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NONCOMMUTATIVE INTEGRATION FOR STATES ON VON NEUMANN ALGEBRAS

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1. INTRODUCTION

The general theory of noncommutative integration has its model in the theory developed for the particular case in which the von Neumann algebra $\mathcal{M}$ (acting on a Hilbert space $\mathcal{H}$) playing the role of the space $L^{\infty}(\Omega, \Sigma, \mu)$ in the abelian case is semifinite while the role of the measure μ is given to a normal faithful semifinite (n.f.s.) trace tr on $\mathcal{M}$. It has been developed by Segal [13], Kunze [10], Dye [6], Stinespring [16], Nelson [11], and it is remarkably close to the ordinary measure and integration theory. In particular it is possible to represent $\mathcal{M}_*$ (the predual of $\mathcal{M}$) as the Banach space (called $L^1(\mathcal{M}, \mathrm{tr})$) of closed, densely defined (in general unbounded) linear operators on $\mathcal{H}$ affiliated with $\mathcal{M}$. It is also possible to define interpolation spaces between $\mathcal{M} \equiv L^{\infty}(\mathcal{M}, \mathrm{tr})$ and $L^1(\mathcal{M}, \mathrm{tr})$ of closed, densely defined operators on $\mathcal{H}$ affiliated with $\mathcal{M}$, which are called $L^p(\mathcal{M}, \mathrm{tr})$, for $1 \leqslant p < +\infty$. For $X \in L^p(\mathcal{M}, \mathrm{tr})$ $(1 \leqslant p < +\infty)$ the $L^p(\mathcal{M}, \mathrm{tr})$ norm is given by

$$\|X\|_{p,\mathrm{tr}} = \mathrm{tr}\,(|X|^p)^{1/p}. \tag{1.1}$$

For $X \in L^p(\mathcal{M}, \mathrm{tr})$ and $Y \in L^{p'}(\mathcal{M}, \mathrm{tr})$ with $\frac{1}{p} + \frac{1}{p'} = 1$, their duality coupling is given by

$$(X, Y) \to \mathrm{tr}(X^+ Y). \tag{1.2}$$

Among all the other features which can be carried over from the classical case to this situation, it is worthwhile to note that $L^{p_1}(\mathcal{M}, \mathrm{tr}) \cap L^{p_2}(\mathcal{M}, \mathrm{tr})$ is norm dense in $L^{p_i}(\mathcal{M}, \mathrm{tr})$ $(i = 1, 2)$ for $1 \leqslant p_i < +\infty$ and weak-* dense in $L^{\infty}(\mathcal{M}, \mathrm{tr})$ if $p_1 = +\infty$.

It would be desirable to obtain a theory as close as possible to this model for the general case of a von Neumann algebra $\mathcal{M}$ on which a n.f.s. weight ω is defined. The problems however start at the very first step, i.e. the representation of the elements of $\mathcal{M}_*^+$, the positive part of $\mathcal{M}_*$, as positive closed densely defined operators on $\mathscr{H}$, which in general is not possible. Various theories for the general noncommutative integration theory have been therefore developed, whose main difference, from our point of view, lies in the choice of the features of the main model which have to be lost.

In the approach due to Haagerup [7] the $L^p(\mathscr{M}, \omega)$ spaces are represented as Banach spaces of closed, densely defined operators, and formulas similar to 1.1 and 1.2 hold, but those operators do not act on the same Hilbert space $\mathscr{H}$ as $\mathscr{M}$ (i.e. the theory is not spatial), and the intersection between any two different L^p spaces is trivial, consisting only of the zero operator.

A spatial theory, closely connected with the preceeding one, has been developed by Connes [5] and Hilsum [8]. Again the L^p spaces are described as Banach spaces of closed, densely defined operators, which now do act on $\mathscr{H}$, and the analogues of 1.1 and 1.2 hold. However the triviality of the intersection permains, and, moreover, the weight involved is defined on the commutant $\mathscr{M}'$ of $\mathscr{M}$ rather than on $\mathscr{M}$ itself.

Another approach has been followed by Sherstnev [14], [15] (and extensive bibliography there quoted), Trunov [20], Trunov and Sherstnev [21] and Zoletarev [22].

In this approach the elements of $\mathscr{M}_*$ are represented as quadratic forms on a suitable dense domain $D(\mathscr{H}, \omega)$ of $\mathscr{H}$, which, however, are not in general the quadratic forms of any closed densely defined linear operator on $\mathscr{H}$. Their intersection with those of the operators in $\mathscr{M}$ is norm dense in $\mathscr{M}_*$ and weak-* dense in $\mathscr{M}$. It is not possible to define L^p spaces for $p \neq 1, +\infty$ (except by using abstract interpolation, [22], or in the case of $\mathscr{M}$ semifinite, where heavy use is made of the Radon-Nikodym theorem for traces [20]) and define directly products or powers of elements of $L^p(\mathscr{M}, \omega)$. So we cannot obtain the analogue of 1.1 and 1.2.

It is also worthwhile to mention an approach due to Araki and Masuda [3], in which they act in the standard representation of $\mathscr{M}$ with respect to a n.f. state ω and the $L^p(\mathscr{M}, \omega)$ spaces are described as subsets or completions of $\mathscr{H}$ with respect to suitable norms.

Finally, the L^p spaces have also been studied as interpolation spaces between $\mathscr{M}$ and $\mathscr{M}_*$ by Kosaki [9], Terp [19] and Zoletarev [22].

The aim of this paper is to present an approach to the theory of noncommutative integration for the case of a general von Neumann algebra $\mathscr{M}$ on which a n.f. state ω is defined which retains the most of the features present in the trace model.

As in [22] the spaces $L(p; \mathscr{M}, \omega)$ are Banach spaces of complex forms, i.e. complex linear combinations of positive forms on a dense subset $D(\mathscr{H}, \omega)$ of the Hilbert space on which $\mathscr{M}$ acts, making thereby the theory spatial.

In general those complex forms are not, as in the case of traces, the "diagonal elements" of any closed, densely defined operator on $\mathscr{H}$; for each $p \in [1, +\infty]$, however, they are closely connected with an operator in the space $L^p(\mathscr{M}, \omega')$ defined in [8] (ω' is a normal faithful semifinite weight on the commutant $\mathcal{M}'$). This enables us to define a product, belonging to $L(p_3; \mathcal{M}, \omega)$, between any element of $L(p_1; \mathcal{M}, \omega)$ and any element of $L(p_2; \mathcal{M}, \omega)$, provided $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3}$, with $1 < p_1, p_2, p_3 < < +\infty$; and so to get a connection with the formulas developed in [8] similiar to 1.1 and 1.2. To do so, we must use, as remarked, an auxiliary n.f.s. weight ω' on $\mathcal{M}'$; it is important to note, however, that all objects which are expected to depend only on $\mathcal{M}$ and ω (i.e. p-norms, products, etc.) in fact do not depend on the particular ω' used. For $p \in (1, +\infty)$ those spaces are defined using interpolation as in [22], and are a concrete representation of the interpolation spaces in [18]. This has a remarkable consequence that the ω conditional expectation defined and studied in [1], [2] and [4] can be lifted to a contraction on $L(p; \mathcal{M}, \omega)$ for $p \in [1, +\infty]$.

Note also that our $L(1; \mathcal{M}, \omega)$ coincides with the one defined in [14], and [15], and if $\mathcal{M}$ is semifinite, our $L(p; \mathcal{M}, \omega)$ spaces coincide with those defined in [20]. Some partial results in the direction of this paper can be found in [4].

This paper is organized as follows: Section 2 contains a short sketch of the theory of noncommutative integration in the approaches of Haagerup [7], Connes [5] and Hilsum [8], since it will be used heavily in the rest of the paper, so as to establish our notations. Also some useful consequences of their treatment are given as lemmas.

Section 3 contains the definition and properties of a Radon-Nikodym derivative of a general element of $\mathcal{M}_*^+$ with respect to a n.f. state ω, while in Section 4, after defining directly the spaces $L(1; \mathcal{M}, \omega)$ and $L(\infty; \mathcal{M}, \omega)$, the spaces $L(p; \mathcal{M}, \omega)$ are defined as interpolation spaces between them, and the explicit formula establishing their connection with the spaces $L^p(\mathcal{M}, \omega')$ is proved. Section 5, after the definition of the product between elements of our spaces $L(p; \mathcal{M}, \omega)$, is devoted to the proof of classical theorems for products and the duality relations.

Finally, in Section 6 we get, as an application, some results on ω-conditional expectations.

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2. PRELIMINARIES AND NOTATIONS

In this paper $\mathcal{M}$ shall denote a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ and $\mathcal{M}'$ its commutant; ω will be a normal, faithful, semifinite (n.f.s.) weight on $\mathcal{M}$ (mostly we shall deal only with a n.f. state) and ω' an n.f.s. weight on $\mathcal{M}'$. The modular automorphism group of ω (ω') on $\mathcal{M}$ ($\mathcal{M}'$) will be σ_ω^t ($\sigma_{\omega'}^t$).

We shall denote by $\mathcal{H}_\omega$ the Hilbert space on which the standard representation π_ω of $\mathcal{M}$ with respect to ω acts and η_ω will denote the immersion mapping of the set $\mathcal{N}_\omega = \{a \in \mathcal{M} : \omega(a^+a) < +\infty, \omega(aa^+) < +\infty\}$. The commutant of $\pi_\omega(\mathcal{M})$

will be denoted by $\pi'_\omega(\mathcal{M})$, the isometric involution by J_ω and the modular operator by Δ_ω. All the above mentioned objects will be endowed with a prime "$'$" if they are referred to the couple $(\mathcal{M}', \omega')$. In the case in which ω is a n.f. state, the Hilbert space $\mathcal{H}_\omega$ has a cyclic and separating vector Ω such that $\omega(a) = \langle \pi_\omega(a)\Omega, \Omega\rangle$ for all a in $\mathcal{M}$, and $\eta_\omega(a) = \pi_\omega(a)\Omega$ for $a \in \mathcal{M}$.

