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PERTURBATIONS OF REFLEXIVE OPERATOR ALGEBRAS

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This paper is a sequel to a joint paper with Man Duen Choi [4] in which perturbation results for finite dimensional operator algebras are discussed. Here we are concerned with infinite dimensional reflexive algebras. The starting point is the extension of a theorem about algebras with finite distributive subspace lattice to algebras containing a maximal σ-complete Boolean algebra of idempotents. Specifically, it is shown that if $\mathscr{B}$ is any norm closed algebra close to a reflexive algebra $\mathscr{A}$ containing a maximal σ-complete Boolean algebra of idempotents, then there is a (complete) lattice isomorphism θ of Lat $\mathscr{A}$ onto Lat $\mathscr{B}$ which is close to the identity. This leads to a number of applications, including a new proof of a (slight strengthening of a) theorem of Lance on perturbations of nest algebras; and its analogue for nest subalgebras of von Neumann algebras.

The study of perturbations of operator algebras started with Kadison and Kastler [12]. Then Christensen [5] showed that for von Neumann algebras of type I, closeness implies unitary equivalence via a unitary near the identity. This will be one of our main tools, together with a theorem of Wermer [19] for orthogonalizing Boolean algebras of idempotents. Again it is crucial to our purposes that Boolean algebras which are already 'almost orthogonal' can be straightened out by a small change.

1. PRELIMINARIES

In this paper, $\mathscr{H}$ will denote a *separable* Hilbert space. The bounded operators on $\mathscr{H}$ are denoted by $\mathscr{B}(\mathscr{H})$. Given a subset of $\mathscr{B}(\mathscr{H})$, Lat $\mathscr{A}$ denotes the complete lattice of subspaces left invariant by each element of $\mathscr{A}$. Dually, if $\mathscr{L}$ is a lattice of subspaces, then Alg $\mathscr{L}$ denotes the weakly closed algebra of all operators which leave each element of $\mathscr{L}$ invariant. An algebra $\mathscr{A}$ is *reflexive* if $\mathscr{A} =$ Alg Lat $\mathscr{A}$.

The word projection will always mean orthogonal projection, in distinction from idempotents. If L is a (closed) subspace of $\mathscr{H}$, $P(L)$ denotes the projection onto L. (We will not make the common notational simplification of identifying L with $P(L)$ as we have need of other idempotents with range L.)

Given two subspaces A and B of a Banach space, define a distance between them by

$$d(A, B) = \max\left\{ \sup_{a \in A \setminus \{0\}} \frac{\|a + B\|}{\|a\|}, \sup_{b \in B \setminus \{0\}} \frac{\|b + A\|}{\|b\|} \right\}.$$

This is equivalent to (but slightly different from) the Hausdorff metric on the unit balls. Two algebras $\mathscr{A}$ and $\mathscr{B}$ on $\mathscr{H}$ are *similar* if there is an invertible operator T on $\mathscr{H}$ such that $T\mathscr{A}T^{-1} = \mathscr{B}$. In this case, T is said to implement the similarity.

A lattice $\mathscr{L}$ of subspaces of $\mathscr{H}$ is called a commutative subspace latice (CSL) if it is complete, and $P(L)P(M) = P(M)P(L)$ for every L and M in $\mathscr{L}$. For the structure of such lattices refer to [1]. Clearly, this class is not invariant under similarity, so we are obliged to enlarge our class of lattices being considered. The key idea exploited in this paper is that the lattice $\mathscr{B}$ of projections in the double commutant $\mathscr{L}''$ of $\mathscr{L}$ is a σ-complete Boolean algebra of projections containing $\{P(L) : L \in \mathscr{L}\}$. Thus, define a lattice $\mathscr{M}$ to be a sub-Boolean lattice if there is a bounded, σ-complete Boolean algebra $\mathscr{B}$ of idempotents containing a family $\{R_M : M \in \mathscr{M}\}$ of idempotents such that the range, Ran R_M, of R_M equals M for each M in $\mathscr{M}$. An algebra of operators $\mathscr{A}$ will be called a *sub-Boolean operator algebra* (SBA) if it is reflexive and Lat $\mathscr{A}$ is sub-Boolean.

