $\mathcal{H}$ is a Hilbert space, $\mathcal{B}(\mathcal{H})$ will denote the von Neumann algebra of all bounded operators on $\mathcal{H}$; $\mathrm{Id}_{\mathcal{H}}$ will be the identity operator and Id_r ($r \in \mathbf{N} \cup \{\infty\}$) will mean $\mathrm{Id}_{\mathcal{H}}$ for some Hilbert space $\mathcal{H}$ of dimension r. By subspace of a Hilbert space we mean closed subspace. Ideals of a C^*-algebra will be closed and two sided. An automorphism α of a C^*-algebra $\mathfrak{C}$ is inner if there is a unitary U in $\mathfrak{C}$ with $\alpha(x) = UxU^* = \mathrm{ad}\, U(x)$ $(x \in \mathfrak{C})$. If $X_1 \xrightarrow{\varphi_1} X_2 \xrightarrow{\varphi_2} X_3 \ldots$ is a sequence of sets and maps, define $\varphi_{mn} = \varphi_{m-1} \ldots \varphi_n$ $(m > n)$.

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1. THE MONOID ASSOCIATED WITH AN ALGEBRA

Let M be the additive monoid $\mathbf{N} \cup \{\infty\} = \{0, 1, \ldots, \infty\}$ where $x + \infty = \infty$ $(x \in M)$. There is also on M an abelian multiplication which distributes over addition where $x \cdot \infty = \infty$ $(x \in M \setminus \{0\})$ and $0 \cdot x = 0$ $(x \in M)$. Given a monoid homomorphism $A : M \to M$ there is an m $(= A(1))$ in M with $A(x) = mx$ $(x \in M)$. In particular $A \equiv 0$ or $A(\infty) = \infty$. Thus a monoid homomorphism $\tau : \overset{r}{\oplus} M \to \overset{s}{\oplus} M$ is described by a matrix $[\tau_{ij}]$ with entries $\tau_{ij} \in M$ (define $\tau_{ij} = p_i \tau(e_j)$ where $p_i : \overset{s}{\oplus} M \to M$ is the i^{th} coordinate map and e_j is the element of $\overset{r}{\oplus} M$ which is one at j and zero elsewhere). Write M^r for $\overset{r}{\oplus} M$.

Let $\mathscr{R} = \bigoplus_{i=1}^{r} \mathscr{R}_i$ and $\mathscr{N} = \bigoplus_{i=1}^{s} \mathscr{N}_i$ be finite direct sums of (countably decomposable) type I_∞ factors and $\varphi : \mathscr{R} \to \mathscr{N}$ a normal homomorphism mapping unit to unit. As in AF theory [2, 4, 5] we associate with φ a monoid homomorphism $\varphi_* : M^r \to M^s$, $\varphi_* = [\varphi(i,j)]$ with $\varphi(i,j) \in M$.

Let p_i, q_i be the units of $\mathscr{N}_i$, $\mathscr{R}_i$ respectively and consider the normal $*$-homomorphism $\gamma : \mathscr{R}_j \to \mathscr{N}_i$ given by mapping $x \in \mathscr{R}_j$ to $p_i \varphi(x)$. As $\mathscr{R}_j$ is a factor this map is either zero, in which case $\varphi(i,j) = 0$, or as now assumed, injective. A representation Γ of $\mathscr{N}_i$ is unitarily equivalent to an isomorphism of $\mathscr{N}_i$ with $B(\mathscr{H}) \otimes \mathrm{Id}_{\mathscr{L}}$ where $\mathscr{H}$ and $\mathscr{L}$ are Hilbert spaces, $\mathscr{H}$ infinite dimensional. Thus $\Gamma\gamma$ is unitarily equivalent to the representation $(\pi \oplus 0) \otimes \mathrm{Id}_{\mathscr{L}}$ with π a nondegenerate normal representation of $\mathscr{R}_j$ on a subspace $\mathscr{H}_1$ of $\mathscr{H}$ and 0 the zero representation of $\mathscr{R}_j$ on $\mathscr{H}_1^\perp$. Now $p_i = \sum_{k=1}^{r} p_i \varphi(q_k)$ as $\varphi(1) = 1$, so $\mathscr{H}_1^\perp$ is either zero or infinite dimensional. We have

$$\varphi(\mathscr{R}_j)' \cap \mathscr{N}_i = \gamma(\mathscr{R}_j)' \cap \mathscr{N}_i \cong \Gamma(\gamma(\mathscr{R}_j)' \cap \mathscr{N}_i) = (\pi(\mathscr{R}_j)' \oplus B(\mathscr{H}_1^\perp)) \otimes \mathrm{Id}_{\mathscr{L}} .$$

The type I_∞ factor $\pi(\mathscr{R}_j)$ has type I_n commutant $(n \in M)$ completely determined by $\varphi(\mathscr{R}_j)' \cap \mathscr{N}_i$. Define $\varphi(i,j) = n$, the multiplicity of $\mathscr{R}_j$ in $\mathscr{N}_i$.

Inner automorphisms of $\mathscr{N}$ applied to φ have no effect on the associated matrix. If $U = \oplus U_i$ is a unitary in $\mathscr{N}$,

$$((\mathrm{ad}\, U)\varphi(\mathscr{R}_j))' \cap \mathscr{N}_i = U_i(\varphi(\mathscr{R}_j)' \cap \mathscr{N}_i)U_i^* \cong \varphi(\mathscr{R}_j)' \cap \mathscr{N}_i .$$

Thus $((\mathrm{ad}\, U)\varphi)_* = \varphi_*$.

