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ON THE COMPUTATION OF INVARIANTS FOR ITPFI FACTORS

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Krieger's theorem [11] states, in part, that the flow of weights considered as a mapping from type III_0 Krieger factors (see terminology) with algebraic isomorphism as the equivalence relation to strictly ergodic flows with conjugacy as the equivalence relation, is one-to-one and onto between equivalence classes. The simplest flows are the pure point spectrum flows ([5]). The corresponding Krieger factors are known to be ITPFI ([4]), and the motivation for the present work was to obtain explicit eigenvalue list constructions of these factors.

This problem leads naturally to the construction of Section 1 where we introduce an invariant $\mathscr{C}(M, T)$ (see below) which can be computed much more easily than the flow of weights, and seems to be very useful. The main result of this section is Theorem 1.10 which is basic for Section 2 and is also used in Sections 3 and 4. This invariant can be understood in terms of the flow of weights as follows (see Remark 1.11). Let M be a factor, (Ω, P, F_t) its flow of weights, and T a subgroup of the Connes invariant $T(M)$ (which is also the L^∞-point spectrum of (Ω, P, F_t)). Let $(f_\theta)_{\theta\in T}$ be a multiplicative choice of eigenfunctions of (Ω, P, F_t). This gives a map $f\colon \Omega \to \hat{T}$ given by $\langle f(\omega), \theta\rangle = f_\theta(\omega)$. The measure $f(P)$ defines a certain equivalence class $\mathscr{C}(M, T)$ of measures on $\hat{T}$ (see Proposition 1.2). The relation with the original problem is as follows. Let M be a Krieger factor and take $T = T(M)$. Then the flow of weights will be a pure point spectrum flow iff the map f is essentially injective and the Haar measure on $\hat{T}$ belongs to $\mathscr{C}(M, T)$.

In [8], Hamachi and Osikawa consider the ITPFI_2 factors $M(L_k, \lambda^{2^k})$, $0 < \lambda < 1$, and prove that for L_k sufficiently large, the flow of weights is pure point spectrum. In Section 3 we study this family of factors. We compute for all sequences L_k the flow of weights by showing that the map f indicated above is essentially injective (Theorem 3.1). We give a condition on the L_k that $\mathscr{C}(M, T)$ contains the Haar measure and hence the flow is pure point spectrum (Proposition 3.8). Proposition 3.4 gives a precise condition for $T(M)$ to be either $\Delta = \{\theta \in R\,;\, \lambda^{i2^k\theta} = 1$ for some $k \in \mathbf{N}\}$ or uncountable. This result gives some insight into the occurrence of uncountable $T(M)$ (Remark 3.11). Finally, we give an outline of the proof that the same situation holds if Δ is replaced by any countable subgroup of the rationals (Remark 3.10, see also [8]).

The best-known flow is perhaps the Kronecker flow (the flow built over an irrational rotation under a constant ceiling function). In Section 4 we construct a family of ITPFI_2 factors $M = M(L_k, \lambda_k)$. We prove that if the L_k are large enough but not too large, then $T(M)$ is the point spectrum of a Kronecker flow (Proposition 4.1). We give a condition that $\mathscr{C}(M, T(M))$ is the Haar measure class on $T(M)^\wedge$. However, we are unable to carry out the ergodic decomposition involved in constructing the flow of weights from the eigenvalue list for M, which is required to show that f is essentially injective. It seems, so far, that only in very special cases one has succeeded in computing an ergodic decomposition. However, our investigations did lead to a number of other interesting results on Krieger factors (see also [6], [7]).

In Section 2 we use Theorem 1.10 to construct an ITPFI factor M which is not a tensor square (Theorem 2.1). Since every ITPFI of bounded type is an ITPFI_2 ([6], Theorem 2.1) and hence a tensor square, M is not of bounded type (Corollary 2.3).

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NOTATION. All the definitions and notation can easily be found in the literature (for instance in [14]). However, let us recall the definitions which are frequently used.

DEFINITIONS. 1. If φ is a normal semi-finite faithful weight on a factor M, then C_φ denotes the *center of the centralizer* $M_\varphi = \{x \in M;\ \sigma_t^\varphi(x) = x,\ t \in \mathbf{R}\}$ of φ ([12], Chapter 10).

2. A *Krieger factor* is the crossed product of an abelian von Neumann algebra by an ergodic automorphism.

3. A factor M is called an *ITPFI* if it is of the form $M = \bigotimes_{k=1}^{\infty} (M_{n_k}(\mathbf{C}), \varphi_k)$ acting on the Hilbert space $\bigotimes_{k=1}^{\infty} (H_k, \xi_k)$ where $M_{n_k}(\mathbf{C})$ denotes the algebra of $n_k \times n_k$ matrices acting on the Hilbert space H_k ($n_k \geqslant 2$) and $\varphi_k(x) = \langle x\xi_k, \xi_k\rangle$.

4. If all the n_k are bounded by a number n, M is said to be an *ITPFI of bounded type*.

5. If all the n_k are equal to 2, M is said to be an $ITPFI_2$.

6. Let $(\varphi_{\lambda_n})_{n\geq 1}$ be a sequence of states on $M_2(\mathbf{C})$ with eigenvalues $\left\{\frac{1}{1+\lambda_n}, \frac{\lambda_n}{1+\lambda_n}\right\}$, $0 < \lambda_n < 1$ and $(L_n)_{n\geq 1}$ be a sequence of positive integers.

Then $M(L_n, \lambda_n) = \bigotimes_{n\geq 1} (M_2(\mathbf{C}), \varphi_{\lambda_n})^{\otimes L_n}$ denotes the *$ITPFI_2$ factor corresponding to $(L_n, \lambda_n)_{n\geq 1}$*.

Finally, $U(1)$ denotes the multiplicative group of complex numbers of modulus 1. If x is a non-negative real number, $[x]$ stands for its integral part.

1. THE INVARIANT $\mathscr{C}(M, T)$

Let M be a factor, T a subgroup of $T(M)$. While the construction of $\mathscr{C}(M, T)$ can be done from the flow of weights (see Remark 1.11), we adopt a somewhat different approach. Let φ be a normal semi-finite faithful weight on M. If $\theta \in T(M)$ there exists a unitary u_θ of C_φ, unique up to a scalar, with $\sigma_\theta^\varphi = \mathrm{Ad}\, u_\theta$. Let $A_\varphi(T)$ be the (abelian) C^*-subalgebra (of C_φ), generated by $\{u_\theta, \theta \in T\}$.

Write $A_\varphi(T) = C(X_\varphi)$, where X_φ is the compact space of characters of $A_\varphi(T)$.

1.1. LEMMA. 1) *If $x, y \in X_\varphi$, then the map $f_{x,y} : T \to U(1)$ defined by $f_{x,y}(\theta) = \langle u_\theta, x\rangle^{-1}\langle u_\theta, y\rangle$, is a character of T.*

2) *The map $f_x : X_\varphi \to \hat{T}$ given by $f_x(y) = f_{x,y}$ is continuous and injective.*

3) *If $\sigma_{\theta_0}^\varphi = 1$, i.e. φ is periodic, then the image of f_x is contained in $(T/\mathbf{Z}\theta_0)^\wedge = \{\chi \in \hat{T} ; \langle \chi, \theta_0\rangle = 1\}$.*

Proof. 1) If $\lambda \in \mathbf{C}$, $|\lambda| = 1$, then $\langle \lambda u_\theta, x\rangle^{-1}\langle \lambda u_\theta, y\rangle = \langle u_\theta, x\rangle^{-1}\langle u_\theta, y\rangle$. This shows that $f_{x,y}$ is well-defined.

2) If $f_x(y) = f_x(z)$, then $\langle u_\theta, y\rangle = \langle u_\theta, z\rangle$. As the u_θ's generate $A_\varphi(T)$, the characters y and z coincide.

3) If $\sigma_{\theta_0}^\varphi = 1$, then $f_{x,y}(\theta_0) = 1$. ▨

The representation of M in the separable Hilbert space H restricts to a representation of $A_\varphi(T)$ and yields a class $\mathscr{D}_\varphi$ of measures on X_φ.

1.2. PROPOSITION. a) *Let $\mathscr{R}$ be the equivalence relation on the space of probability measures on $\hat{T}$ given by: $\mu \mathscr{R} \nu$ iff there exists $\chi \in \hat{T}$ such that μ is equivalent to $\delta_\chi * \nu$ (where δ_χ is the Dirac measure).*

Then the equivalence class $\mathscr{C}_\varphi(T)$ under $\mathscr{R}$ of $f_x(\mu)$ does not depend on the choices of x in X_φ and of the probability measure μ in $\mathscr{Q}_\varphi$.

b) *Let $(u_\theta)_{\theta\in T}$ be a choice of unitaries as above satisfying $u_{\theta+\theta'} = u_\theta u_{\theta'}$ (θ, θ' in T). Let α be a faithful normal state on M. Then there exists a probability measure μ ($= \mu(T, \varphi, \alpha, u)$) on $\hat{T}$, whose Fourier-Stieltjes transform is $\hat{\mu}(\theta) = \alpha(u_\theta)$, and $\mu \in \mathscr{C}_\varphi(T)$.*

Proof. a) follows from the equality $f_x(z) = f_{x,y} + f_y(z)$ (the group $\hat{T}$ is written additively).

b) Note that the choice of an x in X_φ determines a multiplicative choice of u_θ, taking $\langle u_\theta, x\rangle = 1$. Also two multiplicative choices of u_θ's differ by an element of $\hat{T}$. The restriction of the state α to $A_\varphi(T)$ determines a probability measure ν on X_φ whose class is in $\mathscr{Q}_\varphi$. If the choice of the u_θ's is given by x, we have

$$f_x(\nu)^\wedge(\theta) = \int_{X_\varphi} f_{x,y}(\theta)\,\mathrm{d}\nu(y) = \int \langle u_\theta, y\rangle\,\mathrm{d}\nu(y) = \alpha(u_\theta). \qquad ▨$$

1.3. Remarks. 1. Let φ be a normal semifinite faithful weight on M and let α be a normal faithful state. Let T be a subgroup of $T(M)$ and let $(u_\theta)_{\theta\in T}$ be a multiplicative choice of unitaries as above. Let τ be an automorphism of M. By [2], Lemma 1.2.10, $\tau^{-1}(u_\theta)$ is a multiplicative choice of unitaries corresponding to $\varphi\circ\tau$. The equality $\alpha(u_\theta) = \alpha\circ\tau(\tau^{-1}(u_\theta))$ shows that $\mathscr{C}_\varphi(T) = \mathscr{C}_{\varphi\circ\tau}(T)$.

2. If λ is a positive real number, $\sigma_t^{\lambda\varphi} = \sigma_t^\varphi$. Therefore $\mathscr{C}_{\lambda\varphi}(T) = \mathscr{C}_\varphi(T)$.

3. Let $T' \subseteq T$ be two subgroups of $T(M)$ and $i: T' \to T$ be the inclusion. Then $A_\varphi(T') \subseteq A_\varphi(T)$. Therefore we get a surjective map $\pi: X_\varphi^T \to X_\varphi^{T'}$. Moreover, for every $x, y \in X_\varphi$ we have $\hat{i}\circ f_{x,y}^T = f_{\pi(x),\pi(y)}^{T'}$. Hence the class $\mathscr{C}_\varphi(T')$ is equal to $\hat{i}(\mathscr{C}_\varphi(T))$.

Let now M and N be two factors. Let φ, ψ be normal semifinite, faithful weights on M and N respectively. Let T be a subgroup of $T(M) \cap T(N)$.

