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THE NEVANLINNA-PICK PROBLEM FOR MATRIX-VALUED FUNCTIONS

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1. INTRODUCTION

In recent years some complex interpolation problems with a long history have come once again to the fore because of a remarkable diversity of applications in systems engineering. One version of the Nevanlinna-Pick problem is to minimise the supremum norm over the set of bounded analytic functions in the open unit disc U subject to a finite set of interpolation conditions (see [21]). There are applications of the solution of this problem in optimal circuit design going back over 40 years now (see [9]), but the recent heightening of interest was brought about by results of V. M. Adamyan, D. Z. Arov and M. G. Kreĭn [1, 2, 3] on a mathematically equivalent problem formulated in terms of infinite Hankel matrices. Engineers have found uses for these results in the problems of identification and realization [8] and in model reduction and digital filter design [9, 10]. The Nevanlinna-Pick problem plays an important role in J. W. Helton's far-reaching application of non-Euclidean functional analysis to electronics [14, 15]. Evans and Helton have even encountered the problem in modelling fluid retention in the lungs [12].

In consequence both of these developments and of progress in operator theory there have been many papers on Nevanlinna-Pick interpolation recently in both engineering and mathematics journals. There are now several alternative mathematical approaches which give a neat and unified treatment of a wide range of interpolation and approximation problems. A powerful approach is based on the ideas of commutant lifting [19] and contractive intertwining dilations [5], which can be traced back to pioneering work of D. Sarason [20]. A very elegant method [6, 7] is based on the theory of spaces with indefinite inner product, while a more function-theoretic approach, using Hankel operators, stems from Adamyan et al [1, 2, 3]. All of these allow the extension of the original interpolation problem to matrix-valued analytic functions: this is essential for many of the new applications.

To put all this mathematics to work we require efficient numerical algorithms, and the present paper forms part of a project to implement such an algorithm for the Nevanlinna-Pick problem for matrix-valued functions. A. C. Allison and the author [4] have successfully implemented and tested an algorithm for the scalar problem, but the extension to matrix functions makes for quite new difficulties, both mathematical and computational. The main one is the question of uniqueness. A problem which is to be solved numerically should be formulated so as to have a unique solution. In scalar cases which are of practical interest there is always a unique interpolating function of minimal norm; in the matrix case there almost never is. The drift of the theoretical works cited above is to prove existence theorems and to describe the set of all solutions. The purpose of this paper is to prove that there is a natural strengthening of the minimisation condition which restores uniqueness and which enables the unique minimising function to be calculated from an explicit formula in terms of the singular values and vectors of a succession of operators. These operators, here called after Sarason, are compressions of multiplication operators on vector-valued H^2 spaces.

In order to convert the results of this paper into a practical algorithm one has to represent these operators by matrices with respect to orthonormal bases, and it is not obvious that this can be done efficiently. However, F. B. Yeh has shown in his thesis [22] that it can, at least, subject to certain quite mild assumptions on the data, and he has programmed a preliminary version of the algorithm given in Theorem 2 below in the case of 2×2 matrices, with promising results. It takes quite a lot more mathematics to achieve this representation: the details will be published elsewhere.

To state the problem we introduce the space $H^\infty_{m\times n}$ of bounded analytic functions from the open unit disc U to the space $\mathbf{C}^{m\times n}$ of complex $m \times n$ matrices, $m, n \in \mathbf{N}$. $\mathbf{C}^{m\times n}$ carries the Hilbert space operator norm (sometimes called the spectral norm), and $H^\infty_{m\times n}$ carries the norm

$$\|F\|_\infty = \sup_{z\in U} \|F(z)\|_{\mathbf{C}^{m\times n}}.$$

An element of $H^\infty_{m\times n}$ has a radial limit at almost every point of the unit circle by Fatou's theorem (see [16]), and so $H^\infty_{m\times n}$ can be identified isometrically and linearly with a subspace of $L^\infty_{m\times n}$, the space of equivalence classes (modulo equality almost everywhere) of Lebesgue measurable $\mathbf{C}^{m\times n}$-valued functions on ∂U with essential supremum norm. A function $B \in H^\infty_{m\times m}$ is said to be *inner* if $B(z)$ is unitary for almost every $z \in \partial U$.

In this paper the following will be called the Nevanlinna-Pick problem.

PROBLEM. *Let $F \in H^{\infty}_{m\times n}$, $m, n \in \mathbf{N}$, and let B, C be inner functions of types $m \times m$, $n \times n$ respectively. Find a function $G \in F + BH^{\infty}_{m\times n}C$ such that $\|G\|_{\infty}$ is minimised.*

A standard normal families argument shows that the infimum of $\|G\|_{\infty}$ is attained. Here is a very simple illustration of the fact that the solution of the problem is far from being unique in general.

Let $m = n = 2$, let $C = I_2$, the identity matrix, and let

$$F(z) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad B(z) = zI_2 \quad \text{for } z \in U.$$

The coset $F + BH^{\infty}_{2\times 2}C$ consists of all functions

$$G = \begin{bmatrix} g_0 & a \\ b & g_1 \end{bmatrix}$$

such that $g_0(0) = 2$, $g_1(0) = 1$ and $a(0) = 0 = b(0)$. Over such G's the infimum of $\|G\|_{\infty}$ is clearly 2, and the infimum is attained by all functions of the form

$$G = \begin{bmatrix} 2 & 0 \\ 0 & g_1 \end{bmatrix}$$

with $g_1(0) = 1$ and $\|g_1\|_{H^{\infty}} \leqslant 2$, of which there is a profusion.

In the absence of any physical reason for preferring one of the infinitely many solutions, when calculating a minimising function we can either make an arbitrary choice or impose stronger, mathematicaly natural requirements to force uniqueness. S. Y. Kung [17] adopts the first alternative in his implementation of an algorithm based on the work of Adamyan et al.; so does Yeh [22] in an algorithm which is closer to Sarason's approach. However, this arbitrariness has disadvantages beyond its aesthetic repugnance. It is harder to implement than the "strengthened minimisation" algorithm given below, and Yeh's numerical experience suggests that it is also rather unstable. A discussion of numerical aspects and a comparison of the methods will also be presented later. On the assumption that God is an engineer as well as a geometer, I am inclined to expect that the stronger minimisation condition, seeming so mathematically "right", will have physical significance.

The example above illustrates the extra conditions we demand. How can we single out one of the infinitely many minimising functions diag$\{2, g_1\}$? It is in the spirit of the problem to choose $\|g_1\|_{\infty}$ to be as small as possible. This occurs uniquely for $G \equiv$ diag$\{2, 1\}$. Now

$$|g_1(z)| = s_1(G(z)),$$

where, as usual, $s_0(A) \geqslant s_1(A) \geqslant \dots$ denote the singular values or s-numbers of a matrix A (see [13]). Thus, if we ask for an element $G \in F + BH^{\infty}_{2\times 2}C$ of minimal norm which is such that

$$\sup_{z \in U} s_1(G(z)) = \min,$$

then the unique solution is $G(z) = \operatorname{diag}\{2, 1\}$.

It is obvious how to generalize this minimisation condition. For $G \in H^{\infty}_{m\times n}$ write

$$s_j^{\infty}(G) = \sup_{z \in U} s_j(G(z))$$

and

$$s^{\infty}(G) = (s_0^{\infty}(G),\ s_1^{\infty}(G),\ s_2^{\infty}(G), \dots)$$

(this infinite sequence has at most $\min(m, n)$ non-zero terms). Of course $s_0^{\infty}(G) = \|G\|_{\infty}$. We shall simply ask for the solution of the Nevanlinna-Pick problem which minimises not only $\|G\|_{\infty}$ but also $s^{\infty}(G)$, with respect to the lexicographic ordering. We shall find that, as long as B^*FC^* is continuous, this does determine G uniquely, and moreover that G has the striking property that each of the singular values $s_j(G(z))$ is constant a.e. on the unit circle.

2. OPERATOR-THEORETIC PRELIMINARIES

The unique $G \in F + BH^{\infty}_{m\times n}C$ for which $s^{\infty}(G)$ is a minimum will be constructed in terms of the singular value analysis of suitable compressions of multiplication operators on H^2 spaces, but whereas in [20] it sufficed to consider a single such operator, here we require a succession of them. Once again the justification depends on L^1-L^{∞} duality and a factorization theorem.

We depart slightly from the notation of [20]. For $m \in \mathbf{N}$ let L^2_m be the space of square summable Lebesgue measurable functions (modulo equality almost everywhere) of $\mathbf{C}^m$-valued functions on ∂U, with pointwise algebraic operations and inner product

$$(x, y) = \frac{1}{2\pi}\int_0^{2\pi} (x(e^{i\theta}),\ y(e^{i\theta}))_{\mathbf{C}^m} d\theta.$$

L^2_m is a Hilbert space: it will be helpful to think of its elements as column vectors of scalar L^2 functions, as we are going to study elements of $H^{\infty}_{m\times n}$ by making them act by multiplication on L^2_n. For $m, n \in \mathbf{N}$ and $1 \leqslant p \leqslant \infty$ let $L^p_{m\times n}$ denote the space of (equivalence classes of) Lebesgue measurable functions F from ∂U to $\mathbf{C}^{m\times n}$ such that the function

$$\theta \to \|F(e^{i\theta})\|_p$$

belongs to L^p of the circle, where $\|\cdot\|_p$ is the l_p norm on $\mathbf{C}^{m\times n}$:

$$\|A\|_p = \begin{cases} \left\{\sum_{j=0}^{\infty} s_j(A)^p\right\}^{1/p} & \text{if } p < \infty \\ s_0(A) & \text{if } p = \infty. \end{cases}$$

$L^p_{m\times n}$ is a Banach space with respect to the norm

$$\|F\|_p = \begin{cases} \left\{\frac{1}{2\pi}\int_0^{2\pi} \|F(e^{i\theta})\|_p^p d\theta\right\}^{1/p} & \text{if } p < \infty \\ \operatorname{ess\,sup}_\theta \|F(e^{i\theta})\| & \text{if } p = \infty, \end{cases}$$

and $L^2_{m\times n}$ is a Hilbert space with respect to the inner product

$$(F, G) = \frac{1}{2\pi}\int_0^{2\pi} \operatorname{tr}\{G(e^{i\theta})^* F(e^{i\theta})\} d\theta.$$

Any $F \in L^\infty_{m\times n}$ determines a bounded linear functional $\langle\cdot, F\rangle$ on $L^1_{m\times n}$ by

$$\langle G, F\rangle = \frac{1}{2\pi}\int_0^{2\pi} \operatorname{tr}\{F(e^{i\theta})^T G(e^{i\theta})\} d\theta,$$

where the superscript T denotes transposition. The mapping $F \to \langle\cdot, F\rangle$ is an isometric isomorphism of $L^\infty_{m\times n}$ onto $(L^1_{m\times n})^*$ (see [20]).

