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DECOMPOSABLE OPERATORS AND AUTOMATIC CONTINUITY

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INTRODUCTION

This paper introduces a class of decomposable operators for which it is possible to give a very useful algebraic description of the spectral maximal subspaces. This class, the *super-decomposable* operators, is a subset of the strongly decomposable operators.

After developing the basic theory, we relate this notion to the classical ones and present several wide classes of examples, among them multiplication operators.

This leads naturally to questions about multipliers; in Section 3 an investigation is made of some of the relationships between super-decomposability of a multiplier on a Banach algebra and of the corresponding multiplication operator on the algebra of multipliers.

Finally, in Section 4 we present some applications to automatic continuity theory. We give necessary and sufficient conditions on a decomposable operator $T \in \mathfrak{L}(X)$ and a super-decomposable operator $S \in \mathfrak{L}(Y)$ that every linear map $\theta : X \to Y$ for which $\theta T = S\theta$ be automatically continuous. This generalizes work of Vrbová [23, 24]. Among the corollaries of this is the following: if $\theta : L^p(\mathbf{R}) \to L^q(\mathbf{R})$ $(1 \leqslant p,\ q < \infty)$ commutes with some non-trivial translation operator, then θ is continuous.

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1. DECOMPOSABLE OPERATORS AND DIVISIBLE SUBSPACES

Throughout this paper we shall use the standard notions and some basic results of the theory of decomposable operators as presented in [12] and [22]. Let $\mathfrak{F}(\mathbf{C})$ denote the family of all closed subsets of $\mathbf{C}$, and let $\mathfrak{L}(X)$ be the space of all continuous linear operators on a complex Banach space X. Given an operator $T \in \mathfrak{L}(X)$, let $\mathrm{Lat}(T)$ stand for the collection of all closed linear T-invariant subspaces of X.

If T has the single-valued extension property, we are interested in an algebraic representation of the spectral maximal spaces

$$X_T(F) := \{x \in X : \sigma_T(x) \subseteq F\} \quad \text{for } F \in \mathfrak{F}(\mathbf{C}),$$

where $\sigma_T(x)$ denotes the local spectrum of T at the point $x \in X$. Given a linear mapping $T : X \to X$ and a subset F of $\mathbf{C}$, we define the space $E_T(F)$ to be the span of all linear subspaces

$$Y \subseteq X \text{ such that } (T - \lambda)Y = Y \quad \text{for all } \lambda \in \mathbf{C} \setminus F.$$

Thus $E_T(F)$ is simply the largest linear subspace of X sharing this property; this space need not be closed in general. Recall that a linear subspace Y of X is said to be *T-divisible* if

$$(T - \lambda)Y = Y \quad \text{for all } \lambda \in \mathbf{C}.$$

Hence $E_T(\emptyset)$ is exactly the largest T-divisible linear subspace. It is easily seen that $E_T(\emptyset) = E_T(F)$ whenever $\sigma(T) \cap F = \emptyset$; for a proof of this and for further information on divisible subspaces see for instance [10].

1.1. PROPOSITION. *If $T \in \mathfrak{L}(X)$ has the single-valued extension property, then*

$$X_T(F) \subseteq E_T(F) \quad \textit{for all } F \in \mathfrak{F}(\mathbf{C}).$$

Moreover, if $T \in \mathfrak{L}(X)$ is decomposable, then the identity

$$X_T(F) = E_T(F) \quad \textit{for all } F \in \mathfrak{F}(\mathbf{C})$$

holds if and only if for every $F \in \mathfrak{F}(\mathbf{C})$ the linear operator

$$T_F : X/X_T(F) \to X/X_T(F)$$

canonically induced by T on the quotient space $X/X_T(F)$ has no divisible linear subspace different from $\{0\}$.

Proof. The first assertion can be easily deduced from [12, Proposition 1.1.2]. Now, let $T \in \mathfrak{L}(X)$ be decomposable and assume that $F \in \mathfrak{F}(\mathbf{C})$ satisfies $X_T(F) = E_T(F)$. We claim that $\{0\}$ is the only divisible linear subspace for T_F. Given an arbitrary T_F-divisible subspace $Z \subseteq X/X_T(F)$, we define $Y := Q^{-1}(Z) \subseteq X$, where $Q : X \to X/X_T(F)$ denotes the canonical quotient map. The space Y is T-invariant. Furthermore, for all $\lambda \in \mathbf{C} \setminus F$ and $u \in Y$ there exists $v \in Y$ such that $Q(u) = (T_F - \lambda)Q(v)$, which implies $u - (T - \lambda)(v) \in X_T(F)$. Noting again that

$\sigma(T \mid X_T(F)) \subseteq F$, we obtain $u - (T - \lambda)(v) = (T - \lambda)(w)$ for some $w \in X_T(F)$ and therefore $u = (T - \lambda)(v + w) \in (T - \lambda)Y$. We conclude that $Y = (T - \lambda)Y$ for all $\lambda \in \mathbf{C} \setminus F$ and hence $Y \subseteq E_T(F) = X_T(F)$. This forces Z to be trivial.

We finally suppose that none of the respective operators on the quotients has a divisible subspace different from $\{0\}$. Given any $F \in \mathfrak{F}(\mathbf{C})$, let $U, V \subseteq \mathbf{C}$ be open such that $F \subseteq U \subseteq \overline{U} \subseteq V$ and assume that U is connected and unbounded. As T is decomposable and $\mathbf{C} = V \cup (\mathbf{C} \setminus \overline{U})$, we obtain $X = X_T(\overline{V}) + X_T(\mathbf{C} \setminus U)$. Hence, by [2, Lemma 3], the induced linear operator $T_{\overline{V}}$ on the quotient space $X/X_T(\overline{V})$ satisfies $\sigma(T_{\overline{V}}) \subseteq \sigma_f(T \mid X_T(\mathbf{C} \setminus U))$, where σ_f denotes the full spectrum, i.e. the complement of the unbounded component of the resolvent set. Since $\sigma(T \mid X_T(\mathbf{C} \setminus U)) \subseteq \subseteq \mathbf{C} \setminus U$ and since U is connected and unbounded, we conclude that $\sigma(T_{\overline{V}}) \subseteq \subseteq \mathbf{C} \setminus U \subseteq \mathbf{C} \setminus F$. Consider $E_{T_{\overline{V}}}(F) \subseteq X/X_T(\overline{V})$. By the remarks immediately before this proposition, the equation $(T_{\overline{V}} - \lambda)E_{T_{\overline{V}}}(F) = E_{T_{\overline{V}}}(F)$ holds not only for all $\lambda \in \mathbf{C} \setminus F$, but also for all $\lambda \in F$, hence for all $\lambda \in \mathbf{C}$. Consequently, by our present assumption, $E_{T_{\overline{V}}}(F) = \{0\}$. Now

$$QE_T(F) = Q(T - \lambda)E_T(F) = (T_{\overline{V}} - \lambda)QE_T(F) \quad \text{for all } \lambda \in \mathbf{C} \setminus F,$$

where $Q : X \to X/X_T(\overline{V})$ denotes the quotient map. By maximality it follows that $QE_T(F) \subseteq E_{T_{\overline{V}}}(F) = \{0\}$ and thus $E_T(F) \subseteq X_T(\overline{V})$. Taking for U and V the complements of suitable discs in the complex plane, we arrive at

$$E_T(F) \subseteq X_T(\mathbf{C} \setminus D)$$

for every open disc $D \subseteq \mathbf{C}$ with positive distance from F. Since $\mathbf{C} \setminus F$ can be covered by countably many such discs D_n and since $X_T(\cdot)$ preserves countable intersections, we conclude that

$$E_T(F) \subseteq \bigcap_{n=1}^{\infty} X_T(\mathbf{C} \setminus D_n) = X_T\left(\mathbf{C} \setminus \bigcup_{n=1}^{\infty} D_n\right) = X_T(F),$$

which completes the proof.

Unfortunately it is not always easy to decide whether a given decomposable operator satisfies the condition concerning the quotients in Proposition 1.1. We therefore introduce the following class of operators, where the usual decomposition property from spectral theory is slightly strengthened; see for instance [22, Definition IV.4.12]. Let I denote the identity operator on X.