2.1. DEFINITION. $D(\mathcal{H}, \omega) = \{\xi \in \mathcal{H}; \exists\, \alpha > 0 : \|a\xi\|^2 \leqslant \alpha\omega(a^+a)\ \forall\, a \in \mathcal{M}\}$ (see [9], [19], [5], [8], [14]).

The space $D(\mathcal{H}, \omega)$ is a vector space dense in $\mathcal{H}$, and for each $\xi \in D(\mathcal{H}, \omega)$ there is a unique bounded linear operator $R_\omega(\xi): \mathcal{H}_\omega \to \mathcal{H}$ such that $R_\omega(\xi)\eta_\omega(a) = a\xi$. The mapping $\xi \to R_\omega(\xi)$ is linear, and for all $\xi, \eta \in D(\mathcal{H}, \omega)$ the operator $R_\omega(\xi)R_\omega(\eta)^+$ is in $\mathcal{M}'$.

A positive form q on a dense subspace D of a Hilbert space $\mathcal{H}$ is a mapping from D to $[0, +\infty]$ such that

(2.2)

1. $$q(\lambda\xi) = |\lambda|^2 q(\xi) \quad \forall\, \lambda \in \mathbf{C}, \forall\, \xi \in D$$

2. $$q(\xi + \eta) + q(\xi - \eta) = 2\,q(\xi) + 2q(\eta) \quad \forall\, \xi, \eta \in D.$$

The set of vectors on which q is finite (or $\mathscr{D}(q)$) is a linear subspace of D, on which, if it is dense, q defines a positive quadratic form. The mapping $\xi \to q(\xi)$ is lower semicontinuous iff there is a closed positive quadratic form $\tilde{q}$ on $\mathcal{H}$ such that $(\mathrm{Dom}\,\tilde{q}) \cap D = \mathscr{D}(q)$ and $q(\xi) = \tilde{q}(\xi, \xi)$ for $\xi \in \mathscr{D}(q)$. In the above situation, the positive linear operator $T(q)$ associated with the closure of $q|\mathscr{D}(q)$ is the largest positive self-adjoint operator T such that $\|T^{1/2}\xi\|^2 = q(\xi)$ for all $\xi \in \mathscr{D}(q)$. We shall say that q is a complex form on D if it is a complex linear combination of positive forms whose domain is D.

If ω is a n.f.s. weight on $\mathcal{M}$, the equality

$$q_\omega(\xi) = \omega(R_{\omega'}(\xi)R_{\omega'}(\xi)^+) \tag{2.3}$$

defines a lower semicontinuous positive form on $D(\mathcal{H}, \omega')$ with dense domain, and the operator $T(q_\omega) = \dfrac{\mathrm{d}\omega}{\mathrm{d}\omega'}$ is called the spatial derivative of ω with respect to ω'. For a detailed treatment of the material skeched above, see the original paper by Connes [5].

Hilsum [8] now defines the spaces $L^p(\mathcal{M}, \omega')$ for $1 < p < +\infty$.

2.4. DEFINITION. $L^p(\mathcal{M}, \omega')$ is the set of the closed operators T with dense domain on $\mathcal{H}$ such that, given the polar decomposition $T = u|T|$ of T, $u \in \mathcal{M}$ and $|T|^p = \dfrac{\mathrm{d}\omega}{\mathrm{d}\omega'}$ for some ω in the predual $\mathcal{M}_*$ of $\mathcal{M}$.

If $\omega \in \mathcal{M}_*$ with polar decomposition $\omega = u|\omega|$, let $T_{\omega'}(\omega) = u \dfrac{d|\omega|}{d\omega'}$ and $\int T_{\omega'}(\omega) d\omega' = \omega(I)$. Then the spaces $L^p(\mathcal{M}, \omega')$ are Banach spaces under the norm

$$\|T\|_p = \left[\int |T|^p d\omega'\right]^{1/p'} \tag{2.5}$$

if, as we shall always do in this paper, by sum (and, later, product) of unbounded operators we take the strong sum (and strong product), as defined in [13]. It is proved in [8] that many classical theorems (for instance the duality theorem between $L^p(\mathcal{M}, \omega')$ and $L^{p'}(\mathcal{M}, \omega')$) remain true for those spaces. Also $L^1(\mathcal{M}, \omega')$ is isometrically isomorphic to $\mathcal{M}_*$.

2.6. EXAMPLE. Let $\mathcal{M}$ be a von Neumann algebra operating on a Hilbert space *with a cyclic and separating vector* Ω *of norm one*. Let $\omega(a) = \langle a\Omega, \Omega\rangle$ for $a \in \mathcal{M}$ and $\omega'(a') = \langle a'\Omega, \Omega\rangle$ for $a' \in \mathcal{M}'$. Then $D(\mathcal{H}, \omega') = \{a\Omega : a \in \mathcal{M}\}$, $R_{\omega'}(a\Omega) = a$ and $\dfrac{d\omega}{d\omega'} = \Delta_\omega$ the modular operator for ω (cf. [5], proof of Lemma 10a)). So $\Delta^{1/p} \in L^p(\mathcal{M}, \omega')$ for $1 \leqslant p < +\infty$.

The linear mapping $U: T \in L^2(\mathcal{M}, \omega') \to T\Omega$ is a unitary operator, as it is a bijection and $\int |T|^2 d\omega' = \int |T|^2 |R_{\omega'}(\Omega)^+|^2 d\omega' = \||T|\Omega\|^2$ (see also [19], 2.2).

This approach to the theory of noncommutative L^p spaces is closely connected to and partly based upon the approach due to Haagerup [7]. There the L^p spaces consist of closed and densely defined operators affiliated with the crossed product ([17]) $\mathcal{M}_0 = \mathcal{R}(\mathcal{M}, \sigma_\omega^t)$ of $\mathcal{M}$ with the modular automorphism group of ω. If $\mathcal{M}$ is identified with its canonical immersion in $\mathcal{M}_0$, then $\mathcal{M}_0$ is generated by $\mathcal{M}$ and a one parameter group of unitaries $\lambda_\omega(t) = h_\omega^{it}$ (with generator h_ω) such that for $a \in \mathcal{M}$ $\sigma_\omega^t(a) = \lambda_\omega(t) a \lambda_\omega(t)^+$. Let $\theta^s = \theta_\omega^s$ be the dual action on $\mathcal{M}_0$ [17], and T the operator valued weight $T: \mathcal{M}_0^+ \to \mathcal{M}_0^+$ given by $T(a) = \int_{\mathbf{R}} \theta^s(a) ds$. There is a trace τ_ω on $\mathcal{M}_0$ such that $\omega \cdot T = \tau(h_\omega \cdot)$. For $\varphi \in \mathcal{M}_*^+$, set $\tilde{\varphi} = \varphi \circ T$ and let $h_{\omega,\varphi}$ be the τ_ω-measurable (cf. [10]) Radon-Nikodym derivative of $\tilde{\varphi}$ with respect to τ_ω on $\mathcal{M}_0$. So $h_\omega = h_{\omega,\omega}$. The mapping $\varphi \to h_{\omega,\varphi}$ for $\varphi \in \mathcal{M}_*^+$ has a unique extension to a linear map of $\mathcal{M}_*$ onto the set of τ_ω-measurable operators $h_{\omega,\varphi}$ affiliated with $\mathcal{M}_0$, satisfying $\theta^s h_{\omega,\varphi} = e^{-s} h_{\omega,\varphi}$. So it is natural to define $L^1(\mathcal{M})$ as the set of such operators, and define a linear functional tr on $L^1(\mathcal{M})$ by $\mathrm{tr}(h_{\omega,\varphi}) = \varphi(I)$.

Then $L^p(\mathcal{M})$ $(1 \leqslant p < +\infty)$ can be defined as the set of closed, densely defined operators a with polar decomposition $a = u|a|$ affiliated with $\mathcal{M}_0$ such that $|a|^p \in L^1(\mathcal{M})$ and $u \in \mathcal{M}$. The norm of $a \in L^p(\mathcal{M})$ is given by $\|a\|_p = \mathrm{tr}(|a|^p)^{1/p}$.

In [8] (page 153) the algebra $W(\mathcal{M})$ is defined as the von Neumann algebra acting on $\mathcal{H}\otimes L^2(\mathbf{R})$ ($\simeq L^2(\mathbf{R}, \mathcal{H})$) generated by the operators $a\otimes I$ for $a \in \mathcal{M}$ and $\frac{d\omega}{d\omega'} \otimes L_{\mathcal{H}}$ for $\omega \in \mathcal{M}_*^+$, with $L_{\mathcal{H}}$ the generator of the regular representation of $\mathbf{R}$ ($= L_{\mathcal{H}}^{it}$). The operator $(U_{\omega,\omega'}\xi)(s) = \left(\frac{d\omega}{d\omega'}\right)^{is} \xi(s)$ for $\xi \in L^2(\mathbf{R}, \mathcal{H})$ is a unitary operator on this space, and we have the spatial isomorphism $U^+_{\omega,\omega'}\mathcal{M}_0 U_{\omega,\omega'} = W(\mathcal{M})$ (in particular, $U^+_{\omega,\omega'} a U_{\omega,\omega'} = a\otimes I$ for $a \in \mathcal{M}$), while $U^+_{\omega,\omega'} h_{\omega,\varphi} U_{\omega,\omega'} = \frac{d\varphi}{d\omega'} \otimes L_{\mathcal{H}}$ for all $\varphi \in \mathcal{M}_*^+$. This implies that any closed operator T acting on $\mathcal{H}$ is in $L^p(\mathcal{M}, \omega')$ iff $T\otimes L^{1/p} \in L^p(\mathcal{M})$.

The above summarized results imply the following lemma which will be useful later.