For f in $L^p_{m\times n}$ or L^2_m we define the kth Fourier coefficient of f, for $k \in \mathbf{Z}$, to be

$$\hat{f}(k) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-ik\theta} d\theta;$$

this is an element of $\mathbf{C}^{m\times n}$ or $\mathbf{C}^m$ respectively. Then $H^p_{m\times n}$, H^2_m are the subspaces of $L^p_{m\times n}$, L^2_m respectively consisting of those functions whose negative Fourier coefficients vanish. As in the scalar case H^p can be regarded as a space of analytic functions in U via the correspondence

$$f \leftrightarrow \sum_{k=0}^{\infty} \hat{f}(k) z^k.$$

Any $F \in L^\infty_{m\times n}$ induces a *multiplication operator* $M_F : L^2_n \to L^2_m$ in an obvious way:

$$(M_F x)(z) = F(z)x(z)$$

for $x \in L^2_n$, $z \in \partial U$. We shall sometimes write Fx in place of $M_F x$. Clearly $\|M_F\| = \|F\|_\infty$. For $F \in L^\infty_{m\times n}$ we define $\bar{F} \in L^\infty_{m\times n}$, $F^* \in L^\infty_{n\times m}$ by

$$\bar{F}(e^{i\theta}) = F(e^{i\theta})^-,$$

$$F^*(e^{i\theta}) = F(e^{i\theta})^*$$

(if $A \in \mathbf{C}^{m\times n}$, $A = [a_{ij}]$, then $\bar{A} \in \mathbf{C}^{m\times n}$ is defined to be $[\bar{a}_{ij}]$).

We can now give the most straightforward generalization to matrix-valued functions of the relevant results of Sarason. This can be obtained using little beyond the ideas of [20], and is known to many people; however, we prove it in full as a preparation for our main result.

THEOREM 1. *Let $F \in H^\infty_{m\times n}$ and let $B \in H^\infty_{m\times n}$, $C \in H^\infty_{n\times n}$ be inner functions. Let K be a closed subspace of L^2_m containing $FC^*H^2_n + BH^2_m$ and let*

$$T: C^*H^2_n \to K \ominus BH^2_m$$

be defined by

$$T = PM_F \,|\, C^*H^2_n$$

where P is the orthogonal projection operator from L^2_m to $K \ominus BH^2_m$.

Then

$$\|F + BH^\infty_{m\times n}C\|_{H^\infty_{m\times n}/BH^\infty_{m\times n}C} = \|T\|.$$

Furthermore, if G is an element of minimal norm in $F + BH^\infty_{m\times n}C$ and $u \in C^*H^2_n$ is a maximising vector for T then $Gu = Tu$.

By a *maximising vector* for T is meant any vector $u \neq 0$ such that $\|Tu\| = \|T\|\,\|u\|$. This theorem does not require B and C to be rational, but without some restriction we cannot be sure that T will have any maximising vectors, and in fact the second statement can be vacuous. In the case that a unit maximising vector u for T does exist, the pair u, $w = Tu/\|T\|$ (when $T \neq 0$) is called a *Schmidt pair of T corresponding to the singular value* $\|T\|$. The pair u, w satisfies

$$\|u\| = 1 = \|w\|, \quad Tu = \|T\|w.$$

We note that Theorem 1 can be deduced from the commutant lifting theorem but we wish to set up the machinery for Theorem 2.

Note also that a possible choice for K is $(\det C)^- H^2_m$.

Proof. If $F \in BH^\infty_{m\times n}C$ then

$$TC^*H^2_n \subseteq PBH^\infty_{m\times n}\cdot H^2_n \subseteq PBH^2_m = \{0\},$$

so that $T = 0$. It follows that, for any $\tilde{F} \in F + BH^{\infty}_{m\times n}C$, we have $PM_{\tilde{F}} = PM_F$ on $C^*H^2_n$, and hence

$$\|T\| = \|PM_{\tilde{F}}\| \leqslant \|M_{\tilde{F}}\| \leqslant \|\tilde{F}\|_{\infty}.$$

Taking the infimum of the right hand term over all $\tilde{F} \in F + BH^{\infty}_{m\times n}C$ yields

$$\|T\| \leqslant \|F + BH^{\infty}_{m\times n}C\|.$$

To prove the opposite inequality, suppose $\|F + BH^{\infty}_{m\times n}C\| > 1$. It is easy to see that the annihilator of $zH^1_{m\times n}$ in $L^{\infty}_{m\times n}$ (which we identify with the dual space of $L^1_{m\times n}$, as indicated above) is $H^{\infty}_{m\times n}$ (cf [20]). Using the relation

$$\langle f, B^*gC^*\rangle = \langle \bar{B}f\bar{C}, g\rangle$$

for all $f \in L^1_{m\times n}$, $g \in L^{\infty}_{m\times n}$, we infer that the annihilator of $z\bar{B}H^1_{m\times n}\bar{C}$ in $L^{\infty}_{m\times n}$ is $BH^{\infty}_{m\times n}C$. It follows that the dual of $z\bar{B}H^1_{m\times n}\bar{C}$ can be identified with $L^{\infty}_{m\times n}/BH^{\infty}_{m\times n}C$. Our supposition thus implies that there exists $f \in H^1_{m\times n}$ of unit norm such that

$$\langle z\bar{B}f\bar{C}, F\rangle > 1;$$

that is

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{tr}\{e^{i\theta}F(e^{i\theta})^{\mathrm{T}}\bar{B}f\bar{C}(e^{i\theta})\}d\theta > 1. \tag{2.1}$$

We now wish to apply Theorem 4 of [20] to write the H^1 function f as a product of two H^2 functions; however, this is formulated for functions whose values are operators from a Hilbert space to itself. Accordingly we introduce the function

$$f_1 = \begin{bmatrix} 0 & 0 \\ f & 0 \end{bmatrix}$$

of type $(n+m)\times(n+m)$. By the cited theorem of Sarason there exist k_1, $k_2 \in H^2_{(n+m)\times(n+m)}$ such that $f_1 = k_1k_2^{\mathrm{T}}$, $\bar{k}_2k_2^{\mathrm{T}} = (f_1^*f_1)^{1/2}$ and $k_1^*k_1 = k_2^{\mathrm{T}}\bar{k}_2$. Let

$$g_1 = [0 \;\; I_m]k_1 \text{ of type } m\times(n+m),$$

$$g_2 = [0 \;\; I_n]k_2 \text{ of type } n\times(n+m).$$

Then

$$g_1g_2^{\mathrm{T}} = [0 \;\; I_m]f_1\begin{bmatrix} 0 \\ I_n \end{bmatrix} = f$$

and the H^2 norms of g_1, g_2 are at most 1. Thus, on substituting in (2.1), we have

$$1 < \frac{1}{2\pi}\int_0^{2\pi} \operatorname{tr}\{e^{i\theta}F^{\mathrm{T}}\bar{B}g_1g_2^{\mathrm{T}}\bar{C}\}d\theta = \frac{1}{2\pi}\int_0^{2\pi} \operatorname{tr}\{g_2^{\mathrm{T}}\bar{C}F^{\mathrm{T}}e^{i\theta}\bar{B}g_1\}d\theta =$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \operatorname{tr}\{g_2^*CF^*e^{-i\theta}B\bar{g}_1\}d\theta = (FC^*g_2, \bar{z}B\bar{g}_1)_{L^2_{m\times(n+m)}}.$$

The subspace of $L^2_{m\times(n+m)}$ consisting of those functions each of whose columns belongs to K can be identified with (and will be denoted by) $K\otimes\mathbf{C}^{n+m}$. Let y be the orthogonal projection of $\bar{z}B\bar{g}_1$ on this space. Since $FC^*g_2 \in K\otimes\mathbf{C}^{n+m}$, we have

$$(FC^*g_2, y) = (FC^*g_2, \bar{z}B\bar{g}_1) > 1.$$

Now each column of $\bar{z}\bar{g}_1$ is orthogonal to H^2_m, and hence each column of $\bar{z}B\bar{g}_1$ is orthogonal to BH^2_m. As $K \supseteq BH^2_m$, it follows that the projection onto K of any column of $\bar{z}B\bar{g}_1$ lies in $K\ominus BH^2_m$. Thus $y \in (K\ominus BH^2_m)\otimes\mathbf{C}^{n+m}$. We have also

$$\|y\|_{L^2} \leqslant \|\bar{z}B\bar{g}_1\|_{L^2} \leqslant 1.$$

And if we write $x = C^*g_2$, then $x \in C^*H^2_n\otimes\mathbf{C}^{n+m}$ and $\|x\|_{L^2}\leqslant 1$. Thus we have constructed vectors $x \in C^*H^2_n\otimes\mathbf{C}^{n+m}$, $y \in (K\ominus BH^2_m)\otimes\mathbf{C}^{n+m}$ of norm at most one such that

$$(Fx, y)_{L^2_{m\times(n+m)}} > 1.$$

It follows that

$$\|\mathbf{PM}_F\| > 1, \tag{2.2}$$

where $\mathbf{P}$ is the orthogonal projection from $L^2_{m\times(n+m)}$ to $(K\ominus BH^2_m)\otimes\mathbf{C}^{n+m}$ and $\mathbf{M}_F$ is the operation of multiplication on the left by F, acting from $C^*H^2_n\otimes\mathbf{C}^{n+m}$ to $L^2_{m\times(n+m)}$. In terms of the notation of Theorem 1 we have

$$\mathbf{P} = P\otimes I_{n+m},\ \mathbf{M}_F = M_F\otimes I_{n+m},$$

and hence

$$\mathbf{PM}_F = T\otimes I_{n+m}.$$

The relation (2.2) now implies that

$$\|T\| > 1.$$

We therefore have

$$\|T\| = \|F + BH^\infty_{m\times n}C\|.$$

The second statement follows just as in the scalar case. Suppose that $u \in C^* H^2_n$ is a maximising vector for T and that G is an element of minimal norm in $F + BH^\infty_{m\times n}C$. By the first part of the theorem, $\|G\|_\infty = \|T\|$. Hence

$$\|T\|\,\|u\| = \|Tu\| = \|PM_G u\| \leqslant \|Gu\| \leqslant \|G\|_\infty \|u\| \leqslant \|T\|\,\|u\|.$$