1.4. Proposition. *If μ is in $\mathscr{C}_\varphi(T)$ and ν is in $\mathscr{C}_\psi(T)$, then $\mu * \nu$ is in $\mathscr{C}_{\varphi\otimes\psi}(T)$ (and therefore determines this class).*

Proof. Let $(u_\theta)_{\theta\in T}$, $(v_\theta)_{\theta\in T}$ be multiplicative choices of unitaries of C_φ and C_ψ satisfying $\sigma_\theta^\varphi = \mathrm{Ad}\,u_\theta$ and $\sigma_\theta^\psi = \mathrm{Ad}\,v_\theta$. Let α and β be normal faithful states on M and N. As $\alpha\otimes\beta(u_\theta\otimes v_\theta) = \alpha(u_\theta)\beta(v_\theta)$ the result follows from Proposition 1.2.b). ▨

Let ω be the weight on $\mathscr{L}(L^2(\mathbf{R}))$ given by $\omega(x) = \mathrm{Trace}(\rho x)$ $x \in \mathscr{L}(L^2(\mathbf{R}))_+$, where ρ is defined by $\rho f(t) = \mathrm{e}^t f(t)$; $f \in L^2(\mathbf{R})$. Then the (dominant) weight $\varphi\otimes\omega$ on $M\otimes\mathscr{L}(L^2(\mathbf{R}))$ does not depend within unitary equivalence on the normal semi-finite faithful weight φ on M (cf. [2], Lemma 1.2.5; [3], Theorem II.1.1; [4], Section 4).

Let ω_ξ be the weight on $\mathscr{L}(\ell^2(\mathbf{Z}))$ given by $\omega_\xi(x) = \mathrm{Trace}(\rho_\xi x)$ $x \in \mathscr{L}(\ell^2(\mathbf{Z}))_+$, where ρ_ξ is defined by $\rho_\xi(\varepsilon_n) = \mathrm{e}^{-n\xi}\varepsilon_n$ ($(\varepsilon_n)_{n\in\mathbf{Z}}$ denotes the canonical basis of $\ell^2(\mathbf{Z})$). If φ, ψ are normal, semi-finite, faithful weights, with $\sigma^\varphi_{2\pi/\xi} = \sigma^\psi_{2\pi/\xi} = 1$ then there exists $t \in [0, \xi)$ such that $\mathrm{e}^t\varphi \otimes \omega_\xi$ and $\psi \otimes \omega_\xi$ are unitarily equivalent. (t is such that $\mathrm{e}^{2\mathrm{i}\pi t/\xi} = (D\psi : D\varphi)_{2\pi/\xi}$) (cf. [2], Lemma 1.2.5; [3]; [4], Section 5).

1.5. Definition. Let M be a factor.

a) We denote by $\mathscr{C}(M, T)$ the equivalence class $\mathscr{C}_{\varphi\otimes\omega}(T)$.

b) We denote by $\mathscr{C}_\xi(M, T)$ the equivalence class $\mathscr{C}_{\varphi\otimes\omega_\xi}(T)$, where $2\pi/\xi \in T$ and $\sigma^\varphi_{2\pi/\xi} = 1$.

One computes $\mathscr{C}(M, T)$ and $\mathscr{C}_\xi(M, T)$ using Proposition 1.4 and:

1.6. Proposition. a) *Let $h\colon \mathbf{R} \to \hat{T}$ be given by $\langle h_t, \theta\rangle = \mathrm{e}^{it\theta}$ for $t \in \mathbf{R}$ and $\theta \in T$. Then if m is a probability measure in $\mathbf{R}$ equivalent to the Lebesgue measure, $h(m) \in \mathscr{C}_\omega(T)$ and therefore determines this class.*

b) *Let $H_\xi\colon \mathbf{Z} \to \hat{T}$ be given by $H_\xi(n) = h_{n\xi}$. Then if m is a probability measure on $\mathbf{Z}$ with support $\mathbf{Z}$ then $H_\xi(m) \in \mathscr{C}_{\omega_\xi}(T)$ and therefore determines this class.*

Proof. a) Let α be a faithful state on $\mathscr{L}(L^2(\mathbf{R}))$ given such that $\alpha(g) = \int g(x)\,\mathrm{d}m(x)$ for all g in $L^\infty(\mathbf{R})$ considered as a multiplication operator. Let V_s be the multiplication operator by $\mathrm{e}^{is(\cdot)}$. We have $\sigma^\omega_s = \mathrm{Ad}\, V_s$. If $\theta \in T$, we get

$$\alpha(V_\theta) = \int_{\mathbf{R}} \mathrm{e}^{it\theta}\,\mathrm{d}m(t) = \int_{\mathbf{R}} \langle h_t, \theta\rangle\,\mathrm{d}m(t) = \int_{\hat{T}} \langle \chi, \theta\rangle\,\mathrm{d}h(m)(\chi).$$

b) is proved similarly. ▨

1.7. Corollary. a) *Let φ be a faithful, normal state on M. Let $u = (u_\theta)_{\theta\in T}$ be a multiplicative choice of unitaries of C_φ with $\mathrm{Ad}\, u_\theta = \sigma^\varphi_\theta$. Let f be a strictly positive function on $\mathbf{R}$ of Lebesgue integral 1. Then there exists a probability measure μ $(= \mu(T, \varphi, u, f))$ on $\hat{T}$ whose Fourier-Stieltjes transform is $\hat{\mu}(\theta) = \varphi(u_\theta)\cdot\hat{f}(-\theta)$ and $\mu \in \mathscr{C}(M, T)$.*

b) *Let φ be a faithful, normal state on M with $\sigma^\varphi_{2\pi/\xi} = 1$. Let $u = (u_\theta)_{\theta\in T}$ be a multiplicative choice of unitaries as above with $u_{2\pi/\xi} = 1$. Let f be a strictly positive function on $\mathbf{Z}$ with sum 1. Then there exists a probability measure μ $(= \mu_\xi(T, \varphi$ $u, f))$ on T whose Fourier-Stieltjes transform is*

$$\hat{\mu}(\theta) = \varphi(u_\theta)\hat{f}(\mathrm{e}^{-\mathrm{i}\theta\xi}) \quad \textit{and } \mu \in \mathscr{C}_\xi(M, T). \qquad ▨$$

Let $U_t \in \mathscr{L}(L^2(\mathbf{R}))$ be given by $U_t f(s) = f(s + t)$. The flow of weights F_t of M is given by the restriction of $\mathrm{Ad}(1 \otimes U_t)$ to $C_{\varphi\otimes\omega}$ ([3]; [4], § 4).

Let $U \in \mathscr{L}(\ell^2(\mathbf{Z}))$ be given by $Uf(n) = f(n+1)$. Let φ be a normal, semi-finite, faithful weight on M with $\sigma^{\varphi}_{2\pi/\xi} = 1$. The flow of weights of M is built over the base transformation S corresponding to the restriction of $\mathrm{Ad}(1 \otimes U)$ to $C_{\varphi \otimes \omega_\xi}$ and under the constant ceiling function ξ ([3]; [4], § 4).

1.8. REMARKS. a) The equality $(1 \otimes U_t)(u_\theta \otimes V_\theta)(1 \otimes U_t)^{-1} = e^{it\theta}(u_\theta \otimes V_\theta)$ shows that the restriction of the flow F_t to $A_{\varphi\otimes\omega}(T)$ is given by translation by h_t (Proposition 1.6 a)). In particular, if $\mu \in \mathscr{C}(M, T)$, it is (quasi-invariant and) ergodic under the action of $\mathbf{R}$ by addition of h_t ([3], II, Theorem 3.1).

b) If φ is a periodic weight of period $2\pi/\xi$, the restriction of the transformation S to $A_{\varphi\otimes\omega_\xi}(T)$ is given by addition of H_ξ (Proposition 1.6 b)). In particular if $\mu \in \mathscr{C}_\xi(M, T)$, it is H_ξ-(quasi-invariant and) ergodic.

c) It is useful to get rid of the term $\hat{f}(e^{-i\theta\xi})$ in Corollary 1.7 b). Let $\mu \in \mathscr{C}_\varphi(T)$. Let m be a probability measure on $\mathbf{Z}$ with support $\mathbf{Z}$. By Proposition 1.4, $\mu * H_\xi(m) \in \mathscr{C}_{\varphi\otimes\omega_\xi}(T) = \mathscr{C}_\xi(M, T)$. Moreover $\mu \ll \mu * H_\xi(m)$.

Let us recall that a measure ν in $\hat{T}$ (not necessarily H_ξ-quasi-invariant) is said to be H_ξ-ergodic if for every H_ξ-invariant Borel subset E of $\hat{T}$, $\nu(E) = 0$ or $\nu(\hat{T} \setminus E) = 0$.

Let $\mathscr{R}'$ be the equivalence relation on the set of H_ξ-ergodic probability measures on $\hat{T}$ given by: $\mu_1 \mathscr{R}' \mu_2$ iff there exists $\chi \in \hat{T}$ such that $\delta_\chi * \mu_1$ and μ_2 are not mutually singular. Note that with the notations of Proposition 1.2, $\mu_1 \mathscr{R}' \mu_2$ iff $\mu_1 * H_\xi(m) \mathscr{R} \mu_2 * H_\xi(m)$. We can look at $\mathscr{C}_\xi(M, T)$ as the equivalence class under $\mathscr{R}'$ of μ, where $\mu \in \mathscr{C}_\varphi(T)$ satisfies $\hat{\mu}(\theta) = \varphi(u_\theta)$.

We now come to the case of ITPFI factors. We need the following:

1.9. LEMMA. *Let μ be a probability measure on $\hat{T}$ which is H_ξ-quasi-invariant, (ergodic and) approximately transitive ([4]). Then for all probability measures μ' on $\hat{T}$ with $\mu' \ll \mu$ and for all sequences $\theta_n \in T$ with $\lim_{n\to\infty} e^{i\theta_n\xi} = 1$ we have*

$$\lim_{n\to\infty} (\hat{\mu}(\theta_n) - \hat{\mu}'(\theta_n)) = 0.$$

Proof. Let $\varepsilon > 0$. As μ is approximately transitive, there exist probability measures ν, ν' carried by $\mathbf{Z}H_\xi \subset \hat{T}$ and $\mu_0 \ll \mu$ such that $\|\mu - \mu_0 * \nu\| \leqslant \varepsilon/4$ and $\|\mu' - \mu_0 * \nu'\| \leqslant \varepsilon/4$.

As $\hat{\nu}$ and $\hat{\nu}'$ are continuous functions on $\mathbf{R}$ and periodic of period $2\pi/\xi$, there exists N such that $n \geqslant N$ implies $|\hat{\nu}(\theta_n) - 1| < \varepsilon/4$ and $|\hat{\nu}'(\theta_n) - 1| < \varepsilon/4$. If $n \geqslant N$, we have:

$$|\hat{\mu}(\theta_n) - \hat{\mu}'(\theta_n)| \leqslant |\hat{\mu}(\theta_n) - (\mu_0 * \nu)^\wedge(\theta_n)| + |\hat{\mu}_0(\theta_n)(\hat{\nu}(\theta_n) - \hat{\nu}'(\theta_n))| +$$

$$+ |(\mu_0 * \nu')^\wedge(\theta_n) - \hat{\mu}'(\theta_n)| \leqslant \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon.$$

▨

Let M be an ITPFI factor and $T \subseteq T(M)$. Let ξ be a real number with $2\pi/\xi \in T$. Let $\mu \in \mathscr{C}_\xi(M, T)$. By Remark 1.8 b), the transformation $(\hat{T}, \mu, H_\xi)$ is a factor of a base transformation (B, ν, S) over which the flow of weights of M is constructed under the constant ceiling function ξ. Using then Theorem 8.3, Lemma 2.5 and Remark 2.4 of [4], we get that $(\hat{T}, \mu, H_\xi)$ is approximately transitive.