2.7. LEMMA. *Let $\mathcal{M}$, $\mathcal{M}'$, ω, ω' and π_ω, $\mathcal{H}_\omega$, Ω be as defined earlier. For all $\varphi \in \mathcal{M}_*$ let $\pi_\omega(\varphi)$ ($\pi_\omega(\varphi)'$) be the element of $(\pi_\omega(\mathcal{M}))_*$ ($(\pi_\omega\mathcal{M}')_*$) defined as $[\pi_\omega(\varphi)](\pi_\omega(a)) = \varphi(a)$ for $a \in \mathcal{M}$ (respectively $[\pi_\omega(\varphi)'](J_\omega\pi_\omega(a)J_\omega) = \varphi(a)$). Then*

$$\text{a)} \qquad \int \left(\frac{d\omega}{d\omega'}\right)^{1/2} a \left(\frac{d\omega}{d\omega'}\right)^{1/2} b\, d\omega' = \langle \pi_\omega(a)\Omega,\ J_\omega\pi_\omega(b^+)\Omega\rangle$$

for $a, b \in \mathcal{M}$.

$$\text{b)} \qquad \int \left(\frac{d\omega}{d\omega'}\right)^{\alpha} a \left(\frac{d\omega}{d\omega'}\right)^{\beta} b \left(\frac{d\omega}{d\omega'}\right)^{\gamma} c\, d\omega' = \int \Delta_\omega^\alpha \pi_\omega(a)\Delta_\omega^\beta \pi_\omega(b)\Delta_\omega^\gamma \pi_\omega(c)\, d\pi_\omega(\omega)'$$

for $a, b, c \in \mathcal{M}$, $0 \leqslant \alpha, \beta, \gamma \leqslant 1$, $\alpha + \beta + \gamma = 1$.

Proof. The von Neumann algebras $\mathcal{M}_0 = \mathcal{R}(\mathcal{M}, \sigma_\omega^t)$ and $(\pi_\omega(\mathcal{M}))_0 = \mathcal{R}(\pi_\omega(\mathcal{M}), \sigma^t_{\pi_\omega(\omega)})$ are isomorphic by [17], Proposition 3.4 and their isomorphism $\varkappa$ is such that $\varkappa(a) = \pi_\omega(a)$ for all $a \in \mathcal{M}$ and $\varkappa(\lambda_\omega(t)) = \lambda_{\pi_\omega(\omega)}(t)$. This implies $\varkappa(h_\omega^{it}) = h^{it}_{\pi_\omega(\omega)}$.

For a n.f. state φ of $\mathcal{M}$ we have $(t \in \mathbf{R}): \varkappa(h^{it}_{\omega,\varphi}) = \varkappa(h_\omega^{it})\varkappa(h_\omega^{-it}h^{it}_{\omega,\varphi}) = h^{it}_{\pi_\omega(\omega)}(u^t_{(\omega,\varphi)})$, since a straightforward check shows that the one parameter family of operators $h_\omega^{-it}h^{it}_{\omega,\varphi}$ is the family $u^t_{(\omega,\varphi)}$ of unitaries in $\mathcal{M}$ defining the Radon-Nikodym cocycle for the states ω and φ. So the one parameter family of unitaries in $\pi_\omega(\mathcal{M})$, $(u^t_{(\omega,\varphi)})$ must satisfy for $t \in \mathbf{R}$ and all $a \in \mathcal{M}$ the equalities:

$$\varkappa(u^t_{(\omega,\varphi)})\varkappa(a)\varkappa(u^t_{(\omega,\varphi)})^+ = \varkappa(u^t_{(\omega,\varphi)}\, a\, u^t_{(\omega,\varphi)}{}^+) =$$

$$= \varkappa(h_\omega^{-it}h^{it}_{\omega,\varphi} a\, h_\omega^{-it}h^{it}_{\omega,\varphi}) = \varkappa(\sigma_\omega^{-t}(\sigma^t_\varphi(a))).$$

So $\varkappa(u^t_{(\omega,\varphi)})$ is the family of unitaries $u^t_{(\pi_\omega(\omega),\pi_\omega(\varphi))}$ defining the Radon-Nikodym cocycle for the states $\pi_\omega(\omega)$ and $\pi_\omega(\varphi)$ on $\mathcal{M}$, and therefore $\varkappa(h^{it}_{\omega,\varphi}) = h^{it}_{\pi_\omega(\omega),\pi_\omega(\varphi)}$.

Let us note now that, by Theorem 1 in [11], $\varkappa$ can be extended to an isomorphism between the *-algebras of the τ_ω-measurable operators affiliated with $\mathcal{M}_0$ and of the $\tau_{\pi_\omega(\omega)}$-measurable operators affiliated with $(\pi(\mathcal{M}))_0$ by setting $\varkappa(a) = \varkappa(u)\int_0^\infty \lambda\, \mathrm{d}\varkappa(e_\lambda)$ when a has polar decomposition $a = u\int_0^\infty \lambda\, \mathrm{d}e_\lambda$. Since $\varkappa(h^{it}_{\omega,\varphi}) = h^{it}_{\pi_\omega(\omega),\pi_\omega(\varphi)}$, we get $\varkappa(h_{\omega,\varphi}) = h_{\pi_\omega(\omega),\pi_\omega'(\varphi)}$ first for φ a n.f. state, and then, by linearity, for the general $\varphi \in \mathcal{M}_*^+$.

a)

$$\varkappa\left(U_{\omega,\omega'}\left[\left(\frac{\mathrm{d}\omega}{\mathrm{d}\omega'}\right)^{1/2} a\left(\frac{\mathrm{d}\omega}{\mathrm{d}\omega'}\right)^{1/2} \otimes L_{\mathscr{H}}\right] U^+_{\omega,\omega'}\right)$$

$$= \varkappa\left[\left[U_{\omega,\omega'}\left(\frac{\mathrm{d}\omega}{\mathrm{d}\omega'} \otimes L_{\mathscr{H}}\right)U^+_{\omega,\omega'}\right]^{1/2}[U_{\omega,\omega'}(a \otimes I)U^+_{\omega,\omega'}]\left[U_{\omega,\omega'}\left(\frac{\mathrm{d}\omega}{\mathrm{d}\omega'} \otimes L_{\mathscr{H}}\right)U^+_{\omega,\omega'}\right]^{1/2}\right] =$$

$$= \varkappa(h_\omega^{1/2} a\, h_\omega^{1/2}) = h^{1/2}_{\pi_\omega(\omega)}\pi_\omega(a)h^{1/2}_{\pi_\omega(\omega)}.$$

So if $\left(\frac{\mathrm{d}\omega}{\mathrm{d}\omega'}\right)^{1/2} a\left(\frac{\mathrm{d}\omega}{\mathrm{d}\omega'}\right)$ is the representative in $L^1(\mathcal{M}, \omega')$ of $\varphi \in \mathcal{M}$ then $h^{1/2}_{\pi_\omega(\omega)}\pi_\omega(a)\, h^{1/2}_{\pi_\omega(\omega)}$ is the representative of $\pi_\omega(\varphi) \in [\pi_\omega(\mathcal{M})]_*$ in $L^1(\pi_\omega(\mathcal{M}), \pi_\omega(\omega)')$. On the other hand (see Example 2.6)):

$$U_{\pi_\omega(\omega),\pi_\omega(\omega)'}\,[\Delta_\omega^{1/2}\pi_\omega(a)\Delta_\omega^{1/2} \otimes L_{\mathscr{H}_\omega}]U^+_{\pi_\omega(\omega),\pi_\omega(\omega)'} =$$

$$= U_{\pi_\omega(\omega),\pi_\omega(\omega)'}\left(\frac{\mathrm{d}(\pi_\omega(\omega))}{\mathrm{d}(\pi_\omega(\omega)')}\right)^{1/2}\pi_\omega(a)\left(\frac{\mathrm{d}(\pi_\omega(\omega))}{\mathrm{d}(\pi_\omega(\omega)')}\right) \otimes L_{\mathscr{H}_\omega}\Bigg]\, U^+_{\pi_\omega(\omega),\pi_\omega(\omega)'} =$$

$$= h^{1/2}_{\pi_\omega(\omega)}\pi_\omega(a)h^{1/2}_{\pi_\omega(\omega)},$$

and so $\Delta_\omega^{1/2}\pi_\omega(a)\,\Delta_\omega^{1/2}$ represents $\pi_\omega(\varphi)$ in $L^1(\pi_\omega(\mathcal{M}), \pi_\omega(\omega)')$. Therefore

$$\int\left(\frac{\mathrm{d}\omega}{\mathrm{d}\omega'}\right)^{1/2} a\left(\frac{\mathrm{d}\omega}{\mathrm{d}\omega'}\right)^{1/2} b\,\mathrm{d}\omega = \int \Delta_\omega^{1/2}\pi_\omega(a)\,\Delta_\omega^{1/2}\pi_\omega(b)\,\mathrm{d}\pi_\omega(\omega)',$$

for all $a, b \in \mathcal{M}$. If $b > 0$, then $\pi_\omega(b) = |R_{\pi_\omega(\omega)'}(\pi_\omega(b^{1/2})\Omega)^+|^2$; so, for $a > 0$, the last member of the preceding equality is equal to:

$$\int \Delta_\omega^{1/2}\pi_\omega(a)\Delta_\omega^{1/2}|R_{\pi_\omega(\omega)'}(\pi_\omega(b^{1/2}))^+|^2\mathrm{d}\pi_\omega(\omega)' =$$

$$= \|\pi_\omega(a^{1/2})\Delta_\omega^{1/2}\pi_\omega(b^{1/2})\Omega\|_{\mathscr{H}} = \langle\pi_\omega(a)\Omega,\, J_\omega\pi_\omega(b)\Omega\rangle_{\mathscr{H}_\omega},$$

which is the equality in our statement. It follows for the general $a, b \in \mathcal{M}$ by polarization (see also [19]).

b) The proof can again be carried out as the proof of the first part of a).