1.2. Definition. An operator $T \in \mathfrak{L}(X)$ is called *super-decomposable*, if for every pair of open sets $U, V \subseteq \mathbf{C}$ such that $U \cup V = \mathbf{C}$ there exists some $R \in \mathfrak{L}(X)$ such that $RT = TR$, $\sigma(T \mid \overline{R(X)}) \subseteq U$, and $\sigma(T \mid \overline{(I - R)(X)}) \subseteq V$.

This definition makes sense, because the condition $RT = TR$ forces the spaces $\overline{R(X)}$ and $\overline{(I - R)(X)}$ to be T-invariant. We shall see that Proposition 1.1 can be considerably strengthened in the case of a super-decomposable operator. First, however, we show that these operators admit partitions of unity in the sense of spectral theory and note some useful characterizations.

1.3. THEOREM. *Every super-decomposable operator $T \in \mathfrak{L}(X)$ is strongly decomposable. Moreover, for every finite open covering $\{U_1, \ldots, U_m\}$ of* **C** *there exist $R_1, \ldots, R_m \in \mathfrak{L}(X)$ such that $R_1 + \ldots + R_m = I$ as well as*

$$R_k T = T R_k \quad \textit{and} \quad \sigma(T \mid \overline{R_k(X)}) \subseteq U_k \quad \textit{for } k = 1, \ldots, m.$$

Proof. First note that T is decomposable by [2, Corollary 2]; see also [22, Theorem IV. 4.28]. Let $\mathfrak{E}$ denote the corresponding spectral capacity, and consider an open covering $\{U_1, \ldots, U_m\}$ of **C**, where $m \geqslant 2$. Then there exist open sets $V_1, W_1 \subseteq \mathbf{C}$ satisfying

$$\mathbf{C} \setminus (U_2 \cup \ldots \cup U_m) \subseteq W_1 \subseteq \overline{W}_1 \subseteq V_1 \subseteq \overline{V}_1 \subseteq U_1.$$

By assumption, we obtain an operator $S_1 \in \mathfrak{L}(X)$ such that $S_1 T = T S_1$ and $\sigma(T \mid \overline{S_1(X)}) \subseteq V_1$, $\sigma(T \mid \overline{(I - S_1)(X)}) \subseteq \mathbf{C} \setminus \overline{W}_1$. Since $W_1 \cup U_2 \cup \ldots \cup U_m = \mathbf{C}$, an obvious repetition of this argument supplies us with open sets $V_k, W_k \subseteq \mathbf{C}$ and with operators $S_k \in \mathfrak{L}(X)$ commuting with T for $k = 1, \ldots, m$ such that

$$\overline{W}_k \subseteq V_k \subseteq \overline{V}_k \subseteq U_k, \quad W_1 \cup \ldots \cup W_m = \mathbf{C},$$

$$\sigma(T \mid \overline{S_k(X)}) \subseteq V_k, \quad \sigma(T \mid \overline{(I - S_k)(X)}) \subseteq \mathbf{C} \setminus \overline{W}_k$$

for $k = 1, \ldots, m$. We now define

$$R_1 := S_1, \quad R_k := (I - S_1) \ldots (I - S_{k-1}) S_k \quad \text{for } k = 2, \ldots, m.$$

Obviously T commutes with each of the operators $R_1, \ldots, R_m$. Furthermore, one easily verifies by induction that

$$R_1 + \ldots + R_k = I - (I - S_1) \ldots (I - S_k) \quad \text{for } k = 1, \ldots, m.$$

We shall use this for $k = m$. Observe that every $Y \in \mathrm{Lat}(T)$ satisfies $Y \subseteq \mathfrak{E}(\sigma(T \mid Y))$ and that $S\mathfrak{E}(F) \subseteq \mathfrak{E}(F)$ for all $F \in \mathfrak{F}(C)$ and all $S \in \mathfrak{L}(X)$ commuting with T. Hence our construction yields

$$(I - S_1) \ldots (I - S_m)(X) \subseteq \bigcap_{k=1}^{m} \mathfrak{E}(\mathbf{C} \setminus W_k) = \mathfrak{E}\left(\mathbf{C} \setminus \bigcup_{k=1}^{m} W_k\right) = \mathfrak{E}(\emptyset) = \{0\}$$

and consequently $R_1 + \ldots + R_m = I$. Now let $k \in \{1, \ldots, m\}$ be given. Since $\sigma(T \mid \overline{S_k(X)}) \subseteq V_k$, it follows that $\overline{S_k(X)} \subseteq \mathfrak{E}(\overline{V}_k)$. Also

$$R_k(X) = (I - S_1) \ldots (I - S_{k-1}) S_k(X) \subseteq (I - S_1) \ldots (I - S_{k-1}) \mathfrak{E}(\overline{V}_k) \subseteq \mathfrak{E}(\overline{V}_k)$$

so that $\overline{R_k(X)} \subseteq \mathfrak{E}(\overline{V}_k)$. We know that for every fixed $\lambda \in \mathbf{C} \setminus U_k$ the restrictions $(T-\lambda) \mid \mathfrak{E}(\overline{V}_k)$ and $(T-\lambda) \mid \overline{S_k(X)}$ are bijective on $\mathfrak{E}(\overline{V}_k)$ and on $\overline{S_k(X)}$, respectively. This forces $(T-\lambda) \mid \overline{R_k(X)}$ to be bijective on $\overline{R_k(X)}$ as well. Indeed, $T-\lambda$ is one-to-one on $\mathfrak{E}(\overline{V}_k)$ and hence also on $\overline{R_k(X)}$; and if $A \in \mathfrak{L}(\mathfrak{E}(\overline{V}_k))$ denotes the inverse of the operator $(T-\lambda) \mid \mathfrak{E}(\overline{V}_k)$, one easily verifies that $AS_k(X) \subseteq \overline{S_k(X)}$, which implies $AR_k(X) \subseteq \overline{R_k(X)}$ and consequently $A\overline{R_k(X)} \subseteq \overline{R_k(X)}$. Thus $\sigma(T \mid \overline{R_k(X)}) \subseteq U_k$, which settles the last assertion of our theorem. For arbitrary $F \in \mathfrak{F}(\mathbf{C})$ we finally note that $R_k\mathfrak{E}(F) \subseteq \mathfrak{E}(F) \cap \mathfrak{E}(\overline{U}_k) = \mathfrak{E}(F \cap \overline{U}_k)$ for $k = 1, \ldots, m$ and hence

$$\mathfrak{E}(F) \subseteq R_1\mathfrak{E}(F) + \ldots + R_m\mathfrak{E}(F) \subseteq \mathfrak{E}(F \cap \overline{U}_1) + \ldots + \mathfrak{E}(F \cap \overline{U}_m).$$

Of course, the strong decomposability of T is also immediate from the characterization in [22, Theorem IV.4.28], which has to be applied to the restriction of T to an arbitrary spectral maximal space for T.

1.4. THEOREM. *For every $T \in \mathfrak{L}(X)$ the following assertions are equivalent:*

(a) *T is super-decomposable.*

(b) *For every open covering $\{U_1, U_2\}$ of $\mathbf{C}$ there exist spaces $X_1, X_2 \in \mathrm{Lat}(T)$ as well as operators $R_1, R_2 \in \mathfrak{L}(X)$ commuting with T such that $R_1 + R_2 = I$ and $R_j(X) \subseteq X_j$, $\sigma(T \mid X_j) \subseteq U_j$ for $j = 1, 2$.*

(c) *T is decomposable, and for every pair of spectral maximal spaces $Y, Z \in \mathrm{Lat}(T)$ satisfying $\sigma(T \mid Y) \cap \sigma(T \mid Z) = \emptyset$ there exists some $R \in \mathfrak{L}(X)$ commuting with T such that $R \mid Y = 0$ and $(I - R) \mid Z = 0$.*

Proof. (a) $\Rightarrow$ (*b*) is obvious.