The matrix φ_* contains information for a canonical description (cf. [4]) of the map φ. Given γ_i a representation of $\mathcal{N}_i$ as $\mathcal{B}(\mathcal{H}_i)$, let Γ_i be the normal representation of $\mathcal{N}$ defined by mapping x to $\gamma_i(p_i x)$. We have $\mathrm{Id}_{\mathcal{H}_i} = \Gamma_i(p_i) = \sum_k \Gamma_i(\varphi(q_k))$ where $\Gamma_i(\varphi(q_k))$ is a projection corresponding to a zero or infinite dimensional subspace $\mathcal{L}_{ik}$ of $\mathcal{H}_i$. Thus $\Gamma_i\varphi$ is unitarily equivalent to a representation of the form $\bigoplus_k \pi_{ik}$ on $\bigoplus_k \mathcal{L}_{ik}$ where π_{ik} is a nondegenerate representation of $\mathcal{R}_k$ on $\mathcal{L}_{ik}$. Each representation π_{ik} is unitarily equivalent to a representation of $\mathcal{R}_k$ as $\mathcal{B}(\mathcal{P}_{ik}) \otimes \mathrm{Id}_{n_{ik}}$ where $n_{ik} \in M$ and $\mathcal{P}_{ik}$, if not zero, is an infinite dimensional Hilbert space. It follows that $\varphi(i, k) = n_{ik}$.

Using the canonical form of these maps and the fact that two non zero, normal representations π_1, π_2 of a type I factor $\mathcal{R}$ are unitarily equivalent if and only if $\pi_1(\mathcal{R})' \cong \pi_2(\mathcal{R})'$, we have the next result.

PROPOSITION 1.1. *If φ, ψ are two $*$-homomorphisms mapping $\mathcal{R}$ to $\mathcal{N}$ ($\mathcal{R}, \mathcal{N}$ as above and φ, ψ mapping unit to unit) with $\varphi_* = \psi_*$ then there is an inner automorphism α of $\mathcal{N}$ with $\alpha\varphi = \psi$.*

The canonical description also enables us to see that if $\varphi : \mathcal{R} \to \mathcal{N}$ and $\psi : \mathcal{N} \to \mathcal{S}$ are unital homomorphisms of finite sums of type I_∞ factors then $(\psi\varphi)_* = \psi_*\varphi_*$.

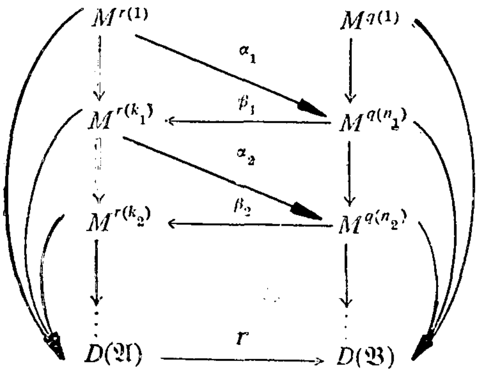
The assumption that $\varphi : \mathcal{R} \to \mathcal{N}$ is unital implies that for each i there is a j with $\varphi(i, j) \neq 0$. If φ is also injective (so $\varphi \mid \mathcal{R}_j$ is injective for all j) then there is an i for each j with $\varphi(i, j) \neq 0$.

PROPOSITION 1.2. *If $\mathcal{R} = \bigoplus^r \mathcal{R}_i$, $\mathcal{N} = \bigoplus^s \mathcal{N}_i$ are finite sums of type I_∞ factors and $[\psi(i, j)]$ is an $s \times r$ matrix (with entries in M) with at least one non zero entry in each row and column then there is an injective unital $*$-homomorphism $\psi : \mathcal{R} \to \mathcal{N}$ with $\psi_* = [\psi(i, j)]$.*

Let $\mathfrak{A}$, $\mathfrak{B}$ be type I_∞ sequence algebras where $\mathfrak{A} = \varinjlim(\mathfrak{A}_n, \varphi_n)$, $\mathfrak{B} = \varinjlim(\mathfrak{B}_n, \psi_n)$ and $\mathfrak{A}_n = \bigoplus_{k=1}^{r(n)} \mathfrak{A}_{nk}$, $\mathfrak{B}_n = \bigoplus_{k=1}^{q(n)} \mathfrak{B}_{nk}$ are sums of type I_∞ factors. The maps φ_n, ψ_n (and thus the canonical maps $i_n : \mathfrak{A}_n \to \mathfrak{A}$, $j_n : \mathfrak{B}_n \to \mathfrak{B}$) are by assumption unital injections. To each type I_∞ sequence algebra $\mathfrak{A}$, associate the monoid $D(\mathfrak{A}) = \varinjlim(M^{r(n)}, (\varphi_n)_*)$. Denote by $(i_n)_* : M^{r(n)} \to D(\mathfrak{A})$ the canonical maps and let $R(n) = \{1, \ldots, r(n)\}$, $Q(n) = \{1, \ldots, q(n)\}$.

PROPOSITION 1.3. *Let $\mathfrak{A}$, $\mathfrak{B}$ be type I_∞ sequence algebras. If $\Gamma : D(\mathfrak{A}) \to D(\mathfrak{B})$ is an isomorphism then there is an isomorphism $\tilde{\Gamma} : \mathfrak{A} \to \mathfrak{B}$.*

Proof. The monoids M^s ($s \in \mathbf{N}$) are finitely generated and so we obtain a commutative diagram of monoid homomorphisms



The commutativity of the diagram ensures that the maps α_i, β_i have at least one non zero entry in each row and column.