3. A RADON-NIKODYM DERIVATIVE

In the following ω will be assumed to be a n.f. state on $\mathcal{M}$, while ω' will be allowed to be a n.f.s. weight on $\mathcal{M}'$.

3.1. PROPOSITION. *Let* $\xi' \in D(\mathcal{H}, \omega')$. *Then* $\left(\frac{d\omega}{d\omega'}\right)^{1/2} \xi' \in D(\mathcal{H}, \omega)$.

Proof. If $a \in \mathcal{M}$, using Lemma 2.7 a) we get,

$$\left\| a\left(\frac{d\omega}{d\omega'}\right)^{1/2} \xi' \right\|^2 = \left\| \left| a\left(\frac{d\omega}{d\omega'}\right)^{1/2} \right| \xi' \right\|^2 = \int \left(\frac{d\omega}{d\omega'}\right)^{1/2} a^+ a \left(\frac{d\omega}{d\omega'}\right)^{1/2} |R_{\omega'}(\xi')^+|^2 d\omega' =$$

$$= \langle \pi_\omega(a^+a)\Omega, J_\omega \pi_\omega(|R_{\omega'}, (\xi')^+|^2)\Omega \rangle \leqslant \| |R_{\omega'}(\xi')^+|^2 \| \, \|\pi_\omega(a)\Omega\|^2_{\mathcal{H}_\omega} =$$

$$= \| \, |R_{\omega'}(\xi')^+|^2 \, \| \omega(a^+a),$$

which implies our statement.

3.2. LEMMA. *Let* $\xi, \eta \in D(\mathcal{H}, \omega)$. *Then* $R_\omega(\eta)^+ R_\omega(\xi) \in (\pi_\omega(\mathcal{M}))'$.

Proof. Let $a_1, a_2, a_3 \in \mathcal{M}$. Then

$$\langle \pi_\omega(a_1)\Omega, R_\omega(\eta)^+ R_\omega(\xi) \, \pi_\omega(a_2)\pi_\omega(a_2)\Omega \rangle_{\mathcal{H}_\omega} =$$

$$= \langle a_1\eta, a_3 a_2 \xi \rangle_{\mathcal{H}} = \langle a_3^+ a_1 \eta, a_2 \xi \rangle_{\mathcal{H}} =$$

$$= \langle R_\omega(\eta) \, \pi_\omega(a_3)^+ \pi_\omega(a_1)\Omega, R_\omega(\xi)\pi_\omega(a_2)\Omega \rangle_{\mathcal{H}} =$$

$$= \langle \pi_\omega(a_1)\Omega, \pi_\omega(a_3) R_\omega(\eta)^+ R_\omega(\xi)\pi_\omega(a_2)\Omega \rangle$$

and the thesis follows by the density of $\pi_\omega(\mathcal{M})\Omega$ in $\mathcal{H}_\omega$.

3.3. DEFINITION. For $\psi \in \mathcal{M}_*^+$, set

$$[q(\omega, \psi)](\xi) = \psi(\pi_\omega^{-1}(J_\omega |R_\omega(\xi)|^2 J_\omega))$$

for all $\xi \in D(\mathcal{H}, \omega)$. We shall call $q(\omega, \psi)$ (or shortly $q(\psi)$) *the Radon-Nikodym derivative of* ψ *with respect to* ω.

It is immediate to check that the mapping $\xi \to [q(\omega, \psi)](\xi)$ is a positive form on $D(\mathscr{H}, \omega)$. It is not in general lower semicontinuous in $\|\cdot\|_{\mathscr{H}}$; we have however, as a substitute for this, the following

3.4. PROPOSITION. *For any* n.f.s. *weight* ω' *on* $\mathscr{M}'$, *the restriction of* $q(\omega, \psi)$ *on* $\left(\frac{\mathrm{d}\omega}{\mathrm{d}\omega'}\right)^{1/2} D(\mathscr{H}, \omega')$ *is lower semicontinuous in the norm* $\xi \to \|\xi\|_1 = \left\|\left(\frac{\mathrm{d}\omega}{\mathrm{d}\omega'}\right)^{-1/2} \xi\right\|$.

We need the following lemma first:

3.5. LEMMA. *Let* $\xi, \eta \in \left(\frac{\mathrm{d}\omega}{\mathrm{d}\omega'}\right)^{1/2} D(\mathscr{H}, \omega')$; *then*

$$R_{\omega'}\left(\left(\frac{\mathrm{d}\omega}{\mathrm{d}\omega'}\right)^{-1/2} \xi\right) R_{\omega'}\left(\left(\frac{\mathrm{d}\omega}{\mathrm{d}\omega'}\right)^{-1/2} \eta\right)^{+} = \pi_{\omega}^{-1}(J_{\omega} R_{\omega}(\eta)^{+} R_{\omega}(\xi) J_{\omega}).$$

Proof. The first member of the above equality is in $\mathscr{M}$ by Proposition 3.1 and [5]; the second by 3.2. It is enough to prove it for $\xi = \eta$ and then use polarization.

Let now $a \in \mathscr{M}_{+}$. Then:

$$\langle a\xi, \xi\rangle_{\mathscr{H}} = \|a^{1/2}\xi\|_{\mathscr{H}} = \|R_{\omega}(\xi)\pi_{\omega}(a^{1/2})\Omega\|_{\mathscr{H}}^{2} = \langle \pi_{\omega}(a)\Omega, |R_{\omega}(\xi)|^{2}\Omega\rangle_{\mathscr{H}} \cdot$$

On the other hand,

$$\langle a\xi, \xi\rangle_{\mathscr{H}} = \left\|\left[a^{1/2}\left(\frac{\mathrm{d}\omega}{\mathrm{d}\omega'}\right)^{1/2}\right]\left(\frac{\mathrm{d}\omega}{\mathrm{d}\omega'}\right)^{-1/2}\xi\right\|_{\mathscr{H}}^{2} = \left\|a^{1/2}\left(\frac{\mathrm{d}\omega}{\mathrm{d}\omega'}\right)^{1/2}\left|\left(\frac{\mathrm{d}\omega}{\mathrm{d}\omega'}\right)^{-1/2}\xi\right\|_{\mathscr{H}}^{2} =$$

$$= \int \left(\frac{\mathrm{d}\omega}{\mathrm{d}\omega'}\right)^{1/2} a \left(\frac{\mathrm{d}\omega}{\mathrm{d}\omega'}\right)^{1/2} R_{\omega'}\left(\left(\frac{\mathrm{d}\omega}{\mathrm{d}\omega'}\right)^{-1/2}\xi\right)^{+}\Big|^{2} \mathrm{d}\omega' =$$

$$= \left\langle \pi_{\omega}(a)\Omega, J_{\omega}\left(\pi_{\omega}\left(\left|R_{\omega'}\left(\left(\frac{\mathrm{d}\omega}{\mathrm{d}\omega'}\right)^{-1/2}\xi\right)^{+}\right|^{2}\right)\Omega\right\rangle_{\mathscr{H}_{\omega}}$$

by Lemma 2.7 a) and the statement follows.

Proof of Proposition 3.4. By Lemma 3.5

$$[q(\omega, \psi)](\xi) = \psi\left(\left|R_{\omega'}\left(\left(\frac{\mathrm{d}\omega}{\mathrm{d}\omega'}\right)^{-1/2}\xi\right)^{+}\right|^{2}\right);$$

hence by [5], the mapping $\left(\frac{\mathrm{d}\omega}{\mathrm{d}\omega'}\right)^{-1/2}\xi \to [q(\omega, \psi)](\xi)$ is lower semicontinuous in $\mathscr{H}$ and our statement follows.

3.6. REMARK. Proposition 3.1, Lemma 3.5 and [5] imply that $[q(\omega,\psi)](\xi) = \left\| \left(\frac{d\psi}{d\omega'}\right)^{1/2} \left(\frac{d\omega'}{d\omega}\right)^{1/2} \xi \right\|^2$ for $\xi \in \left(\frac{d\omega}{d\omega'}\right)^{1/2} D(\mathcal{H}, \omega')$.

3.7. PROPOSITION. *Let $\psi, \psi_1, \psi_2 \in \mathcal{M}_*^+$. Then*

a) *If $q(\psi_1)$ and $q(\psi_2)$ coincide on $\left(\frac{d\omega}{d\omega'}\right)^{1/2} D(\mathcal{H}, \omega')$, then $\psi_1 = \psi_2$;*

b) *$q(\alpha\psi_1 + \beta\psi_2) = \alpha q(\psi_1) + \beta q(\psi_2)$ for $\alpha, \beta > 0$;*

c) *$[q(\omega)](\xi) = \|\xi\|^2$ for $\xi \in \left(\frac{d\omega}{d\omega'}\right)^{1/2} D(\mathcal{H}, \omega')$;*

d) *for all $a \in \mathcal{M}_+$ there is a unique $\psi_a \in \mathcal{M}_*^+$ such that $[q(\psi_a)](\xi) = \|a^{1/2}\xi\|$ for all $\xi \in \left(\frac{d\omega}{d\omega'}\right)^{1/2} D(\mathcal{H}, \omega')$; moreover the following statements are equivalent:*

e′) *$\psi \leqslant \alpha\omega$ for some $\alpha > 0$;*

e″) *$\psi = \psi_a$ for some $a \in \mathcal{M}$.*

Proof. a) follows from the weak density in $\mathcal{M}$ of the set of the elements of the form $\left| R_{\omega'}\left(\left(\frac{d\omega}{d\omega'}\right)^{-1/2} \xi\right)^+\right|^2$ (see [5]).

b) is obvious.

c) $[q(\omega)](\xi) = \omega\left(\left|R_{\omega'}\left(\left(\frac{d\omega}{d\omega'}\right)^{-1/2}\xi\right)^+\right|^2\right) = \|\xi\|^2$.