1.10. THEOREM. *Let M be an* ITPFI *factor and let T be a subgroup of $T(M)$. Let ξ be a real number with $2\pi/\xi \in T$. Let φ and ψ be two normal, faithful, periodic states on M with period $2\pi/\xi$ and let $(u_\theta)_{\theta\in T}$, $(v_\theta)_{\theta\in T}$ be unitaries of M ($u_\theta \in C_\varphi$, $v_\theta \in C_\psi$) with $\sigma_\theta^\varphi = \mathrm{Ad}\, u_\theta$, $\sigma_\theta^\psi = \mathrm{Ad}\, v_\theta$. Then for every sequence $(\theta_n)_{n\geqslant 1}$ with $\theta_n \in T$ and $\lim_{n\to\infty} e^{i\theta_n\xi} = 1$, we have $\lim_{n\to\infty} (|\varphi(u_{\theta_n})| - |\psi(v_{\theta_n})|) = 0$.*

Proof. We may assume that the choices u_θ and v_θ are multiplicative, and $u_{2\pi/\xi} = v_{2\pi/\xi} = 1$. By Proposition 1.2. b), there exist measures $\mu \in \mathscr{C}_\varphi(T)$ and $\nu \in \mathscr{C}_\psi(T)$ with $\hat{\mu}(\theta) = \varphi(u_\theta)$, $\hat{\nu}(\theta) = \psi(v_\theta)$. By Remark 1.8. c), there exist measures $\mu', \nu' \in \mathscr{C}_\xi(M, T)$ with $\mu \ll \mu'$ and $\nu \ll \nu'$. As $\mu' \mathscr{R} \nu'$, there exists $\chi \in \hat{T}$ with $\delta_\chi * \mu' \sim \nu'$. Therefore $\delta_\chi * \mu \ll \nu'$. As ν' is H_ξ-approximately transitive, we get $\lim_{n\to\infty} (\hat{\nu}'(\theta_n) - (\delta_\chi * \mu)^\wedge(\theta_n)) = 0$ and $\lim_{n\to\infty} (\hat{\nu}'(\theta_n) - \hat{\nu}(\theta_n)) = 0$ (Lemma 1.9). The result follows from the equality $|(\delta_\chi * \mu)^\wedge(\theta)| = |\hat{\mu}(\theta)|$. ▨

1.11. REMARK. a) The invariants $\mathscr{C}(M, T)$ and $\mathscr{C}_\xi(M, T)$ can be presented in the following way:

Let (Ω, P, F_t) be an ergodic flow. Let $T \subseteq \mathbf{R}$ be a subgroup of its L^∞-point spectrum. For all $\theta \in T$, let $g_\theta \in L^\infty(\Omega, P)$, $|g_\theta| = 1$ such that $g_\theta \circ F_t = e^{i\theta t} g_\theta$ for all t in $\mathbf{R}$. Let χ be a character of the von Neumann algebra $L^\infty(\Omega, P)$. Put $f_\theta = \chi(g_\theta)^{-1} g_\theta$. We then have for all θ and θ' in T, $f_\theta \cdot f_{\theta'} = f_{\theta+\theta'}$ (cf. also [5], Chapter 12). Let now $f : \Omega \to \hat{T}$ be given by $\langle f(\omega), \theta \rangle = f_\theta(\omega)$.

If M is a factor of type III and (Ω, P, F_t) is its flow of weights, then the measure $f(P)$ belongs to the class $\mathscr{C}(M, T)$ and therefore determines this class.

Assume that F_t is constructed over the base transformation (B, ν, S) under the constant ceiling function ξ. Let $T' \subseteq U(1)$ be a subgroup of the point spectrum of S.

Let $(f_u)_{u\in T'}$ be a multiplicative choice of eigenfunctions for S of modulus 1.

Let $T = \{\theta \in \mathbf{R};\, e^{i\theta\xi} \in T'\}$. Define $g : B \to \hat{T}$ by $\langle g(b), \theta \rangle = f_{\exp(i\theta\xi)}(b)$. Then the measure $g(\nu)$ belongs to the class $\mathscr{C}_\xi(M, T)$ and therefore determines this class.

b) Such a construction can be done for an ergodic action of any locally compact abelian group.

2. AN ITPFI FACTOR WHICH IS NOT A TENSOR SQUARE

We use here the results of Section 1 to construct an ITPFI factor M which is not a tensor square. As every ITPFI of bounded type is an ITPFI_2 ([6]) and hence a tensor square, M is not of bounded type (Corollary 2.2).

Let $(p_k)_{k\geqslant 1}$ be a sequence of positive integer multiples of 8, (for instance $p_k = 8$, $k \geqslant 1$). Let $q_n = \prod_{k=1}^{n} p_k$.

For $n \geqslant 1$, let $\varphi_n = \mathrm{Tr}(h_n \cdot)$ be the state on $M_{1+2^{q_n}}(\mathbf{C})$, where h_n is diagonal and has coefficients: $1/2$ with multiplicity 1 and 2^{-q_n-1} with multiplicity 2^{q_n}.

2.1. THEOREM. *The factor* $M = \bigotimes_{n\geqslant 1} (M_{1+2^{q_n}}(\mathbf{C}), \varphi_n)$ *is not a tensor square.*

Proof. Let $T = \{\theta \in \mathbf{R};\ 2^{\mathrm{i}\theta q_n} = 1$ for some $n \geqslant 1\}$. For $\theta \in T$ and $n \geqslant 1$ let $u_{\theta,n}$ be the unitary in $M_{1+2^{q_n}}(\mathbf{C})$, given by $u_{\theta,n} = (2^{\frac{q_n+2}{2}} h_n)^{\mathrm{i}\theta}$. We have: $\sigma_\theta^{\varphi_n} = \mathrm{Ad}\, u_{\theta,n}$. If $\theta \in T$ and n is large enough, $u_{\theta,n} = 1$. Set $u_\theta = \bigotimes_{n\geqslant 1} u_{\theta,n} \in M$ and $\varphi = \bigotimes_{n\geqslant 1} \varphi_n$. We have $\sigma_\theta^\varphi = \mathrm{Ad}\, u_\theta$. Hence $T \subseteq T(M)$.

We will show that there exists no probability measure ν on $\hat{T}$, such that $\nu * \nu \in \mathscr{C}_{\mathrm{Log}\,2}(M, T)$; Proposition 1.4 will then imply that M is not a tensor square.

Let μ_0 be the probability measure on $\hat{T}$, whose Fourier-Stieltjes coefficients are $\hat{\mu}_0(\theta) = \varphi(u_\theta)$. If $\mu \in \mathscr{C}_{\mathrm{Log}\,2}(M, T)$, then by Remark 1.8. c) and by the definition of the equivalence relation $\mathscr{R}$ (1.2), there exists $\chi \in \hat{T}$ such that $\mu_0 \ll \delta_\chi * \mu$. By Theorem 1.10, we get that $\lim_{n\to\infty} (|\hat{\mu}_0(\theta_n)| - |\hat{\mu}(\theta_n)|) = 0$, for every sequence $\theta_n \in T$, with $\theta_n \to 0$.

Let $\theta = \dfrac{2\pi j}{q_n \,\mathrm{Log}\, 2}$, $j \in \mathbf{Z}$. Then $\theta \in T$ and

$$\varphi(u_\theta) = \prod_{k=1}^{n} \varphi_k(u_{\theta,k}) = \prod_{k=1}^{n} \cos\left(\frac{\pi j q_k}{q_n}\right).$$

In particular, if

$$\theta_n = \frac{\pi}{q_n \,\mathrm{Log}\, 2}, \quad \varphi(u_{\theta_n}) = \prod_{k=1}^{n} \cos\left(\frac{\pi q_k}{2q_n}\right) = 0.$$

If $\theta'_n = \dfrac{\pi}{2q_n \,\mathrm{Log}\, 2}$ and $k \leqslant n-1$,

$$\varphi_k(u_{\theta'_n k}) = \cos\frac{\pi q_k}{4q_n} \geqslant \cos\left(\frac{\pi}{4\times 8^{n-k}}\right) \geqslant 1 - \frac{\pi^2}{32} \times 8^{2(k-n)}.$$

Hence

$$\prod_{k=1}^{n-1} \varphi_k(u_{\theta'_n k}) \geqslant 1 - \frac{\pi^2}{32} \sum_{j=1}^{+\infty} 8^{-2j} = 1 - \frac{\pi^2}{2016}$$

so that

$$\varphi(u_{\theta'_n}) \geqslant \frac{1}{\sqrt{2}} \left(1 - \frac{\pi^2}{2016}\right) > \frac{7}{10}.$$

If $\nu * \nu \in \mathscr{C}_{\mathrm{Log}\,2}(M, T)$, then for n large enough, $|\hat{\nu}(\theta_n)| < 1/10$ and $|\hat{\nu}(\theta'_n)|^2 > > 6/10$. As ν is a positive measure, $\hat{\nu}$ is positive definite and the matrix

$$A := \begin{bmatrix} \hat{\nu}(1) & \hat{\nu}(\theta_n) & \hat{\nu}(\theta'_n) \\ \hat{\nu}(-\theta_n) & \hat{\nu}(1) & \hat{\nu}(\theta'_n - \theta_n) \\ \hat{\nu}(-\theta'_n) & \hat{\nu}(\theta_n - \theta'_n) & \hat{\nu}(1) \end{bmatrix}$$

is hermitian positive. But as $\hat{\nu}(1) = 1$

$$\det A = 1 + 2\,\mathrm{Re}(\hat{\nu}(\theta_n)\overline{\hat{\nu}(-\theta'_n)^2}) - 2|\hat{\nu}(\theta'_n)|^2 - |\hat{\nu}(\theta_n)|^2 < 0.$$ ▨

2.2. COROLLARY. *The* ITPFI *factor M is not of bounded type.*

Proof. By Proposition 1.1 of [6], every ITPFI_2 factor can be written in the form $N = M(L_k, \lambda_k)$, with $\sum_{k \geqslant 1} \lambda_k < \infty$. Put $L'_k = \left[\frac{L_k}{2}\right]$. Then we have: $M(L'_k, \lambda_k)^{\otimes 2} \cong N$. Since every ITPFI of bounded type is an ITPFI_2 ([6], Theorem 2.1), the result follows. ▨

2.3. REMARK. Using exactly the same proof, we can show that for every $p \geqslant 2$, the ITPFI factor M is not a p^{th} tensor power. It can also be seen that $M \otimes M$ is not a p^{th} power, if $p \geqslant 3$ (by the same argument!).

A natural invariant appears to be $R(M) = \{p \in \mathbf{N} \setminus \{0\}$; there exists N, with $N^{\otimes p} \cong M\}$. If M is an ITPFI_2 factor, then $R(M) = \mathbf{N} \setminus \{0\}$.

3. AN EXAMPLE OF HAMACHI-OSIKAWA

In [8], Hamachi and Osikawa consider the ITPFI_2 factors $M = M(L_k, \lambda^{2^k})$, $0 < \lambda < 1$. They prove that for L_k large enough, the flow of weights of M has pure point spectrum. We study here this family of factors. We compute for all sequences L_k the flow of weights of M. We then give estimates on the growth of the L_k's for this flow to have pure point spectrum.

Let λ be a real number in (0,1). In this section we take $\lambda_k = \lambda^{2^k}$, $k \geqslant 0$ and consider type III_0 ITPFI_2 factors $M = \bigotimes_{k\geqslant 0} (M_2(\mathbf{C}), \varphi_k)^{\otimes L_k} = M(L_k, \lambda_k)$ (cf. notation).

Let $\Delta = \{\theta \in \mathbf{R};\ \lambda^{i2^k\theta} = 1, \text{ for some } k \in \mathbf{N}\}$. Let $\theta \in \Delta$. As $\sigma_\theta^{\varphi_k} = 1$ for k large enough, we have $\Delta \subseteq T(M)$ ([2], Théorème 1.3.7 (a)).