(b) $\Rightarrow$ (c). Again, it follows easily from [2, Corollary 2] or [22, Theorem IV.4.28] that T is decomposable. Now, let $Y, Z \in \mathrm{Lat}(T)$ be spectral maximal spaces for T such that $\sigma(T \mid Y)$ and $\sigma(T \mid Z)$ are disjoint. Then $U_1 := \mathbf{C} \setminus \sigma(T \mid Y)$ and $U_2 := \mathbf{C} \setminus \sigma(T \mid Z)$ are open with $U_1 \cup U_2 = \mathbf{C}$. Let X_1, X_2 and R_1, R_2 be chosen according to condition (b). In order to show that $R_1 \mid Y = 0$, we fix an arbitrary $y \in Y$. Then $\sigma_T(R_1 y) \subseteq \sigma_T(y) \subseteq \sigma(T \mid Y) = \mathbf{C} \setminus U_1$ by [12, Proposition 1.1.2]. On the other hand, we conclude from $R_1(X) \subseteq X_1 \subseteq X_T(\sigma(T \mid X_1))$ that $\sigma_T(R_1 y) \subseteq \sigma(T \mid X_1) \subseteq U_1$. It follows that $\sigma_T(R_1 y) = \emptyset$, which implies $R_1 y = 0$, again by [12, Proposition 1.1.2]. Thus $R_1 \mid Y = 0$, and the same reasoning shows that $R_2 \mid Z = 0$. Consequently $R := R_1$ has the desired properties.

(c) $\Rightarrow$ (a). Given an arbitrary open covering U, V of $\mathbf{C}$, we choose open sets $U_1, U_2, V_1, V_2 \subseteq \mathbf{C}$ such that $U_1 \cup V_1 = \mathbf{C}$ and

$$U_1 \subseteq \overline{U}_1 \subseteq U_2 \subseteq \overline{U}_2 \subseteq U, \quad V_1 \subseteq \overline{V}_1 \subseteq V_2 \subseteq \overline{V}_2 \subseteq V.$$

Then $F_1 := \mathbf{C} \setminus U_1$ and $F_2 := \mathbf{C} \setminus V_1$ are closed and disjoint. Hence condition (c) supplies us with some $R \in \mathfrak{L}(X)$ commuting with T such that $R \mid X_T(F_1) = 0$ and

$(I-R)\,|\,X_T(F_2)=0$. We want to show that $\sigma(T\,|\,R(X))\subseteq \overline{U}_2$. Since $U_2\cup(\mathbf{C}\setminus\overline{U}_1)=\mathbf{C}$, we have the splitting

$$X=X_T(\overline{U}_2)+X_T(\overline{\mathbf{C}\setminus\overline{U}_1})=X_T(\overline{U}_2)+X_T(F_1).$$

Since $R\,|\,X_T(F_1)=0$, we have $R(X)=R(X_T(\overline{U}_2))\subseteq X_T(\overline{U}_2)$. Now take any $\lambda\in\mathbf{C}\setminus\overline{U}_2$ and consider the operator $S:=((T-\lambda)\,|\,X_T(\overline{U}_2))^{-1}$ on $X_T(\overline{U}_2)$. It suffices to show that $SR(X)\subseteq R(X)$, since this implies $S(\overline{R(X)})\subseteq\overline{R(X)}$ so that the restriction $S\,|\,\overline{R(X)}$ will be the inverse of $(T-\lambda)\,|\,\overline{R(X)}$. Given an arbitrary $x\in R(X)$, we have $x=Ry$ for some $y\in X_T(\overline{U}_2)$ and hence

$$Sx=SRy=SR(T-\lambda)Sy=S(T-\lambda)RSy=RSy\in R(X).$$

We have shown that $\sigma(T\,|\,\overline{R(X)})\subseteq\overline{U}_2\subseteq U$. By the same reasoning, we conclude that $\sigma(T\,|\,\overline{(I-R)(X)})$ is contained in V. This completes the proof.

The concept in condition (c) has been studied earlier by Apostol [9]; we shall return to this in Section 3. Condition (b) has been considered by Wang [25] who also notes the equivalence of (b) and (c).

1.5. Proposition. *Let $T\in\mathfrak{L}(X)$ be super-decomposable and assume that $\{0\}$ is the only T-divisible linear subspace of X. Then $X_T(F)=E_T(F)$ for all $F\in\mathfrak{F}(\mathbf{C})$.*

Proof. Given a closed subset F of $\mathbf{C}$, it suffices to show that $E_T(F)\subseteq X_T(\overline{V})$ for an arbitrary open neighborhood V of F, since $X_T(\cdot)$ is known to preserve countable intersections. We choose an open subset U of $\mathbf{C}$ such that $F\subseteq U\subseteq\overline{U}\subseteq V$ and choose $R\in\mathfrak{L}(X)$ commuting with T such that $\sigma(T\,|\,\overline{R(X)})\subseteq\mathbf{C}\setminus\overline{U}\subseteq\mathbf{C}\setminus F$ and $\sigma(T\,|\,\overline{(I-R)(X)})\subseteq\overline{V}$. The last inclusion implies that $(I-R)(X)\subseteq X_T(\overline{V})$. Hence $E_T(F)\subseteq X_T(\overline{V})$ will follow as soon as $R(E_T(F))$ is seen to be $=\{0\}$. To this end, let Z denote the largest linear subspace of $R(X)$ such that $(T-\lambda)Z=Z$ for all $\lambda\in\mathbf{C}\setminus F$. Since $\sigma(T\,|\,\overline{R(X)})\subseteq\mathbf{C}\setminus F$, the space Z is actually T-divisible. Hence our assumption on T forces Z to be trivial. On the other hand, we have $R(E_T(F))=(T-\lambda)R(E_T(F))$ for all $\lambda\in\mathbf{C}\setminus F$ and consequently $R(E_T(F))\subseteq Z=\{0\}$. The assertion follows.

1.6. Remark. Super-decomposable operators may well have non-trivial divisible subspaces. Indeed, in the following section we shall see that compact operators as well as quasi-nilpotent operators are necessarily super-decomposable, but if $X=C([0,1])$ and if $T\in\mathfrak{L}(X)$ denotes the Volterra operator given by

$$(Tf)(s):=\int_0^s f(t)\,dt \quad \text{for all } f\in C([0,1]) \text{ and } s\in[0,1],$$

then T is both compact and quasi-nilpotent and has the following non-trivial divisible linear subspace

$$Y := \{f \in C^\infty([0, 1]) : f^{(k)}(0) = 0 \quad \text{for all } k = 0, 1, 2, \dots\}.$$

On the other hand, it will become clear that in many important cases non-trivial divisible subspaces do not exist.

1.7. REMARK. The algebraic representation $X_T(F) = E_T(F)$ from Proposition 1.5 should be compared with the structure of the spectral maximal spaces for generalized scalar operators, which is of a similar flavor. We first note that there is an alternative description of the space $E_T(F)$ for an arbitrary linear mapping $T: X \to X$ and $F \subseteq \mathbf{C}$: consider the transfinite sequence of spaces given by $Y(0) := X$;

$$Y(\alpha + 1) := \bigcap_{\lambda \in \mathbf{C} \setminus F} (T - \lambda) Y(\alpha),$$

and if α is a limit ordinal

$$Y(\alpha) := \bigcap_{\beta < \alpha} Y(\beta).$$

Then it is easy to see that this transfinite sequence is eventually constant and that this constant value coincides with the space $E_T(F)$. Now, for a generalized scalar operator $T \in \mathfrak{L}(X)$ Vrbová [24, Theorem 1.2] proved the existence of some $p \in \mathbf{N}$ such that

$$X_T(F) = \bigcap_{\lambda \in \mathbf{C} \setminus F} (T - \lambda)^p X \quad \text{for all } F \in \mathfrak{F}(\mathbf{C});$$

moreover, in the special case of a normal operator on a Hilbert space p can be taken to be 1 by a theorem of Pták and Vrbová [19]. In light of the preceding discussion, this representation is certainly more precise than the identity $X_T(F) = E_T(F)$ for all $F \in \mathfrak{F}(\mathbf{C})$. In particular, it follows that generalized scalar operators do not have divisible subspaces different from $\{0\}$. Again, we shall see that all generalized scalar operators are super-decomposable.