Define unital injections $\tilde{\alpha}_1 : \mathfrak{A}_1 \to \mathfrak{B}_{n_1}$, $\tilde{\beta}_1 : \mathfrak{B}_{n_1} \to \mathfrak{A}_{k_1}$ with $(\tilde{\alpha}_1)_* = \alpha_1$ and $(\tilde{\beta}_1)_* = \beta_1$. Proposition 1.1 states that there is an inner automorphism ξ of $\mathfrak{A}_{k_1}$ with $\xi\tilde{\beta}_1\tilde{\alpha}_1 = \varphi_{k_1 1}$, so renaming $\xi\tilde{\beta}_1$ as $\tilde{\beta}_1$ we have $\tilde{\beta}_1\tilde{\alpha}_1 = \varphi_{k_1 1}$. Continuing in this manner we arrive at a sequence of compatible unital injections $\tilde{\alpha}_{j+1} : \mathfrak{A}_{k_j} \to \mathfrak{B}_{n_{j+1}}$ ($j \in \mathbf{N}$) defining a $*$-isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$. ∎

We proceed to show the converse: an isomorphism of two type I_∞ sequence algebras yields an isomorphism of the associated monoids. For $g = (g_1, \ldots, g_{r(n)}) \in M^{r(n)}$ write $\bar{g}$ for the element $(i_n)_* g$ of $D(\mathfrak{A})$. If $h \in \mathfrak{A}_n$ is a projection with $\operatorname{rank}(h \cdot \mathrm{Id}_{\mathfrak{A}_{nk}}) = g_k$ ($k \in R(n)$), write $\operatorname{rank} h = g$.

We shall make use of some standard results (see [4] for example). If $x \in \mathfrak{A}$ is a projection (unitary, respectively) and $\varepsilon > 0$, there is an $n \in \mathbf{N}$ and there is a projection (unitary) y in $\mathfrak{A}_n$ with $\|x - i_n(y)\| < \varepsilon$. If e, f are projections in a unital C^*-algebra $\mathfrak{C}$ with $\|e - f\| < 1$ then there is a unitary $U \in \mathfrak{C}$ with $\operatorname{ad} U(e) = f$ and $\|U - 1\| < 2\|e - f\|$. The following is also true [cf. 7, 4].

Lemma 1.4. *Let $\mathfrak{C}$ be a C^*-subalgebra of a C^*-algebra $\mathfrak{F}$. If there is a partial isometry $u \in \mathfrak{F}$ with $u^*u = p \in \mathfrak{C}$ and $uu^* = q \in \mathfrak{C}$ and there is $a \in \mathfrak{C}$ with $\|a\| < 1$ and $\|a - u\| < \varepsilon$ $(< 1/2)$ then there is a partial isometry $w \in \mathfrak{C}$ with $w^*w = p$, $ww^* = q$ and $\|w - u\| < 3\varepsilon$.*

For each projection p in $\mathfrak{A}$ define as follows $[p]$ in $D(\mathfrak{A})$. There is an $n \in \mathbf{N}$ for which we can choose a projection p_0 in $\mathfrak{A}_n$ with $\|i_n(p_0) - p\| < 1/4$. Let $[p] = \bar{g}_0$ where $g_0 = \operatorname{rank} p_0 \in M^{r(n)}$. If p_1 in $\mathfrak{A}_m$ is another projection with $\|i_m(p_1) - p\| < 1/4$ then (if $m > n$)

$$\|\varphi_{mn}(p_0) - p_1\| = \|i_m(p_0) - i_n(p_1)\| < 1/2$$

and the projections $\varphi_{mn}(p_0)$ and p_1 are unitarily equivalent in $\mathfrak{A}_m$. Thus $\bar{g}_1 = \bar{\text{rank}}(\varphi_{mn}(p_0)) = (\varphi_{mn})_* g_0$, $\bar{g}_0 = \bar{g}_1$ and $[p]$ is well defined. The next proposition makes clear the relationship of the monoid and the map $p \to [p]$ to the K_0 group and the dimension function ([4], [5]).

PROPOSITION 1.5. *If p_1, p_2 are projections in $\mathfrak{A}$ then $[p_1] = [p_2]$ if and only if there is a partial isometry v in $\mathfrak{A}$ with $v^*v = p_1$, $vv^* = p_2$.*

Proof. Choose projections q_1, q_2 in $\mathfrak{A}_n$ with $\|i_n(q_k) - p_k\| < 1/4$ $(k = 1, 2)$. We have $[i_n(q_k)] = [p_k]$ and $i_n(q_k)$ is unitarily equivalent (via a unitary in $\mathfrak{A}$) to p_k $(k = 1, 2)$. It is therefore enough to prove the result for the projections $i_n(q_1)$ and $i_n(q_2)$. If $[i_n(q_1)] = [i_n(q_2)]$, i.e. $\overline{\text{rank}\, q_1} = \overline{\text{rank}\, q_2}$, then there is an $m \geqslant n$ with $(\varphi_{mn})_* \text{rank}\, q_1 = (\varphi_{mn})_* \text{rank}\, q_2$ and there is a partial isometry v in $\mathfrak{A}_m$ with initial projection $(\varphi_{mn})q_1$ and final projection $(\varphi_{mn})q_2$. Conversely, suppose there is a partial isometry v in $\mathfrak{A}$ with $v^*v = i_n(q_1)$ and $vv^* = i_n(q_2)$. By Lemma 1.4 there is a partial isometry w in $\mathfrak{A}_m$ $(m \geqslant n)$ with $w^*w = \varphi_{mn}(q_1)$ and $ww^* = \varphi_{mn}(q_2)$. Thus $\text{rank}\, \varphi_{mn}(q_1) = \text{rank}\, \varphi_{mn}(q_2)$ (in $\mathfrak{A}_m$), $(\varphi_{mn})_* \text{rank}\, q_1 = (\varphi_{mn})_* \text{rank}\, q_2$ and $[i_n(q_1)] = [i_n(q_2)]$. ▨

It follows that if p, q are projections in $\mathfrak{A}$ with $\|p - q\| < 1$ then $[p] = [q]$. It also follows from Proposition 1.5 and the Murray-von Neumann additivity of equivalence that $[p + q] = [p] + [q]$ for orthogonal projections $p, q \in \mathfrak{A}$.