Let $(\Omega, \mu) = \prod_{k\geqslant 0} (\{0,1, \ldots, L_k\}, \mu_k)$, where μ_k is the measure on $\{0,1, \ldots, L_k\}$, given by $\mu_k(j) = \dfrac{L_k!}{j!(L_k - j)!} \dfrac{\lambda_k^j}{(1+\lambda_k)^{L_k}}$. Let β be a probability measure on $\mathbf{Z}$, with support $\mathbf{Z}$. Let $\mathscr{R}$ be the equivalence relation on $(\Omega \times \mathbf{Z}, \mu \times \beta)$, given by $(\omega, n)\mathscr{R}(\omega', m)$ iff $\omega_k = \omega'_k$ for all but finitely many k's and

$$\sum_{k\geqslant 0} (\omega_k - \omega'_k)\, 2^k = m - n.$$

Let $(B, \nu) = (\Omega\times \mathbf{Z}, \mu\times\beta)/\mathscr{R}$ (this quotient stands for the ergodic decomposition).

Let S be the transformation of (B, ν) induced by the addition of 1 (in $\mathbf{Z}$) on $(\Omega\times \mathbf{Z}, \mu\times\beta)$.

The flow of weights of M is built over the base transformation (B, ν, S) under the (constant) ceiling function $-\mathrm{Log}\,\lambda$ ([3], Corollary II.6.4; cf. also Appendix of [6]).

Let $\psi_0 : \Omega \to \mathbf{Z}_2$ be given by $\psi_0(\omega) = \psi_0((\omega_k)_{k\geqslant 1}) = \sum_{k\geqslant 0} \omega_k 2^k$ and $\psi : B \to \mathbf{Z}_2$ be induced by the map $(\omega, n) \mapsto \psi_0(\omega) + n$ from $\Omega\times\mathbf{Z}$ to $\mathbf{Z}_2$.

The main feature of this example is coming from:

3.1. THEOREM. *The map $\psi : B \to \mathbf{Z}_2$, defined above, is essentially injective. In particular, the flow of weights of M can be built over the base transformation $(\mathbf{Z}_2, \psi(\nu), H)$ under the (constant) ceiling function $-\mathrm{Log}\,\lambda$, where H denotes the addition of 1 in $\mathbf{Z}_2$.*

For the proof, we need two straightforward technical lemmas. Let μ and μ' be two probability measures on a standard Borel space X. As in [10], we set

$$\rho(\mu, \mu') = \int_X (\mathrm{d}\mu(x))^{1/2}(\mathrm{d}\mu'(x))^{1/2} = \int_X \left(\frac{\mathrm{d}\mu}{\mathrm{d}m}(x)\right)^{1/2}\cdot\left(\frac{\mathrm{d}\mu'}{\mathrm{d}m}(x)\right)^{1/2} \mathrm{d}m(x),$$

where m is a measure on X, with $\mu \ll m$ and $\mu' \ll m$.

3.2. LEMMA. *Let p be a positive real number. Let $L \in \mathbf{N}$, $L \geqslant 4$, $k \in \mathbf{Z}$ and $\xi \in \mathbf{R}$ with $0 < \xi \leqslant (p+2)^{-2}$. Let μ, μ' be the measures on $\mathbf{Z}$ given by*

$$\mu(j) = \frac{L!}{j!(L-j)!}\frac{\xi^j}{(1+\xi)^L} \quad (\mu(j) = 0 \text{ if } j < 0 \text{ or } j > L),$$

$$\mu'(j) = \mu(j+k).$$

Let $\sigma = \frac{\sqrt{L\xi}}{1+\xi}$ *be the standard deviation of* μ. *If* $|k| \leqslant p\sigma$, *then* $\rho(\mu, \mu') \geqslant (3/4)\, e^{-p^2}$.

Proof. We may assume $k \geqslant 0$ (interchanging if necessary the roles of μ and μ'). We have $\frac{\mu(k+j)}{\mu(j)} = \prod_{i=1}^{k} \frac{\xi(L-j-k+i)}{j+i}$ and

$$\operatorname{Log} \frac{\mu(k+j)}{\mu(j)} = \sum_{i=1}^{k} \operatorname{Log} \frac{\xi(L-j-k+i)}{j+i} \geqslant \sum_{i=1}^{k} \frac{(E-j)(1+\xi)-(k-i)\xi-i}{E-(j-E+k-i)\xi}$$

$\left(\text{using the inequality } \operatorname{Log}\frac{a}{b} \geqslant \frac{a-b}{a}\right)$, where $E = \frac{L\xi}{1+\xi}$. If $j \leqslant E + 2\sigma$ and $k \leqslant p\sigma$, then

$$\operatorname{Log} \frac{\mu(k+j)}{\mu(j)} \geqslant \sum_{i=1}^{k} \frac{(E-j)(1+\xi)-(k-i)\xi-i}{E-(2+p)\xi\sigma} \geqslant \sum_{i=1}^{k} \frac{(E-j)(1+\xi)-k}{E-(2+p)\xi\sigma}.$$

Hence,

$$\frac{\mu(k+j)^{1/2}}{\mu(j)^{1/2}} \geqslant \exp\left(-\frac{k^2}{2(E-(2+p)\xi\sigma)}\right)\left(1+\frac{1}{2}\cdot\frac{(E-j)(1+\xi)}{E-(2+p)\xi\sigma}\right).$$

Note that

$$0 = \sum_{j=0}^{L} (E-j)\,\mu(j) \leqslant \sum_{j=0}^{[E+2\sigma]} (E-j)\mu(j).$$

Therefore,

$$\rho(\mu,\mu') \geqslant \sum_{j=0}^{[E+2\sigma]} \frac{\mu(k+j)^{1/2}}{\mu(j)^{1/2}}\,\mu(j) \geqslant \mu(\{0,1,\ldots,[E+2\sigma]\}) \exp\left(-\frac{k^2}{2(E-(2+p)\xi\sigma)}\right).$$

By Tchebyshev's inequality we have $\mu(\{0,1,\ldots,[E+2\sigma]\}) \geqslant 3/4$. As $E = (1+\xi)\sigma^2$ and as $\frac{(2+p)\xi}{\sigma(1+\xi)} = \frac{(2+p)\xi^{1/2}}{\sqrt{L}} \leqslant \frac{1}{\sqrt{L}}$ by assumption, the above inequality reads $\rho(\mu,\mu') \geqslant 3/4 \exp\left(-\frac{k^2}{2\sigma^2\left(1-\frac{1}{\sqrt{L}}\right)(1+\xi)}\right)$ and as $k \leqslant p\sigma$ and $1 - \frac{1}{\sqrt{L}} \geqslant 1/2$, we get the result. ▨

3.3. Lemma. *Let* μ_1, μ_2 *be probability measures on a Borel space* X *with* $\rho(\mu_1,\mu_2) = \alpha > 0$. *Let* C_1, C_2 *be Borel subsets of* X. *If* $\mu_j(C_j) > 1 - \alpha^2/4$, $j = 1, 2$, *then* $C_1 \cap C_2 \neq \emptyset$.

Proof. Let $m = \mu_1 + \mu_2$. The Cauchy-Schwarz inequality gives

$$\int_{C_j^c} \left(\frac{d\mu_1}{dm}(x) \cdot \frac{d\mu_2}{dm}(x) \right)^{1/2} dm(x) \leqslant (\mu_1(C_j^c)\mu_2(C_j^c))^{1/2} < \frac{\alpha}{2} \quad (j = 1,2).$$

Hence $C_1^c \cup C_2^c \neq X$. ▨

Proof of Theorem 3.1. Since M is of type III and $M(1, \lambda_k)$ is type I_∞, we have $M \cong M \otimes M(1, \lambda_k) \cong M(L_k + 1, \lambda_k)$. Hence we may assume that $L_k \geqslant 1$ for all k. Then for a.e. $(\omega, n) \in \Omega \times \mathbf{Z}$, there exists $\omega' \in \Omega$ such that $(\omega, n)\mathscr{R}(\omega', 0)$ Thus it suffices to prove that the map $\varphi : \Omega/\mathscr{R}' \to \mathbf{Z}_2$ is essentially injective, where φ is the map induced by ψ_0 and $\mathscr{R}'$ is the equivalence relation on Ω given by $\omega\mathscr{R}'\omega'$ iff $(\omega, 0)\mathscr{R}(\omega', 0)$ (i.e. $\omega_k = \omega_k'$ for all but finitely many k's and $\sum_{k \geqslant 0} (\omega_k - \omega_k') 2^k = 0$).

Let $\mathscr{A}_n$ be the σ-algebra on Ω generated by the ω_k, $k = 1, \ldots, n-1$ and let $\mathscr{A} = \bigvee_n \mathscr{A}_n$. If $g \in L^1(\Omega, \mathscr{A}, \mu)$, then $g = \lim_n E^{\mathscr{A}_n}(g)$ where $E^{\mathscr{A}_n}(g)$ is the conditional expectation of g with respect to $\mathscr{A}_n$. Let $\mathscr{B}_n$ denote the σ-algebra on Ω generated by $X_n = \sum_{k=0}^{n-1} \omega_k 2^k$ and ω_k, $k \geqslant n$. Let $\mathscr{B} = \bigcap_n \mathscr{B}_n$. Let $\mathscr{D}_n$ denote the σ-algebra on Ω, generated by X_n modulo 2^n, and let $\mathscr{D} = \bigvee_n \mathscr{D}_n$. Note that $\mathscr{A} \supset \mathscr{B}_n \supset \mathscr{D}_n$, that if $f \in L^1(\Omega, \mathscr{A}, \mu)$ is invariant with respect to $\mathscr{R}'$, then $f \in L^1(\Omega, \mathscr{B}, \mu)$, and that $\mathscr{D}$ is the σ-algebra on Ω generated by the map φ. Thus the problem is to show that $\mathscr{B}$ and $\mathscr{D}$ coincide.

To prove this, let $\varepsilon > 0$, $f \in L^1(\Omega, \mathscr{B}, \mu)$, $\|f\|_\infty \leqslant 1$. Then it suffices to show that there exists some $m < \infty$ and $f_0 \in L^1(\Omega, \mathscr{D}_m, \mu)$ such that $\|f - f_0\|_1 < \varepsilon$.

Since $E^{\mathscr{A}_n}(f)$ is measurable with respect to $\mathscr{B}_n$, there exists a function g on $\left\{0, 1, \ldots, \sum_{k=0}^{n-1} L_k 2^k\right\}$ such that $E^{\mathscr{A}_n}(f) = g \circ X_n$. (In the following we will consider g as a function on $\mathbf{Z}$.)

Choose $N < \infty$ such that for all $n \geqslant N$

$$\|E^{\mathscr{A}_n}(f) - f\|_1 < \varepsilon_1$$

where $\varepsilon_1 = \frac{3}{2^{12}} \varepsilon^2 \exp\left(-\frac{32}{\varepsilon}\right)$. Since $\sigma^2(X_n) = \sum_{k=0}^{n-1} \frac{L_k \lambda_k}{(1 + \lambda_k)^2} 2^{2k}$, we have

$$\sum_{n \geqslant 0} 2^{-2n} \sigma^2(X_{n+1}) \geqslant \sum_{n \geqslant 0} \frac{L_n \lambda_n}{(1 + \lambda_n)^2} = \infty.$$

By the ratio test there exist infinitely many n such that

$$(*) \qquad \sigma^2(X_{n+1}) \geqslant 2\sigma^2(X_n)$$

which gives $2^{2n}\sigma^2(\omega_n) = 2^{2n}\dfrac{L_n\lambda_n}{(1+\lambda_n)^2} \geqslant \sigma^2(X_n)$. Let $p = 4\varepsilon^{-1/2}$. Choose $m \geqslant N$ such that equation $(*)$ is satisfied and $\lambda_m \leqslant (p+2)^{-2}$ and $L_m \geqslant 4$.

Write $E^{\mathscr{A}_{m+1}}(f) = g \circ X_{m+1}$ and $E^{\mathscr{A}_m}(f) = h \circ X_m$. Let $P_n = X_n(\mu)$ be the distribution of the random variable X_n and let μ_n be the binomial distribution

$$\mu_n(j) = \frac{L_n!}{(L_n-j)!\,j!}\,\frac{\lambda_n^j}{(1+\lambda_n)^{L_n}}, \qquad j = 0, 1, \ldots, L_n.$$

Note that $P_{n+1} = P_n * \mu_n$.