An important tool for dealing with σ-complete Boolean algebras is a lemma of Wermer [19].

LEMMA 1.1. *Let $\mathscr{B}$ be a σ-complete Boolean algebra of idempotents bounded by $1 + r$. Then $\mathscr{B}$ is similar to a Boolean algebra of projections. Furthermore, the invertible operator S implementing the similarity may be taken to satisfy $\|S - I\| < 2r$.*

The next result shows that sub-Boolean operator algebras form the smallest similarity invariant class of algebras containing all CSL algebras. Condition iii) is the direct analogue of the fact that CSL algebras are the reflexive algebras containing a maximal abelian von Neumann algebra.

PROPOSITION 1.2. *For an algebra of operators $\mathscr{A}$, the following are equivalent:*

i) *$\mathscr{A}$ is a sub-Boolean operator algebra;*

ii) *$\mathscr{A}$ is similar to a CSL algebra;*

iii) *$\mathscr{A}$ is reflexive, and contains a maximal bounded Boolean algebra of idempotents.*

Proof. Assuming i), apply Lemma 1.1 to the σ-complete Boolean algebra containing Lat $\mathscr{A}$. Then $S\,\mathrm{Lat}\,\mathscr{A}\,S^{-1} = \mathrm{Lat}\,S\mathscr{A}S^{-1}$ is contained in a Boolean algebra of projections, and thus is abelian. So $S\mathscr{A}S^{-1}$ is a CSL algebra. Conversely, (ii) implies (i) is immediate.

Likewise, any CSL algebra contains a maximal abelian von Neumann algebra $\mathcal{M}$, and S the projections $\mathcal{B}$ of $\mathcal{M}$ form a maximal abelian Boolean algebra of projections. So ii) implies iii). Conversely, Lemma 1.1 shows that if $\mathcal{A}$ satisfies iii), then $\mathcal{A}$ is similar to a reflexive algebra $\mathcal{A}'$ containing a maximal abelian Boolean algebra of projections $\mathcal{B}$. Since span($\mathcal{B}$) is a maximal abelian von Neumann algebra, $\mathcal{A}'$ is a CSL. ▨

2. THE MAIN RESULT

Because of Proposition 1.2, we will consider perturbations only of CSL algebras. It is for this class that best norm estimates can be obtained. The first result shows that two nearby sub-Boolean algebras are similar via an invertible near the identity exactly when they have nearby Boolean algebras as well. Unfortunately, the Boolean algebra containing a given lattice is not unique unless the lattice is complemented. This is a major source of difficulty. Indeed, in the nest algebra case, the nearby Boolean algebra is only deduced after similarity is established.

PROPOSITION 2.1. *Let $\mathcal{A}$ be a* CSL *algebra, and let $\mathcal{B}$ be the Boolean algebra of projections generated by* Lat $\mathcal{A}$. *Let $\mathcal{A}_1$ be another sub-Boolean operator algebra such that*

$$d(\mathcal{A}, \mathcal{A}_1) < \frac{1}{2}.$$

i) *If* Lat $\mathcal{A}_1$ *is contained in a Boolean algebra $\mathcal{B}_1$ with*

$$d(\mathcal{B}, \mathcal{B}_1) < \varepsilon \leqslant \frac{1}{1200},$$

then there is an invertible operator S with $\|S - I\| < 20\varepsilon$ such that $S\mathcal{A}_1S^{-1} = \mathcal{A}$.

ii) *Conversely, if there is an invertible S with $\|S-I\|<\varepsilon$ such that $S\mathcal{A}_1S^{-1}=\mathcal{A}$, then* Lat $\mathcal{A}_1$ *is contained in a Boolean algebra $\mathcal{B}_1$ such that*

$$d(\mathcal{B}, \mathcal{B}_1) < \frac{2\varepsilon}{1-\varepsilon}.$$

Proof. Firstly, $\mathcal{B}_1$ is bounded by $1 + \varepsilon$. So Lemma 1.1 provides an invertible T with $\|T - I\| < 2\varepsilon$ which orthogonalizes $\mathcal{B}_1$. Let $\mathcal{A}_2 = T\mathcal{A}_1T^{-1}$ and $\mathcal{B}_2 = T\mathcal{B}_1T^{-1}$. Note that

$$\|A - TAT^{-1}\| = \|(1 - T)A + TAT^{-1}(T - I)\| <$$

$$< 2\varepsilon\|A\|\left[1 + \frac{1+2\varepsilon}{1-2\varepsilon}\right] = \frac{4\varepsilon}{1-2\varepsilon}\|A\|.$$