Proposition 1.5 is also related to [15, Proposition 3.1].

2. EXAMPLES OF SUPER-DECOMPOSABLE OPERATORS

In this section, we present some important classes of super-decomposable operators. Our arguments will sometimes be easier than the corresponding classical proofs concerning decomposability, as given for instance in [12]. This simplification is due to the very useful characterization of decomposable operators from [22, Theorem IV.4.28], which is in some sense incorporated in our definition of super-decomposable operators.

Our first result is an immediate consequence of the definition of spectral operators in the sense of Dunford on an arbitrary complex Banach space X.

2.1. PROPOSITION. *Every spectral operator $T \in \mathfrak{L}(X)$ is super-decomposable.*

Next, let Ω be a non-empty subset of $\mathbf{C}$, and let $\mathfrak{A}$ be an admissible algebra of complex-valued function on Ω in the sense of [12, Definition 3.1.2]; standard examples are the algebras $C^k(\Omega)$ and $C^k(\overline{\Omega})$, where Ω is an open (and bounded) subset of $\mathbf{C}$ and $k \in \{0, 1, \ldots, \infty\}$. An operator $T \in \mathfrak{L}(X)$ is called $\mathfrak{A}$-*scalar*, if there exists an algebraic homomorphism $\Phi: \mathfrak{A} \to \mathfrak{L}(X)$ such that $\Phi(1) = I$ and $\Phi(Z) = T$, where Z denotes the identity function on Ω. By [5, Proposition 7], such a homomorphism is necessarily an $\mathfrak{A}$-spectral function in the sense of [12, Definition 3.1.3]. Consequently, the present definition of an $\mathfrak{A}$-scalar operator coincides with the usual terminology from [12, Definition 3.1.18]. Moreover, $T \in \mathfrak{L}(X)$ is called $\mathfrak{A}$-*spectral* [12, Definition 3.3.5], if there exists an $\mathfrak{A}$-spectral function $\Phi : \mathfrak{A} \to \mathfrak{L}(X)$ commuting with T such that T is quasi-nilpotent equivalent to $\Phi(Z)$.

Every operator $T \in \mathfrak{L}(X)$ with a totally disconnected spectrum is $\mathfrak{A}$-scalar for some admissible algebra $\mathfrak{A}$ [12, Example 3.1.20]; this holds, in particular, for all compact and all quasi-nilpotent operators. Another important example is furnished by the generalized scalar operators: recall from [12, Definition 4.1.2] that $T \in \mathfrak{L}(X)$ is *generalized scalar*, if there exists a continuous homomorphism $\Phi : C^\infty(\mathbf{C}) \to \mathfrak{L}(X)$ satisfying $\Phi(1) = I$ and $\Phi(Z) = T$. We mention in passing that, according to a recent automatic continuity result from [8, Theorem 6.5], the continuity of such a functional calculus $\Phi : C^\infty(\mathbf{C}) \to \mathfrak{L}(X)$ for T is equivalent to the absence of non-trivial T-divisible subspaces.

2.2. PROPOSITION. *Every $\mathfrak{A}$-spectral operator $T \in \mathfrak{L}(X)$ is super-decomposable. In particular, all generalized scalar operators as well as all operators with totally disconnected spectrum are super-decomposable.*

Proof. Let $\Phi : \mathfrak{A} \to \mathfrak{L}(X)$ be a homomorphism satisfying $\Phi(1) = I$ and $\Phi(f)T = T\Phi(f)$ for all $f \in \mathfrak{A}$ such that T is quasi-nilpotent equivalent to $S := \Phi(Z)$. Now, given an arbitrary open covering $\{U, V\}$ of $\mathbf{C}$, there exist $f, g \in \mathfrak{A}$ such that $\operatorname{supp} f \subseteq U$, $\operatorname{supp} g \subseteq V$, and $f + g = 1$ on Ω. Obviously $R := \Phi(f) \in \mathfrak{L}(X)$ commutes with S and T. To prove that $\sigma(S \mid \overline{R(X)}) \subseteq U$, fix some $k \in \mathfrak{A}$ such that $\operatorname{supp} k \subseteq U$ and $k = 1$ on an open neighborhood of $\operatorname{supp} f$. Since $\mathfrak{A}$ is admissible, for every $\lambda \in \mathbf{C} \setminus U$ there exists some $k_\lambda \in \mathfrak{A}$ satisfying $(Z - \lambda)k_\lambda = k$ on Ω. Because of $\Phi(k)\Phi(f) = \Phi(f)$, we conclude that

$$(S - \lambda)\Phi(k_\lambda)x = \Phi(k_\lambda)(S - \lambda)x = \Phi(k)x = x \quad \text{for all } x \in \overline{R(X)}.$$

Moreover, clearly $\Phi(k_\lambda)x \in \overline{R(X)}$ for all such x. Thus $\sigma(S \mid \overline{R(X)}) \subseteq U$ and similarly $\sigma(S \mid \overline{(I - R)(X)}) \subseteq V$ so that S is super-decomposable. From [12, Theorem 2.2.1] we conclude that T is decomposable and satisfies $X_T(F) = X_S(F)$ for all $F \in \mathfrak{F}(\mathbf{C})$.

Since $R(X) \subseteq X_S(\overline{U}) = X_T(\overline{U})$ and $(I - R)(X) \subseteq X_S(\overline{V}) = X_T(\overline{V})$, we finally deduce from Theorem 1.4 that T is super-decomposable.

2.3. THEOREM. *Let A be a commutative Banach algebra over $\mathbf{C}$ and assume either that the spectrum $\mathfrak{M}(A)$ is totally disconnected or that A is semi-simple and regular. Then, for every $a \in A$ and every algebraic homomorphism $\Phi : A \to \mathfrak{L}(X)$, the operator $\Phi(a) \in \mathfrak{L}(X)$ is super-decomposable. Moreover, the operators $R \in \mathfrak{L}(X)$ occurring in Definition 1.2 may be chosen in the unitization of $\Phi(A)$ in $\mathfrak{L}(X)$.*

Proof. (i) We first take the case that $\mathfrak{M} := \mathfrak{M}(A)$ is totally disconnected. Let us also assume that A has no identity element, which is slightly more involved than the case of a unital Banach algebra. Consider the unitization $\tilde{A} := A \oplus \mathbf{C}1$ of the Banach algebra A and the canonical extension $\tilde{\Phi} : \tilde{A} \to \mathfrak{L}(X)$ of the homomorphism Φ given by $\Phi(u + \lambda 1) := \Phi(u) + \lambda I$ for all $u \in A$, $\lambda \in \mathbf{C}$. Then $\tilde{\mathfrak{M}} := \mathfrak{M}(\tilde{A})$ is the one-point compactification of the locally compact space $\mathfrak{M}$ with the Gelfand topology. Since an operator $T \in \mathfrak{L}(X)$ is super-decomposable if and only if $T - \mu I$ is super-decomposable for some $\mu \in \mathbf{C}$, it suffices to prove the assertion for an operator $\tilde{\Phi}(a) \in \mathfrak{L}(X)$, where $a \in \tilde{A}$ satisfies $0 \notin \sigma(a)$. Now, given open sets $U, V \subseteq \mathbf{C}$ such that $U \cup V = \mathbf{C}$, we may assume that $\mu \in V$, where μ is the complex number for which $a - \mu 1 \in A$. Then $K := \tilde{\mathfrak{M}} \setminus \hat{a}^{-1}(V)$ is a compact subset of $\mathfrak{M}$ and $\hat{a}^{-1}(U) \cap \mathfrak{M}$ is an open neighborhood of K in $\mathfrak{M}$, where $\hat{a} : \tilde{\mathfrak{M}} \to \mathbf{C}$ denotes the Gelfand transform of a on $\tilde{\mathfrak{M}}$. As $\mathfrak{M}$ is locally compact, there exists a compact neighborhood L of K in $\mathfrak{M}$ such that $L \subseteq \hat{a}^{-1}(U)$. Since L is compact and totally disconnected, we may apply [26, Theorem 6.2.6] to obtain a compact and open subset C of $\mathfrak{M}$ such that $K \subseteq C \subseteq L \subseteq \hat{a}^{-1}(U)$. By the Shilov idempotent theorem [20], there exists an idempotent $e \in \tilde{A}$ such that $\hat{e} = 1$ on C and $\hat{e} = 0$ on $\tilde{\mathfrak{M}} \setminus C$. We claim that the operator $R := \tilde{\Phi}(e) \in \mathfrak{L}(X)$ satisfies the conditions of Definition 1.2 for $T := \tilde{\Phi}(a)$. Certainly R and T commute. Moreover, given any $\lambda \in \mathbf{C} \setminus U$, since $\sigma(ae) = \hat{a}\hat{e}(\tilde{\mathfrak{M}}) \subseteq U \cup \{0\}$, there exists some $e_\lambda \in \tilde{A}$ satisfying $(ae - \lambda)e_\lambda = e$, at least if $\lambda \neq 0$. Since $a \in \tilde{A}$ is invertible and $e \in \tilde{A}$ is idempotent, we arrive at $(a - \lambda)e_\lambda e = e$ for some suitable $e_\lambda \in \tilde{A}$ including the case $\lambda = 0$. It follows that