We have:

$$2\varepsilon_1 > \|E^{\mathscr{A}_m}(f) - E^{\mathscr{A}_{m+1}}(f)\|_1 = \iint |h(x) - g(x + 2^m j)|\, \mathrm{d}P_m(x)\, \mathrm{d}\mu_m(j).$$

Let $A = \left\{x \in \mathbf{Z};\, \int |h(x) - g(x+2^m j)|\, \mathrm{d}\mu_m(j) < \dfrac{24\varepsilon_1}{\varepsilon}\right\}$. Then $P_m(A) > 1 - \varepsilon/12$. For $x \in A$, let $B_x = \{j \in \{0, \ldots, L_m\};\, |h(x) - g(x+2^m j)| \leqslant \varepsilon/8\}$. We have

$$\mu_m(B_x) > 1 - \frac{24\varepsilon_1}{\varepsilon}\cdot\frac{8}{\varepsilon} = 1 - \frac{9}{64}\exp\left(-\frac{32}{\varepsilon}\right).$$

Let $x, x' \in A$ with $x - x' = k2^m$ where k is an integer $\leqslant p\sigma$ $\left(p = 4\varepsilon^{-1/2},\right.$ $\left.\sigma = \sigma(\omega_m) = \dfrac{\sqrt{L_m\lambda_m}}{1+\lambda_m}\right)$. Let μ'_m be the probability measure on $\mathbf{Z}$ given by $\mu'_m(j) = \mu_m(k+j)$. By Lemma 3.2 we have

$$\rho(\mu_m, \mu'_m) = \alpha \geqslant \frac{3}{4}\,\mathrm{e}^{-p^2} = \frac{3}{4}\,\mathrm{e}^{-16/\varepsilon}.$$

As $\mu'_m(B_{x'} - k) = \mu_m(B_{x'}) > 1 - \alpha^2/4$ and $\mu_m(B_x) > 1 - \alpha^2/4$ we get $B_x \cap (B_{x'} - k) \neq \emptyset$ (Lemma 3.3). Hence there exists j such that $|h(x) - g(x+2^m j)| \leqslant \varepsilon/8$ and $|h(x') - g(x' + 2^m(j+k))| \leqslant \varepsilon/8$. Therefore $|h(x) - h(x')| \leqslant \varepsilon/4$.

Let $A' = A \cap \{x \in \mathbf{Z};\, |x - E(X_m)| < 2^{m-1}p\sigma\}$. For every class d of integers modulo 2^m, such that $d \cap A' \neq \emptyset$, choose x_d in this intersection and put $g_0(d) = h(x_d)$. If this intersection is empty, put $g_0(d) = 0$. For $x \in \mathbf{Z}$ we put $h_0(x) = g_0(d)$ where d is the class of x modulo 2^m.

If $x \in A'$, then $|h(x) - h_0(x)| \leqslant \varepsilon/4$.

As $\sigma^2(X_m) \leqslant 2^{2m}\sigma^2$, we get using Tchebyshev's inequality: $P_m(A') \geqslant 1 - \varepsilon/12 - 4/p^2 \geqslant 1 - \varepsilon/3$. Hence

$$\int |h(x) - h_0(x)|\, \mathrm{d}P_m(x) \leqslant 2(1 - P_m(A')) + \frac{\varepsilon}{4} \leqslant \frac{2\varepsilon}{3} + \frac{\varepsilon}{4} \leqslant \frac{11}{12}\,\varepsilon.$$

Put $f_0 = h_0 \circ X_m$. Note that f_0 is measurable with respect to $\mathcal{G}_m$ and

$$\|f - f_0\|_1 \leqslant \|f - h \circ X_m\|_1 + \|h \circ X_m - h_0 \circ X_m\|_1 < \varepsilon.$$

Although the flow of weights of M is, for all choices of L_k's, given by rotation in $\hat{\Delta}$, $T(M)$ can be larger than Δ. We next give a necessary and sufficient condition to have $T(M) = \Delta$.

Note that every $x \in [0,1]$ admits a decomposition $x = \sum_{0 \leqslant j < p_x} (-1)^j 2^{-l_j}$ where $(l_j)_{0 \leqslant j < p_x}$ is an increasing sequence of nonnegative integers ($p_x \in \mathbf{N} \cup \{+\infty\}$).

3.4. PROPOSITION. *For* $n \geqslant 1$, *let* $V_n = \sum_{k=0}^{n-1} \frac{L_k \lambda_k}{(1+\lambda_k)^2} 2^{2(k-n)} = \sigma^2(2^{-n} X_n)$.

a) *If* $\liminf_n V_n > 0$, *then* $T(M) = \Delta$.

b) *If* $\liminf_n V_n = 0$, *then* $T(M)$ *is uncountable.*

Proof. Recall that $\frac{\theta}{\operatorname{Log} \lambda} \in T(M)$ iff $\sum_{k \geqslant 0} L_k \lambda_k (1 - \cos 2^k \theta) < \infty$ ([2], Corollaire 1.3.9).

a) Let x be in $[0,1]$. Write $x = \sum_{0 \leqslant j < p_x} (-1)^j 2^{-l_j}$. Note that the closest integer to $2^n x$ is $e^n = \sum_{l_i \leqslant n} (-1)^i 2^{n-l_i}$. Put for convenience $l_{-1} = 0$. If $l_{j-1} \leqslant n < l_j$, we have $2^{n-l_j-1} \leqslant |2^n x - e_n| \leqslant 2^{n-l_j}$. As for $|t| \leqslant 1/4$, $1 - \cos 2\pi t \geqslant 16 t^2$, we get:

$$1 - \cos 2\pi(2^n x) \geqslant 1 - \cos 2\pi(2^{n-l_j-1}) \geqslant 16(2^{n-l_j-1})^2 = 2^{2(n-l_j+1)}.$$

Hence

$$\sum_{k \geqslant 0} L_k \lambda_k (1 - \cos 2\pi(2^k x)) \geqslant \sum_{0 \leqslant j < p_x} \sum_{k=l_{j-1}}^{l_j - 1} L_k \lambda_k 2^{2(k-l_j+1)}. \tag{1}$$

Set $a_j = \sum_{k=l_{j-1}}^{l_j-1} \frac{L_k \lambda_k}{(1+\lambda_k)^2} 2^{2(k-l_j)}$. We have:

$$V_{l_j} = a_j + V_{l_{j-1}} 2^{2(l_{j-1}-l_j)} = \sum_{k=0}^{j} a_k 2^{2(l_k - l_j)} \leqslant \sum_{k=0}^{j} a_k 2^{2(k-j)}.$$

Hence

$$\sum_{0 \leqslant j < p_x} V_{l_j} \leqslant \sum_{0 \leqslant k \leqslant j < p_x} a_k 2^{2(k-j)} \leqslant \frac{4}{3} \sum_{0 \leqslant k < p_x} a_k.$$

By (1), if $\frac{2\pi x}{\operatorname{Log}\lambda} \in T(M)$, then $\sum_{0 \leqslant j < p_x} a_j < \infty$. Therefore $\sum_{0 < j < p_x} V_{l_j} < \infty$ and as $\liminf_n V_n > 0$, this sum has to be finite, i.e. $\frac{2\pi x}{\operatorname{Log}\lambda} \in \Delta$.

b) If $\liminf_n V_n = 0$, let $(l_j)_{j \geqslant 1}$ be increasing and $V_{l_j} < 2^{-j}$. If $x = \sum_{j \geqslant 1} \varepsilon_j 2^{-l_j}$, $\varepsilon_j \in \{0, 1\}$, then if $l_{j-1} \leqslant k < l_j$, we have: $2^k x - [2^k x] \leqslant 2^{k+1-l_j}$ and $1 - \cos 2\pi(2^k x) \leqslant 2\pi^2 2^{2(k+1-l_j)}$. Hence

$$\sum_{k \geqslant 0} L_k \lambda_k (1 - \cos 2\pi(2^k x)) \leqslant 8\pi^2 \sum_{j \geqslant 1} \sum_{k = l_{j-1}}^{l_j - 1} L_k \lambda_k 2^{2(k-l_j)} \leqslant$$

$$\leqslant 32\pi^2 \sum_{j=1} \sum_{k=0}^{l_j - 1} \frac{L_k \lambda_k}{(1+\lambda_k)^2} 2^{2(k-l_j)} = 32\pi^2 \sum_{j \geqslant 1} V_{l_j} < \infty.$$

Therefore for all choices of ε_j's, $\frac{2\pi x}{\operatorname{Log}\lambda} \in T(M)$; hence, $T(M)$ is uncountable. ▨

3.5. Remark. If x is a non-dyadic rational number, then there exists $\alpha > 0$ such that $d(2^k x, \mathbf{Z}) \geqslant \alpha$ for all $k \geqslant 0$ (this follows from the periodicity of the dyadic expansion of x).

Let $M = M(L_k, \lambda_k)$ be a type III factor. Then as $\sum_{k \geqslant 0} L_k \lambda_k \, d(2^k x, \mathbf{Z}) \geqslant \alpha^2 \sum_{k \geqslant 0} L_k \lambda_k = \infty$, $\frac{2\pi}{\operatorname{Log}\lambda} x \notin T(M)$. Therefore the dyadic numbers are the only rationals in $\frac{\operatorname{Log}\lambda}{2\pi} \cdot T(M)$.

By Theorem 3.1 the isomorphism class of the factor M is completely determined by the measure $\psi(\nu)$. Our next goal is to find conditions for the measure $\psi(\nu)$ to be equivalent to the Haar measure. For $k \geqslant 0$ and $\theta \in \Delta$, let $u_{\theta,k}$ be the unitary in $M_2(\mathbf{C})$, $u_{\theta,k} = \begin{bmatrix} 1 & 0 \\ 0 & \lambda^{i2^k\theta} \end{bmatrix}$. We have $\sigma_\theta^{\varphi_k} = \operatorname{Ad} u_{\theta,k}$. Put $U_\theta = \bigotimes_{k \geqslant 0} (u_{\theta,k})^{\otimes L_k} \in M$ (note that $u_{\theta,k} = 1$ for k large enough). Let φ be the state on M, $\varphi = \bigotimes_{k \geqslant 0} \varphi_k^{\otimes L_k}$.

3.6. Remark. If $\theta \in \Delta$ and $x \in \mathbf{Z}_2$, then $\lambda^{i\theta x}$ has a natural meaning. Using this pairing, $\mathbf{Z}_2$ identifies with the subgroup $\left(\Delta \Big/ \frac{2\pi}{\operatorname{Log}\lambda} \mathbf{Z}\right)^\wedge$ of $\hat{\Delta}$. We can now look at the measure $\psi(\nu)$ as a measure on $\hat{\Delta}$. We have with the notations of Corollary 1.7 b) $\psi(\nu) = \mu_{-\operatorname{Log}\lambda}(\Delta, \varphi, U, B)$. Therefore $\psi(\nu) \in \mathscr{C}_{-\operatorname{Log}\lambda}(M, \Delta)$. Theorem 3.1 asserts that $\mathscr{C}_{-\operatorname{Log}\lambda}(M; \Delta)$ is a complete invariant.

By Remark 1.8 c) there exists a probability measure P on $\hat{\Delta}$ ($P = \mu(\Delta, \varphi, \rho, U)$ cf. Proposition 1.2 b)) with $P \ll \psi(v)$ and $\hat{P}(\theta) = \varphi(U_\theta)$. Note also that $P = \psi_0(\mu)$ (with the notations above Theorem 3.1).