Thus

$$d(\mathscr{A}_1, \mathscr{A}_2) < \frac{4\varepsilon}{1-2\varepsilon} \cdot \frac{1+2\varepsilon}{1-2\varepsilon} < 5\varepsilon$$

and

$$d(\mathscr{B}_1, \mathscr{B}_2) < \frac{4\varepsilon}{1-2\varepsilon} < 5\varepsilon.$$

So $d(\mathscr{A}, \mathscr{A}_2) < 1/2 + 5\varepsilon$ and $d(\mathscr{B}, \mathscr{B}_2) < 6\varepsilon$.

Now $\mathscr{B}$ and $\mathscr{B}_2$ span abelian von Neumann algebras $\mathscr{M}$ and $\mathscr{M}_2$. Furthermore, the unit balls of $\mathscr{M}$ and $\mathscr{M}_2$ are the absolutely convex hulls of $\mathscr{B}$ and $\mathscr{B}_2$ respectively. Hence $d(\mathscr{M}, \mathscr{M}_2) < 6\varepsilon$. By [5], there is a unitary U with $\|U - I\| < 18\varepsilon$ such that $U\mathscr{M}_2U^* = \mathscr{M}$.

Let $S = UT$. Then $S\mathscr{A}_1S^{-1}$ contains $\mathscr{M}$, and $\operatorname{Lat} S\mathscr{A}_1S^{-1}$ is contained in $\mathscr{M}$. Also

$$d(\mathscr{A}_2, U\mathscr{A}_2U^*) \leqslant 2\|U - I\| < 36\varepsilon$$

and thus

$$d(\mathscr{A}, S\mathscr{A}_1S^{-1}) < \frac{1}{2} + 41\varepsilon < 1.$$

Let P be the orthogonal projection onto an invariant subspace of $\mathscr{A}$. Note that P belongs to $\mathscr{M}$, and thus to $S\mathscr{A}_1S^{-1}$. If P is not invariant for $S\mathscr{A}_1S^{-1}$ as well, there is an operator $A_1 = P^\perp A_1 P$ of norm one in $S\mathscr{A}_1S^{-1}$. Choose A in $\mathscr{A}$ with $\|A - A_1\| < 1$. Then

$$1 = \|A\| = \|P^\perp(A - A_1)P\| \leqslant \|A - A_1\| < 1.$$

Thus P belongs to $\operatorname{Lat}(S\mathscr{A}_1S^{-1})$ as well. The same argument works in reverse, so $\operatorname{Lat}\mathscr{A} = \operatorname{Lat} S\mathscr{A}_1S^{-1}$. As both algebras are reflexive, they must be equal. Finally,

$$\|I - S\| = \|I - U + U(I - T)\| < 20\varepsilon.$$

Part (ii) is a simple estimate that has been used in the proof above of (i). ▨

The main result of this paper shows that any algebra close to a sub-Boolean operator algebra has an isomorphic invariant subspace lattice nearby. Unfortunately, we are unable to show that it is sub-Boolean. This result is the infinite dimensional analogue of [4, Theorem 4.4].

THEOREM 2.2. *Let* $\mathscr{A} = \operatorname{Alg}\mathscr{L}$ *be a* CSL *algebra. Let* $\mathscr{A}_1$ *be a norm closed algebra such that*

$$d(\mathscr{A}, \mathscr{A}_1) < \varepsilon \leqslant 0.01 .$$

Then there is a complete lattice isomorphism θ *of* $\mathscr{L}$ *onto* $\operatorname{Lat}\mathscr{A}_1$ *such that* $\|\theta - \mathrm{id}\| < 4\varepsilon$.

Proof. The proof of Theorem 4.4 of [4] makes no use of finite dimensionality whatsoever. Thus there is a lattice isomorphism θ of $\mathscr{L}$ onto $\mathscr{L}_1 = \operatorname{Lat}\mathscr{A}_1$ with $\|\theta - \mathrm{id}\| < 4\varepsilon$ and $\|\theta^{-1} - \mathrm{id}\| < 4\varepsilon$. To complete the proof, it must be shown that θ respects arbitrary sups and infs.