$$(T - \lambda)\tilde{\Phi}(e_\lambda e)x = \tilde{\Phi}(e_\lambda e)(T - \lambda)x = \tilde{\Phi}(e)x = x$$

and $\tilde{\Phi}(e_\lambda e)x \in \overline{R(X)}$ for all $x \in \overline{R(X)}$, which implies $\sigma(T \mid \overline{R(X)}) \subseteq U$. A similar argument ensures that $\sigma(T \mid \overline{(I - R)(X)}) \subseteq V$, which settles the first half of the theorem.

(ii) We now consider the case that A is semi-simple and regular. Without loss of generality, we may also assume that A has a unit element $1 \in A$ and that $\Phi(1) = I$, since otherwise the following argument can be applied to the unitization $\tilde{A}$ of A and to the canonical extension $\tilde{\Phi}$ of Φ. Given an arbitrary open covering $\{U, V\}$ of $\mathbf{C}$, we choose open sets $U_1, V_1 \subseteq \mathbf{C}$ such that $\overline{U}_1 \subseteq U$, $\overline{V}_1 \subseteq V$, and $U_1 \cup V_1 = \mathbf{C}$. Then $\hat{a}^{-1}(U_1)$ is an open neighborhood of the compact set $K := \mathfrak{M}(A) \setminus \hat{a}^{-1}(V_1)$

in $\mathfrak{M}(A)$. By the regularity of A, there exists some $b \in A$ such that $\hat{b} = 1$ on a neighborhood of K as well as $\operatorname{supp} \hat{b} \subseteq \hat{a}^{-1}(U_1)$. Using the semi-simplicity and again the regularity of A, we may successively choose $c, d \in A$ such that $bc = c$, $cd = d$ and $\hat{c} = 1$, $\hat{d} = 1$ on certain neighborhoods of K. We assert that $R := \Phi(c) \in \mathfrak{L}(X)$ has the desired super-decomposability properties for $T := \Phi(a)$. Clearly $RT = TR$. Since $\operatorname{supp} \hat{b} \subseteq \hat{a}^{-1}(U_1)$ and $\overline{U}_1 \subseteq U$, [12, Theorem 6.2.5] supplies us, for every $\lambda \in \mathbf{C} \setminus U$, with some $b_\lambda \in A$ satisfying $(a - \lambda)b_\lambda = b$. Taking the identity $bc = c$ into account, we arrive at

$$(T - \lambda)\Phi(b_\lambda)x = \Phi(b_\lambda)(T - \lambda)x = \Phi(b)x = x$$

for all $x \in R(X)$. Moreover, we have $\Phi(b_\lambda)x \in R(X)$ for all such x and therefore $\sigma(T \mid R(X)) \subseteq U$. On the other hand, we know that $\operatorname{supp}(1 - \hat{d}) \subseteq \hat{a}^{-1}(\overline{V}_1)$, $\overline{V}_1 \subseteq V$, and $(1 - d)(1 - c) = 1 - c$. Hence the same method yields the inclusion $\sigma(T \mid \overline{(I - R)(X)}) \subseteq V$, which completes the proof of the theorem.

The preceding result is related to [3, Corollary 4.7] concerning systems of operators with non-analytic functional calculi. The following easy consequence strengthens a classical result on the decomposability of certain multiplication operators from [12, Theorem 6.2.6].

2.4. COROLLARY. *Let X be a commutative, semi-simple and regular Banach algebra over* **C**. *Then every multiplication operator on X is super-decomposable and has no divisible subspace different from* $\{0\}$.

Proof. For each $a \in X$, let $T_a \in \mathfrak{L}(X)$ denote the corresponding multiplication operator on X given by $T_a(x) := ax$ for all $x \in X$. Then Theorem 2.3 applies to the left regular representation Φ of X given by $\Phi(a) := T_a$ for all $a \in X$. In order to prove the last assertion, we observe that every divisible linear subspace Y of X for such an operator T_a is obviously contained in the radical of X. Hence the semi-simplicity of X forces Y to be trivial.

Unfortunately, this result does *not* carry over to the case of multipliers on regular Banach algebras and to the case of multiplication operators on semi-simple Banach algebras. We shall investigate the important examples of the Banach algebras $L^1(G)$ and $M(G)$, where G denotes an arbitrary locally compact abelian group. The algebras of all absolutely continuous and all discrete measures on G will be denoted by $M_a(G)$ and $M_d(G)$, respectively. Thus $M_a(G) \cong L^1(G)$ in the canonical way; and $M_a(G) + M_d(G)$ is exactly the subalgebra of all measures on G, whose continuous part is even absolutely continuous.

Now, for each $\mu \in M(G)$ let $T_\mu : L^1(G) \to L^1(G)$ denote the corresponding *convolution operator* given by $T_\mu(f) := \mu * f$ for all $f \in L^1(G)$. Thus the operators T_μ for $\mu \in M(G)$ are precisely the multipliers of the Banach algebra $L^1(G)$; see [18, Theorem 0.1.1]. We shall also consider the convolution operators $\tilde{T}_\mu : M(G) \to M(G)$ given by $\tilde{T}_\mu(\nu) := \mu * \nu$ for all $\mu, \nu \in M(G)$, which are just the multiplication operators on the Banach algebra $M(G)$.

It is well-known that $M_a(G)$ and $M_d(G)$ are both semi-simple and regular Banach algebras acting on $L^1(G)$ and on $M(G)$ by convolution. Hence Theorem 2.3 ensures that T_μ and $\tilde{T}_\mu$ are super-decomposable, whenever $\mu \in M_a(G) \cup M_d(G)$. Recently, it has been observed independently by Albrecht [4, Lemma 3.2] and by Eschmeier [14, Corollary 3] that for every non-discrete locally compact abelian group G there exists $\mu \in M(G)$ such that neither T_μ nor $\tilde{T}_\mu$ is decomposable; thus answering in the negative a question of Colojoară and Foiaş [12]. On the other hand, it is shown in [4, Theorem 3.1] and [14, Corollary 12] that T_μ is decomposable at least for all measures $\mu \in M_a(G) + M_d(G)$. We now give a short proof of a slight extension of this result, which will be useful for us in connection with certain automatic continuity problems. A precise characterization of those $\mu \in M(G)$, for which T_μ and $\tilde{T}_\mu$ are (super)-decomposable, is still missing.