Thus $\|P(Gu)\| = \|Gu\|$, and so

$$Gu = P(Gu) = Tu.$$

If $n = 1$ then u is a scalar function and we may divide through, as in the scalar case, to obtain a formula for G

$$G = Tu/u,$$

so that G is determined uniquely when T has a maximising vector. As we have seen in the example in Section 1, such is not the case when $m, n > 1$. Roughly speaking, the relation $Gu = Tu$ determines only a rank 1 part of the function G, and it seems to be a non-trivial problem to find the rest of G even for a single element of minimal norm in the coset in question. The path followed by Adamyan et al. is to observe that if we can find n pointwise independent maximising vectors $u_1, \ldots, u_n$ for T then we shall have

$$G[u_1 \ldots u_n] = [Tu_1 \ldots Tu_n],$$

from which G may be determined. Our example of non-uniqueness shows that such independent maximising vectors do not exist in general, so what they do is prove that there is a function $\tilde{F} \in F + BH^\infty_{m\times n}C$ such that the T operator corresponding to the problem $\tilde{F} + zBH^\infty_{m\times n}C$ has n independent maximising vectors (actually they work in terms of equivalent statements about block Hankel operators). They characterize all such functions $\tilde{F}$: in using this approach for numerical computation there seems to be no alternative to making a choice of a single $\tilde{F}$ quite arbitrarily. Furthermore, the calculation of such an $\tilde{F}$ is quite a substantial step numerically, whether by Kung's [17] or by Yeh's [22] method, and Yeh's experience suggests that it is none too stable. I believe that the formula which follows in Section 4 not only gives greater insight but also makes for a simpler and more stable numerical algorithm.

3. DIAGONALIZATION LEMMAS

In this section we derive some technical results which will be needed for the proof of the main theorem, in which an algorithm for the solution of the strengthened Nevanlinna-Pick problem is presented. The proof for $m \times n$ matrices is by induction on n, which necessitates peeling off a dimension, and this is most perspicuously accomplished by reducing the minimising function to block diagonal form.

Recall that the Nevanlinna class N can be regarded as the space of functions f, analytic in U, expressible in the form g/h where $g, h \in H^\infty$ and $h \neq 0$ (see [11]). Each of g, h has a factorization of the form BSF where B is a Blaschke product, S is a singular inner function and F is an outer function. We say that $f \in N^+$ if $f \in N$ and f can be written in the form g/h where $g, h \in H^\infty$ and the singular inner factor of h is 1. The importance of N^+ here is that $N^+ \cap L^\infty = H^\infty$ ([11, Theorem 2.11]). We denote by $(N^+)^m$ the space of column vectors with m components, each belonging to N.

LEMMA 1. *Let φ be a scalar inner function and let $v \in H_n^2 \ominus \varphi H_n^2$, $v \neq 0$. There exist an outer function a and an inner function ψ such that $|a(z)| = \|v(z)\|^{-1}$ for almost all $z \in \partial U$, $\psi\bar{a} \in N^+$ and*

$$\psi\bar{a} = \varphi a \quad \text{a.e. } on\ \partial U.$$

Proof. Since $v \perp \varphi H_n^2$, $\bar{v} \perp \bar{\varphi}(L_n^2 \ominus zH_n^2)$ and so $\varphi\bar{v} \in zH_n^2$. Thus if $v = [v^1, \ldots, v^n]^{\mathrm{T}}$,

$$\varphi(z)\|v(z)\|^2 = \sum_i v^i(z)\varphi\bar{v}^i(z),$$

and the latter is clearly an H^1 function. It therefore has a factorization [11, Theorem 2.8]

$$\varphi\|v(\cdot)\|^2 = \psi g$$

where ψ is inner and g is outer, $g \in H^1$. Outer functions, being exponentials, have square roots which are again outer. Let a be the reciprocal of an outer square root of g. Then a is outer (for the class N, in the terminology of [11]), and

$$\varphi\|v(\cdot)\|^2 = \psi a^{-2}.$$

Taking moduli we obtain $\|v(\cdot)\|^{-1} = |a|$ a.e. on ∂U, as required. Moreover

$$a\bar{a} = \|v(\cdot)\|^{-2} = \varphi a^2/\psi,$$

so that

$$\psi\bar{a} = \varphi a \in N^+.$$

For $x, y \in L_n^2$, we shall say that x is *pointwise orthogonal to* y if $x(z) \perp y(z)$ for almost every $z \in \partial U$.

LEMMA 2. *Let φ be a scalar inner function and let $v \in H_n^2 \ominus \varphi H_n^2$, $v \neq 0$. Let $a \in N$ be outer and satisfy $|a(z)| = \|v(z)\|^{-1}$ a.e.. There exist $v_1', \ldots, v_{n-1}' \in H_n^\infty$ such that $a(z)v(z)$, $v_1'(z), \ldots, v_{n-1}'(z)$ constitutes an orthonormal basis of $\mathbf{C}^n$ for almost all $z \in \partial U$.*

Proof. As in Lemma 1 we have $\varphi\bar{v} \in zH_n^2$ and hence $a\varphi v^* \in H_{1\times n}^\infty$. By Lemma 2.1 of [18] there exists $\theta \in H_{n\times(n-1)}^\infty$ which is "left inner" (i.e. isometric a.e. on ∂U) such that

$$\operatorname{Ker} M_{a\varphi v^*} = \theta H_{n-1}^2.$$

Thus $v^*\theta = 0$, so that v is pointwise orthogonal to the columns of θ, and the same is therefore true of av. As θ is isometric a.e., the columns of θ constitute $n-1$ orthonormal vectors in $\mathbf{C}^n$ for almost all $z \in \partial U$, and hence we may take $v_1', \ldots, v_{n-1}'$ to be the columns of θ.

We now prove the main diagonalization lemma. This states, roughly, that in Theorem 1 we can take both u and Gu to have all components except the first identically equal to zero, and that G will then have block diagonal form.

LEMMA 3. *Let F, B, C, K and T be as in Theorem 1, and let $m, n > 1$. Let $t_0 = \|T\|$ and let $v_0 \in C^*H_n^2$, $w_0 \in K \ominus BH_m^2$ be a Schmidt pair for t_0 (so that v_0, w_0 are unit vectors and $Tv_0 = t_0 w_0$). Let G be an element of minimal H^∞ norm in the coset $F + BH_{m\times n}^\infty C$. Then*

(i) $\|v_0(z)\| = \|w_0(z)\|$ *for almost all* $z \in \partial U$;

(ii) $\|G(z)\| = t_0$ a.e. *on* ∂U;

(iii) *there exist $V \in L_{n\times n}^\infty$, $W \in L_{m\times m}^\infty$ such that CV and W^* are inner functions,*

$$V^*v_0 = [f\ 0\ \ldots\ 0]^{\mathrm{T}} \tag{3.1}$$

and

$$W^*w_0 = [\chi f\ 0\ \ldots\ 0]^{\mathrm{T}} \tag{3.2}$$

for some $f \in L^2$ and scalar inner χ, and such that

$$W^*GV,\ W^*FV \in H_{m\times n}^\infty.$$

Moreover,

$$W^*GV = \begin{bmatrix} g_0 & 0 \\ 0 & g_1 \end{bmatrix}$$

for some $g_0 \in H^\infty$ of constant (a.e.) *modulus t_0 on ∂U, and some $g_1 \in H_{(m-1)\times(n-1)}^\infty$ such that $\|g_1\|_\infty \leqslant t_0$.*

Proof. (i) Certainly $\|G(z)v_0(z)\| \leqslant t_0\|v_0(z)\|$ a.e. since, by Theorem 1, $\|G\|_\infty = t_0$. If strict inequality holds on a set of positive measure then, for some $c < 1$ and all z in a set of positive measure,

$$\|G(z)v_0(z)\|^2 \leqslant c^2 t_0^2 \|v_0(z)\|^2.$$

On integrating both sides round ∂U we obtain

$$\|Gv_0\|_{L_m^2}^2 < t_0^2 \|v_0\|_{L_n^2}^2 = t_0^2.$$

However, $Gv_0 = t_0 w_0$ and so $\|Gv_0\| = t_0$. Thus equality holds a.e., and so

$$t_0\|w_0(z)\| = \|G(z)v_0(z)\| = t_0\|v_0(z)\| \tag{3.3}$$

a.e.. This proves (i), as the result holds trivially if $t_0 = 0$.

(ii) From (3.3), $\|G(z)\| \geqslant t_0$ a.e.. From Theorem 1, $\|G\|_\infty = t_0$, and hence $\|G(z)\| \leqslant t_0$ a.e..