Given a $*$-homomorphism $\Phi : \mathfrak{A} \to \mathfrak{B}$ of type I_∞ sequence algebras, define a map $\Phi_* : D(\mathfrak{A}) \to D(\mathfrak{B})$ by $\varphi_* g = [\Phi(p)]$ where p is any projection in $\mathfrak{A}$ with $[p] = g$. Such projections abound, for if $g_0 \in M^{r(n)}$ with $\bar{g}_0 = g$ then $[i_n(p_0)] = g$ where p_0 is a projection in $\mathfrak{A}_n$ with $\text{rank}\, p_0 = g_0$. Proposition 1.5 implies that Φ_* is a well defined map.

PROPOSITION 1.6. *If $\Phi : \mathfrak{A} \to \mathfrak{B}$ is a $*$-homomorphism of type I_∞ sequence algebras then there is a monoid homomorphism $\Phi_* : D(\mathfrak{A}) \to D(\mathfrak{B})$. If $\Psi : \mathfrak{B} \to \mathfrak{C}$ is another such $*$-homomorphism, $(\Psi\Phi)_* = \Psi_* \Phi_*$. If Φ is an isomorphism, Φ_* is a monoid isomorphism.*

Proof. We need only check that Φ_* is a monoid homomorphism. This follows from the fact that given $g_1, g_2 \in M^{r(n)}$ we may choose orthogonal projections p_1, p_2 in $\mathfrak{A}_n$ with $\text{rank}\, p_1 = g_1$ and $\text{rank}\, p_2 = g_2$. ▨

If $\Gamma : D(\mathfrak{A}) \to D(\mathfrak{B})$ is an isomorphism then the proof of Proposition 1.3 yields an isomorphism $\tilde{\Gamma} : \mathfrak{A} \to \mathfrak{B}$. We have $(\tilde{\Gamma})_* = \Gamma$. The isomorphism $\tilde{\Gamma}$ is "local", that is for each $n \in \mathbf{N}$ there is an m $(= m_n) \in \mathbf{N}$ with $\tilde{\Gamma}(i_n(\mathfrak{A}_n)) \subseteq j_m(\mathfrak{B}_m)$.

Recall that an automorphism α of a C^*-algebra (with unit) is approximately inner if and only if there is a net of inner automorphisms converging pointwise in norm to α.

PROPOSITION 1.7. *If α is an approximately inner automorphism of a type I_∞ sequence algebra $\mathfrak{A}$ then $\alpha_* = \mathrm{Id}_{D(\mathfrak{A})}$.*

Proof. If $\varepsilon < 1$ and p is a projection in $\mathfrak{A}$ then there is a unitary U in $\mathfrak{A}$ with $\|\mathrm{ad}\, U(p) - \alpha(p)\| < \varepsilon$. We have $\alpha_*[p] = [\alpha(p)] = [\mathrm{ad}\, U(p)] = [p]$. ▨

Although the isomorphism constructed in Proposition 1.3 is not necessarily unique, we can conclude that any two local isomorphisms so constructed are approximately inner equivalent.

PROPOSITION 1.8. *If α is a local automorphism of $\mathfrak{A}$ with $\alpha_* = \mathrm{Id}$ then α is approximately inner.*

Proof. If $\alpha(i_n(\mathfrak{A}_n)) \subseteq i_m(\mathfrak{A}_m)$, define $\alpha_n^m : \mathfrak{A}_n \to \mathfrak{A}_m$ by $i_m\alpha_n^m(x) = \alpha(i_n(x))$ $(x \in \mathfrak{A}_n)$. If $r > m$ then $\alpha_n^r = \varphi_{rm}\alpha_n^m$. For a projection $p \in \mathfrak{A}_n$, $\overline{\mathrm{rank}\, p} = \alpha_* \overline{\mathrm{rank}\, p} = \overline{\mathrm{rank}\, \alpha_n^m(p)} = (\alpha_n^m)_* \overline{\mathrm{rank}\, p}$. Thus, as $M^{r(n)}$ is finitely generated, there is an $\tilde{m} \geq m$ with $(\varphi_{\tilde{m}n})_* = (\alpha_n^{\tilde{m}})_*$. Proposition 1.1 yields a unitary $U_{\tilde{m}} \in \mathfrak{A}_{\tilde{m}}$ with $(\mathrm{ad}\, U_{\tilde{m}})\varphi_{\tilde{m}n} = \alpha_n^{\tilde{m}}$. It follows that there are $n_j \in \mathbf{N}$ and unitaries $U_j \in \mathfrak{A}_{n_j}$ with $(\mathrm{ad}\, U_j)\varphi_{n_j n_{j-1}} = \alpha_{n_{j-1}}^{n_j}$ $(j \in \mathbf{N})$. We have $(\mathrm{ad}\, i_{n_j})(U_j) = \alpha$ on $i_{n_p}(\mathfrak{A}_{n_p})$ $(p \leq j)$ and thus $(\mathrm{ad}\, i_{n_j})(U_j) \to \alpha$ pointwise in norm. ▨

2. IDEAL STRUCTURE

We describe the ideal structure in a manner closely resembling the AF algebra situation [2]. If J is an ideal of a direct limit C^*-algebra $\mathfrak{F} = \varinjlim(\mathfrak{F}_n, \varphi_n)$, we have $J = \overline{\cup\, i_n(i_n^{-1}(J))}$ where $i_n : \mathfrak{F}_n \to \mathfrak{F}$ are the canonical maps ([3]). If J_n is the ideal $i_n^{-1}(J)$ of $\mathfrak{F}_n$ then $\varphi_n^{-1}(J_{n+1}) = J_n$. Conversely, if ideals I_n of $\mathfrak{F}_n$ are specified with $\varphi_n^{-1}(I_{n+1}) = I_n$ then $I = \overline{\cup\, i_n(I_n)}$ is an ideal of $\mathfrak{F}$ and $i_n^{-1}(I) = I_n$.