Recall in particular that for each M in $\mathscr{L}_1$, there is an idempotent in $\mathscr{A}_1$ with range M. This idempotent Q may be chosen so that

$$\|P(\theta^{-1}M) - Q\| < 18\varepsilon.$$

If Q_1 and Q_2 are idempotents with range M_1 and M_2 respectively then Q_1Q_2 and $Q_1 + Q_2 - Q_2Q_1$ are idempotents with ranges $M_1 \wedge M_2$ and $M_1 \vee M_2$, respectively.

Let L_α belong to $\mathscr{L}$ and let $M_\alpha = \theta(L_\alpha)$ in $\mathscr{L}_1$. Let $P_\alpha = P(L_\alpha)$ and for each finite subset σ, let

$$L_\sigma = \bigwedge_{\alpha\in\sigma} L_\alpha \quad \text{and} \quad P_\sigma = P(L_\sigma).$$

Similarly, let

$$M_\sigma = \bigwedge_{\alpha\in\sigma} M_\alpha = \theta(L_\sigma).$$

Pick idempotents Q_σ in $\mathscr{A}_1$ with range M_σ such that $\|P_\sigma - Q_\sigma\| < 3\varepsilon$. Also, let

$$L_0 = \bigwedge L_\alpha, \quad M_0 = \bigwedge M_\alpha$$

and $P_0 = P(L_0)$. Notice that P_σ tends to P_0 in the weak topology.

Let R be a weak limit of a convergent subnet of $\{Q_\sigma\}$. For all x in M_0, $Q_\sigma x = x$ for all σ and thus $Rx = x$. Also, for each σ, and $\nu \geqslant \sigma$ $\operatorname{Ran} Q_\nu = M_\nu$ is contained in M_σ. So $\operatorname{Ran} R$ is contained in M_σ for every σ, and thus R is an idempotent in $\mathscr{A}_1$ with range M_0.

Since $\|Q_\sigma - P_\sigma\| < 18\varepsilon$ for all σ, it follows that $\|R - P_0\| \leqslant 18\varepsilon$, and hence

$$\|P(M_0) - P_0\| \leqslant \|P(M_0) - R\| + \|R - P_0\| =$$

$$= \|R^* - R\| + \|R - P_0\| < 54\varepsilon.$$

So $M_0 = \theta(L_0)$ as required.

The argument for sups follows analogously. ▨

REMARK 2.3. It is an important consequence of the proof that each invariant subspace L of $\mathscr{A}_1$ is the range of an idempotent in $\mathscr{A}_1$. Suppose L_1 and L_2 are two invariant subspaces, and P_1 and P_2 belong to $\mathscr{A}_1$ with range L_1 and L_2 respectively. Let

$$Q_2 = P_1P_2P_1 + (1 - P_1)P_2(1 - P_1) = P_2P_1 + (1 - P_1)P_2.$$

This is an idempotent in $\mathscr{A}_1$ with range L_2 which commutes with P_1. Consequently, for any finite sublattice of $\mathscr{L}_1$, one can find commuting idempotents in $\mathscr{A}_1$ with range corresponding to the elements of the lattice. This can readily be extended to a Boolean algebra of idempotents in $\mathscr{A}_1$. That is, every finite sublattice of $\mathscr{L}_1$ is sub-Boolean. Unfortunately there is no canonical way to construct these algebras because the idempotents are not unique. Indeed, an idempotent Q with range L is unique precisely when $\mathrm{Ran}(1 - Q)$ is also in $\mathrm{Lat}\,\mathscr{A}_1$.

It remains an important question for the perturbation problem as to whether $\mathscr{A}_1$ need be a sub-Boolean operator algebra.

3. APPLICATIONS

In one very special class of algebras, it is easy to find an appropriate Boolean algebra. Say that a CSL lattice $\mathscr{L}$ has *co-rank n* if it contains elements $L_1, \ldots, L_n$ such that $\{\mathscr{L}, L_1^{\perp}, \ldots, L_n^{\perp}\}$ generate a Boolean algebra. One can generalize this notion to arbitrary distributive lattices, since any such lattice embeds into a Boolean algebra. This property is clearly preserved by isomorphism.