(iii) Let $\beta = \det B$, $\gamma = \det C$, so that β, γ are scalar inner functions. We can suppose $t_0 \neq 0$. We wish to apply Lemma 2 with $v = \gamma v_0$. As v_0 is a singular value of T corresponding to the non-zero singular value t_0, $v_0 \perp \operatorname{Ker} T$. If $x \in \beta H_n^2$ then $Fx \in \beta H_m^2 = B(\operatorname{adj} B)H_m^2 \subseteq BH_m^2$, so that the projection Tx of Fx on $L_m^2 \ominus BH_m^2$ is zero. This shows that $\beta H_n^2 \subseteq \operatorname{Ker} T$, and hence $v_0 \in L_n^2 \ominus \beta H_n^2$. Furthermore $v_0 \in C^* H_n^2 \subseteq \gamma H_n^2$, and hence $\gamma v_0 \in H_n^2 \ominus \beta\gamma H_n^2$. By Lemma 1 we may choose an outer function $a_0 \in N$ such that $|a_0| = \|v_0(\cdot)\|^{-1}$, and by Lemma 2 there exist $v_1', \ldots, v_{n-1}' \in H_n^\infty$ such that $a_0(z)v_0(z), v_1'(z), \ldots, v_{n-1}'(z)$ is an orthonormal basis of $\mathbf{C}^n$ for almost all z.

By Theorem 1 $Gv_0 = t_0 w_0$ and hence

$$\gamma w_0 = t_0^{-1} G\gamma v_0 \in H_m^2.$$

As also $w_0 \in (BH_m^2)^\perp \subseteq (\beta H_m^2)^\perp$, we have $\gamma w_0 \in H_m^2 \ominus \beta\gamma H_m^2$. Applying Lemma 2 again we deduce that there exist $w_1', \ldots, w_{m-1}' \in H_m^\infty$ such that $a_0(z)w_0(z)$, $w_1'(z), \ldots, w_{m-1}'(z)$ is an orthonormal basis of $\mathbf{C}^m$ for almost all z. Now if Φ is any $m \times m$ inner function, the relation

$$z(\det \Phi)\Phi^* = z \operatorname{adj} \Phi$$

shows that every column of Φ belongs to $H_m^2 \ominus z(\det \Phi)H_m^2$. Taking Φ to be the inner function

$$[a_0\gamma w_0 \ w_1' \ \ldots \ w_{m-1}']$$

we infer that, for a suitable scalar inner function φ, $w_j' \in H_m^2 \ominus \varphi H_m^2$, $1 \leqslant j \leqslant m-1$. This implies that $\varphi \bar{w}_j' \in H_m^2$. We have observed that $w_0 \perp \beta H_m^2$, so that $\beta w_0 \in H_m^2$, and, by Lemma 1, $\psi \bar{a}_0 \in N^+$ for a suitable scalar inner function ψ. It follows that $\psi \bar{a}_0 \beta \bar{w}_0 \in (N^+)^m$. As $\psi \bar{a}_0 \beta \bar{w}_0$ has norm 1 a.e. on ∂U we have $\psi \bar{a}_0 \beta \bar{w}_0 \in H_m^\infty$. Write $\chi = \beta\psi$: then χ is a scalar inner function. Let

$$V = [a_0 v_0 \ v_1' \ \ldots \ v_{n-1}'],$$

$$W = [\overline{\chi a_0 w_0} \ \overline{\varphi w_1'} \ \ldots \ \overline{\varphi w_{m-1}'}].$$

Both these matrix-valued functions are unitary a.e. on ∂U. As $\chi \bar{a}_0 w_0^*$, $w_j'^* \in H_{1\times m}^\infty$ we have $W^* \in H_{m\times m}^\infty$, and so W^* is inner. As $v_0 \in C^* H_n^2$,

$$CV = [a_0 C v_0 \ C v_1' \ \ldots \ C v_{n-1}']$$

is in $H_{n\times n}^2$ and so is also inner.

We next show that $W^*GV \in H^\infty_{m\times n}$. The $(j+1)$th column of this function, $0 \leqslant j \leqslant n-1$, is W^*Gv_j', which clearly belongs to H^∞_m. We have also

$$W[1\ 0\ \dots\ 0]^{\mathrm{T}} = a_0\bar{\chi}w_0,$$

whence

$$W^*a_0w_0 = \chi[1\ 0\ \dots\ 0]^{\mathrm{T}}. \tag{3.4}$$

The first column of W^*GV is thus

$$W^*Ga_0v_0 = W^*a_0t_0w_0 = t_0W^*a_0w_0 = t_0\chi[1\ 0\ \dots\ 0]^{\mathrm{T}}.$$

Hence $W^*GV \in H^\infty_{m\times n}$. Furthermore, since W^*FV differs from W^*GV by an element of $W^*BH^\infty_{m\times n}CV$, which is a subspace of $H^\infty_{m\times n}$, W^*FV also lies in $H^\infty_{m\times n}$.

Equation (3.4) and a similar calculation for V show that (3.1) and (3.2) hold with $f = 1/a_0$.

Lastly, the relation

$$W^*GV(V^*v_0) = t_0W^*w_0$$

yields

$$W^*GV\begin{bmatrix} f \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} t_0\chi f \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

whence W^*GV has the form

$$W^*GV = \begin{bmatrix} t_0\chi & * \\ 0 & g_1 \end{bmatrix}$$

for some $g_1 \in H^\infty_{(m-1)\times(n-1)}$. By (ii) the Euclidean norm of the first row is at most t_0 for almost all z, and hence the first row must be $[t_0\chi\ 0\ \dots\ 0]$.

For spaces of functions E, F on ∂U we make the definition

$$\begin{bmatrix} E \\ F \end{bmatrix} = \left\{\begin{bmatrix} x \\ y \end{bmatrix} : x \in E,\ y \in F\right\}.$$

LEMMA 4. *Let m, $n > 1$ and let $\hat{B} \in H^\infty_{m\times m}$, $\hat{C} \in H^\infty_{n\times n}$ be inner functions. There exist inner functions $\tilde{B}$ and $\tilde{C}$, $(m-1)$-square and $(n-1)$-square respectively, such that*

$$\begin{bmatrix} 0 \\ \tilde{C}^{\mathrm{T}}H^2_{n-1} \end{bmatrix} = \begin{bmatrix} 0 \\ H^2_{n-1} \end{bmatrix} \cap \hat{C}^{\mathrm{T}}H^2_n \tag{3.5}$$

$$\begin{bmatrix} 0 \\ \tilde{B}H^2_{m-1} \end{bmatrix} = \begin{bmatrix} 0 \\ H^2_{m-1} \end{bmatrix} \cap \hat{B}H^2_m. \tag{3.6}$$

The following relations hold.

(i) $\hat{B}H^{\infty}_{m\times n}\hat{C} \cap \begin{bmatrix} 0 & 0 \\ 0 & H^{\infty}_{(m-1)\times(n-1)} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{B}H^{\infty}_{(m-1)\times(n-1)}\tilde{C} \end{bmatrix}$;

(ii) $\begin{bmatrix} 0 \\ \tilde{C}^* H^2_{n-1} \end{bmatrix}$ *is the orthogonal projection on* $\begin{bmatrix} 0 \\ L^2_{n-1} \end{bmatrix}$ *of* $\hat{C}^* H^2_n$;

(iii) $\begin{bmatrix} 0 \\ L^2_{m-1} \ominus \tilde{B}H^2_{m-1} \end{bmatrix}$ *is the orthogonal projection on* $\begin{bmatrix} 0 \\ L^2_{m-1} \end{bmatrix}$ *of* $L^2_m \ominus \hat{B}H^2_m$.

(iv) *Suppose further that*

$$\hat{G} = \begin{bmatrix} g_0 & \\ & g_1 \end{bmatrix} \in H^{\infty}_{m\times n},$$

with g_0, g_1 *of types* 1×1, $(m-1)\times(n-1)$ *respectively, and that* $\hat{K}$ *is a closed subspace of* L^2_m *containing* $\hat{G}\hat{C}^* H^2_n + \hat{B}H^2_m$. *Then the closure* $\tilde{K}$ *of the orthogonal projection on* $\begin{bmatrix} 0 \\ L^2_{m-1} \end{bmatrix}$ *of* $\hat{K}$ *is a closed subspace of* L^2_{m-1} *containing* $g_1\tilde{C}H_{n-1} + \tilde{B}H^2_{m-1}$.

Proof. The space $E = \left\{ f \in H^2_{n-1} : \begin{bmatrix} 0 \\ f \end{bmatrix} \in \hat{C}^{\mathrm{T}} H^2_n \right\}$ is closed and z-invariant in H^2_{n-1}, and hence, by the Lax-Beurling theorem [16] is of the form ΦH^2_k for some $k \leqslant n-1$ and some $\Phi \in H^{\infty}_{(n-1)\times k}$ which is isometric a.e. on ∂U. By considering columns 2 to n in the relation

$$\hat{C}^{\mathrm{T}}(\operatorname{adj} \hat{C}^{\mathrm{T}}) = (\det \hat{C}) I_n$$

we perceive that E contains non-zero elements $f_1, \ldots, f_{n-1}$ which are pointwise mutually orthogonal. Hence we must have $k = n-1$, so that Φ is in fact inner. Take $\tilde{C}$ to be Φ^{T}.