3.7. LEMMA. a) *For* $x = \frac{j}{2^n}$, *put* $\theta = -\frac{2\pi x}{\operatorname{Log} \lambda}\cdot$ *We get*

$$\sum_{k=0}^{n-1} \frac{L_k \lambda_k}{(1+\lambda_k)^2}(1 - \cos(2\pi 2^k x)) \leqslant -\operatorname{Log}|\varphi(U_\theta)| \leqslant \sum_{k=0}^{n-1} \frac{L_k \lambda_k}{1-\lambda_k}(1 - \cos(2\pi 2^k x)).$$

b) *If* $x = \sum_{j=0}^{m} (-1)^j 2^{-l_j}$, *put* $\theta = -\frac{2\pi x}{\operatorname{Log} \lambda}\cdot$ *Then*

$$-\operatorname{Log}|\varphi(U_\theta)| \geqslant 3 \sum_{j=0}^{m} V_{l_j}.$$

Proof. a) We have $\varphi_k(u_{\theta,k}) = \frac{1 + \lambda_k e^{-2i\pi 2^k x}}{1+\lambda_k}$. As $|\varphi_k(u_{\theta,k})| \geqslant \operatorname{Re}(\varphi_k(u_{\theta,k})) = \frac{1 + \lambda_k \cos 2\pi 2^k x}{1 + \lambda_k}$ we get, using the inequality $\operatorname{Log}\frac{a}{b} \geqslant \frac{a-b}{a}$,

$$\operatorname{Log}|\varphi_k(u_{\theta,k})| \geqslant -\frac{\lambda_k(1-\cos 2\pi 2^k x)}{1+\lambda_k \cos 2\pi 2^k x} \geqslant -\frac{\lambda_k}{1-\lambda_k}(1 - \cos 2\pi 2^k x).$$

On the other hand, $|\varphi_k(u_{\theta,k})|^2 = 1 - \frac{2\lambda_k}{(1+\lambda_k)^2}(1 - \cos 2\pi 2^k x)$, so that

$$\operatorname{Log}|\varphi_k(u_{\theta,k})|^2 \leqslant -\frac{2\lambda_k}{(1+\lambda_k)^2}(1 - \cos 2\pi 2^k x).$$

b) Let now $x = \sum_{j=0}^{m} (-1)^j 2^{-l_j}$. If $l_{j-1} \leqslant k < l_j$, we have

$$2^{2(k-l_j+1)} \leqslant 1 - \cos 2\pi 2^k x \leqslant 2\pi^2 2^{2(k-l_j)}.$$

Now, $-\log|\varphi(U_\theta)| = -\sum_{k \geqslant 0} L_k \operatorname{Log}|\varphi_k(u_{\theta,k})|$. Therefore,

$$-\operatorname{Log}|\varphi(U_\theta)| \geqslant \sum_{j=0}^{m} \sum_{k=l_{j-1}}^{l_j - 1} \frac{L_k \lambda_k}{(1+\lambda_k)^2} 2^{2(k-l_j+1)} \qquad (l_{-1} = 0).$$

We put $a_j = \sum_{k=l_{j-1}}^{l_j-1} \frac{L_k\lambda_k}{(1+\lambda_k)^2} 2^{2(k-l_j)}$. We know by the proof of Proposition 3.4 that $\sum_{j=0}^{m} V_{l_j} \leqslant \frac{4}{3} \sum_{j=0}^{m} a_k$. We hence get

$$-\mathrm{Log}|\varphi(U_0)| \geqslant 4 \sum_{j=0}^{m} a_j \geqslant 3 \sum_{j=0}^{m} V_{l_j}. \qquad ▧$$

3.8. PROPOSITION. (cf. [8], Theorem 2). a) *If* $\liminf V_n < +\infty$, *then the measure* $\psi(v)$ *is not equivalent to the Haar measure on* $\mathbf{Z}_2$.

b) *If* $\sum_{n\geqslant 1} e^{-6V_n} < +\infty$, *then the measure* $\psi(v)$ *is equivalent to the Haar measure of* $\mathbf{Z}_2$ *and the flow of weights of M has pure point spectrum.*

Proof. Put $D = \Lambda \Big/ \left(\frac{2\pi}{\mathrm{Log}\,\lambda}\right)\mathbf{Z}$. By Remark 3.6, $\hat{D} \subseteq \hat{\Lambda}$ identifies with $\mathbf{Z}_2$.

a) Let P be the measure on $\hat{\Lambda}$ with $P \ll \psi(v)$ and $\hat{P}(\theta) = \varphi(U_\theta)$, (cf. Remark 3.6). If $\psi(v)$ is equivalent to the Haar measure of $\mathbf{Z}_2$, then $(\hat{P}(\theta))_{\theta\in D} \in C_0(D)$. Therefore $\lim_{n\to\infty} \hat{P}\left(-\frac{2\pi 2^{-n}}{\mathrm{Log}\,\lambda}\right) = 0$. But by Lemma 3.7 (a), $\left|\hat{P}\left(-\frac{2\pi 2^{-n}}{\mathrm{Log}\,\lambda}\right)\right| \geqslant e^{-2\pi^2 \frac{(1+\lambda)^2}{1-\lambda} V_n}$. Hence $\lim_{n\to\infty} V_n = +\infty$.

b) Let F denote the set of finite subsets of $\mathbf{N}$. If A belongs to F, we set $m_A = \# A - 1$ (*) and we denote by $l_{A,0} < l_{A,1} < \ldots < l_{A,m_A}$ the elements of A. Put then $\theta_A = -\frac{2\pi}{\mathrm{Log}\,\lambda}\left(\sum_{j=0}^{m_A} (-1)^j 2^{-l_{A,j}}\right)$. For every element θ of D, there exist exactly two A's with $\theta_A = \theta$ (e.g. for $\theta = 0$, $A = \emptyset$ or $\{0\}$; for $\theta = -\frac{\pi}{\mathrm{Log}\,\lambda}$, $A = \{1\}$ or $\{0, 1\}$). We therefore have

$$\sum_{\theta\in D} |\hat{P}(\theta)|^2 = \frac{1}{2} \sum_{A\in F} |\hat{P}(\theta_A)|^2 \leqslant \frac{1}{2} \sum_{A\in F} \prod_{l\in A} e^{-6V_l}$$

(using Lemma 3.7 b)). But by definition of F and by assumption, we have:

$$\sum_{A\in F} \prod_{l\in A} e^{-6V_l} = \sum_{n\geqslant 1} (1 + e^{-6V_n}) < +\infty.$$

Therefore, $\hat{P} \in \ell^2(D)$ and P is absolutely continuous with respect to the Haar measure of $\mathbf{Z}_2$. As $P \ll \psi(v)$ and as both $\psi(v)$ and the Haar measure are $\mathbf{Z}$-quasi-invariant and ergodic, they are equivalent.

(*) $\# A$ denotes the cardinality of A.

Using Theorem 3.1, we now get that the flow of weights of M has pure point spectrum. ▨

In particular if $\liminf V_n < +\infty$ the factors $M([tL_k], \lambda_k)_{t\in \mathbf{R}_+^*}$ are pairwise non isomorphic ($[tL_k]$ is the integral part of tL_k cf. [7], Corollary 2.6). This is a partial converse to Theorem 3.1 of [6]. Note that as condition C ([6], Definition 4.1) is not satisfied and we can not apply Proposition 4.4 of [6].

We now give another partial converse to Theorem 3.1 of [6]:

3.9. PROPOSITION. *Let L_k and L'_k be two sequences of integers. If* $\limsup L_k\lambda_k < +\infty$ *and if* $M = M(L_k, \lambda_k)$ *and* $N = M(L'_k, \lambda_k)$ *are isomorphic, then* $\lim (L_k - L'_k)\lambda_k = 0$.

Proof. Let f, f', g and g' be defined for $k \geqslant 0$ by

$$f_k = \frac{2L_k\lambda_k}{(1+\lambda_k)^2}, \quad f'_k = \frac{2L'_k\lambda_k}{(1+\lambda_k)^2}, \quad g_k = \frac{2L_k\lambda_k}{1-\lambda_k} \quad \text{and} \quad g'_k = \frac{2L'_k\lambda_k}{1-\lambda_k}.$$

By assumption $f, g \in \ell^\infty(\mathbf{N})$.

Let $a \in \ell^1(\mathbf{N})$ be given by $a_k = \dfrac{1-\cos \pi 2^{-k}}{2}$ $(k \geqslant 0)$.

For $n \geqslant 0$, let $\theta_n = -\dfrac{2\pi 2^{-n}}{\operatorname{Log}\lambda}$ and U_{θ_n}, U'_{θ_n} be the unitaries corresponding to M and N (cf. Lemma 3.7). The algebra $\ell^1(\mathbf{N})$ acts by convolution on the Banach space $\ell^\infty(\mathbf{N})$. Lemma 3.7 reads:

$$(f * a)_{n-1} \leqslant -\operatorname{Log}|\varphi(U_{\theta_n})| \leqslant (g * a)_{n-1} \quad \text{and} \tag{1}$$
$$(f' * a)_{n-1} \leqslant -\operatorname{Log}|\varphi'(U'_{\theta_n})| \leqslant (g' * a)_{n-1}.$$

As M and N are isomorphic, we get by Theorem 1.10,

$$\lim_{n\to\infty} (|\varphi(U_{\theta_n})| - |\varphi'(U'_{\theta_n})|) = 0.$$

As $g*a \in \ell^\infty(\mathbf{N})$, we get $\limsup(-\operatorname{Log}|\varphi(U_{\theta_n})|) < +\infty$. Hence $\liminf_n |\varphi'(U'_{\theta_n})| = \liminf_n |\varphi(U_{\theta_n})| > 0$. As $\dfrac{2L'_k\lambda_k}{(1+\lambda_k)^2} \leqslant -\operatorname{Log}|\varphi(U'_{\theta_n})|$, we derive that $f' \in \ell^\infty(\mathbf{N})$ and $g' \in \ell^\infty(\mathbf{N})$.

Now $(g-f) \in C_0(\mathbf{N})$ and $(g'-f') \in C_0(\mathbf{N})$. We derive that $(g-f)*a \in C_0(\mathbf{N})$ and $(g'-f')*a \in C_0(\mathbf{N})$ and by (1)

$$\lim_{n\to\infty}((f*a)_{n-1} + \operatorname{Log}|\varphi(U_{\theta_n})|) = 0 \quad \text{and} \quad \lim_{n\to\infty}((f'*a)_{n-1} + \operatorname{Log}|\varphi'(U'_{\theta_n})|) = 0.$$

As $\lim_{n\to\infty} (\text{Log}\,|\varphi(U_{\theta_n})| - \text{Log}\,|\varphi'(U'_{\theta_n})|) = 0$, we have $(f - f') * a \in C_0(\mathbf{N})$. Moreover as $a_{k-1} = \sin^2 \pi 2^{-k} = 4a_k \cos^2 \pi 2^{-k-1}$ and $a_1 = 1/2$, we have for $k \geqslant 2$, $a_{k-1} \geqslant 4a_k \cos^2 \frac{\pi}{8} \geqslant 3a_k$. Hence $\sum_{k=1}^{\infty} a_k \leqslant \frac{1}{2} \cdot \sum_{k=0}^{\infty} 3^{-k} = 3/4$. We get $\|a - \delta\|_1 \leqslant 3/4 < 1$, where δ is the unit of $\ell^1(\mathbf{N})$ and a is invertible in $\ell^1(\mathbf{N})$.

As $C_0(\mathbf{N}) * \ell^1(\mathbf{N}) \subseteq C_0(\mathbf{N})$, we have $f - f' = (f - f') * a * a^{-1} \in C_0(\mathbf{N})$. ▨

3.10. Remark. Let now $(p_k)_{k\geqslant 1}$ be a sequence of integers $p_k \geqslant 2$. Put $q_n = \prod_{k=1}^{n} p_k$, $\lambda_k = \lambda^{q_k}$. Let $M = M(L_k, \lambda_k)$ be the corresponding factor. Let also $\Delta = \{\theta \in \mathbf{R}; \lambda^{iq_k\theta} = 1 \text{ for some } k \in \mathbf{N}\} \subseteq T(M)$. The whole section can be rewritten in this context. We obtain:

1) The flow of weights of M is given by the action h_t of $\mathbf{R}$ (notations of Remark 1.8 (a)) on $\hat{\Delta}$ (Theorem 3.1).