d) Let $a \in \mathcal{M}_+$ and $\xi \in \left(\frac{d\omega}{d\omega'}\right)^{1/2} D(\mathcal{H}, \omega')$. Then

$$\|a^{1/2}\xi\| = \left\| \left[\left(\frac{d\omega}{d\omega'}\right)^{1/2} a \left(\frac{d\omega}{d\omega'}\right)^{1/2}\right]^{1/2} \left[\left(\frac{d\omega}{d\omega'}\right)^{-1/2} \xi\right] \right\|^2.$$

So ψ_a is the unique element of $\mathcal{M}_*$ represented by the operator $\left(\frac{d\omega}{d\omega'}\right)^{1/2} a \left(\frac{d\omega}{d\omega'}\right)^{1/2}$ in $L^1(\mathcal{M}, \omega')$. For $b \in \mathcal{M}_+$, we have:

$$\psi_a(b) = \int \left(\frac{d\omega}{d\omega'}\right)^{1/2} a \left(\frac{d\omega}{d\omega'}\right)^{1/2} b \, d\omega' = \int \left(\frac{d\omega}{d\omega'}\right)^{1/2} b \left(\frac{d\omega}{d\omega'}\right)^{1/2} a \, d\omega' \leqslant$$

$$\leqslant \|a\| \left\| \left(\frac{d\omega}{d\omega'}\right)^{1/2} b \left(\frac{d\omega}{d\omega'}\right)^{1/2} \right\|_{L^1(\mathcal{M}, \omega')} = \|a\| \int \left(\frac{d\omega}{d\omega'}\right)^{1/2} b \left(\frac{d\omega}{d\omega'}\right)^{1/2} d\omega' = \|a\| \omega(b).$$

Conversely, if $\psi \leqslant \alpha\omega$ for some $\alpha > 0$, then $[q(\psi)](\xi) \leqslant \alpha\|\xi\|^2$ and there is therefore an $a \in \mathcal{M}_+$ such that $[q(\psi)](\xi) = \|a^{1/2}\xi\|$.

3.8. REMARK. A remnant of the chain rule of the classical Radon-Nikodym derivatives can be found in the following relation

$$[q(\omega_1,\psi)](\xi)=[q(\omega_2,\psi)]\left(\left(\frac{\mathrm{d}\omega_2}{\mathrm{d}\omega'}\right)^{1/2}\left(\frac{\mathrm{d}\omega'}{\mathrm{d}\omega_1}\right)^{1/2}\xi\right),$$

true for all n.f. states ω_1, ω_2 on $\mathcal{M}$, $\psi\in\mathcal{M}_*^+$ and $\xi\in\left(\frac{\mathrm{d}\omega}{\mathrm{d}\omega'}\right)^{1/2}D(\mathcal{H},\omega')$.

3.9. PROPOSITION. *The mapping* $a\to\psi_a$ *defined in Proposition* 3.7 a) *for* $a\in\mathcal{M}^+$ *coincides with the restriction to* $\mathcal{M}^+$ *of the mapping defined with the same notation in* [19], *Definition* 1.

Proof. Let $a\in\mathcal{M}^+$, $\xi\in D(\mathcal{H},\omega)$. Then

$$\psi_a\left(\left|R_{\omega'}\left(\left(\frac{\mathrm{d}\omega}{\mathrm{d}\omega'}\right)^{-1/2}\xi\right)^+\right|^2\right)=\int\left(\frac{\mathrm{d}\omega}{\mathrm{d}\omega'}\right)^{1/2}a\left(\frac{\mathrm{d}\omega}{\mathrm{d}\omega'}\right)^{1/2}\left|R_{\omega'}\left(\left(\frac{\mathrm{d}\omega}{\mathrm{d}\omega'}\right)^{-1/2}\xi\right)^+\right|^2\mathrm{d}\omega'=$$

$$=\left\langle\pi_\omega(a)\Omega,J_\omega\left|R_\omega\left(\left(\frac{\mathrm{d}\omega}{\mathrm{d}\omega'}\right)^{-1/2}\xi\right)^+\right|^2\Omega\right\rangle=$$

$$=\left\langle J_\omega\pi_\omega(a)J_\omega\pi_\omega\left(\left|R_{\omega'}\left(\left(\frac{\mathrm{d}\omega}{\mathrm{d}\omega'}\right)^{-1/2}\xi\right)^+\right|\right)\Omega,\pi_\omega\left(\left|R_{\omega'}\left(\left(\frac{\mathrm{d}\omega}{\mathrm{d}\omega'}\right)^{-1/2}\xi\right)^+\right|\right)\Omega\right\rangle$$

by Lemma 2.7 a). Now, as $\left\{\left(\frac{\mathrm{d}\omega}{\mathrm{d}\omega'}\right)^{-1/2}\xi:\xi\in D(\mathcal{H},\omega)\right\}=D(\mathcal{H},\omega')$ if ω is a state, by linearity and an application of [5], Proposition 3 we obtain our statement.

In the following we shall use the notation ψ_a for $a\in\mathcal{M}$ with the same meaning as in [19].

4. THE SPACES $L(p;\mathcal{M},\omega)$

The preceding section suggests the following definitions:

4.1. DEFINITION. $L(1;\mathcal{M},\omega)$ is the Banach space of the complex forms on $D(\mathcal{H},\omega)$ of the type:

$$[q(\omega,\psi)](\xi)=\psi(\pi_\omega^{-1}(J_\omega|R_\omega(\xi)|^2J_\omega))\quad(\xi\in D(\mathcal{H},\omega))$$

for some $\psi\in\mathcal{M}_*$ with the norm $\|q(\omega,\psi)\|_{L(1;\mathcal{M},\omega)}=\|\psi\|_{\mathcal{M}_*}$. If $\psi\in\mathcal{M}_*$, and $q=q(\omega,\psi)$, we set $\psi=\psi(q)$.

4.2. DEFINITION. $L(\infty; \mathcal{M}, \omega)$ is the Banach space of the complex forms on $D(\mathcal{H}, \omega)$ of the type

$$[q(\omega, a)](\xi) = \langle \xi, a\xi \rangle \quad (\xi \in D(\mathcal{H}, \omega))$$

for some $a \in \mathcal{M}$, with the norm $\|q(\omega, \psi)\|_{L(\infty; \mathcal{M}, \omega)} = \|a\|_{\mathcal{M}}$. If $a \in \mathcal{M}$, $q = q(\omega, a)$ we set $a = a(q)$.

4.3. LEMMA. *$L(\infty; \mathcal{M}, \omega) \subseteq L(1; \mathcal{M}, \omega)$; moreover $q(\omega, a) = q(\omega, \psi_a)$ for $a \in \mathcal{M}$.*

Proof. The statement follows immediately from Proposition 3.7 and linearity.

4.4. REMARK. By Proposition 3.8 the preceding lemma implies that the embedding of $L(\infty; \mathcal{M}, \omega)$ into $L(1; \mathcal{M}, \omega)$ defined by the inclusion implements the linear norm decreasing injection $a \to \varphi_a$ from L ($\equiv \mathcal{M}$, as ω is a state) to $\mathcal{M}_*$ defined in [19], Proposition 2.

It is also possible to check that the space L_1 defined above coincides with the one defined by Sherstnev, which is obtained by completion of $L(\infty; \mathcal{M}, \omega)$ with respect to the $L(1; \mathcal{M}, \omega)$ norm (see for instance [15]).

The above remark and [19], §1, imply immediately the following.

4.5. LEMMA. *The Banach spaces $L(\infty; \mathcal{M}, \omega)$ and $L(1; \mathcal{M}, \omega)$ are a compatible couple (in the sense of the theory of interpolation).*

4.6. DEFINITION. For each $p \in]1, \infty[$ we define $L(p; \mathcal{M}, \omega)$ to be the complex interpolation space at $c = \frac{1}{p}$ relative to the couple $L(\infty; \mathcal{M}, \omega)$ and $L(1; \mathcal{M}, \omega)$.

The second part of Remark 4.4 implies that the spaces $L(p; \mathcal{M}, \omega)$ coincide with those defined in [22]. For the particular case of a semifinite von Neumann algebra they have been studied also in [20].

In the following we will set as in [19]

$$v_T(\xi, \eta) = \langle U|T|^{1/2}\xi, |T^+|^{1/2}\eta \rangle$$

if T is a closely, densely defined operator on $\mathcal{H}$, with polar decomposition $T = U|T|$, and $\xi \in \mathcal{D}(|T|^{1/2})$, $\eta \in \mathcal{D}(|T^+|^{1/2})$.

4.7. THEOREM. *Let ω be a fixed* n.f.s. *weight on $\mathcal{M}$ and let $p \in [1, \infty]$. Then $L(p; \mathcal{M}, \omega)$ coincides with the set of quadratic forms $q \in L(1; \mathcal{M}, \omega)$, whose restriction to $\left(\frac{\mathrm{d}\omega}{\mathrm{d}\omega'}\right)^{1/2} D(\mathcal{H}, \omega')$ is of the form*

$$(*) \qquad q(\xi) = v_T\left(\left(\frac{\mathrm{d}\omega}{\mathrm{d}\omega'}\right)^{-1/2p}\xi, \left(\frac{\mathrm{d}\omega}{\mathrm{d}\omega'}\right)^{-1/2p}\xi\right)$$

for some $T \in L^p(\mathcal{M}, \omega')$. *Moreover the correspondence between forms in* $L(p;\mathcal{M}, \omega)$ *and operators in* $L^p(\mathcal{M}, \omega')$ *given by* $(*)$ *defines a linear isometry of* $L^p(\mathcal{M}, \omega')$ *onto* $L(p;\mathcal{M}, \omega)$.