An interesting class of CSL algebras of co-rank $(n-1)$ have the form

$$\mathscr{A} = \left\{ \begin{bmatrix} D_1 & & S_{ij} \\ & D_2 & \\ & & \ddots & \\ 0 & & & D_n \end{bmatrix} : D_i \in \mathscr{M},\ S_{ij} \in \mathscr{S}_{ij} \right\}$$

where $\mathscr{M}$ is a maximal abelian von Neumann algebra, and $\mathscr{S}_{ij}$ are weakly closed $\mathscr{M}$ bi-modules satisfying

$$\mathscr{S}_{ij}\mathscr{S}_{jk} \subseteq \mathscr{S}_{ik}, \quad 1 \leqslant i < j < k \leqslant n.$$

For $1 \leqslant i \leqslant n-1$, let

$$P_i = I \oplus \ldots \oplus I \oplus 0 \oplus \ldots \oplus 0 = I^{(i)} \oplus 0^{(n-i)}.$$

Then P_i belongs to $\mathscr{L} = \mathrm{Lat}\,\mathscr{A}$, and $\{\mathscr{L}, P_1^{\perp}, \ldots, P_{n-1}^{\perp}\}$ generate the Boolean algebra of projections in the maximal abelian von Neumann algebra

$$\mathscr{A} \cap \mathscr{A}^* = \mathscr{M} \oplus \ldots \oplus \mathscr{M}.$$

This class includes the so called tridiagonal algebras [11], which are co-rank 1.

COROLLARY 3.1. *Let $\mathscr{A}$ be a sub-Boolean operator algebra such that $\mathscr{L} = \mathrm{Lat}\,\mathscr{A}$ has co-rank n. Then there is an $\varepsilon_0 > 0$ and a constant C (which depend only on n and* *the bound $M = \sup\{\|P\| : P \in \mathscr{B}\}$ where $\mathscr{B}$ is a Boolean algebra of idempotents in $\mathscr{A}$ containing $\mathscr{L}$) with the property: if $\mathscr{A}_1$ is a norm closed algebra with*

$$d(\mathscr{A}, \mathscr{A}_1) = \varepsilon < \varepsilon_0$$

then $\mathscr{A}_1$ is similar to $\mathscr{A}$ via an invertible operator S such that

$$\|S - I\| < C\varepsilon.$$

Proof. By Lemma 1.1 and Proposition 1.2, after a similarity T with $\|T - I\| < 2(M - 1)$, one may assume that $\mathscr{A}$ is a CSL algebra. Let $L_1, \ldots, L_n$ be the elements of $\mathscr{L}$ such that $\{\mathscr{L}, L_1^\perp, \ldots, L_n^\perp\}$ generate the Boolean algebra. Let $P_i = P(L_i)$. Then $\mathscr{B}$ is the Boolean algebra of projections generated by $P_1, \ldots, P_n$ and $\{P(L) : L \in \mathscr{L}\}$. Let $\varepsilon_0 = (6000n8^n)^{-1}$.

Given an algebra $\mathscr{A}_1$, pick idempotents Q_i in $\mathscr{A}_1$ with $\|Q_i - P_i\| < 3\varepsilon$ and thus $\operatorname{Ran} Q_i = \theta(L_i)$. As in Remark 2.3, one can modify Q_i to commuting idempotents R_i in $\mathscr{A}_1$ with the same ranges. The precise calculations will be omitted, but for ε_0 sufficiently small, a crude estimate yields

$$\|P_i - R_i\| \leqslant 4^n\varepsilon, \quad 1 \leqslant i \leqslant n.$$

For convenience, write $Q^{(1)} = Q$ and $Q^{(-1)} = I - Q$ for any idempotent Q. For each n-tuple $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ of $\pm$ 1's, let

$$E_\varepsilon = \prod P_i^{(\varepsilon_i)} \quad \text{and} \quad F_\varepsilon = \prod R_i^{(\varepsilon_i)}.$$

The non-zero elements of $\{E_\varepsilon\}$, respectively $\{F_\varepsilon\}$, are the atoms of the Boolean algebras generated by $\{P_i\}$ and $\{R_i\}$. One obtains

$$\|E_\varepsilon - F_\varepsilon\| \leqslant n(1 + 4^n\varepsilon)^{n-1}4^n\varepsilon \leqslant A_n\varepsilon$$

for $0 < \varepsilon \leqslant \varepsilon_0$ where $A_n = 2n4^n$. This ensures that $A_n\varepsilon_0 \leqslant 2^{-n}$, so that in particular $E_\varepsilon = 0$ if and only if $F_\varepsilon = 0$.