Let $\mathscr{R} = \oplus \mathscr{R}_i$ be a finite direct sum of (countably decomposable) type I_∞ factors. An ideal J of $\mathscr{R}$ is given by $\oplus J_k$ where J_k if not zero is either $\mathscr{R}_k$ or the ideal of compact operators K_k of $\mathscr{R}_k$. Define $\Lambda(J) = \{k \mid J_k \neq 0\}$. If $\mathscr{N} = \oplus \mathscr{N}_i$ is also a finite direct sum of type I_∞ factors and $\varphi : \mathscr{R} \to \mathscr{N}$ a unital injection, define $S(k) = \{q \mid \varphi(q, k) \neq 0\}$ and $S'(k) = \{q \mid \varphi(q, k) = \infty\}$.

LEMMA 2.1. *With $\varphi : \mathscr{R} \to \mathscr{N}$ as above and J, I ideals of $\mathscr{R}, \mathscr{N}$ respectively we have $\varphi^{-1}(I) = J$ if and only if the following conditions are satisfied.*

i) *If $k \in \Lambda(J)$ and $q \in S(k)$ then $q \in \Lambda(I)$. If in addition $J_k = \mathscr{R}_k$ or $q \in S'(k)$ then $I_q = \mathscr{N}_q$.*

ii) *If $S(k) \subset \Lambda(I)$ and $I_q = \mathscr{N}_q$ $(q \in S'(k))$ then $k \in \Lambda(J)$. If in addition $I_q = \mathscr{N}_q$ $(q \in S(k))$ then $J_k = \mathscr{R}_k$.*

Proof. First assume $\varphi^{-1}(I) = J$. If e_q is the identity of $\mathcal{N}_q$ and x is a non zero projection in J_k ($k \in \Lambda(J)$) then for $q \in S(k)$ we have $0 \neq e_q\varphi(x) \in I_q$ and $q \in \Lambda(I)$. If $J_k = \mathcal{R}_k$ then an infinite projection x in $\mathcal{R}_k$ yields an infinite projection $e_q\varphi(x)$ in I_q and so $I_q = \mathcal{N}_q$. Condition ii) follows from $\varphi(K_k) \subseteq \bigoplus_q \{K_q \mid q \in S(k)\setminus S'(k)\} \oplus \{\mathcal{R}_q \mid q \in S'(k)\} \subseteq I$ and $\mathcal{R}_k \subseteq \{\mathcal{R}_q \mid q \in S(k)\} \subseteq I$.

We show condition i) implies $J \subseteq \varphi^{-1}(I)$. For x a non zero projection in J_k we have $\varphi(x) = \sum e_q\varphi(x) \in I$ by showing $e_q\varphi(x) \in I_q$. If $q \in S(k)$ then $I_q \neq 0$. If $q \in S'(k)$ or if $J_k = \mathcal{R}_k$ then $\mathcal{N}_q = I_q$ and $e_q\varphi(x) \in I_q$. Otherwise, x and $e_q\varphi(x)$ are finite projections and $e_q\varphi(x) \in I_q$.

To show $\varphi^{-1}(I) \subseteq J$ it is enough to show $x_k = x \cdot \mathrm{Id}_{\mathcal{R}_k} \in J_k$ for x (and therefore x_k) a projection in $\varphi^{-1}(I)$. For $q \in S(k)$ and $x_k \neq 0$ we have $0 \neq e_q\varphi(x_k) \in I_q$ and $q \in \Lambda(I)$. If $q \in S'(k)$ then $e_q\varphi(x_k)$ is an infinite projection in I_q, $I_q = \mathcal{N}_q$ and condition ii) implies $k \in \Lambda(J)$. Thus $x_k \in J_k$ if x_k is a finite projection. If x_k is an infinite projection then $I_q = \mathcal{N}_q$ ($q \in S(k)$) and condition ii) implies $J_k = \mathcal{R}_k$ and $x_k \in J_k$. ▨

PROPOSITION 2.2. *Let J be an ideal of the type I_∞ sequence algebra $\mathfrak{A} = \varinjlim(\mathfrak{A}_n, \varphi_n)$. The ideals $i_n^{-1}(J)$ and $i_m^{-1}(J)$ of $\mathfrak{A}_n$, $\mathfrak{A}_m$ ($m > n$) respectively satisfy both conditions of Lemma* 2.1 *(where $\varphi = \varphi_{mn}$). Conversely, given ideals I_n of $\mathfrak{A}_n$ ($n \in \mathbf{N}$) such that the ideals I_n and I_m satisfy both conditions of Lemma* 2.1, *$I = \overline{\cup I_n}$ is an ideal of $\mathfrak{A}$ with $i_n^{-1}(I) = I_n$.*

The ideal structure of a type I_∞ sequence algebra $\mathfrak{A} = \varinjlim(\mathfrak{A}_n, \varphi_n)$ is reflected in the order structure of the monoid $D(\mathfrak{A}) = \varinjlim(M^{r(n)}, (\varphi_n)_*)$. First observe that there is an order $\leqslant$ on $D(\mathfrak{A})$ arising from the obvious coordinatewise partial ordering on $M^{r(n)}$ and the order preserving monoid homomorphisms $(\varphi_n)_*$. The monoid $D(\mathfrak{A})$ with this ordering is a partially ordered monoid [1], with greatest element $e = (i_n)_*(\infty, \ldots, \infty)$. If Γ is a monoid homomorphism of monoids associated with type I_∞ sequence algebras then Γ is a partially ordered monoid homomorphism.