Proof. Let us note first that, for $T \in L^p(\mathcal{M}, \omega')$, the formula $(*)$ above makes sense, as $\left(\frac{d\omega}{d\omega'}\right)^{-1/2p} \xi \in \left(\frac{d\omega}{d\omega'}\right)^{1/2p'} D(\mathcal{H}, \omega') \subseteq \mathcal{D}(S)$ for all $\xi \in \left(\frac{d\omega}{d\omega'}\right)^{1/2} D(\mathcal{H}, \omega')$ and all $S \in L^{2p}(\mathcal{M}, \omega')$. For $p = 1$ the statement follows from Definition 4.1 and Remark 3.6 for positive forms, and then by linearity for all complex forms in $L(1;\mathcal{M}, \omega)$. As ω is a state, the diagram (55) in [19] reduces to the following:

$$\mathcal{M} \xrightarrow{\mu_p} L^p(\mathcal{M}, \omega') \xrightarrow{\nu_p} \mathcal{M}_*$$

where μ_p, by [19], Theorem 27 has the explicit form

$$\mu_p(a) = \left(\frac{d\omega}{d\omega'}\right)^{1/2p} a \left(\frac{d\omega}{d\omega'}\right)^{1/2p} \quad \text{for } a \in \mathcal{M},$$

and

$$\nu_p(T) = \iota^{-1}\left(\left(\frac{d\omega}{d\omega'}\right)^{1/2p'} T \left(\frac{d\omega}{d\omega'}\right)^{1/2p'}\right) \quad \text{for } T \in L^p(\mathcal{M}, \omega').$$

In the preceding formula ι is the natural mapping from $\mathcal{M}_*$ to $L^1(\mathcal{M}, \omega')$. The spaces L^p defined in [19], § 5 are the image of $L^p(\mathcal{M}, \omega')$ in $\mathcal{M}_*$ under the injection ν_p. Now, if φ belongs to L^p, by Lemma 4.3, Remark 4.4 and [19], Theorem 36, $q(\omega, \varphi) \in L(p;\mathcal{M}, \omega)$, and the mapping $\varphi \to q(\omega, \varphi)$ from L^p to $L(p;\mathcal{M}, \omega)$ is a linear norm preserving bijection.

So, if $q \in L(p;\mathcal{M}, \omega)$, then $q = q(\omega, \varphi)$ for some $\varphi \in L^p$, and therefore, if $\nu_p(T) = \varphi$, we have:

$$q(\xi) = [q(\omega, \varphi)](\xi) = v_{\iota(\nu_p(T))}\left(\left(\frac{d\omega}{d\omega'}\right)^{1/2} \xi, \left(\frac{d\omega}{d\omega'}\right)^{1/2} \xi\right) =$$

$$= v_{\left(\frac{d\omega}{d\omega'}\right)^{1/2p'} T \left(\frac{d\omega}{d\omega'}\right)^{1/2p'}}\left(\left(\frac{d\omega}{d\omega'}\right)^{-1/2} \xi, \left(\frac{d\omega}{d\omega'}\right)^{-1/2} \xi\right) =$$

$$= v_T\left(\left(\frac{d\omega}{d\omega'}\right)^{-1/2p} \xi, \left(\frac{d\omega}{d\omega'}\right)^{-1/2p} \xi\right), \quad \text{for } \xi \in \left(\frac{d\omega}{d\omega'}\right)^{1/2} D(\mathcal{H}, \omega').$$

The above proof shows also that the mapping $T \to q(\xi)$ is a linear and isometrical bijection.

4.8. DEFINITION. The isometrical bijection from $L(p;\mathcal{M},\omega)$ to $L^p(\mathcal{M},\omega')$ defined by the equality (*) in Theorem 4.7 will be denoted by $\lambda_p^{\omega'}$.

4.9. REMARK. From the proof of the above theorem, it follows also that, as for $1 \leqslant p_1 < p_2 \leqslant +\infty$ $L(p_1;\mathcal{M},\omega) \supseteq L(p_2;\mathcal{M},\omega)$,

$$\lambda_{p_1}^{\omega'}(q) = \left(\frac{d\omega}{d\omega'}\right)^{\frac{1}{2}\left(\frac{1}{p_2'}-\frac{1}{p_1'}\right)} \lambda_{p_2}^{\omega'}(q)\left(\frac{d\omega}{d\omega'}\right)^{\frac{1}{2}\left(\frac{1}{p_2'}-\frac{1}{p_1'}\right)}$$

for $q = q(\omega, a) \in L(\infty;\mathcal{M},\omega)$, $\lambda_\infty^{\omega'}(q) = a$.

5. SOME CLASSICAL THEOREMS

Let $q_1 \in L(p_1;\mathcal{M},\omega)$, $q_2 \in L(p_2;\mathcal{M},\omega)$, $\dfrac{1}{p_1}+\dfrac{1}{p_2}=\dfrac{1}{r}$, $1 \leqslant p_1, p_2, r \leqslant +\infty$. Then, if ω' is a n.f.s. weight on $\mathcal{M}'$, $\lambda_{p_1}^{\omega'}(q_1)\lambda_{p_2}^{\omega'}(q_2)$ is an element of $L^r(\mathcal{M},\omega')$, and, so, there is a unique $q_3 \in L(r;\mathcal{M},\omega')$ such that $\lambda_{p_1}^{\omega'}(q_1)\lambda_{p_2}^{\omega'}(q_2) = \lambda_r^{\omega'}(q_3)$. The following proposition shows that the solution q_3 of the above equation does not depend on ω'.

5.1. LEMMA. *Let $q_1, q_2, p_1, p_2, r, \omega$ be as above, and ω_1', ω_2' n.f.s. weights on $\mathcal{M}'$. Then the equality*

$$\lambda_{p_1}^{\omega_1'}(q_1)\lambda_{p_2}^{\omega_1'}(q_2) = \lambda_r^{\omega_1'}(q_3)$$

holds for some $q_3 \in L(r;\mathcal{M},\omega)$ iff the equality

$$\lambda_{p_1}^{\omega_2'}(q_1)\lambda_{p_2}^{\omega_2'}(q_2) = \lambda_r^{\omega_2'}(q_3)$$

holds for the same q_3.

Proof. The symmetry of the statement makes it sufficient to prove only its if part. By Theorem 4.7 and Remark 4.9 it is enough to prove that $\left(\dfrac{d\omega}{d\omega'}\right)^{1/2-1/2r}\lambda_r^{\omega_1'}(q_3)\left(\dfrac{d\omega}{d\omega'}\right)^{1/2-1/2r}$ represents in $L^1(\mathcal{M},\omega_1')$ the same element of $\mathcal{M}_*$ as $\left(\dfrac{d\omega}{d\omega'}\right)^{1/2-1/2r}\lambda_r^{\omega_2'}(q_3)\left(\dfrac{d\omega}{d\omega_2'}\right)^{1/2-1/2r}$ in $L^1(\mathcal{M},\omega_2')$. As $L(\infty;\mathcal{M},\omega)$ is p norm dense in $L(p;\mathcal{M},\omega)$ for all $1 < p < +\infty$ by the continuity of products in the norms of the spaces $L^p(\mathcal{M},\omega_1')$ and $L^p(\mathcal{M},\omega_2')$, we see that it is enough to show it in the particular case in which $q_1, q_2 \in L(\infty;\mathcal{M},\omega)$ or, equivalently, that for all

$a_1, a_2, a_3 \in \mathscr{M}$,

$$\int \left[\left(\frac{\mathrm{d}\omega}{\mathrm{d}\omega_1'}\right)^{1/2p_2'} a_1 \left(\frac{\mathrm{d}\omega}{\mathrm{d}\omega_1'}\right)^{1/2r} a_2 \left(\frac{\mathrm{d}\omega}{\mathrm{d}\omega_1'}\right)^{1/2p_1'} \right] a_3 \, \mathrm{d}\omega_1' =$$

$$= \int \left[\left(\frac{\mathrm{d}\omega}{\mathrm{d}\omega_2'}\right)^{1/2p_2'} a_1 \left(\frac{\mathrm{d}\omega}{\mathrm{d}\omega_2'}\right)^{1/2r} a_2 \left(\frac{\mathrm{d}\omega}{\mathrm{d}\omega_2'}\right)^{1/2p_2'} \right] a_3 \, \mathrm{d}\omega_2'.$$

Here $\frac{1}{p_i} + \frac{1}{p_i'} = 1$ $(i = 1, 2)$. The above equality, however, is an immediate consequence of Lemma 2.7 b).

By the preceding discussion and the above proved lemma, we can give the following:

5.2. Definition. We define, using the usual notation, the product $q_1(p_1)q_2(p_2)$ as the unique element in $L(r; \mathscr{M}, \omega)$ satisfying the equation

$$\lambda_{p_1}^{\omega'}(q_1)\lambda_{p_2}^{\omega'}(q_2) = \lambda_r^{\omega'}(q_1(p_1)q_2(p_2)).$$

5.3. Remark. Note that if $q_i \in L(p_i; \mathscr{M}, \omega)$ $\left(i = 1,2;\right.$ we drop here the assumption $\left.\frac{1}{p_1} + \frac{1}{p_2} < 1\right)$, then by Theorem 5.6a) it makes sense to consider the products $q_1(p_3)q_2(p_4)$ whenever $p_1 \geqslant p_3 \geqslant 1$, $p_2 \geqslant p_4 \geqslant 1$, $\frac{1}{p_3} + \frac{1}{p_4} \leqslant 1$. It is easy to check, however, that in general all those products are different. Note also that if $q_1(\xi) = \langle \xi, a_1\xi \rangle$, $q_2(\xi) = \langle \xi, a_2\xi \rangle$ for $\xi \in D(\mathscr{H}, \omega)$, then $[q_1(\infty)\, q_2(\infty)]\,(\xi) = \langle \xi a_1, a_2\xi \rangle$.