For each M in $\mathscr{L}_1 = \operatorname{Lat} \mathscr{A}_1$, define

$$Q(M) = \sum_\varepsilon F_\varepsilon Q F_\varepsilon$$

where Q is any idempotent in $\mathscr{A}_1$ with range M. It will be shown that $Q(M)$ is well defined, and that the Boolean algebra $\mathscr{B}_1$ generated by $\{R_1, \ldots, R_n, Q(M) : M \in \mathscr{L}_1\}$ satisfies

$$d(\mathscr{B}, \mathscr{B}_1) < C_1\varepsilon$$

where C_1 is a constant depending only on n.

Fix $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$, and let

$$L_\varepsilon^+ = \inf\{L \in \mathscr{L} : P(L) \geqslant E_\varepsilon\}, \quad P_\varepsilon^+ = P(L_\varepsilon^+)$$

$$L_\varepsilon^- = \sup\{L \in \mathscr{L} : L \leqslant L_\varepsilon^+ \text{ and } L \perp E_\varepsilon\}, \quad P_\varepsilon^- = P(L_\varepsilon^-).$$

It is easy to see that $E_\varepsilon = P_\varepsilon^+ - P_\varepsilon^-$. Also, let $M_\varepsilon^\pm = \theta(L_\varepsilon^\pm)$ and let $Q_\varepsilon^\pm$ be idempotents in $\mathscr{A}_1$ with ranges $M_\varepsilon^\pm$ which commute with $R_1, \ldots, R_n$. Such idempotents can be chosen so that

$$\| P_\varepsilon^\pm - Q_\varepsilon^\pm \| \leqslant A_n \varepsilon,$$

and $F_\varepsilon = Q_\varepsilon^+ - Q_\varepsilon^-$.

It follows from the hypotheses on $\mathscr{L}$ that $E_\varepsilon \mathscr{L} | E_\varepsilon \mathscr{H}$ is a complemented lattice. So given L_1 in $\mathscr{L}$, there is an L_2 in $\mathscr{L}$ so that

$$L_1 \wedge L_2 = P_\varepsilon^- \quad \text{and} \quad L_1 \vee L_2 = P_\varepsilon^+ .$$

Now suppose M_1 is any element of $\mathscr{L}_1$, and $L_1 = \theta^{-1}(M_1)$. Let $M_2 = \theta(L_2)$ where L_2 is chosen as above. Then

$$M_1 \wedge M_2 = Q_\varepsilon^- \quad \text{and} \quad M_1 \vee M_2 = Q_\varepsilon^+ .$$

Thus if Q_1 and Q_2 are any idempotents in $\mathscr{A}_1$ with range M_1 and M_2 respectively,

$$Q_\varepsilon^+ Q_1 Q_2 Q_\varepsilon^+ = Q^-$$

and

$$Q_\varepsilon^+ (Q_1 + Q_2 - Q_1 Q_2) Q_\varepsilon^+ = Q_\varepsilon^+ .$$

In particular,

$$F_\varepsilon Q_1 F_\varepsilon + F_\varepsilon Q_2 F_\varepsilon = F_\varepsilon .$$

It is now immediate that $F_\varepsilon Q_1 F_\varepsilon$ is well defined, and that $F_\varepsilon \mathscr{L}_1 F_\varepsilon | F_\varepsilon \mathscr{H}$ is a complemented lattice isomorphic to $E_\varepsilon \mathscr{L}_1 \,|\, E_\varepsilon \mathscr{H}$.