2) With $V_n = \sum_{k=0}^{n-1} \frac{L_k\lambda_k}{(1 + \lambda_k)^2} \frac{q_k^2}{q_n^2}$, we have: $T(M) = \Delta$ iff $\liminf_n V_n > 0$; if $T(M) \neq \Delta$, then $T(M)$ is uncountable (Proposition 3.4).

3) If $\liminf_n V_n < \infty$, the Haar measure on $\hat{\Delta}$ does not belong to $\mathscr{C}(M, \Delta)$.

If the sequence $(V_n)_{n\geqslant 1}$ goes to infinity quickly enough, then the factor M has a pure point spectrum flow of weights (Proposition 3.8).

To have an estimate for the growth of V_n, one may use a (weakened) version of Lemma 3.7:

For $x = \frac{j}{q_n}$, $0 < j < q_n$ and $\frac{j}{p_n} = \frac{jq_{n-1}}{q_n} \notin \mathbf{N}$, and $\theta = -\frac{2\pi x}{\text{Log}\,\lambda}$, we get

$$\sum_{k=0}^{n-1} \frac{L_k\lambda^{q_k}}{(1 + \lambda^{q_k})^2} (1 - \cos 2\pi q_k x) \leqslant -\text{Log}\,|\varphi(U_\theta)| \leqslant \sum_{k=0}^{n-1} \frac{L_k\lambda^{q_k}}{1 - \lambda^{q_k}} (1 - \cos 2\pi q_k x)$$

and as $-\text{Log}\,|\varphi(U_{\theta_n})| \leqslant -\text{Log}\,|\varphi(U_\theta)|$, where $\theta_n = -\frac{2\pi}{q_n \text{Log}\,\lambda}$, we get $-\text{Log}\,|\varphi(U_\theta)| \geqslant \geqslant 8V_n$. Therefore, $\sum_{\substack{0\leqslant\theta<1\\ \theta\in T}} |\varphi(U_\theta)|^2 \leqslant \sum_{n\geqslant 1} (q_n - q_{n-1}) \exp(-16V_n)$. If this sum is finite, then as in Proposition 3.8, the Haar measure on $\hat{\Delta}$ belongs to $\mathscr{C}(M, \Delta)$.

In particular if θ_0 is a real number and D is a subgroup of $\mathbf{Q}$, then by (1t) and (3), the factors, whose flow of weights is pure point spectrum with point spectrum $\theta_0 D$, are ITPFI$_2$ (cf. [8]).

4) If $\limsup_k \frac{L_k\lambda_k}{p_{k+1}^2} < \infty$ and if $M(L_k, \lambda_k)$ and $M(L'_k, \lambda_k)$ are isomorphic, then $\lim(L_k - L'_k)\lambda_k = 0$ (Proposition 3.9, cf. proof of Proposition 4.4 below).

3.11. Remark. The occurrence of uncountable $T(M)$ (for factors M acting on a separable Hilbert space) was somewhat unexpected (cf. [13]). Proposition 3.4 and Remark 3.10.2 state that $T(M)$ is uncountable iff the multiplicities are not too large. This suggests the following interpretation of this phenomenon. Among the type III factors, the type III_0 factors are "closer" to the semifinite factors. Similarly, among the type III_0 factors, those with uncountable $T(M)$ are "closer" to the semifinite factors.

4. ANOTHER EXAMPLE

Let b be an irrational number, $0 < b < 1$, and let $(p_k/q_k)_{k\geqslant 1}$ be the sequence of its best rational approximations (see [9], Chapter 10). Let λ be a real number with $0 < \lambda < 1$ and $(L_k)_{k\geqslant 1}$ be a sequence of positive integers. Let $M = M(L_k, \lambda^{q_k})$ be the associated ITPFI$_2$ factor.

We first focus on Connes' invariant $T(M)$. We know ([3], Theorem 1.3.7(b)) that for all sequences L_k, $-\dfrac{2\pi}{\operatorname{Log}\lambda} \in T(M)$.

4.1. Proposition. a) *We have* $\dfrac{2\pi b}{\operatorname{Log}\lambda} \in T(M)$ *iff* $\sum_{k\geqslant 1} \dfrac{L_k\lambda^{q_k}}{q_{k+1}^2} < +\infty$.

b) *If* $\liminf_k L_k\lambda^{q_k} \dfrac{q_k^2}{q_{k+1}^2} > 0$, *then* $T(M) \subseteq \dfrac{2\pi}{\operatorname{Log}\lambda}(\mathbf{Z} \oplus b\mathbf{Z})$.

Proof. a) By corollaire 1.3.9 of [2] we have $\dfrac{2\pi b}{\operatorname{Log}\lambda} \in T(M)$ iff $\sum_{k\geqslant 1} L_k\lambda^{q_k}(1 - \cos 2\pi b q_k) < +\infty$. But for all $k \geqslant 1$, $\dfrac{1}{2q_{k+1}} \leqslant |bq_k - p_k| \leqslant \dfrac{1}{q_{k+1}}$. Since for $|t| < 1/2$, $8t^2 \leqslant 1 - \cos 2\pi t \leqslant 2\pi^2 t^2$, we get $\dfrac{2}{q_{k+1}^2} \leqslant 1 - \cos 2\pi b q_k \leqslant \dfrac{2\pi^2}{q_{k+1}^2}$ and $\sum_{k\geqslant 1} L_k\lambda^{q_k}(1 - \cos 2\pi b q_k) < +\infty$ iff $\sum_{k\geqslant 1} \dfrac{L_k\lambda^{q_k}}{q_{k+1}^2} < +\infty$.

b) Assume that $\liminf L_k\lambda^{q_k} \dfrac{q_k^2}{q_{k+1}^2} > 0$. If $\dfrac{2\pi\theta}{\operatorname{Log}\lambda} \in T(M)$, then $\sum_{k\geqslant 1} \dfrac{L_k\lambda^{q_k}q_k^2}{q_{k+1}^2} \cdot \dfrac{q_{k+1}^2}{q_k^2}(1 - \cos 2\pi\theta q_k) < +\infty$. Therefore $\sum_{k\geqslant 1} \dfrac{q_{k+1}^2}{q_k^2}(1 - \cos 2\pi\theta q_k) < +\infty$. In particular, $\lim \dfrac{q_{k+1}^2}{q_k^2}(1 - \cos 2\pi\theta q_k) = 0$. Let r_k be the closest integer to θq_k $(k \geqslant 1)$. We get $\lim \dfrac{q_{k+1}^2}{q_k^2}(\theta q_k - r_k)^2 = 0$, i.e. $\lim \dfrac{q_{k+1}}{q_k}(\theta q_k - r_k) = 0$, hence there exists K such that for $k \geqslant K$, $\dfrac{q_{k+1}}{q_k}|\theta q_k - r_k| < 1/4$. Write $q_{k+1} = a_k q_k + q_{k-1}$ (cf [9]).

If $k \geqslant K+1$, we have $\theta(a_k q_k + q_{k-1}) - a_k r_k - r_{k-1} = a_k(\theta q_k - r_k) + \theta q_{k-1} - r_{k-1}$ and as $a_k < \frac{q_{k+1}}{q_k}$ and $\frac{q_k}{q_{k-1}} > 1$, we have

$$|\theta(a_k q_k + q_{k-1}) - a_k r_k - r_{k-1}| < \frac{q_{k+1}}{q_k} |\theta q_k - r_k| + \frac{q_k}{q_{k-1}} |\theta q_{k-1} - r_{k-1}| < \frac{1}{2}.$$

Therefore, $a_k r_k + r_{k-1}$ is the closest integer to θq_{k+1}, i.e. $r_{k+1} = a_k r_k + r_{k-1}$, for $k \geqslant K+1$. As the matrix $\begin{bmatrix} p_K & q_K \\ p_{K+1} & q_{K+1} \end{bmatrix}$ is invertible in $M_2(\mathbf{Z})$ (cf. [9], Chapter 10) there exist two integers m and n with $\begin{bmatrix} r_K \\ r_{K+1} \end{bmatrix} = \begin{bmatrix} p_K & q_K \\ p_{K+1} & q_{K+1} \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix}$. One then has by induction $r_k = mp_k + nq_k$, for all $k \geqslant K$. Taking the limit of $\frac{r_k}{q_k}$, we get $0 = mb + n$. ▨

From now on we will assume that the two conditions of Proposition 4.1 are satisfied, namely $\sum_{k \geqslant 1} \frac{L_k \lambda^{q_k}}{q_{k+1}^2} < +\infty$ and $\liminf_k L_k \lambda^{q_k} \frac{q_k^2}{q_{k+1}^2} > 0$. Let $T = \frac{2\pi}{\operatorname{Log} \lambda} \cdot (\mathbf{Z} \oplus b\mathbf{Z})$ $(= T(M))$.

We want now to find conditions for the Haar measure on $\hat{T}$ to be in $\mathscr{C}(M, T)$ (Definition 1.5 a)).

For $k \geqslant 0$, put $u_k = c_k \begin{bmatrix} 1 & 0 \\ 0 & e^{2\pi i b q_k} \end{bmatrix}$, where $c_k = \frac{|1 + \lambda^{q_k} e^{2\pi i b q_k}|}{1 + \lambda^{q_k} e^{2\pi i b q_k}}$. Let φ_k be the state $\varphi_{\lambda^{q_k}}$. We have

$$\varphi_k(u_k) = \frac{|1 + \lambda^{q_k} e^{2\pi i b q_k}|}{1 + \lambda^{q_k}} \geqslant \frac{\operatorname{Re}(1 + \lambda^{q_k} e^{2\pi i b q_k})}{1 + \lambda^{q_k}} \geqslant 1 - \frac{\lambda^{q_k}}{1 + \lambda^{q_k}} (1 - \cos 2\pi b q_k).$$

As $\sum_{k \geqslant 1} \frac{L_k \lambda^{q_k}}{1 + \lambda^{q_k}} (1 - \cos 2\pi b q_k) < \infty$, we know (cf. [1]) that $U = \bigotimes_{k \geqslant 1} (u_k^{\otimes L_k})$ makes sense in $M = \bigotimes_{k \geqslant 1} (M_2(\mathbf{C}), \varphi_k)^{\otimes L_k}$. Moreover, $\sigma^{\varphi}_{2\pi b/\operatorname{Log}\lambda} = \operatorname{Ad} U$ (as for all k, $\sigma^{\varphi_k}_{2\pi b/\operatorname{Log}\lambda} = \operatorname{Ad} u_k$ (cf. [2], Lemma 1.3.8)). Also $\sigma^{\varphi}_{2\pi/\operatorname{Log}\lambda} = 1$.

By Remark 1.8.c), there exists a probability measure μ on $\hat{T}$ $\mu = \mu(T, \varphi, \varphi, U)$ (cf. Proposition 1.2 b)) in $\mathscr{C}_\varphi(T)$, with $\hat{\mu}\left(\frac{2\pi}{\operatorname{Log}\lambda}(n + bm)\right) = \varphi(U^m)$. The support of μ is carried by $G = \{x \in \hat{T};\ \langle x, 2\pi/\operatorname{Log}\lambda \rangle = 1\}$ (Lemma 1.1 3)).