5.4. Example. Let ω be a tracial n.f. state. Then we have

$$\lambda_r^{\omega'}(q_1(p_1)q_2(p_2)) = \lambda_{p_1}^{\omega'}(q_1)\lambda_{p_2}^{\omega'}(q_2) =$$

$$= \left(\frac{\mathrm{d}\omega}{\mathrm{d}\omega'}\right)^{1/2p_1} T_1 \left(\frac{\mathrm{d}\omega}{\mathrm{d}\omega'}\right)^{1/2r} T_2 \left(\frac{\mathrm{d}\omega}{\mathrm{d}\omega'}\right)^{1/2p_2} = \left(\frac{\mathrm{d}\omega}{\mathrm{d}\omega'}\right)^{1/2r} T_1 T_2 \left(\frac{\mathrm{d}\omega}{\mathrm{d}\omega'}\right)^{1/2r},$$

as $\frac{\mathrm{d}\omega}{\mathrm{d}\omega'}$ commutes with both T_1 and T_2. This shows that the product $q_1(p_1)q_2(p_2)$ is the old product between operators lifted to their "diagonal matrix elements".

5.5. Definition. Let $q \in L(1; \mathscr{M}, \omega)$. We set $\int q \, \mathrm{d}\omega$ to be the value in the identity of the element of $\mathscr{M}_*$ corresponding to q via the canonical isometric isomorphism.

To complete the analogy with the usual algebraic structure we give also the following:

5.6. DEFINITION. Let $q \in L(1; \mathcal{M}, \omega)$. We define q^+ by setting $q^+(\xi) = \overline{q(\xi)}$ for $\xi \in D(\mathcal{H}, \omega)$.

It is immediate to check that if $q \in L(p; \mathcal{M}, \omega)$, then for each n.f.s. weight ω' on $\mathcal{M}'$ we have $\lambda_p^{\omega'}(q_{\omega'}^+) = \lambda_p^{\omega'}(q_{\omega'})^+$.

5.7. THEOREM. *Let* $q_i \in L(p_i; \mathcal{M}, \omega)$, $+\infty \geqslant p_i \geqslant 1$. *Then*

a)
$$[q_1(p_1)q_2(p_2)]\,(r_{1,2})q_3(p_3) = q_1(p_1)[q_2(p_2)q_3(p_3)](r_{2,3})$$

for $\dfrac{1}{p_1} + \dfrac{1}{p_2} + \dfrac{1}{p_3} \leqslant 1$, $\dfrac{1}{p_1} + \dfrac{1}{p_2} = \dfrac{1}{r_{1,2}}$, $\dfrac{1}{p_2} + \dfrac{1}{p_3} = \dfrac{1}{r_{2,3}}$.

b) *The product* $q_1(p_1)q_2(p_2)$ *is linear in both the first and the second member.*

c)
$$\|q_1(p_1)q_2(p_2)\|_{L(r;\mathcal{M},\omega)} \leqslant \|q_1(p_1)\|_{L(p_1;\mathcal{M},\omega)}\, \|q_2(p_2)\|_{L(p_2;\mathcal{M},\omega)}$$

if $\dfrac{1}{p_1} + \dfrac{1}{p_2} = \dfrac{1}{r} \leqslant 1$.

d) *For* $+\infty \geqslant p > 1$, $\dfrac{1}{p} + \dfrac{1}{p'} = 1$, $L(p; \mathcal{M}, \omega)$ *is the dual space of* $L(p'; \mathcal{M}, \omega)$ *and the duality mapping is given by* $(q_1, q_2) \to \int q_1^+(p)q_2(p')\mathrm{d}\omega$.

Proof. All the above statements are immediate consequences of Definitions 5.2 and 5.5, Theorem 4.7 and their analogues for the spaces $L^p(\mathcal{M}, \omega')$ proved in [8].

The following proposition gets back part of the classical properties shown to be lost in Remark 5.3.

5.8. PROPOSITION. *Let* $q_i \in L(p_i; \mathcal{M}, \omega)$ $(i = 1, 2)$, *with* $1 \leqslant p_1, p_2, r \leqslant +\infty$. $\dfrac{1}{p_1} + \dfrac{1}{p_2} = \dfrac{1}{r}$. *Then*

$$\int q_1(p_3)q_2(p_3')\mathrm{d}\omega = \int q_1(p_4)q_2(p_4')\mathrm{d}\omega$$

whenever $p_2' < p_3 < p_4 < p_1$ *and* $\dfrac{1}{p_i} + \dfrac{1}{p_i'} = 1$ $(i = 3, 4)$.

Proof. For ω' any n.f.s. weight on $\mathcal{M}$, we have (cf. 5.2, 4.7, and [8])

$$\int q_1(p_3)q_2(p_3')\mathrm{d}\omega = \int \lambda_{p_3}^{\omega'}(q_1)\lambda_{p_3'}^{\omega'}(q_2)\mathrm{d}\omega' =$$

$$= \int \left(\frac{\mathrm{d}\omega}{\mathrm{d}\omega'}\right)^{\frac{1}{2}\left(\frac{1}{p_3}-\frac{1}{p_1}\right)} \lambda_{p_1}^{\omega'}(q_1) \left(\frac{\mathrm{d}\omega}{\mathrm{d}\omega'}\right)^{\frac{1}{2}\left(\frac{1}{p_1}-\frac{1}{p_2}\right)} \lambda_{p_2}^{\omega'}(q_2) \left(\frac{\mathrm{d}\omega}{\mathrm{d}\omega'}\right)^{\frac{1}{2}\left(\frac{1}{p_3'}-\frac{1}{p_2}\right)} \mathrm{d}\omega' =$$

$$= \int \left(\frac{\mathrm{d}\omega}{\mathrm{d}\omega'}\right)^{\frac{1}{2}\left(\frac{1}{p_3}-\frac{1}{p_4}\right)} \lambda_{p_4}^{\omega'}(q_1) \left(\frac{\mathrm{d}\omega}{\mathrm{d}\omega'}\right)^{\frac{1}{2}\left(\frac{1}{p_4'}-\frac{1}{p_2}\right)} \lambda_{p_2}^{\omega'}(q_2) \left(\frac{\mathrm{d}\omega}{\mathrm{d}\omega'}\right)^{\frac{1}{2}\left(\frac{1}{p_3'}-\frac{1}{p_2}\right)} \mathrm{d}\omega' =$$

$$= \int \lambda_{p_4}^{\omega'}(q_1) \left(\frac{\mathrm{d}\omega}{\mathrm{d}\omega'}\right)^{\frac{1}{2}\left(\frac{1}{p_4'}-\frac{1}{p_2}\right)} \lambda_{p_2}^{\omega'}(q_2) \left(\frac{\mathrm{d}\omega}{\mathrm{d}\omega'}\right)^{\frac{1}{2}\left(\frac{1}{p_4'}-\frac{1}{p_2}\right)} \mathrm{d}\omega' =$$

$$= \int \lambda_{p_4}^{\omega'}(q_1)\lambda_{p_4'}^{\omega'}(q_2)\mathrm{d}\omega' = \int q_1(p_4)q_2(p_4')\mathrm{d}\omega.$$

The following theorem shows that any isomorphism (not necessarily spatial) between two von Neumann algebras $\mathcal{M}_1$ and $\mathcal{M}_2$ intertwining their weights ω_1 and ω_2 can be lifted to the algebraic structure of their $L(p;\mathcal{M}_1,\omega_1)$ and $L(p;\mathcal{M}_2,\omega_2)$ spaces.

5.9. THEOREM. *Let K be an isomorphism $K:\mathcal{M}_1\to\mathcal{M}_2$ with $\mathcal{M}_1$ and $\mathcal{M}_2$ von Neumann algebras. Let ω_1 (ω_2) be a* n.f. *state on $\mathcal{M}_1$ ($\mathcal{M}_2$) such that for $a\in\mathcal{M}_1$ $\omega_2(K(a))=\omega_1(a)$. Let us set, for $q\in L(\infty;\mathcal{M}_1,\omega_1)$*

$$[K(q)](\eta) = \langle\eta, \varkappa(a)\eta\rangle \qquad \eta\in D(\mathcal{M}_2,\omega_2)$$

if $q(\xi) = \langle\xi, a\xi\rangle$ for $\xi\in D(\mathcal{M}_1,\omega_1)$. Then K can be extended to a linear isometry from $L(1;\mathcal{M}_1,\omega_1)$ to $L(1;\mathcal{M}_2,\omega_2)$ such that

a) *$K|L(p;\mathcal{M}_1,\omega_1)$ is a linear isometry from $L(p;\mathcal{M}_1,\omega_1)$ to $L(p;\mathcal{M}_2,\omega_2)$ for $1\leqslant p\leqslant +\infty$.*

b) $[K(q_1)](p_1)[K(q_2)](p_2) = K(q_1(p_1)q_2(p_2))$ $1\leqslant p_1, p_2, r\leqslant +\infty$ $\dfrac{1}{p_1}+\dfrac{1}{p_2}= \dfrac{1}{r}$, $q_1\in L(p_1;\mathcal{M}_1,\omega_1)$, $q_2\in L(p_2;\mathcal{M}_1,\omega_1)$.

c) $K(q)^+ = K(q^+)$.

Proof. The mapping K is clearly a linear isometry from $L(\infty;\mathcal{M}_1,\omega_1)$ to $L(\infty;\mathcal{M}_2,\omega_2)$;

$$\|K(q)\|_{L(p;\mathcal{M}_2,\omega_2)} = \|q\|_{L(p;\mathcal{M}_1,\omega_1)} \quad \text{for } 1\leqslant p<+\infty.$$

So by the density of $L(\infty; \mathcal{M}, \omega)$ in $L(p; \mathcal{M}, \omega)$ we get a). To prove b) note that the equality in the statement follows from Lemma 2.7 b) if $q_1, q_2 \in L(\infty; \mathcal{M}_1, \omega_1)$ and can then be extended to the general case by density and Theorem 5.7 c).