2.5. THEOREM. *For every $\mu \in M(G)$, the convolution operators T_μ and $\tilde{T}_\mu$ on $L^1(G)$ and $M(G)$, respectively, do not have any divisible subspace different from $\{0\}$. Moreover, $T_\mu \in \mathfrak{L}(L^1(G))$ and $\tilde{T}_\mu \in \mathfrak{L}(M(G))$ are super-decomposable, whenever $\mu \in M_a(G) + M_d(G)$.*

Proof. For $f \in L^1(G)$ and $\mu \in M(G)$, let $\hat{f}, \hat{\mu} : \hat{G} \to \mathbf{C}$ denote the corresponding Fourier and Fourier-Stieltjes transform on the dual group $\hat{G}$ of G. Consider a T_μ-divisible subspace $Y \subseteq L^1(G)$ and fix an arbitrary $f \in Y$. Then, given $\gamma \in \hat{G}$, we have $f = \mu * g - \hat{\mu}(\gamma)g$ for some suitable $g \in Y$ and consequently $\hat{f}(\gamma) = 0$. This implies $\hat{f} = 0$ for all $f \in Y$ and hence $Y = \{0\}$. Next observe that $\tilde{T}_\mu$ is a multiplication operator on a commutative and semi-simple Banach algebra. Thus every $\tilde{T}_\mu$-divisible subspace $Y \subseteq M(G)$ has to be trivial, as well. For the main assertion here is a proof which is completely different from the corresponding arguments in [4] and [14]: Since $M(G)$ is a commutative semi-simple Banach algebra with unit, there exists a closed regular subalgebra A of $M(G)$, which contains all closed regular subalgebras of $M(G)$. This interesting result was obtained by Albrecht [4, Theorem 2.4] using the theory of decomposable operators. Now, $M_a(G)$ and $M_d(G)$ are certainly closed regular subalgebras of $M(G)$ so that $M_a(G) + M_d(G) \subseteq A$. Hence the assertion follows immediately from Theorem 2.3: simply apply this result to the representation Φ of A on $L^1(G)$ and $M(G)$, respectively, given by convolution.

Let us finally note that, in general, the sum of two super-decomposable operators may be far from being super-decomposable, as can be easily inferred from [22, Example V.6.29]. Thus the preceding result is not a completely trivial consequence of Theorem 2.3.

3. SUPER-DECOMPOSABLE MULTIPLICATION OPERATORS

We now continue our investigation of multiplication operators. The emphasis will be on the relations between the concept of super-decomposability and some aspects from the theory of decomposable multiplication operators developed by Apostol [9].

Again, let X be a complex Banach space, and consider a closed sub-algebra B of $\mathfrak{L}(X)$ containing the identity operator I. Recall from [9, Definition 2.7] that B is *normal* with respect to a given operator $T \in B$, if for every pair of spectral maximal spaces $Y,\ Z \in \mathrm{Lat}(T)$ satisfying $\sigma(T \mid Y) \cap \sigma(T \mid Z) = \emptyset$ there exists $R \in B$ commuting with T such that $R \mid Y = 0$ and $(I - R) \mid Z = 0$. With this notion, the essential part of Theorem 1.4 may be rephrased as follows:

3.1. THEOREM. *An operator $T \in \mathfrak{L}(X)$ is super-decomposable if and only if T is decomposable and $\mathfrak{L}(X)$ is normal with respect to T.*

In [9], Apostol is primarily interested in an operator $T \in B$ belonging to the center $Z(B)$ of the algebra B. In this situation, he studies the corresponding multiplication operator $\tilde{T} : B \to B$ given by $\tilde{T}(S) := TS$ for all $S \in B$. Notable among his results is the following [9, Theorem 2.10]: If $T \in Z(B)$, then $\tilde{T} \in \mathfrak{L}(B)$ is decomposable if and only if $T \in \mathfrak{L}(X)$ is decomposable and B is normal with respect to T. We now prove an extension of this.

3.2. THEOREM. *For every $T \in Z(B)$ the following assertions are equivalent:*

(i) *For every open covering $\{U, V\}$ of $\mathbf{C}$ there exists $R \in B$ commuting with T such that $\sigma(T \mid \overline{R(X)}) \subseteq U$ and $\sigma(T \mid \overline{(I-R)(X)}) \subseteq V$.*

(ii) *For every open covering $\{U_1, U_2\}$ of $\mathbf{C}$ there exist spaces $X_1, X_2 \in \mathrm{Lat}(T)$ and operators $R_1, R_2 \in B$ commuting with T such that $R_1 + R_2 = I$ and $R_j(X) \subseteq X_j$, $\sigma(T \mid X_j) \subseteq U_j$ for $j = 1, 2$.*

(iii) *$T \in \mathfrak{L}(X)$ is decomposable, and B is normal with respect to T.*

(iv) *$T \in \mathfrak{L}(X)$ is super-decomposable, and B is normal with respect to T.*

(v) *$\tilde{T} \in \mathfrak{L}(B)$ is decomposable.*

(vi) *$\tilde{T} \in \mathfrak{L}(B)$ is super-decomposable.*

Proof. The equivalence of the assertions (i), (ii), (iii) is immediate from the proof of Theorem 1.4. By (i), it is clear that (iii) is equivalent to the formally stronger statement (iv). The equivalence of (iv) and (v) is the content of [9, Theorem 2.10], and the implication (vi) $\Rightarrow$ (v) is obvious. Finally suppose that (i) — (v) are fulfilled. We shall use Theorem 1.4 to show that $\tilde{T} \in \mathfrak{L}(B)$ is super-decomposable.

Consider an arbitrary open covering $\{U_1, U_2\}$ of $\mathbf{C}$ and choose open sets $V_1, V_2 \subseteq \mathbf{C}$ such that $V_1 \cup V_2 = \mathbf{C}$ and $\overline{V}_j \subseteq U_j$ for $j = 1, 2$. By (i) we obtain operators $R_1, R_2 \in B$ commuting with T such that $R_1 + R_2 = I$ and $\sigma(T \mid \overline{R_j(X)}) \subseteq V_j$ for $j = 1, 2$. The corresponding (left) multiplication operators $\tilde{R}_1, \tilde{R}_2 \in \mathfrak{L}(B)$ satisfy $\tilde{R}_1 + \tilde{R}_2 = I_B$, the identity operator on B. Moreover, since T is decomposable on X, $R_j(X) \subseteq X_T(\overline{V}_j)$ for $j = 1, 2$. We also know that T is decomposable on B, and by [9, Lemma 2.9] the spectral maximal spaces for $\tilde{T}$ are given by

$$B_{\tilde{T}}(F) = \{S \in B : S(X) \subseteq X_T(F)\} \quad \text{for all } F \in \mathfrak{F}(\mathbf{C}).$$

Consequently, if $X_j := B_{\tilde{T}}(\overline{V}_j)$ for $j = 1, 2$, it is clear that $X_j \in \mathrm{Lat}(\tilde{T})$, $\sigma(\tilde{T} \mid X_j) \subseteq \subseteq \overline{V}_j \subseteq U_j$, and $\tilde{R}_j(B) \subseteq X_j$ for $j = 1, 2$. By (b) of Theorem 1.4 it follows that $\tilde{T} \in \mathfrak{L}(B)$ is super-decomposable.

An obvious combination of Theorems 2.3 and 3.2 leads to the following result:

3.3. Corollary. *Let A be a commutative complex Banach algebra and assume that the spectrum of A is totally disconnected or that A is semi-simple and regular. Let $a \in A$, consider an algebraic homomorphism $\Phi : A \to \mathfrak{L}(X)$ and suppose that B is a closed subalgebra of $\mathfrak{L}(X)$ such that $I \in B$, $\Phi(A) \subseteq B$, and $T := \Phi(a) \in Z(B)$; for instance, B may be taken to be the closed subalgebra of $\mathfrak{L}(X)$ generated by I and $\Phi(A)$. Then the corresponding multiplication operator $\tilde{T} \in \mathfrak{L}(B)$ is super-decomposable.*

To give another typical application of the preceding theorem we recall some notions from the elementary theory of multipliers [18, Chapter 1]. Let A be a complex Banach algebra *without order*, which means that if $Ax = \{0\}$ or if $xA = \{0\}$ then $x = 0$. A map $T : A \to A$ is a *multiplier* on A if $xT(y) = (Tx)y$ for all $x, y \in A$. The set $M(A)$ of all multipliers on A is a commutative closed subalgebra of $\mathfrak{L}(A)$ containing the identity operator [18, Theorem 1.1.1]. Hence we obtain from Theorem 3.2:

3.4. Corollary. *A multiplier T on a complex Banach algebra A without order is super-decomposable on A, if the corresponding multiplication operator $\tilde{T} : M(A) \to \to M(A)$ is decomposable on $M(A)$.*

We finally consider multiplication operators on $B = \mathfrak{L}(X)$. This case is not covered by the theory of Apostol [9], but it turns out that some of his techniques can be extended to this setting. We start with the following observation, which may be viewed as a counterpart of [9, Lemma 2.9].