The Boolean algebra $\mathscr{B}$ consists of all projections of the form

$$\{\sum_\varepsilon E_\varepsilon P_\varepsilon : P_\varepsilon \in P(\mathscr{L})\}$$

and $\mathscr{B}_1$ consists of all idempotents of the form

$$\{\sum_\varepsilon F_\varepsilon Q_\varepsilon F_\varepsilon : Q_\varepsilon \in \mathscr{A}_1,\ Q_\varepsilon \mathscr{H} \in \mathscr{L}_1\}.$$

There is a natural extension $\hat{\theta}$ of θ to an isomorphism of $\mathscr{B}$ onto $\mathscr{B}_1$. As easy estimate shows that if $\| P - Q \| < 3\varepsilon$, then

$$\| E_\varepsilon P - F_\varepsilon Q F_\varepsilon \| < \frac{5}{2} A_n \varepsilon .$$

So for any element P of $\mathscr{B}$, one obtains

$$\|P - \hat{\theta}(P)\| < 2^n \cdot \frac{5}{2} A_n \varepsilon.$$

So $d(\mathscr{B}, \mathscr{B}_1) < 5n8^n\varepsilon$.

Now $5n8^n\varepsilon_0 = 1/1200$, so by the proof of Theorem 2.1, there is an invertible operator S with

$$\|S - I\| < 100n8^n\varepsilon$$

which takes $\mathscr{B}_1$ onto $\mathscr{B}$ and implements θ^{-1}. So $S\mathscr{A}_1 S^{-1}$ is a subalgebra of $\mathscr{A}$ which is closer than 1, so $S\mathscr{A}_1 S^{-1}$ equals $\mathscr{A}$ and $\mathscr{A}_1$ is indeed reflexive, so is a sub-Boolean algebra. ▧

Another class of algebras of which more information can be squeezed out are the *completely distributive lattices.* All sub-Boolean algebras are distributive, but to be completely distributive, a lattice must satisfy a distributive law over sets of arbitrary cardinality (see [18] for specifics). For our purposes, a Theorem due to Laurie and Longstaff [17] is needed. For L in $\mathscr{L}$, let

$$L_- = \bigvee \{M \in \mathscr{L} : L \nleq M\}.$$

LEMMA 3.2. [17]. *If a subspace lattice $\mathscr{L}$ is completely distributive and sub-Boolean, then* $\mathrm{Alg}\,\mathscr{L}$ *is the weakly closed span of*

$$\bigcup_{L \in \mathscr{L}} P(L)\mathscr{B}(\mathscr{H})P(L_-)^\perp.$$

COROLLARY 3.3. *Let $\mathscr{A} = \mathrm{Alg}\,\mathscr{L}$ be a sub-Boolean operator algebra with completely distributive invariant subspace lattice $\mathscr{L}$. Then there is an $\varepsilon_0 > 0$ so that if $\mathscr{A}_1$ is a weakly closed algebra with sub-Boolean lattice and $d(\mathscr{A}, \mathscr{A}_1) < \varepsilon_0$, then $\mathscr{A}_1$ is reflexive.*

Proof. By Proposition 1.2, one may assume that $\mathscr{L}$ is a CSL lattice. By Theorem 2.2, there is an isomorphism θ of $\mathscr{L}$ onto $\mathscr{L}_1 = \mathrm{Lat}\,\mathscr{A}_1$. So $\mathscr{L}_1$ is also completely distributive, and $\theta(L_-) = \theta(L)_-$ for all L in $\mathscr{L}$. By Lemma 3.2, it suffices to show that for every M in $\mathscr{L}_1$,

$$P(M)\mathscr{B}(\mathscr{H})P(M_-)^\perp$$

is contained in $\mathscr{A}_1$.

So fix M in $\mathscr{L}_1$, and let Q and Q_- be idempotents in $\mathscr{A}_1$ with range M and M_- respectively such that

$$\|Q - P\| < 3\varepsilon \quad \text{and} \quad \|Q_- - P_-\| < 3\varepsilon$$

where $P = P(\theta^{-1}(M))$ and $P_- = P(\theta^{-1}(M_-))$. If $A = PAP_-^\perp$ has norm one, there is an element B of $\mathscr{A}_1$ such that $\|A - B\| < \varepsilon$. So $QB(1 - Q_-)$ belongs to $\mathscr{A}_1$, and