Call a submonoid Q of a partially ordered monoid S an order ideal if given $g, h \in S$ with $g \leqslant h$ and $h \in Q$ then $g \in Q$. If $J_n = \bigoplus_{k=1}^{r(n)} J_{nk}$ is a two sided ideal of $\mathfrak{A}_n$, define an order ideal $Q_n = \bigoplus_{k=1}^{r(n)} Q_{nk}$ of $M^{r(n)}$ by

$$Q_{nk} = \begin{cases} 0 & \text{if and only if } J_{nk} = 0 \\ \mathbf{N} & \text{if and only if } J_{nk} = K_{nk} \\ M & \text{if and only if } J_{nk} = \mathfrak{A}_{nk}. \end{cases}$$

Any order ideal Q_n of $M^{r(n)}$ is of this form and we can define the corresponding ideal J_n of $\mathfrak{A}_n$. Given ideals J_n of $\mathfrak{A}_n$ ($n \in \mathbf{N}$) and corresponding order ideals Q_n of $M^{r(n)}$ then $(\varphi_n)_*^{-1}Q_{n+1} = Q_n$ if and only if $(\varphi_n)^{-1}J_{n+1} = J_n$. If Q_n is an order ideal of $M^{r(n)}$ ($n \in \mathbf{N}$) with $(\varphi_n)_*^{-1}Q_{n+1} = Q_n$ then $Q = \cup (i_n)_* Q_n$ is an order ideal of $D(\mathfrak{A})$ and $(i_n)_*^{-1}Q = Q_n$. These remarks imply that there is a one-to-one correspondence between order ideals of $D(\mathfrak{A})$ and ideals of $\mathfrak{A}$. If J_1, J_2 are ideals of $\mathfrak{A}$ with corresponding order ideals Q_1, Q_2 respectively, then $J_1 \subseteq J_2$ if and only if $Q_1 \subseteq Q_2$. Thus this one-to-one correspondence of ideals is a lattice isomorphism of (complete) lattices (where the partial ordering is defined by set inclusion).

The partially ordered monoid $D(\mathfrak{A})$ has "the" Riesz decomposition property (cf. [4]), namely if $x, y_1, y_2 \in D(\mathfrak{A})$ with $x \leqslant y_1 + y_2$ then there are $x_1, x_2 \in D(\mathfrak{A})$ with $x = x_1 + x_2$ and $x_1 \leqslant y_1$, $x_2 \leqslant y_2$. Thus, if Q_1, Q_2 are two order ideals of $D(\mathfrak{A})$ then $Q_1 + Q_2$ is an order ideal of $D(\mathfrak{A})$.

The set of idempotents $\tilde{M} = \{0, \infty\}$ of the monoid M is a submonoid. If $\mathfrak{A} = \varinjlim(\mathfrak{A}_n, \varphi_n)$ is a type I_∞ sequence algebra then $\tilde{D}(\mathfrak{A}) = \varinjlim(\tilde{M}^{r(n)}, (\varphi_n)_*)$ is (isomorphic to) the submonoid of all idempotents of $D(\mathfrak{A})$ and forms a join-semilattice with $g \vee h = g + h$ $(g, h \in \tilde{D}(\mathfrak{A}))$ [1]. Multiplication coordinatewise by ∞ gives rise to a partially ordered monoid homomorphism of $M^{r(n)}$ onto $\tilde{M}^{r(n)}$ which extends to a partially ordered monoid homomorphism L of $D(\mathfrak{A})$ onto $\tilde{D}(\mathfrak{A})$ with $L \mid \tilde{D}(\mathfrak{A}) = \mathrm{Id}_{\tilde{D}(\mathfrak{A})}$. If $\mathfrak{A}$ and $\mathfrak{B}$ are two type I_∞ sequence algebras and $\Gamma : D(\mathfrak{A}) \to D(\mathfrak{B})$ is a monoid homomorphism, then $\Gamma L = L\Gamma$.

PROPOSITION 2.3. *There is an injective join morphism I from $\tilde{D}(\mathfrak{A})$ to the lattice of order ideals.*

Proof. For $p \in \tilde{D}(\mathfrak{A})$ let $I(p)$ be the order ideal $\{x \in D(\mathfrak{A}) \mid x \leqslant p\}$. The Riesz decomposition property implies that $I(p) + I(q) = I(p + q)$ for $p, q \in \tilde{D}(\mathfrak{A})$ and thus I is a join morphism. If $I(p) = I(q)$ then $p \leqslant q$, $q \leqslant p$ and $p = q$. Note also that $I(0) = 0$ and $I(e) = D(\mathfrak{A})$. ∎

The join-semilattice $\tilde{D}(\mathfrak{A})$ has a universal lower bound (namely 0) so if every ascending chain in $\tilde{D}(\mathfrak{A})$ is finite then $\tilde{D}(\mathfrak{A})$ is actually a lattice [1]. In general, $\tilde{D}(\mathfrak{A})$ is not a lattice. For example, choose $\mathfrak{B}$ with $\tilde{D}(\mathfrak{B}) = \varinjlim(\tilde{M}^{r(n)}, (\varphi_n)_*)$ where $r(n) = n + 1$ and $(\varphi_n)(i, j) = 1$ if $(i, j) = (1, 1), (2, n + 1)$ or the form $(j + 1, j)$, otherwise 0. If $(i_1)_*(\infty, 0) = g$ and $(i_1)_*(0, \infty) = h$ then $\{g, h\}$ has no greatest lower bound in $\tilde{D}(\mathfrak{B})$. The join-semilattice $\tilde{D}(\mathfrak{A})$ is distributive however, and thus is classified by its Stone space [8].