To prove (i), consider any $h \in H^{\infty}_{m\times n}$ such that

$$\hat{B}h\hat{C} = \begin{bmatrix} 0 & 0 \\ 0 & g \end{bmatrix}$$

for some $g \in H^{\infty}_{(m-1)\times(n-1)}$. Since $\hat{B}$ is inner the first column of $h\hat{C}$ is zero, and we have $h\hat{C} = [0 \;\; k]$ for some $k \in H^{\infty}_{m\times(n-1)}$. Thus

$$\hat{C}^{\mathrm{T}} h^{\mathrm{T}} = \begin{bmatrix} 0 \\ k^{\mathrm{T}} \end{bmatrix},$$

and on applying (3.5) to each column of the right hand side in turn, we infer that $k^{\mathrm{T}} = \tilde{C}^{\mathrm{T}} f^{\mathrm{T}}$ for some $f^{\mathrm{T}} \in H^2_{(n-1)\times m}$. Since k^{T} is bounded and $\tilde{C}^{\mathrm{T}}$ is inner f^{T} is also bounded. Thus $k = f\tilde{C}$ where $f \in H^{\infty}_{m\times(n-1)}$. Thus

$$\begin{bmatrix} 0 & 0 \\ 0 & g \end{bmatrix} = \hat{B}h\hat{C} = \hat{B}[0 \;\; f\tilde{C}] = [0 \;\; \hat{B}f\tilde{C}].$$

Hence

$$\hat{B}f\tilde{C} = \begin{bmatrix} 0 \\ g \end{bmatrix},$$

so that

$$\hat{B}f = \begin{bmatrix} 0 \\ g\tilde{C}^* \end{bmatrix}.$$

On applying (3.6) to each column of this equation in turn we deduce that $g\tilde{C}^* = \tilde{B}\varphi$ for some $\varphi \in H^2_{(m-1)\times(m-1)}$. As above, since g is bounded and $\tilde{B}$ and $\tilde{C}$ are inner, φ must be bounded, so that $g \in \tilde{B}H^\infty_{(m-1)\times(n-1)}C$. This proves one inclusion in (i).

To prove the opposite inclusion, consider any $\varphi \in H^\infty_{(m-1)\times(m-1)}$. By (3.6) applied to the columns of $\begin{bmatrix} 0 \\ \tilde{B}\varphi \end{bmatrix}$, there exists $f \in H^\infty_{m\times(n-1)}$ such that

$$\hat{B}f = \begin{bmatrix} 0 \\ \tilde{B}\varphi \end{bmatrix}.$$

Likewise there exists $h^{\mathrm{T}} \in H^\infty_{n\times m}$ such that

$$\hat{C}^{\mathrm{T}}h^{\mathrm{T}} = \begin{bmatrix} 0 \\ \tilde{C}^{\mathrm{T}}f^{\mathrm{T}} \end{bmatrix}.$$

Then $h\hat{C} = [0 \ f\tilde{C}]$ and so

$$\hat{B}h\hat{C} = [0 \ \ \hat{B}f\tilde{C}] = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{B}\varphi\tilde{C} \end{bmatrix}.$$

This concludes the proof of (i).

(ii) Taking bars in (3.5) and multiplying by $\bar{z}$ we have

$$\begin{bmatrix} 0 \\ \bar{z}\,\tilde{C}^*\bar{H}^2_{n-1} \end{bmatrix} = \begin{bmatrix} 0 \\ \bar{z}\,\bar{H}^2_{n-1} \end{bmatrix} \cap \bar{z}\hat{C}^*\bar{H}^2_n.$$

Taking orthogonal complements in L^2_n we get

$$\begin{bmatrix} L^2 \\ \tilde{C}^*H^2_{n-1} \end{bmatrix} = \begin{bmatrix} L^2 \\ H^2_{n-1} \end{bmatrix} + \hat{C}^*H^2_n = \begin{bmatrix} L^2 \\ 0 \end{bmatrix} + \hat{C}^*H^2_n,$$

since $\hat{C}^*H^2_n \supseteq H^2_n$. Projecting on $\begin{bmatrix} 0 \\ L^2_{n-1} \end{bmatrix}$ gives us statement (ii).

(iii) On taking orthogonal complements in L^2_m in (3.6) we obtain

$$\begin{bmatrix} L^2 \\ L^2_{m-1} \ominus \tilde{B}H^2_{m-1} \end{bmatrix} = \begin{bmatrix} L^2 \\ 0 \end{bmatrix} + (H^2_m)^\perp + (BH^2_m)^\perp.$$

Since $H_m^2 \supseteq BH_m^2, (H_m^2)^\perp \subseteq (BH_m^2)^\perp$ and hence

$$\begin{bmatrix} L^2 \\ L_{m-1}^2 \ominus \tilde{B}H_{m-1}^2 \end{bmatrix} = \begin{bmatrix} L^2 \\ 0 \end{bmatrix} + (L_m^2 \ominus BH_m^2).$$

Projecting on $\begin{bmatrix} 0 \\ L_{m-1}^2 \end{bmatrix}$ gives us (iii).

(iv) Since $\hat{K} \supseteq \hat{B}H_m^2$, $\tilde{K}$ contains the projection of $\hat{B}H_m^2$ on $\begin{bmatrix} 0 \\ L_{m-1}^2 \end{bmatrix}$, which certainly contains the intersection of these two spaces, which is $\tilde{B}H_{m-1}^2$, by (3.6). Thus $\tilde{K} \supseteq \tilde{B}H_{m-1}^2$ (we are identifying $x \in L_{m-1}^2$ with $\begin{bmatrix} 0 \\ x \end{bmatrix}$, where it is safe to do so).

We wish to show that $g_1C^*H_{n-1}^2 \subseteq \tilde{K}$. Consider $y \in \tilde{C}^*H_{n-1}^2$. By statement (ii) there exists $x \in L^2$ such that $\begin{bmatrix} x \\ y \end{bmatrix} \in C^*H_n^2$. Since $\hat{K} \supseteq \hat{G}\hat{C}^*H_n^2$, we have $\begin{bmatrix} g_0 & x \\ g_1 & y \end{bmatrix} \in \hat{K}$.

Projecting gives $g_1 y \in \tilde{K}$, as desired.

We need the following elementary observation.

LEMMA 5. *Let x, y be column vectors in $\mathbf{C}^n$ such that $y \neq 0$ and*

$$x^T y = c, \quad |c| = \|x\| \, \|y\|,$$

where $\|\cdot\|$ is the Euclidean norm. Then

$$x^T = \frac{c}{\|y\|^2} y^*.$$

Proof. We have $|(y, \bar{x})| = \|y\| \|\bar{x}\|$, and hence $\bar{x} = \lambda y$, for some $\lambda \in \mathbf{C}$. Then

$$c = (y, \bar{x}) = (y, \lambda y) = \bar{\lambda} \|y\|^2,$$

and so

$$x^T = \bar{\lambda} y^* = \frac{c}{\|y\|^2} y^*.$$

4. SOLUTION OF THE STRENGTHENED NEVANLINNA-PICK PROBLEM

In Section 1 we saw that minimising the $H_{m\times n}^\infty$ norm over a coset $F + BH_{m\times n}^\infty C$ will hardly ever determine a unique function in the non-scalar case, and that a plausible attempt to restore uniqueness is to impose the stronger requirement that the infinite sequence

$$s^\infty(G) = (s_0^\infty(G), s_1^\infty(G), s_2^\infty(G), \ldots)$$

be a minimum with respect to the lexicographic ordering of $\mathbf{R}^N$. We are now ready for the main result of the paper, which states that for a wide class of F, B and C (including the rational case) there is indeed a unique element which minimises s^∞, and that furthermore this extremal element can be written down explicitly in terms of singular values and vectors of a succession of Sarason-type operators.

We denote by $\mathscr{C}_{m\times n}$ the space of continuous $\mathbf{C}^{m\times n}$-valued functions on ∂U. In the algorithm and proof which follow the reader is recommended to take K and all the spaces K_j to be L^2_m to begin with. Using the more general K causes complications, but is necessary for the practical implementation of the algorithm.

For any sets E, $F \subseteq L^2_n$ the *pointwise orthogonal complement of E in F* is defined to be

$$\{f \in F : f(z) \perp e(z) \text{ in } \mathbf{C}^n \text{ for all } e \in E \text{ and almost all } z \in \partial U\}.$$

THEOREM 2. *Let $F \in H^\infty_{m\times n}$ and let $B \in H^\infty_{m\times m}$, $C \in H^\infty_{n\times n}$ be inner functions such that $B^*FC^* \in H^\infty_{m\times n} + \mathscr{C}_{m\times n}$. The minimum of $s^\infty(G)$ over all G in the coset $F + BH^\infty_{m\times n}C$, with respect to the lexicographic ordering, is attained for a unique element G in the coset. This function G can be obtained by the following algorithm.*

Algorithm

Let K be a closed subspace of L^2_m such that $K \supseteq FC^*H_n + BH_m$.

Define an integer $r \geqslant 0$, subspaces V_j of L^2_n and K_j, E_j, W_j of L^2_m, operators $T_j\colon V_j \to W_j$, $0 \leqslant j \leqslant r$, positive numbers t_j and vectors $v_j \in V_j$, $w_j \in W_j$, $0 \leqslant j \leqslant r-1$ inductively as follows.

Let $V_0 = C^*H^2_n$, $K_0 = K$, $E_0 = BH^2_m$ and $W_0 = K_0 \ominus E_0$. Let

$$T_0 = P_0M_F\colon V_0 \to W_0$$

where $M_F\colon V_0 \to L^2_m$ is the operation of multiplication by F and $P_0\colon L^2_m \to W_0$ is the orthogonal projection operator.

If $T_0 = 0$ then $r = 0$ and the desired function G is identically zero. Otherwise, let $t_0 = \|T_0\| > 0$ and let $v_0 \in V_0$, $w_0 \in W_0$ be unit vectors such that $T_0v_0 = t_0w_0$.

Now suppose that V_i, K_i, E_i, W_i, v_i, w_i and t_i have been defined for $0 \leqslant i \leqslant j-1$. Let V_j be the orthogonal projection of V_{j-1} onto the pointwise orthgonal complement of v_{j-1} in L^2_n. Let K_j be the closure of the orthogonal projection of K_{j-1} onto the pointwise orthogonal complement of w_{j-1} in L^2_m. Let E_j be the pointwise orthogonal complement of w_{j-1} in E_{j-1}, and let $W_j = K_j \ominus E_j$. Let

$$T_j = P_jM_F\colon V_j \to W_j$$

where $M_F\colon V_j \to L^2_m$ is a multiplication operator and $P_j\colon L^2_m \to W_j$ is the orthogonal projection.