4.2. Lemma. *For* $k \in \mathbf{N}$ *put* $v_k = \dfrac{L_k \lambda^{q_k}}{(1+\lambda^{q_k})^2} \dfrac{q_k^2}{q_{k+1}^2}$. *We have*

$$\sum_{n=\left[\frac{q_k}{2}\right]+1}^{\left[\frac{q_{k+1}}{2}\right]} |\varphi(U^n)|^2 \leqslant \frac{\exp(-2v_k)}{1-\exp\left(-\dfrac{4v_k}{q_k}\right)}.$$

Proof. We have $|\varphi(U^n)| \leqslant |\varphi_k(u_k^n)|^{L_k}$ and

$$|\varphi_k(u_k^n)| = \frac{|1+\lambda^{q_k} e^{2\pi i n b q_k}|}{1+\lambda^{q_k}} = \left(1 - \frac{2\lambda^{q_k}}{(1+\lambda^{q_k})^2}(1-\cos 2\pi nbq_k)\right)^{1/2}.$$

Hence

$$\operatorname{Log} |\varphi_k(u_k^n)| \leqslant -\frac{1}{2}\left(\frac{2\lambda^{q_k}}{(1+\lambda^{q_k})^2}(1-\cos 2\pi nbq_k)\right).$$

As $\dfrac{n}{2q_{k+1}} \leqslant n|bq_k - p_k| \leqslant \dfrac{n}{q_{k+1}}$ we get for $\dfrac{q_k}{2} \leqslant n \leqslant \dfrac{q_{k+1}}{2}$,

$$1-\cos 2\pi nbq_k - 1 - \cos\frac{\pi n}{q_{k+1}} \geqslant \frac{4n^2}{q_{k+1}^2} \geqslant \frac{2nq_k}{q_{k+1}^2}.$$

Hence

$$\sum_{n=\left[\frac{q_k}{2}\right]+1}^{\left[\frac{q_{k+1}}{2}\right]} |\varphi(U^n)|^2 \leqslant \sum_{n=\left[\frac{q_k}{2}\right]+1}^{+\infty} \exp\left(-4v_k \frac{n}{q_k}\right) \leqslant \frac{\exp(-2v_k)}{1-\exp\left(-\dfrac{4v_k}{q_k}\right)}. \quad ▨$$

4.3. Proposition. *If* $\liminf_k \dfrac{L_k\lambda^{q_k}}{(1+\lambda^{q_k})^2}\dfrac{q_k}{q_{k+1}^2} > 0$, *then the Haar measure* m *of* $G = \{x \in \hat{T};\ \langle x, 2\pi/\operatorname{Log}\lambda\rangle = 1\}$ *belongs to* $\mathscr{C}_{-\operatorname{Log}\lambda}(M, T)$.

Proof. By Lemma 4.2, for k large enough,

$$\sum_{n=\left[\frac{q_k}{2}\right]+1}^{\left[\frac{q_{k+1}}{2}\right]} \left|\hat{\mu}\left(n\frac{2\pi b}{\operatorname{Log}\lambda}\right)\right|^2 \leqslant \frac{2\exp(-cq_k)}{1-\exp(-2c)},$$

where c is a positive constant. As $q_k \geqslant k$ for all k,

$$\sum_{n\in\mathbf{Z}}\left|\hat{\mu}\left(n\frac{2\pi b}{\operatorname{Log}\lambda}\right)\right|^2 = 1 + 2\sum_{n\geqslant 1}\left|\hat{\mu}\left(n\frac{2\pi b}{\operatorname{Log}\lambda}\right)\right|^2 < \infty.$$

Therefore $\mu \ll m$. By Remark 1.8 c), m belongs to $\mathscr{C}_{-\operatorname{Log}\lambda}(M, T)$. ▨

We now consider two factors M and N corresponding to the sequences (L_k, λ^{q_k}) and (L'_k, λ^{q_k}). We have:

4.4. PROPOSITION. *Assume that* $\limsup_k L_k\lambda^{q_k}\frac{q_k^2}{q_{k+1}^2} < +\infty$ *and that* $q_{k+1} \geqslant 3q_k$ *for all* k. *If* M *and* N *are isomorphic, then* $\lim_{k\to\infty}(L_k - L'_k)\lambda^{q_k}\frac{q_k^2}{q_{k+1}^2} = 0.$

Proof. Let $\varphi = \bigotimes_{k\geqslant 1}(\varphi_{\lambda^{q_k}}^{\otimes L_k})$ and $\varphi' = \bigotimes_{k\geqslant 1}(\varphi_{\lambda^{q_k}}^{\otimes L'_k})$ be periodic states on M and N and let U and U' be the corresponding unitaries of M and N (cf. notations of Lemma 4.2). As in Lemma 3.7, one checks easily:

$$\sum_{k\geqslant 1}\frac{L_k\lambda^{q_k}}{(1+\lambda^{q_k})^2}(1-\cos 2\pi b q_k q_n) \leqslant -\operatorname{Log}|\varphi(U^{q_n})| \leqslant \sum_{k\geqslant 1}\frac{L_k\lambda^{q_k}}{1-\lambda^{q_k}}(1-\cos 2\pi b q_k q_n)$$

and

$$\sum_{k\geqslant 1}\frac{L'_k\lambda^{q_k}}{(1+\lambda^{q_k})^2}(1-\cos 2\pi b q_k q_n) \leqslant -\operatorname{Log}|\varphi'(U'^{q_n})| \leqslant \sum_{k\geqslant 1}\frac{L'_k\lambda^{q_k}}{1-\lambda^{q_k}}(1-\cos 2\pi b q_k q_n).$$

For all $k \geqslant 1$, put $\theta_k = 2\pi(bq_k - p_k)$. We have

$$\frac{4\pi}{3q_{k+1}} < \frac{2\pi}{q_{k+1}} - \frac{2\pi}{q_{k+2}} < |\theta_k| < \frac{2\pi}{q_{k+1}} \qquad \text{(cf. [9], Chapter 10).}$$

Let f, f', g and g' be defined for $k \geqslant 1$ by

$$f_k = \frac{L_k\lambda^{q_k}}{(1+\lambda^{q_k})^2}(1-\cos 2\pi b q_k^2), \quad g_k = \frac{L_k\lambda^{q_k}}{1-\lambda^{q_k}}(1-\cos 2\pi b q_k^2),$$

$${}'_k = \frac{L'_k\lambda^{q_k}}{(1+\lambda^{q_k})^2}(1-\cos 2\pi b q_k^2), \quad g'_k = \frac{L'_k\lambda^{q_k}}{1-\lambda^{q_k}}(1-\cos 2\pi b q_k^2).$$

Let A be the infinite matrix $(a_{j,k})_{j,k}$ where $a_{j,k} = \frac{1-\cos 2\pi b q_j q_k}{1-\cos 2\pi b q_k^2}$. If $j < k$, we have $a_{j,k} = \frac{1-\cos q_j\theta_k}{1-\cos q_k\theta_k}$ and $\frac{a_{j,k}}{a_{j+1,k}} = \frac{1-\cos q_j\theta_k}{1-\cos q_{j+1}\theta_k}$. As $q_{j+1} > 3q_j$ and $|\theta_k| < \frac{2\pi}{q_{k+1}} < \frac{\pi}{q_k} \leqslant \frac{\pi}{q_{j+1}}$, we get $\frac{a_{j,k}}{a_{j+1,k}} < \frac{1-\cos(\pi/3)}{1-\cos\pi} = \frac{1}{4}$. Hence $a_{j,k} < 4^{j-k}$. If

$j \geqslant k$, we have $a_{j,k} = \dfrac{1-\cos q_k\theta_j}{1-\cos q_k\theta_k}$ and $\dfrac{a_{j+1,k}}{a_{j,k}} = \dfrac{1-\cos q_k\theta_{j+1}}{1-\cos q_k\theta_j}$. We have $\left|\dfrac{\theta_{j+1}}{\theta_j}\right| <$ $< \dfrac{2\pi}{q_{j+2}} \cdot \dfrac{3q_{j+1}}{4\pi} < \dfrac{1}{2}$ and as $|q_k\theta_j| \leqslant |q_k\theta_k| < \dfrac{2\pi q_k}{q_{k+1}} < \dfrac{2\pi}{3}$, we get $\dfrac{a_{j+1,k}}{a_{j,k}} <$ $< \dfrac{1-\cos\dfrac{\pi}{3}}{1-\cos\dfrac{2\pi}{3}} = \dfrac{1}{3}$. Hence for $j > k$, $a_{j,k} < 3^{k-j}$.

As $f_k \leqslant g_k \leqslant \dfrac{L_k\lambda^{q_k}}{1-\lambda^{q_k}} \cdot \dfrac{\theta_k^2 q_k^2}{2}$ and $\dfrac{L_k\lambda^{q_k}}{1-\lambda^{q_k}} \cdot \dfrac{2\pi^2 q_k^2}{q_{k+1}}$ is bounded by assumption, we get that the sequence $-\operatorname{Log}|\varphi(U^{q_k})| \leqslant \sum_{j\geqslant 1} a_{j,k}g_k$ is bounded. By Theorem 1.10, we derive that $\lim_{k\to\infty}(-\operatorname{Log}|\varphi(U^{q_k})| + \operatorname{Log}|\varphi'(U'^{q_k})|) = 0$, hence $f'_k \leqslant \sum_{j\geqslant 1} a_{j,k}f'_k \leqslant$ $\leqslant -\operatorname{Log}|\varphi'(U'^{q_k})|$ is bounded and as $g'_k < \dfrac{(1+\lambda)^2}{1-\lambda} f_k$, we get that $f', g' \in \ell^\infty(\mathbf{N})$.

For an infinite matrix $D = (d_{j,k})$, put $\|D\| = \sup_{j\geqslant 1} \sum_{k\geqslant 1} |d_{j,k}|$. Let $\mathscr{A} = \{D\,;\, \|D\| < +\infty\}$. Then $(\mathscr{A}, \|\cdot\|)$ is a Banach algebra and acts naturally on $\ell^\infty(\mathbf{N})$. Let $\mathscr{B} \subseteq \mathscr{A}$ be the closed subalgebra $\mathscr{B} = \{D \in \mathscr{A}\,;\, Dh \in C_0(\mathbf{N})$ if $h \in C_0(\mathbf{N})\}$, i.e.

$$\mathscr{B} = \left\{D \in \mathscr{A}\,;\, \lim_{j\to\infty} \sum_{k=1}^{n} |d_{j,k}| = 0 \text{ for all } n \in \mathbf{N}\right\}.$$

Then $A = (a_{j,k}) \in \mathscr{B}$. Moreover for all $j \geqslant 1$,

$$\sum_{k\neq j} |a_{j,k}| + \sum_{k=1}^{j-1} 3^{k-j} + \sum_{k=j+1}^{\infty} 4^{j-k} \leqslant \frac{5}{6}.$$

Therefore $\|A - 1\| \leqslant 5/6 < 1$ and A is invertible in $\mathscr{B}$.

As $g - f \in C_0(\mathbf{N})$, we get that $Ag - Af \in C_0(\mathbf{N})$. Therefore $Af - h \in C_0(\mathbf{N})$, where $h_k = -\operatorname{Log}|\varphi(U^{q_k})|$. In the same way, $Af - h' \in C_0(\mathbf{N})$, where $h'_k = -\operatorname{Log}|\varphi'(U'^{q_k})|$. By Theorem 1.10, we get that $h - h' \in C_0(\mathbf{N})$. Hence, $A(f - f') \in C_0(\mathbf{N})$. As A is invertible in $\mathscr{B}$, $f - f' \in C_0(\mathbf{N})$. ▨

4.5. Remark. We have actually just proved the following somehow stronger statement than Proposition 4.4:

If $\limsup_k L_k\lambda^{q_k} \dfrac{q_k^2}{q_{k+1}^2} < +\infty$, $q_{k+1} \geqslant 3q_k$ and if $\mathscr{C}_{-\operatorname{Log}\lambda}(M, T) = \mathscr{C}_{-\operatorname{Log}\lambda}(N, T)$, then $\lim_{k\to\infty}(L_k - L'_k)\lambda^{q_k} \dfrac{q_k^2}{q_{k+1}^2} = 0$.

In particular, in that case

$$\mathscr{C}_{-\mathrm{Log}\,\lambda}(M, T) \neq \mathscr{C}_{-\mathrm{Log}\,\lambda}(M \otimes M, T) = \mathscr{C}_{-\mathrm{Log}\,\lambda}(M, T) * \mathscr{C}_{-\mathrm{Log}\,\lambda}(M, T),$$

and the Haar measure m of G does not belong to $\mathscr{C}_{-\mathrm{Log}\,\lambda}(M, T)$ (cf. Proposition 4.3).

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