3.5. Lemma. *Suppose that $T \in \mathfrak{L}(X)$ has the single-valued extension property and that the spaces $X_T(F)$ are closed for every $F \in \mathfrak{F}(\mathbf{C})$. Then the corresponding multiplication operator $\tilde{T} \in \mathfrak{L}(\mathfrak{L}(X))$ has the single-valued extension property, its local spectrum is given by*

$$\sigma_{\tilde{T}}(S) = \overline{\bigcup_{x \in X} \sigma_T(Sx)} \quad \textit{for every } S \in \mathfrak{L}(X),$$

and we have the representation

$$\mathfrak{L}_{\tilde{T}}(F) := \mathfrak{L}(X)_{\tilde{T}}(F) = \{S \in \mathfrak{L}(X) : S(X) \subseteq X_T(F)\} \quad \textit{for every } F \in \mathfrak{F}(\mathbf{C});$$

in particular, $\mathfrak{L}_{\tilde{T}}(F)$ is spectral maximal for $\tilde{T}$.

Proof. It is routine to check that if T has the single-valued extension property then so does $\tilde{T}$. Also, given $S \in \mathfrak{L}(X)$, the definition of local spectrum immediately yields that $\sigma_T(Sx) \subseteq \sigma_{\tilde{T}}(S)$ for every $x \in X$. Thus, if $K := \overline{\bigcup_{x \in X} \sigma_T(Sx)}$, then $K \subseteq \sigma_{\tilde{T}}(S)$.

For the converse inclusion consider $X_T(K)$ and suppose $\lambda \notin K$. Since $X_T(K)$ is closed, [12, 1.3.8] shows that $A_\lambda := ((\lambda - T) \mid X_T(K))^{-1}$ exists. Moreover, for every $x \in X$, $Sx \in X_T(K)$ (by definition of K and of $X_T(K)$) and hence $A_\lambda Sx$ is well-defined. It is clear that $\lambda \to A_\lambda S$ defines an analytic function on $\mathbf{C} \setminus K$ and since $(\lambda - T)A_\lambda S = S$ for every $\lambda \notin K$, it follows that $\lambda \in \rho_{\tilde{T}}(S)$. Thus $\sigma_{\tilde{T}}(S) \subseteq K$.

If $F \in \mathfrak{F}(\mathbf{C})$ then $\sigma_{\tilde{T}}(S) \subseteq F$ if and only if $\sigma_T(Sx) \subseteq F$ for all $x \in X$, hence $S \in \mathfrak{L}_{\tilde{T}}(F)$ if and only if $Sx \in X_T(F)$ for all $x \in X$. This proves the formula given for $\mathfrak{L}_{\tilde{T}}(F)$. The rest is immediate from [12, 1.3.8].

3.6. THEOREM. *If $T \in \mathfrak{L}(X)$ is super-decomposable, then the corresponding multiplication operator $\tilde{T} \in \mathfrak{L}(\mathfrak{L}(X))$ is super-decomposable.*

Proof. Proceed as in the second part of the proof of Theorem 3.2, with Lemma 3.5 used instead of [9, Lemma 2.9].

4. APPLICATIONS TO PROBLEMS OF AUTOMATIC CONTINUITY

Let X and Y be complex Banach spaces and consider decomposable operators $T \in \mathfrak{L}(X)$ and $S \in \mathfrak{L}(Y)$. It follows easily from the definition of spectral maximal spaces that every *continuous* linear map $\theta : X \to Y$ intertwining T and S in the sense that $\theta T = S\theta$ satisfies

$$\theta X_T(F) \subseteq Y_S(F) \quad \text{for all } F \in \mathfrak{F}(\mathbf{C}).$$

In the theory of automatic continuity, it is of some importance to know whether this inclusion holds without any continuity assumption on the intertwining operator θ. This problem was posed by Jewell [11, Problem 22, p. 200] and can also be found in [13, Problem 20, p. 464]. The results of Section 1 admit a partial positive solution which suffices for the applications we have in mind. In general the answer turns out to be negative; we give a counterexample based on an idea communicated to us by Barry Johnson.

4.1. PROPOSITION. *Assume that $T \in \mathfrak{L}(X)$ has the single-valued extension property and that $S \in \mathfrak{L}(Y)$ is super-decomposable without any non-trivial divisible subspace. Then every linear transformation $\theta : X \to Y$ with the property $\theta T = S\theta$ necessarily satisfies $\theta X_T(F) \subseteq Y_S(F)$ for all $F \in \mathfrak{F}(\mathbf{C})$.*

Proof. By the first part of Proposition 1.1

$$\theta X_T(F) \subseteq \theta E_T(F) = \theta(T - \lambda)E_T(F) = (S - \lambda)\theta E_T(F) \quad \text{for every } \lambda \in \mathbf{C} \setminus F.$$

This shows that $\theta E_T(F) \subseteq E_S(F)$ and since $E_S(F) = Y_S(F)$, by Proposition 1.5, the proof is complete.

4.2. EXAMPLE. There exist a super-decomposable operator $T \in \mathfrak{L}(X)$ on some Banach space X and a discontinuous linear transformation $\theta : X \to X$ commuting with T such that $\theta X_T(F) \subseteq X_T(F)$ does *not* hold for all $F \in \mathfrak{F}(\mathbf{C})$.

Proof. Let $T_0 \in \mathfrak{L}(X_0)$ be a quasi-nilpotent operator on a Banach space X_0 with a non-trivial divisible linear subspace Y_0 for T_0; for instance, T_0 may be taken to be the Volterra operator considered in Remark 1.6. Let X_1, Y_1, T_1 be copies of X_0, Y_0, T_0, respectively, and consider the operator $T \in \mathfrak{L}(X)$ given by

$$T := T_0 \oplus (I - T_1) \quad \text{on} \quad X := X_0 \oplus X_1.$$

Then it is clear that $\sigma(T) = \{0, 1\}$. Hence T is decomposable, since there is an obvious way of defining a spectral capacity $\mathfrak{E}$ for T, which takes only the values $\{0\}$, X_0, X_1, and X. Moreover, it is easily seen directly and can also be deduced from Proposition 2.2 that T is actually super-decomposable. Since Y_0 is a T_0-divisible inear subspace different from $\{0\}$, the restriction $T \mid X_0 = T_0$ cannot be algebraic; also the copy Y_1 is a non-trivial divisible subspace for the restriction $T \mid X_1 = I - T_1$. Hence, by [21, Theorem 3.6] there exists a discontinuous linear mapping $\theta : X_0 \to X_1$ such that $\theta T_0 = (I - T_1)\theta$. Define $\theta_0 : X \to X$ by $\theta_0(x_0, x_1) = (0, \theta x_0)$; direct computation shows that $\theta_0 T = T\theta_0$. Now observe that $X_T(\{0\}) = \mathfrak{E}(\{0\}) = X_0$. Suppose that $\theta_0 X_0 \subseteq X_0$. Then actually $\theta_0 X_0 \subseteq X_0 \cap X_1 = \{0\}$, which is impossible because of the discontinuity of θ on X_0. This contradiction completes the proof.

We next turn to our main result on the continuity of intertwining operators. This theorem covers the case of generalized scalar operators considered by Vrbová [23, Theorem 3.5], [24, Theorem 1.4] as well as the case of certain operators with a spectral reduction considered by Johnson and Sinclair [17, Theorem 4.3]. Moreover, since operators with a countable spectrum are super-decomposable by Proposition 2.2, our theorem is closely related to a basic automatic continuity result due to Johnson and Sinclair; see [17, Theorem 3.3] and also [21, Theorem 4.1]. In the latter results, the spectrum of the operator S on the range space Y is assumed to be countable, whereas the operator T on the domain space X is not restricted to be decomposable. Thus the following theorem is not quite a generalization of these classical results, but it is suitable for a number of applications.