Finally c) is trivial for $q \in L(\infty; \mathcal{M}_1, \omega_1)$ and can be extended to the general case by noting that the mapping $q \to q^+$ (resp. $K(q) \to K(q)^+$) preserves the norm of $L(p; \mathcal{M}_1, \omega_1)$ (resp. $L(p; \mathcal{M}_2, \omega_2)$) for $1 \leqslant p \leqslant +\infty$.

6. AN APPLICATION

In this section $\mathcal{M}_1$ and $\mathcal{M}_2$ ($\mathcal{M}_2 \subset \mathcal{M}_1$) shall be two von Neumann algebras, and ω_1 (resp. $\omega_2 = \omega_1 | \mathcal{M}_2$) a n.f. state on $\mathcal{M}_1$ (resp. $\mathcal{M}_2$). We shall denote by E the ω_1 conditional expectation from $\mathcal{M}_1$ to $\mathcal{M}_2$. For its definition and properties see [1], [2] and [4]. We shall define a linear contraction ε from $L(\infty; \mathcal{M}_1, \omega_1)$ to $L(\infty; \mathcal{M}_2, \omega_2)$ by setting

$$(*) \qquad [\varepsilon(q)](\eta) = \langle \eta, E(a)\eta \rangle$$

for $\eta \in D(\mathcal{M}_2, \omega_2)$ and $q(\xi) = \langle \xi, a\xi \rangle$ ($a \in \mathcal{M}$, $\xi \in D(\mathcal{M}_1, \omega_1)$).

6.1. PROPOSITION. *The mapping ε can be extended to a linear contraction ε_1 from $L(1; \mathcal{M}_1, \omega_1)$ to $L(1; \mathcal{M}_2, \omega_2)$. Moreover, if q is the representative in $L(1, \mathcal{M}_1, \omega_1)$ of $\varphi(q) \in \mathcal{M}_{1*}$, then $\varepsilon_1(q)$ is the representative in $L(1; \mathcal{M}_2, \omega_2)$ of $\varphi(q) | \mathcal{M} = \varphi(\varepsilon_1(q)) \in \mathcal{M}_{2*}$.*

Proof. Let $q_1 \in L(\infty; \mathcal{M}_1, \omega_1)$, ω_1' (ω_2') a n.f.s. weight on $\mathcal{M}_1'$ ($\mathcal{M}_2'$) and $a \in \mathcal{M}_2$. Then by Lemma 2.7 a), and 3.18, 3.20 in [1], we have:

$$[\varphi(q_1)](a) = \int \lambda^{\omega_1'}_{(1; \mathcal{M}_1, \omega_1)}(q_1) a \, \mathrm{d}\omega_1' =$$

$$= \int \left(\frac{\mathrm{d}\omega_1}{\mathrm{d}\omega_1'}\right)^{1/2} \lambda^{\omega_1'}_{(\infty; \mathcal{M}_1, \omega_1)}(q_1) \left(\frac{\mathrm{d}\omega_1}{\mathrm{d}\omega_1'}\right)^{1/2} a \, \mathrm{d}\omega_1' =$$

$$= \langle \pi_{\omega_1, \mathcal{M}_1}(\lambda^{\omega_1'}_{(\infty; \mathcal{M}_1, \omega_1)}(q_1))\Omega, J_{\omega_1, \mathcal{M}_1} \pi_{\omega_1, \mathcal{M}_1}(a^+)\Omega \rangle =$$

$$= \langle \pi_{\omega_2, \mathcal{M}_2}(E(\lambda^{\omega_1'}_{(\infty; \mathcal{M}_1, \omega_1)}(q_1)))\Omega, J_{\omega_2, \mathcal{M}_2} \pi_{\omega_2, \mathcal{M}_2}(a^+)\Omega \rangle =$$

$$= \langle \pi_{\omega_2, \mathcal{M}_2}(\lambda^{\omega_2'}_{(\infty; \mathcal{M}_2, \omega_2)}(\varepsilon(q_1)))\Omega, J_{\omega_2, \mathcal{M}_2} \pi_{\omega_2, \mathcal{M}_2}(a^+)\Omega \rangle =$$

$$= \int \left(\frac{\mathrm{d}\omega_2}{\mathrm{d}\omega_2'}\right)^{1/2} \lambda^{\omega_2'}_{(\infty; \mathcal{M}_2, \omega_2)}(\varepsilon(q_1)) \left(\frac{\mathrm{d}\omega_2}{\mathrm{d}\omega_2'}\right)^{1/2} a \, \mathrm{d}\omega_2' =$$

$$= \int \lambda^{\omega_2'}_{(1; \mathcal{M}_2, \omega_2)}(\varepsilon(q_1)) a \, \mathrm{d}\omega_2' = [\varphi(\varepsilon(q))](a).$$

So $\varphi(\varepsilon(q)) = \varphi(q)|\mathcal{M}_2$ for $q \in L(\infty; \mathcal{M}_1, \omega_1)$. So $\|q\|_{L(1;\mathcal{M}_1,\omega_1)} = \|\varphi(q)\|_{\mathcal{M}_{1*}} \geqslant \geqslant \|\varphi(q)|\mathcal{M}_2\|_{\mathcal{M}_{1*}} = \|\varphi(\varepsilon(q))\|_{\mathcal{M}_{2*}} = \|\varepsilon(q)\|_{L(1;\mathcal{M}_2,\omega_2)}$ and the mapping $q \to \varepsilon(q)$ is a contraction also from the norm $L(1; \mathcal{M}_1, \omega_1)$ to the norm $L(1; \mathcal{M}_2, \omega_2)$. So using again density of $L(\infty; \mathcal{M}, \omega)$ in $L(p; \mathcal{M}, \omega)$ the statement is proved.

NOTE. A version of the preceding proposition has been already proved in [2].

6.2. COROLLARY. *$\varepsilon|L(p; \mathcal{M}_1, \omega_1)$ is a linear contraction from $L(p; \mathcal{M}_1, \omega_1)$ to $L(p; \mathcal{M}_2, \omega_2)$.*

Proof. The statement follows from the Calderon-Lions interpolation theorem (cf. for instance [12], page 37), Theorem 6.1 and the fact, proved in [4], that E is a contraction on $\mathcal{M}$.

6.3. THEOREM. *Let $\mathcal{M}_1, \mathcal{M}_2, \omega_1$ and ω_2 be as above. Then the ω_1-conditional expectation can be characterized as the unique linear contraction $E: \mathcal{M}_1 \to \mathcal{M}_2$ such that the associated mapping $\varepsilon: L(\infty; \mathcal{M}_1, \omega_1) \to L(\infty; \mathcal{M}_2, \omega_2)$ defined as in $(*)$ at the beginning of the section has an extension $\varepsilon_1: L(1; \mathcal{M}_1, \omega_1) \to L(1; \mathcal{M}_2, \omega_2)$ which is again a linear contraction and satisfies the equality*

$$(**) \qquad \varepsilon_1(q_1(1)\, q_2(\infty)) = [\varepsilon_1(q_1)](1) q_2(\infty)$$

for $q_1 \in L(\infty; \mathcal{M}_1, \omega_1)$, $q_2 \in L(\infty; \mathcal{M}_2, \omega_2)$. Here we identify and denote by q_2 the complex form in $L(\infty; \mathcal{M}_1, \omega_1)$ corresponding to $a \in \mathcal{M}_2$ with the complex form in $L(\infty; \mathcal{M}_2, \omega_2)$ corresponding to the same $a \in \mathcal{M}_2$.

Proof. By Proposition 6.1 if E is the ω_1 conditional expectation from $\mathcal{M}_1$ to $\mathcal{M}_2$, we have, for ω_i' a n.f.s. weight on $\mathcal{M}_i'$ $(i = 1, 2)$, $a \in \mathcal{M}_2$:

$$\int \lambda^{\omega_2'}_{(1;\mathcal{M}_2,\omega_2)}(\varepsilon_1(q_1(1)q_2(\infty)))a\, \mathrm{d}\omega_2' =$$

$$= \int \lambda^{\omega_1'}_{(1;\mathcal{M}_1,\omega_1)}(q_1(1)q_2(\infty))a\, \mathrm{d}\omega_1' =$$

$$= \int \lambda^{\omega_1'}_{(1;\mathcal{M}_1,\omega_1)}(q_1)\lambda^{\omega_1'}_{(\infty;\mathcal{M}_1,\omega_1)}(q_2)a\, \mathrm{d}\omega_1' =$$

$$= \int \lambda^{\omega_2'}_{(1;\mathcal{M}_2,\omega_2)}(\varepsilon_1(q_1))\lambda^{\omega_2'}_{(\infty;\mathcal{M}_2,\omega_2)}(q_2)a\, \mathrm{d}\omega_2' =$$

$$= \int \lambda^{\omega_2'}_{(1;\mathcal{M}_2,\omega_2)}([\varepsilon_1(q_1)](1)q_2(\infty))a\, \mathrm{d}\omega_2',$$

which implies $(**)$.

On the other hand it can be seen similarly that if E satisfies the conditions in the statement then it satisfies the conditions of Proposition 6.1. The uniqueness of the mapping satisfying them, as well as the existence and uniqueness of the ω_1-conditional expectation yield our statement.

6.4. REMARK. The equality $(**)$ above is the natural generalization of the relation $E(fg) = E(f)g$ $(f \in \mathcal{M}_1,\ g \in \mathcal{M}_2)$ typical for the conditional expectation in the classical case as well as when ω_1 is a trace. In the above cases indeed, the products $q_1(1)q_2(\infty)$ and $q_1(\infty)q_2(\infty)$ coincide. In this situation we are led to the well known characterization of the conditional expectation as a norm one projection; the same remains true also in the situation, studied by Takesaki in [18], in which $\sigma^t_{\omega_2}(\mathcal{M}_2) \subseteq \mathcal{M}_2$ for all $t \in \mathbf{R}$.

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