If $T_j = 0$ let $r = j$ (in which case the construction stops). Otherwise let $t_j = \|T_j\|$ and let $v_j \in V_j$, $w_j \in W_j$ be unit vectors such that $T_jv_j = t_jw_j$.

The construction terminates, with $r \leqslant \min\{m, n\}$. *The minimising function* G *is given by*

$$G(z) = \sum_{j=0}^{r-1} \frac{t_j}{\| v_j(z) \|^2} w_j(z) v_j(z)^* \tag{4.1}$$

for almost all $z \in \partial U$. *Moreover,* $s_j(G(z))$ *is constant* a.e. *and equal to* t_j *on* ∂U, *so that*

$$s^\infty(G) = (t_0, t_1, t_2, \ldots, t_{r-1}, 0, 0, \ldots).$$

Proof of existence. From the fact that any element of $\mathscr{C}_{m\times n}$ can be uniformly approximated by trigonometric polynomials it is easily shown that $P_0 M_F : C^* H_n^2 \to W_0$ is compact (this is a standard argument: see [20, § 6]). So therefore is its compression T_j, and the unit maximising vector v_j does exist.

Let us show that the minimum of $s^\infty(G)$ is attained. There is a sequence (h_i) in $H^\infty_{m\times n}$ such that

$$\|F + Bh_i C\| \to \|F + BH^\infty_{m\times n} C\|$$

as $i \to \infty$. The sequence (h_i) is bounded in $H^\infty_{m\times n}$, and so, by Montel's theorem, it has a subsequence which converges to an element $h \in H^\infty_{m\times n}$ uniformly on compact subsets of U. It is clear that

$$\|F + BhC\|_\infty = \|F + BH^\infty_{m\times n} C\|.$$

Now let $\mathscr{G}_1$ denote the set of all $h \in H^\infty_{m\times n}$ for which this equality holds. There is a sequence (h_{1i}) in $\mathscr{G}_1$ such that

$$s_1^\infty(F + Bh_{1i}C) \to \inf_{h\in\mathscr{G}_1} s_1^\infty(F + BhC)$$

as $i \to \infty$. A second application of Montel's theorem shows that $s_1^\infty(F + BhC)$ does attain its infimum as h varies over $\mathscr{G}_1$. Continuing inductively we obtain the desired conclusion after $\min\{m, n\}$ steps.

The proof of the formula for the extremal function in the general case is quite complicated, and I believe it will be a help to exhibit the idea of the proof in a special case. Take $m = n = 2$, $C = I$, so that we may choose $K = H_2^2$ in the theorem. Suppose further that the Schmidt vectors v_0, w_0 of T_0 corresponding to the largest singular value t_0 have the form

$$v_0 = \begin{bmatrix} a \\ 0 \end{bmatrix}, \quad w_0 = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

for some $a, b \in H^2$. If G is an element of minimal norm in $F + BH^\infty_{2\times 2}$ then, by Theorem 1, $\|G\| = \|T\| = t_0$ and

$$Gv_0 = Tv_0 = t_0 w_0.$$

Hence

$$G = \begin{bmatrix} t_0 b/a & * \\ 0 & * \end{bmatrix}.$$

By Lemma 3(i), $|a(z)| = |b(z)|$ a.e., and so the (1, 2) entry of G must be identically zero. Write

$$G = \begin{bmatrix} t_0\psi_0 & 0 \\ 0 & g \end{bmatrix}$$

where ψ_0 is inner, $g \in H^\infty$ and $\|g\|_\infty \leqslant t_0$. We now know the first column of G; it remains to find the second column. Write $F = [F_0 \quad F_1]$ with $F_j \in H^\infty_{2\times 1}$; then

$$\begin{bmatrix} 0 \\ g \end{bmatrix} \in F_1 + BH^\infty_{2\times 1}. \tag{4.2}$$

Since $s_1^\infty(G) = \|g\|_\infty$, the second column of the desired G is precisely the element of smallest norm in

$$\begin{bmatrix} 0 \\ H^\infty \end{bmatrix} \cap (F_1 + BH^\infty_{2\times 1}).$$

Pick any element $[0 \quad f]^{\mathrm{T}}$ of the latter set: we know from (4.2) there is at least one such. We have

$$\begin{bmatrix} 0 \\ H^\infty \end{bmatrix} \cap (F_1 + BH_{2\times 1}) = \begin{bmatrix} 0 \\ f \end{bmatrix} + \begin{bmatrix} 0 \\ H^\infty \end{bmatrix} \cap BH^\infty_{2\times 1} =$$

$$= \begin{bmatrix} 0 \\ f \end{bmatrix} + \begin{bmatrix} 0 \\ \tilde{B}H^\infty \end{bmatrix} = \begin{bmatrix} 0 \\ f + \tilde{B}H^\infty \end{bmatrix}$$

for some scalar inner function $\tilde{B}$, by Lemma 4. We have therefore reduced the determination of the second column of G to a scalar Nevanlinna-Pick problem. We could of course treat this as an independent problem and solve it by known method, but it is clearly preferable to express its solution in terms of the operator T_0 constructed in the first stage of the solution.

We are looking for the element g_1 of smallest norm in $f + \tilde{B}H^\infty$. By Theorem 1,

$$g_1 \tilde{v}_1 = \tilde{t}_1 \tilde{w}_1 \tag{4.3}$$

where $\langle \tilde{v}_1, \tilde{w}_1 \rangle$ is a Schmidt pair corresponding to the largest singular value $\tilde{t}_1$ of the operator

$$\tilde{T}_1 = P_{H^2 \ominus \tilde{B}H^2} M_f : H^2 \to H^2 \ominus \tilde{B}H^2.$$

We must relate $\tilde{T}_1$ to the operator T_1 described in Theorem 2. Here

$$T_1 = T_0 \Big| \begin{bmatrix} 0 \\ H^2 \end{bmatrix}.$$

For any $h \in H^2$ we have, in a self-explanatory notation,

$$\begin{bmatrix} 0 & \\ & \tilde{T}_1 \end{bmatrix}\begin{bmatrix} 0 \\ h \end{bmatrix} = \begin{bmatrix} 0 \\ P_{H^2 \ominus BH^2}(fh) \end{bmatrix} = P_{\begin{bmatrix} 0 \\ H^2 \ominus \tilde{B}H^2 \end{bmatrix}} \begin{bmatrix} 0 \\ fh \end{bmatrix} =$$

$$= P_{\begin{bmatrix} H^2 \\ H^2 \ominus \tilde{B}H^2 \end{bmatrix}} \begin{bmatrix} 0 \\ fh \end{bmatrix} = P_{H^2_2 \ominus \begin{bmatrix} 0 \\ \tilde{B}H^2 \end{bmatrix}} \begin{bmatrix} 0 \\ fh \end{bmatrix} =$$

$$= P_{H^2_2 \ominus \left(BH^2_2 \cap \begin{bmatrix} 0 \\ H^2 \end{bmatrix}\right)} \begin{bmatrix} 0 \\ fh \end{bmatrix} = P_{H^2_2 \ominus BH^2_2} + P_{BH^2_2 \cap \begin{bmatrix} H^2 \\ 0 \end{bmatrix}} \begin{bmatrix} 0 \\ fh \end{bmatrix} =$$

$$= P_{H^2_2 \ominus BH^2_2} F \begin{bmatrix} 0 \\ h \end{bmatrix} = T_0 \begin{bmatrix} 0 \\ h \end{bmatrix}.$$

Thus

$$T_1 = T_0 \Big| \begin{bmatrix} 0 \\ H_2 \end{bmatrix} = \begin{bmatrix} 0 & \\ & \tilde{T}_1 \end{bmatrix} \Big| \begin{bmatrix} 0 \\ H_2 \end{bmatrix}.$$

It follows that $t_1 = \tilde{t}_1$ and that

$$v_1 = \begin{bmatrix} 0 \\ \tilde{v}_1 \end{bmatrix}, \quad w_1 = \begin{bmatrix} 0 \\ \tilde{w}_1 \end{bmatrix}$$

constitutes a Schmidt pair corresponding to the singular value t_1 of T_1. By (4.3), $g_1 v_1 = t_1 w_1$ and so, for the extremal $G \in F + BH^\infty_{2\times 2}$,

$$G[v_0 \quad v_1] = [t_0 w_0 \quad t_1 w_1]. \tag{4.4}$$

As $v_0(z)$, $v_1(z)$ are orthogonal vectors in $\mathbf{C}^2$ for allmost all z,

$$[v_0(z) \quad v_1(z)]^{-1} = \begin{bmatrix} \|v_0(z)\|^{-2} & \\ & \|v_1(z)\|^{-2} \end{bmatrix} \begin{bmatrix} v_0(z)^* \\ v_1(z)^* \end{bmatrix},$$

and hence (4.4) yields

$$G(z) = \sum_{j=0}^{1} \frac{t_j}{\|v_j(z)\|^2} w_j(z) v_j(z)^*$$

for almost every $z \in \partial U$.

The diagonalization lemmas of Section 3 enable us to convert the general problem to a form in which the above idea, together with an induction argument, gives the desired result,

Proof of the general case. By Theorem 1, we may assume $t_0 \neq 0$.

Step 1. The theorem holds when $n = 1$.

In this case v_0 is a non-zero scalar function in H^2, so that the relation $Gv_0 = t_0w_0$, which we know from Theorem 1, can be solved to give $G = t_0w_0v_0^{-1}$. To see that this agrees with the prescription in Theorem 2 observe that $V_1 = \{0\}$ and hence $r = 1$, so that (4.1) gives

$$G(z) = \frac{t_0}{\| v_0(z)\|^2} w_0(z)v_0(z)^*$$

a.e. on ∂U. Since $v_0(z)^*/\| v_0(z)\|^2 = v_0(z)^{-1}$ the two formulae for $G(z)$ agree.

Step 2. The theorem holds when $m = 1$.