COROLLARY 2.4. *Let $\mathfrak{A}$ be a type I_∞ sequence algebra. If $D(\mathfrak{A}) = \tilde{D}(\mathfrak{A})$ and $D(\mathfrak{A})$ has finitely many elements then the map I of Proposition 2.3 is an isomorphism of lattices.*

Proof. It is sufficient to show I is onto. If Q is an order ideal of $D(\mathfrak{A})$, then $Q = \vee\{I(g) \mid g \in Q\} = I(\vee\{g \mid g \in Q\})$. ▨

In general, the map I need not be onto, even if $D(\mathfrak{A}) = \tilde{D}(\mathfrak{A})$. For example, let $\mathfrak{A}_n = \bigoplus_k \{\mathfrak{A}_{nk} \mid k = 1, 2, \ldots, 2^n\}$ and φ_n be such that $(\varphi_n)(i,j) = \infty$ if $i = 2$ or $2j - 1$, 0 otherwise. Then $D(\mathfrak{A}) = \tilde{D}(\mathfrak{A})$ and the order ideal generated by $\{(i_{n+3})_*(0, \ldots, 0, \infty, 0, 0, 0) \mid n \in \mathrm{N}\}$ is not of the form $I(g)$ for some g in $\tilde{D}(\mathfrak{A})$.

If Q is an order ideal of $D(\mathfrak{A})$, denote by $\bar{Q}$ the order ideal $\{x \in D(\mathfrak{A}) \mid x \leqslant h$ $(h \in L(Q))\}$ containing Q.

PROPOSITION 2.5. *There is an order ideal Q of $D(\mathfrak{A})$ with $Q \neq \bar{Q}$ if and only if $\tilde{D}(\mathfrak{A}) \neq D(\mathfrak{A})$.*

Proof. If $\tilde{D}(\mathfrak{A}) = D(\mathfrak{A})$ then $Q = \bar{Q}$ for any order ideal Q of $D(\mathfrak{A})$. Conversely, if $g \in D(\mathfrak{A}) \setminus \tilde{D}(\mathfrak{A})$ then $ng \neq L(g)$ for $n \in \mathrm{N}$. Thus $L(g)$ is a member of $\bar{Q}$ but not of Q where Q is the order ideal $\{h \in D(\mathfrak{A}) \mid h \leqslant ng$ for some $n \in \mathrm{N}\}$. ▨

Consider the compact topological space $S = \times_{n \in \mathrm{N}} R(n)$ where $R(n)$ has the discrete topology and S the product topology. An element c in S is called a *path* (in $\mathfrak{A}$) if and only if $\varphi_n(c(n+1), c(n)) \neq 0$ $(n \in \mathrm{N})$. The set of paths P is closed and thus compact in S. We may define a path inductively as φ_n $(n \in \mathrm{N})$ is an injection, thus P is not the empty set.

LEMMA 2.6. *Let T_n be a non empty subset of $R(n)$ $(n \in \mathrm{N})$ such that given $b \in T_n$ there is $d \in T_{n-1}$ with $\varphi_{n-1}(b, d) \neq 0$ $(n > 0)$. Then there is a path c with $c(n) \in T_n$ $(n \in \mathrm{N})$.*

Proof. For $n \in \mathrm{N}$ define $P_n = \{p \in P \mid p(i) \in T_i$ for $i = 0, \ldots, n\}$. We may define a path $c \in P_n$ by defining $c(m)$ $(m \leqslant n)$ using the hypothesis and defining $c(m)$ $(m > n)$ inductively. Thus P_n $(n \in \mathrm{N})$ is not the empty set. We have $P_n \supseteq P_{n+1}$, P_n is closed and there is an element $c \in \bigcap P_n$. ▨

Given $g \in D(\mathfrak{A}) \setminus \tilde{D}(\mathfrak{A})$, choose $g_0 \in M^{r(n)}$ with $(i_n)_* g_0 = g$. For each $m \geqslant n$ let T_m be the non empty set $\{j \in R(m) \mid$ the j^{th} coordinate of $(\varphi_{mn})_* g_0 \notin \tilde{M}\}$. By Lemma 2.6 there is a path c with $c(m) \in T_m$ $(m \geqslant n)$. In other words, there is a "finite path" in $\mathfrak{A}$ if $D(\mathfrak{A}) \neq \tilde{D}(\mathfrak{A})$.

The non zero idempotents of $D(\mathfrak{A})$ correspond to (properly) infinite projections of $\mathfrak{A}$, and so may be called infinite elements of $D(\mathfrak{A})$. Define g in $D(\mathfrak{A})$ to be finite if $h \leqslant g$ and $h \in \tilde{D}(\mathfrak{A})$ implies $h = 0$. Note that $g \in D(\mathfrak{A})$ is finite if and only if $g_0 \in \mathrm{N}^{r(n)}$ for g_0 in $M^{r(n)}$ with $(i_n)_* g_0 = g$. Thus the set of finite elements $F(\mathfrak{A})$ is an order ideal of $D(\mathfrak{A})$.

If J is an ideal of $\mathfrak{A}$ with corresponding order ideal Q of $D(\mathfrak{A})$, then J is separable if and only if $Q \subseteq F(\mathfrak{A})$. In this case, J is a direct limit of finite sums of compact algebras and so is an AF algebra. Let $K(\mathfrak{A})$ denote the ideal of $\mathfrak{A}$ corresponding to the order ideal $F(\mathfrak{A})$. It is the unique maximal AF ideal in $\mathfrak{A}$.

A type I_∞ sequence algebra $\mathfrak{A} = \lim(\mathfrak{A}_n, \varphi_n)$ is said to be *finitely embedded* if there is an m (without loss of generality, $m = 1$) such that $(\varphi_n)_*$ is a matrix with entries in $\mathbf{N}$ (i.e., all entries are finite) for $n \geqslant m$. In this case $(\varphi_n)_* \mathbf{N}^{r(n)} \subseteq \mathbf{N}^{r(n+1)}$ and $\underline{\lim}(\mathbf{N}^{r(n)}, (\varphi_n)_*)$ is (isomorphic to) the order ideal $F(\mathfrak{A})$ of $D(\mathfrak{A})$.