Recall that a complex number $\lambda \in \mathbf{C}$ is a *critical eigenvalue* of the pair (T, S), if λ is an eigenvalue of S and if the codimension of $(T - \lambda)(X)$ in X is infinite.

4.3. THEOREM. *Assume that* $T \in \mathfrak{L}(X)$ *is decomposable and that* $S \in \mathfrak{L}(Y)$ *is super-decomposable. Then the following assertions are equivalent:*

(a) *Every linear transformation* $\theta : X \to Y$ *for which* $\theta T = S\theta$ *is necessarily continuous.*

(b) *The pair* (T, S) *has no critical eigenvalue, and either* T *is algebraic or* S *has no divisible subspace different from* $\{0\}$.

Proof. The implication (a) ⇒ (b) is well-known and does not depend on decomposability properties of T or S; see [21, Lemma 3.2 and Theorem 3.6]. We now assume that condition (b) is fulfilled, and consider an arbitrary linear mapping $\theta : X \to Y$ satisfying $\theta T = S\theta$. To prove the continuity of θ, it suffices to construct a non-trivial polynomial $p \in \mathbf{C}[\chi]$ such that $p(S)\mathfrak{S} = \{0\}$, where $\mathfrak{S}$ is the separating space of the mapping θ given by

$$\mathfrak{S} := \{y \in Y : \text{there exist } x_n \in X \text{ with } x_n \to 0 \text{ and } \theta(x_n) \to y\}.$$

Indeed, this is standard: cancel from p all those factors $\chi - \lambda$ for which $S - \lambda I$ is one-to-one, i.e. suppose that all the zeros of p are eigenvalues of S. Since (T, S) has no critical eigenvalue, this means that $p(T)X$ is of finite codimension in X. Hence the open mapping theorem implies that $p(T)X$ is closed and that $p(T)$ is an open mapping from X onto $p(T)X$. Since $p(S)\mathfrak{S} = \{0\}$, it follows [21, Lemma 1.3] that $p(S)\theta = \theta p(T)$ is continuous on X, and hence that θ is continuous.

Now, if T is algebraic, we choose a non-zero polynomial $p \in \mathbf{C}[\chi]$ satisfying $p(T) = 0$ and observe that

$$p(S)\mathfrak{S} \subseteq p(S)\,\theta(X) \subseteq \overline{\theta p(T)\,(X)} = \{0\}.$$

It remains to consider the case that S has no divisible linear subspace other than $\{0\}$. From Proposition 4.1 we infer that $\theta X_T(F) \subseteq Y_S(F)$ for all $F \in \mathfrak{F}(\mathbf{C})$, and since $X_T(\cdot)$ and $Y_S(\cdot)$ are spectral capacities, this enables us to use the automatic continuity theory for generalized local linear operators [6], [7]: by [6, Theorem 3.7] or [7, Theorem 4.3] there is a finite subset Λ of $\mathbf{C}$ such that $\mathfrak{S}$ is contained in $Y_S(\Lambda)$. Let $Z \subseteq Y_S(\Lambda)$ denote the largest linear subspace of $Y_S(\Lambda)$ such that $(S - \lambda)Z = Z$ for all $\lambda \in \Lambda$. Because $\sigma(S \mid Y_S(\Lambda)) \subseteq \Lambda$, we obtain $(S - \lambda)Z = Z$ for every $\lambda \in \mathbf{C}$ and therefore $Z = \{0\}$ by our assumption on S. Consequently, if $\mathfrak{S}_\infty \subseteq \mathfrak{S}$ denotes the largest linear subspace of $\mathfrak{S}$ for which $(S - \lambda)\mathfrak{S}_\infty = \mathfrak{S}_\infty$ for all $\lambda \in \Lambda$, we have $\mathfrak{S}_\infty \subseteq Z$ and hence $\mathfrak{S}_\infty = \{0\}$. Since the set Λ is finite, we may apply [17, Lemma 3.1] and [17, Theorem 3.2] to obtain some non-trivial polynomial $p \in \mathbf{C}[\chi]$ having all its roots in Λ such that $p(S)\mathfrak{S} = \{0\}$. The assertion follows.

We close with three typical applications of this theorem. Our first result is related to [16, Theorem 7.4], but here we do not need a hermitian involution. Of course, the assertion can be easily extended to the case of an intertwining operator defined only on a closed ideal of the given Banach algebra. The proof follows immediately from Theorem 4.3 combined with Corollary 2.4.

4.4. COROLLARY. *Let X and Y be commutative semi-simple and regular Banach algebras, and consider a pair (T_x, T_y) of multiplication operators on X and on Y, respectively, which has no critical eigenvalues. Then every linear transformation $\theta : X \to Y$ satisfying $\theta T_x = T_y \theta$ is automatically continuous.*

The next result provides a positive partial answer to a problem posed by Johnson [16, p. 98]. We do not know how far the condition on the measure μ can be relaxed in this context: it is clear that the result remains valid whenever the corresponding convolution operator T_μ on $L^1(G)$ is super-decomposable, but as mentioned earlier, this condition is not fulfilled in general.

4.5. COROLLARY. *Let G be a locally compact abelian group and consider a measure $\mu \in M_a(G) + M_d(G)$, whose Fourier-Stieltjes transform $\hat{\mu}$ is non-constant on every non-empty open subset of the dual group $\hat{G}$. Then every linear transformation $\theta : L^1(G) \to L^1(G)$ satisfying $\mu * \theta(f) = \theta(\mu * f)$ for all $f \in L^1(G)$ is necessarily continuous.*

Proof. Note that the condition on $\hat{\mu}$ guarantees that the corresponding convolution operator T_μ on $L^1(G)$ has no eigenvalues. Hence the assertion follows from Theorem 4.3 in connection with Theorem 2.5.

We finally consider periodically invariant linear operators between L^p-spaces on the real axis. Given $\alpha \in \mathbf{R}$ and $1 \leqslant p < \infty$, let T_α denote the translation operator on the space $L^p(\mathbf{R})$ given by $(T_\alpha f)(t) := f(t - \alpha)$ for all $f \in L^p(\mathbf{R})$ and $t \in \mathbf{R}$. Then we have:

4.6. COROLLARY. *Let $1 \leqslant p, q < \infty$ and consider a linear transformation $\theta : L^p(\mathbf{R}) \to L^q(\mathbf{R})$ such that $\theta T_\alpha = T_\alpha \theta$ for some $\alpha \in \mathbf{R} \setminus \{0\}$. Then θ is automatically continuous.*

Proof. First note that T_α has no eigenvalues. Moreover, it is not hard to see that T_α is a generalized scalar operator on $L^p(\mathbf{R})$. Indeed, since $\| T_\alpha^k \| = 1$ for all $k \in \mathbf{Z}$, a functional calculus $\Phi : C^\infty(\mathbf{C}) \to \mathfrak{L}(X)$ for T_α on $X = L^p(\mathbf{R})$ is obviously given by

$$\Phi(f) := \sum_{k=-\infty}^{\infty} c_k(f) T_\alpha^k \quad \text{for all } f \in C^\infty(\mathbf{C}),$$

where $c_k(f) \in \mathbf{C}$ denotes the k-th Fourier coefficient of the restriction of f to the unit circle. In particular, it follows that T_α is super-decomposable and has no divisible subspace different from $\{0\}$. Hence Theorem 4.3 shows θ to be continuous.

We conclude by mentioning a few *open problems:*

First of all there is Barry Johnson's question [16, p. 98] to which Corollary 4.5 is a partial answer: For what measures μ is Corollary 4.5 valid?

The results of Section 3 bring the following to mind: with the notation from there, if $\tilde{T} : \mathfrak{L}(X) \to \mathfrak{L}(X)$ is super-decomposable, is $T : X \to X$ decomposable?

Is there a converse to Corollary 3.4?

And finally, although the super-decomposable operators do not appear to form an algebraically "nice" set, as it was noted at the end of Section 2, the results of Albrecht [3], notably [3, Theorem 2.6], may lend some hope for a positive answer to this question: do the super-decomposable multipliers form a Banach algebra? Since the notion of super-decomposability may depend on the space on which the multiplier acts, this (loosely phrased) question probably contains several distinct versions.

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