As before, $Gv_0 = t_0w_0$. By Lemma 3(i) and (ii) we have, for almost all $z \in \partial U$,

$$\|G(z)\| = t_0, \quad \|v_0(z)\| = \|w_0(z)\|.$$

When $m = 1$ $G(z)$ is a row vector, so we may apply Lemma 5 with $x^{\mathrm{T}} = G(z)$, $y = v_0(z)$ to obtain

$$G(z) = \frac{t_0}{\|v_0(z)\|^2} w_0(z)v_0(z)^*.$$

This does agree with (4.1) since clearly $W_1 = \{0\}$ and hence $r = 1$.

Now consider a fixed $m > 1$ and proceed by induction on n.

Step 3. Diagonalization.

Suppose $n > 1$ and let G be any element of $F + BH^{\infty}_{m\times n}C$ for which $\|G\|_\infty$ is a minimum (and hence equal to t_0).

According to Lemma 3(iii) we can choose $W \in L^{\infty}_{m\times m}$, $V \in L^{\infty}_{n\times n}$ such that W^* and CV are inner, $W^*GV \in H^{\infty}_{m\times n}$ and

$$W^*GV = \begin{bmatrix} g_0 & 0 \\ 0 & g_1 \end{bmatrix}$$

where $g_0 \in H^\infty$, $g_1 \in H^{\infty}_{(m-1)\times(n-1)}$ and $\|g_0\|_\infty = \|G\|_\infty = t_0$. Let us write $\hat{G} = W^*GV$, $\hat{B} = W^*B$, $\hat{C} = CV$, and consider the strengthened Nevanlinna-Pick problem $\hat{G} + \hat{B}H^{\infty}_{m\times n}\hat{C}$. Since W^* and V are unitary a.e., $s^\infty(h) = s^\infty(W^*hV)$ for any $h \in H^{\infty}_{m\times n}$. Furthermore the mapping $h \to W^*hV$ is a one-one correspondence between the cosets $F + BH^{\infty}_{m\times n}C$ and $\hat{G} + \hat{B}H^{\infty}_{m\times n}\hat{C}$, so that if $\hat{h}$ is an element of the latter coset for which s^∞ is minimised, the $W\hat{h}V^*$ is a solution of the original strengthened Nevanlinna-Pick problem, and conversely.

Now suppose that $G \in F + BH^{\infty}_{m\times n}C$ is a solution of the strengthened Nevanlinna-Pick problem, so that $s^{\infty}(G)$ is a minimum for the lexicographic ordering. Then

$$\hat{G} = \begin{bmatrix} g_0 & 0 \\ 0 & g_1 \end{bmatrix}, \quad g_0 \in H^{\infty}, g_1 \in H^{\infty}_{(m-1)\times(n-1)}$$

and $s^{\infty}(\hat{G}) = (t_0, s^{\infty}(g_1))$ is a minimum over $\hat{G} + \hat{B}H^{\infty}_{m\times n}\hat{C}$. Introduce the set $\mathcal{M}$ of all functions of the form $\operatorname{diag}\{g_0, h\}$, with $h \in H^{\infty}_{(m-1)\times(n-1)}$, which belong to $\hat{G} + \hat{B}H^{\infty}_{m\times n}\hat{C}$. Since $\mathcal{M} \subset \hat{G} + \hat{B}H^{\infty}_{m\times n}\hat{C}$, the minimum of s^{∞} over $\mathcal{M}$ is greater than or equal to its minimum over $\hat{G} + \hat{B}H^{\infty}_{m\times n}\hat{C}$; on the other hand, since the minimum over the latter coset is actually attained at $\hat{G}$, which belongs to $\mathcal{M}$, it follows that the two minima coincide. Thus, to find g_1, we need to find $h \in H^{\infty}_{(m-1)\times(n-1)}$ such that

$$\begin{bmatrix} g_0 & 0 \\ 0 & h \end{bmatrix} \in \begin{bmatrix} g_0 & 0 \\ 0 & g_1 \end{bmatrix} + \hat{B}H^{\infty}_{m\times n}\hat{C} = W^*FV + \hat{B}H^{\infty}_{m\times n}\hat{C} \tag{4.5}$$

and $s^{\infty}(h)$ is minimised.

Step 4. The induction step.

Suppose the theorem holds for all F of type $k\times(n-1)$, any $k \in \mathbf{N}$. Statement (4.5) is equivalent to

$$\begin{bmatrix} 0 & 0 \\ 0 & g_1 - h \end{bmatrix} \in \hat{B}H^{\infty}_{m\times n}\hat{C}, \tag{4.6}$$

and, by Lemma 4, there exist inner functions $\tilde{B}, \tilde{C}$ of types $(m-1)\times(m-1)$ and $(n-1)\times(n-1)$ respectively, such that (4.6) is equivalent to $h \in g_1 + \tilde{B}H^{\infty}_{(m-1)\times(n-1)}\tilde{C}$. By the inductive hypothesis the minimum of s^{∞} over the latter coset is attained at a unique function. By the principle of induction we infer the uniqueness assertion in Theorem 2.

We now turn to the description of the extremal function. By the inductive hypothesis g_1 is obtained as follows.

Let $\tilde{K}$ be a closed subspace of L^2_{m-1} containing $g_1\tilde{C}^*H^2_{n-1} + \tilde{B}H^2_{m-1}$, let $\tilde{V}_0 = \tilde{C}^*H^2_{n-1}$, let $\tilde{W}_0 = \tilde{K} \ominus \tilde{B}H^2_{m-1}$ and let

$$\tilde{T}_0 = \tilde{P}_0 M_{g_1}: \tilde{V}_0 \to \tilde{W}_0,$$

where M_{g_1} is the multiplication operator and $\tilde{P}_0: L^2_{m-1} \to \tilde{W}_0$ is the orthogonal projection. If $\tilde{T}_0 = 0$ then $\tilde{r} = 0$ and $g_1 = 0$. Otherwise we follow the prescription given in Theorem 2, with the obvious modifications, to construct an integer $\tilde{r} \geqslant 1$, subspaces $\tilde{V}_j$ of $\tilde{C}^*H^2_{n-1}$, $\tilde{K}_j$, $\tilde{E}_j$ and $\tilde{W}_j$ of L^2_{m-1}, operators $\tilde{T}_j: \tilde{V}_j \to \tilde{W}_j$, $0 \leqslant j \leqslant r$, positive numbers $\tilde{t}_j$ and unit vectors $\tilde{v}_j \in \tilde{V}_j$, $\tilde{w}_j \in \tilde{W}_j$, $0 \leqslant j \leqslant \tilde{r} - 1$, such that

$\tilde{t}_j = \|\tilde{T}_j\|$ and $\tilde{T}_j\tilde{v}_j = \tilde{t}_j\tilde{w}_j$. We have then

$$g_1(z) = \sum_{j=0}^{\tilde{r}-1} \frac{\tilde{t}_j}{\|\tilde{v}_j(z)\|^2}\, \tilde{w}_j(z)\tilde{v}_j(z)^* \tag{4.7}$$

a.e.. Moreover, $s_j(g_1(z))$ is constant and equal to t_j a.e., and $\tilde{r} \leqslant \min\{m-1, n-1\}$.

From a knowledge of g_1 we can obtain the desired minimising function G, for

$$G = W\hat{G}V^* = W\begin{bmatrix} g_0 & 0 \\ 0 & g_1 \end{bmatrix} V^*.$$

We wish to express the right hand side in terms of the original data (F, B and C), and we have to do this through the intermediary of the "diagonalised data" ($\hat{G}$, $\hat{B}$ and $\hat{C}$). Accordingly let $\hat{V}_j$, $\hat{W}_j$, $\hat{T}_j$ etc. be the analogues of V_j, W_j, T_j etc. constructed by the inductive procedure in Theorem 2 with F, B, C replaced by $\hat{G}$, $\hat{B}$, $\hat{C}$; in the construction take $\hat{K} = W^*K$, so that $\hat{K} \supseteq \hat{G}\hat{C}^*H_n^2 + \hat{B}H_m^2$. By construction of V and W (see Lemma 3), we can choose the Schmidt pair $\hat{v}_0$, $\hat{w}_0$ of $\hat{T}_0$ to be

$$\begin{aligned} \hat{v}_0 &= V^*v_0 = [f \quad 0 \ldots 0]^{\mathrm{T}}, \\ \hat{w}_0 &= W^*w_0 = [\chi f \quad 0 \ldots 0]^{\mathrm{T}} \end{aligned} \tag{4.8}$$

for some unit vector $f \in L^2$ and some scalar inner function χ. From the relations $\hat{G} = \operatorname{diag}\{g_0, g_1\}$, $\hat{G}_0\hat{v}_0 = \hat{t}_0\hat{w}_0$ we infer that $g_0 f = \hat{t}_0\chi f$, and hence $g_0 = \hat{t}_0\chi$. Thus for almost all $z \in \partial U$,

$$\begin{bmatrix} g_0(z) & 0 \\ 0 & 0 \end{bmatrix} = \frac{\hat{t}_0}{\|\hat{v}_0(z)\|^2}\, \hat{w}_0(z)\hat{v}_0(z)^*. \tag{4.9}$$

Take $\tilde{K}$ to be the closure of the orthogonal projection of $\hat{K}$ on $\begin{bmatrix} 0 \\ L_{m-1}^2 \end{bmatrix}$; by Lemma 4(iv) this does contain $g_1\tilde{C}^*H_{n-1}^2 + \tilde{B}H_{m-1}^2$, and so can be used in the foregoing construction. We claim that

$$\hat{V}_1 = \begin{bmatrix} 0 \\ V_0 \end{bmatrix}, \quad \hat{W}_1 = \begin{bmatrix} 0 \\ W_0 \end{bmatrix} \tag{4.10}$$

and

$$\hat{T}_1 = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{T}_0 \end{bmatrix} \Bigg| \hat{V}_1. \tag{4.11}$$