PROPOSITION 2.7. *The type I_∞ sequence algebra $\mathfrak{A}$ is finitely embedded if and only if $D(\mathfrak{A}) = \overline{F(\mathfrak{A})}$.*

Proof. If $\mathfrak{A}$ is finitely embedded then $\overline{F(\mathfrak{A})} = D(\mathfrak{A})$. Conversely, $e \in \overline{F(\mathfrak{A})}$ and there is a $g \in F(\mathfrak{A})$ with $L(g) = e$. If $g_0 \in \mathbf{N}^{r(p)}$ with $(i_p)_* g_0 = g$ then there is an $n \geqslant p$ with $\infty \cdot (\varphi_{np})_*(g_0) = (\infty, \ldots, \infty)$ and thus $(\varphi_{np})_*(g_0) = g_1$ has no zero coordinates. We have $(\varphi_{mn})_* g_1 \in \mathbf{N}^{r(m)}$ for all $m > n$ and $(\varphi_{mn})_*$ has entries in $\mathbf{N}$ for all $m > n$. ∎

COROLLARY 2.8. *If $\Phi : \mathfrak{A} \to \mathfrak{B}$ is an isomorphism of type I_∞ sequence algebras with $\mathfrak{A}$ finitely embedded then $\mathfrak{B}$ is finitely embedded.*

Proof. The partially ordered monoid isomorphism $\Phi_* : D(\mathfrak{A}) \to D(\mathfrak{B})$ preserves finite elements and $L\Phi_* = \Phi_* L$. ∎

If $\mathfrak{A}$ is finitely embedded then the order ideal $F(\mathfrak{A})$ of finite elements is given by $\underline{\lim}\,(\mathbf{N}^{r(n)}, (\varphi_n)_*)$ and the corresponding AF ideal $K(\mathfrak{A})$ of $\mathfrak{A}$ satisfies $i_n^{-1}(K(\mathfrak{A})) = \oplus \{K_{nk} \mid k \in R(n)\}$. The dimension group $\mathrm{K}_0(K(\mathfrak{A})) = \underline{\lim}\,(\mathrm{K}_0(i_n^{-1}(K(\mathfrak{A}))), (\varphi_n)_*) = \underline{\lim}\,(\mathbf{Z}^{r(n)}, (\varphi_n)_*)$ ([4, 5]) which is the group completion of the partially ordered monoid $F(\mathfrak{A})$.

PROPOSITION 2.9. *Two finitely embedded type I_∞ sequence algebras $\mathfrak{A}$, $\mathfrak{B}$ are $*$-isomorphic if and only if their maximal* AF *ideals $K(\mathfrak{A})$, $K(\mathfrak{B})$ are $*$-isomorphic.*

Proof. Any isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$ restricts to an isomorphism of $K(\mathfrak{A})$ onto $K(\mathfrak{B})$. Conversely, if $\varphi : K(\mathfrak{A}) \to K(\mathfrak{B})$ is an isomorphism then there is an ordered group isomorphism $\mathrm{K}_0(\varphi) : \mathrm{K}_0(K(\mathfrak{A})) \to \mathrm{K}_0(K(\mathfrak{B}))$. As in Proposition 1.3 this yields a local isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$. ∎

Note that if $\mathfrak{A}$ is a finitely embedded type I_∞ algebra then it follows that there is a unital AF algebra A with $\mathfrak{A} \cong A \otimes \mathfrak{M}$ ($\mathfrak{M}$ a type I_∞ factor). The maximal AF ideal is $A \otimes K$.

PROPOSITION 2.10. *If $\mathfrak{A}$ is not finitely embedded then $\tilde{D}(\mathfrak{A}) = \{0, e\}$ if and only if $\mathfrak{A}$ is simple.*

Proof. If $p \in \tilde{D}(\mathfrak{A}) \setminus \{0, e\}$ then $I(p)$ is a non trivial order ideal of $D(\mathfrak{A})$. Conversely, suppose $\tilde{D}(\mathfrak{A}) = \{0, e\}$. The result follows from Corollary 2.4 if $D(\mathfrak{A})$

$= \tilde{D}(\mathfrak{A})$. Proposition 2.7 implies $\overline{F(\mathfrak{A})} = 0$ and so $F(\mathfrak{A}) = 0$. Thus, if $g \in D(\mathfrak{A}) \setminus \tilde{D}(\mathfrak{A})$ then g is not finite and there is a nonzero h in $\tilde{D}(\mathfrak{A})$ with $h \leqslant g$. It follows that $g = e$ and $D(\mathfrak{A}) = \tilde{D}(\mathfrak{A})$. ▨

PROPOSITION 2.11. *If $\mathfrak{A}$ is finitely embedded then $\tilde{D}(\mathfrak{A}) = \{0, e\}$ if and only if $\mathfrak{A}$ has exactly one nontrivial ideal. In this case the ideal is $K(\mathfrak{A})$.*

Proof. Let $\tilde{D}(\mathfrak{A}) = \{0, e\}$ and choose g a non zero element of an order ideal Q of $D(\mathfrak{A})$. If g is not finite, then, as in Proposition 2.10, $g = e$ and $Q = D(\mathfrak{A})$. Assume $Q \subseteq F(\mathfrak{A})$. The equality $L(g) = e$ implies that there is an n with $(i_n)_*(\mathbf{N}^{r(n)}) \subseteq Q$ and thus $F(\mathfrak{A}) \subseteq Q$. Conversely, let $p \in \tilde{D}(\mathfrak{A}) \setminus \{0, e\}$ and form the non zero order ideal $K(p) = \{g \in D(\mathfrak{A}) \mid g \text{ finite}, g \leqslant p\}$ contained in $F(\mathfrak{A})$. It follows that $L(K(p)) \subseteq \subseteq \{x \in \tilde{D}(\mathfrak{A}) \mid x \leqslant p\} \subsetneq \tilde{D}(\mathfrak{A}) = L(F(\mathfrak{A}))$ and $K(p) \subsetneq F(\mathfrak{A})$. ▨

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BERNDT BRENKEN
Department of Mathematics,
University of Toronto,
Toronto M5S 1A1,
Ontario,
Canada.

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