Indeed, it is clear from (4.8) that the pointwise orthogonal complements of $\hat{v}_0$, $\hat{w}_0$ are $\begin{bmatrix} 0 \\ L_{n-1}^2 \end{bmatrix}$, $\begin{bmatrix} 0 \\ L_{m-1}^2 \end{bmatrix}$ respectively. $\hat{V}_1$ is thus the orthogonal projection of

$\hat{V}_0 = \hat{C}^* H_n$ onto $\begin{bmatrix} 0 \\ L^2_{n-1} \end{bmatrix}$, and this is $\begin{bmatrix} 0 \\ \tilde{V}_0 \end{bmatrix}$, by Lemma 4(ii). $\hat{K}_1$ is the closure of the projection of $\hat{K}$ onto $\begin{bmatrix} 0 \\ L^2_{m-1} \end{bmatrix}$, which is $\begin{bmatrix} 0 \\ \tilde{K}_0 \end{bmatrix}$, by choice. And

$$\hat{E}_1 = \hat{E}_0 \cap \begin{bmatrix} 0 \\ L^2_{m-1} \end{bmatrix} = \hat{B}H^2_{m-1} \cap \begin{bmatrix} 0 \\ L^2_{m-1} \end{bmatrix} = \begin{bmatrix} 0 \\ \tilde{B}H^2_{m-1} \end{bmatrix} = \begin{bmatrix} 0 \\ \tilde{E}_0 \end{bmatrix}.$$

Thus

$$\hat{W}_1 = \hat{K}_1 \ominus \hat{E}_1 = \begin{bmatrix} 0 \\ \tilde{K}_0 \ominus \tilde{E}_0 \end{bmatrix} = \begin{bmatrix} 0 \\ \tilde{W}_0 \end{bmatrix}.$$

In view of (4.10) we have, for any $x \in L^2_{m-1}$,

$$\hat{P}_1 \begin{bmatrix} 0 \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ \tilde{P}_0 x \end{bmatrix}.$$

Thus, for any $h \in \tilde{V}_0$,

$$\begin{bmatrix} 0 & 0 \\ 0 & \tilde{T}_0 \end{bmatrix} \begin{bmatrix} 0 \\ h \end{bmatrix} = \begin{bmatrix} 0 \\ \tilde{P}_0(g_1 h) \end{bmatrix} = \hat{P}_1 \begin{bmatrix} 0 \\ g_1 h \end{bmatrix} =$$

$$= \hat{P}_1 \left(\hat{G} \begin{bmatrix} 0 \\ h \end{bmatrix} \right) = \hat{T}_1 \begin{bmatrix} 0 \\ h \end{bmatrix}.$$

This proves (4.11).

The relation (4.11) shows that we can take the Schmidt pair $\tilde{v}_0, \tilde{w}_0$ of $\tilde{T}_0$ corresponding to the largest singular value $\tilde{t}_0 = \hat{t}_1$ to be such that

$$\hat{v}_1 = \begin{bmatrix} 0 \\ \tilde{v}_0 \end{bmatrix}, \quad \hat{w}_1 = \begin{bmatrix} 0 \\ \tilde{w}_0 \end{bmatrix}.$$

It follows at once that

$$\hat{V}_2 = \begin{bmatrix} 0 \\ \tilde{V}_1 \end{bmatrix}.$$

Furthermore, $\hat{K}_2$ is the closure of the projection onto the pointwise orthogonal complement of $\hat{w}_1$ $\left(= \begin{bmatrix} 0 \\ \tilde{w}_0 \end{bmatrix} \right)$ of $\hat{K}_1$ $\left(= \begin{bmatrix} 0 \\ \tilde{K}_0 \end{bmatrix} \right)$, hence is $\begin{bmatrix} 0 \\ \tilde{K}_1 \end{bmatrix}$. Likewise $\hat{E}_2$ is the pointwise orthogonal complement of $\hat{w}_1$ in $\hat{E}_1$ $\left(= \begin{bmatrix} 0 \\ \tilde{E}_0 \end{bmatrix} \right)$, hence is $\begin{bmatrix} 0 \\ \tilde{E}_1 \end{bmatrix}$.

It follows that

$$\hat{W}_2 = \hat{K}_2 \ominus \hat{E}_2 = \begin{bmatrix} 0 \\ \tilde{K}_1 \ominus \tilde{E}_1 \end{bmatrix} = \begin{bmatrix} 0 \\ \tilde{W}_1 \end{bmatrix}.$$

For any $h \in \tilde{V}_1$,

$$\begin{bmatrix} 0 & 0 \\ 0 & \tilde{T}_1 \end{bmatrix}\begin{bmatrix} 0 \\ h \end{bmatrix} = \begin{bmatrix} 0 \\ \tilde{P}_1(g_1 h) \end{bmatrix} = \hat{P}_2 \begin{bmatrix} 0 \\ g_1 h \end{bmatrix} =$$

$$= \hat{P}_2\left(\hat{G}\begin{bmatrix} 0 \\ h \end{bmatrix}\right) = \hat{T}_2 \begin{bmatrix} 0 \\ h \end{bmatrix}.$$

Thus $\hat{T}_2 = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{T}_1 \end{bmatrix} \Big| \hat{V}_2$.

Proceeding in this way by induction we infer that $\tilde{r} = \hat{r} - 1$ and, for relevant $j \geqslant 1$,

$$\hat{V}_j = \begin{bmatrix} 0 \\ \tilde{V}_{j-1} \end{bmatrix}, \quad \hat{W}_j = \begin{bmatrix} 0 \\ \tilde{W}_{j-1} \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 \\ 0 & \tilde{T}_{j-1} \end{bmatrix} = \hat{T}_j | \hat{V}_j, \quad \hat{t}_j = \tilde{t}_{j-1}, \tag{4.12}$$

$$\hat{v}_j = \begin{bmatrix} 0 \\ \tilde{v}_{j-1} \end{bmatrix}, \quad \hat{w}_j = \begin{bmatrix} 0 \\ \tilde{w}_{j-1} \end{bmatrix}.$$

We have then, for $1 < j \leqslant r$,

$$\begin{bmatrix} 0 & 0 \\ 0 & \tilde{w}_{j-1}(z)\tilde{v}_{j-1}(z)^* \end{bmatrix} = w_j(z) v_j(z)^*,$$

which, in conjunction with (4.7) yields

$$\begin{bmatrix} 0 & 0 \\ 0 & g_1(z) \end{bmatrix} = \sum_{j=1}^{\hat{r}-1} \frac{\hat{t}_j}{\|\hat{v}_j(z)\|^2} \hat{w}_j(z) \hat{v}_j(z)^*$$

a.e.. Together with (4.9) this implies

$$\hat{G}(z) = \begin{bmatrix} g_0(z) & 0 \\ 0 & g_1(z) \end{bmatrix} = \sum_{j=0}^{\hat{r}-1} \frac{\hat{t}_j}{\|\hat{v}_j(z)\|^2} \hat{w}_j(z) \hat{v}_j(z)^*. \tag{4.13}$$

The final stage is to remove the hats. The relationship between the hatted and unhatted data is best expressed in a diagram

$$\begin{array}{ccccc}
V_0 = C^* H_n^2 & \xrightarrow{M_G} & L_m^2 & \xrightarrow{P_0} & K \ominus BH_n^2 = W_0 \\
\downarrow{\scriptstyle M_{V^*}} & & \downarrow{\scriptstyle M_{W^*}} & & \downarrow{\scriptstyle M_{W^*}} \\
\hat{V}_0 = \hat{C}^* H_n & \xrightarrow[M_{\hat{G}}]{} & L_m^2 & \xrightarrow[\hat{P}_0]{} & \hat{K} \ominus \hat{B} H_n = \hat{W}_0 .
\end{array}$$

Recalling that $\hat{G} = W^*GV$, $\hat{B} = W^*B$, $\hat{C} = CV$ and $\hat{K} = W^*K$, we can check that the diagram commutes. It follows that

$$T_0 = M_W \hat{T}_0 M_{V^*},$$

and hence we may choose the Schmidt pair $\hat{v}_0, \hat{w}_0$ for $\hat{T}_0$ as

$$\hat{v}_0 = V^* v_0, \quad \hat{w}_0 = W^* w_0.$$

Since $V(z), W(z)$ are unitary for almost all z, M_{V^*} maps the pointwise orthogonal complement of v_0 onto the pointwise orthogonal complement of $\hat{v}_0$, so that $M_{V^*}V_1 = \hat{V}_1$. Likewise $M_{W^*}K_1 = \hat{K}_1$, $M_{W^*}E_1 = \hat{E}_1$ and so $M_{W^*}W_1 = \hat{W}_1$. Moreover $T_1 = M_W \hat{T}_1 M_{V^*}$. Continuing inductively we conclude that $\hat{r} = r$, $\hat{t}_j = t_j$ and $\hat{v}_j = V^* v_j$, $\hat{w}_j = W^* w_j$. Hence, from (4.13) we have

$$G(z) = W\hat{G}V^*(z) = \sum_{j=0}^{r-1} \frac{t_j}{\|v_j(z)\|^2} W\hat{w}_j(z)\hat{v}_j^* V^*(z) =$$

$$= \sum_{j=0}^{r-1} \frac{t_j}{\|v_j(z)\|^2} w_j(z)v_j(z)^*.$$

We have also

$$r = \hat{r} = \tilde{r} + 1 \leqslant \min\{m-1, n-1\} + 1 = \min\{m, n\}.$$

Finally, $s_0(G(z)) = t_0$ a.e., by Theorem 1, while, for $1 \leqslant j \leqslant r-1$, we have

$$s_j(G(z)) = s_j(\hat{G}(z)) = s_j\left(\begin{bmatrix} g_0(z) & 0 \\ 0 & g_1(z) \end{bmatrix}\right) =$$

$$= s_{j-1}(g_1(z)) \overset{\text{a.e.}}{=} \tilde{t}_{j-1} = \hat{t}_j = t_j.$$

Thus $s_j(G(z))$ is constant a.e. and equal to t_j, $0 \leqslant j \leqslant r-1$. This completes the inductive step and so proves Theorem 2.

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