$$\|A - QB(1 - Q_-)\| = \|(P - Q)AP_-^\perp + Q(A - B)P_-^\perp + QB(P_-^\perp + Q_- - I)\| \leqslant$$

$$\leqslant 3\varepsilon + (1 + 3\varepsilon)\varepsilon + (1 + 3\varepsilon)(1 + \varepsilon)3\varepsilon = 7\varepsilon + 15\varepsilon^2 + 3\varepsilon^3 < 10\varepsilon$$

provided $\varepsilon \leqslant 1/6$. On the other hand, if $B = QB(1 - Q_-)$ has norm one, then $\mathscr{A}$ contains PBP_-, and

$$\|B - PBP_-^\perp\| = \|(Q - P)B + PB(P_- - Q_-)\| < 3\varepsilon + 3\varepsilon = 6\varepsilon.$$

Thus

$$d(P\mathscr{A}P_-^\perp,\ Q\mathscr{A}_1(1 - Q_-)) < 10\varepsilon.$$

Since, $P\mathscr{A}P_-^\perp = P\mathscr{B}(\mathscr{H})P_-^\perp$, one similarly obtains

$$d(P\mathscr{A}P_-^\perp,\ Q\mathscr{B}(\mathscr{H})(1 - Q_-)) < 6\varepsilon.$$

So

$$d(Q\mathscr{A}_1(1 - Q_-),\ Q\mathscr{B}(\mathscr{H})(1 - Q_-)) < 16\varepsilon \leqslant 1$$

provided $\varepsilon \leqslant 1/16$. But as $Q\mathscr{A}_1(1 - Q_-)$ is a subspace of $Q\mathscr{B}(\mathscr{H})(1 - Q_-)$, they must be equal.

It is routine to verify that $QP(M) = P(M)$ and $P(M_-)^\perp(1 - Q_-) = P(M_-)^\perp$. Hence $\mathscr{A}_1$ contains

$$QP(M)\mathscr{B}(\mathscr{H})P(M_-)^\perp(1 - Q_-) = P(M)\mathscr{B}(\mathscr{H})P(M_-)^\perp$$

as desired. As $\mathscr{A}_1$ is weakly closed and $\mathscr{L}_1$ is completely distributive, it follows from Lemma 3.2 that $\mathscr{A}_1 = \operatorname{Alg}\mathscr{L}_1$. ▨

4. NEST ALGEBRAS

A nest is a complete, linearly ordered subspace lattice. In [13], Lance shows that two nest algebras are close if and only if their corresponding nests are close and they are in fact similar via an invertible near I. We present a modest improvement of these results here. As well, a different proof of Lance's results is given based on the similarity theory of nests [7, 15, 16]. This avoids the cohomology approach, but relies on some very deep results about nest algebras.

THEOREM 4.1. [13]. *Let $\mathscr{N}$ and $\mathscr{M}$ be two nests on a separable Hilbert space $\mathscr{H}$, and let θ be an order isomorphism of $\mathscr{N}$ onto $\mathscr{M}$ such that $\|\theta - \mathrm{id}\| = \varepsilon < 1/2$. Then there is an invertible operator S with $\|S - I\| < 2\varepsilon$ which implements θ.*

Proof. Notice that θ preserves dimension. For if $N_1 < N_2$ in $\mathcal{N}$, let $M_i = \theta(N_i)$. Then

$$d(N_2 \ominus N_1, M_2 \ominus M_1) \leqslant d(N_2, M_2) + d(N_1, M_1) \leqslant 2\|\theta - \mathrm{id}\| < 1.$$

So $\dim N_2 \ominus N_1 = \dim M_2 \ominus M_1$. Using [7], one obtains an invertible operator T which implements θ. Furthermore, if $\delta > 0$ is given, T can be chosen so that there is a unitary U such that $\|T - U\| < \delta$. Take δ so small that $(\varepsilon + \delta)(1 - \varepsilon - \delta)^{-1} < 2\varepsilon$.

Apply Arveson's distance formula [2] to T:

$$d(T, \mathrm{Alg}\mathcal{N}) = \sup\{\|P(N)^{\perp}TP(N)\| : N \in \mathcal{N}\} =$$

$$= \sup\{\|P(N)^{\perp}P(\theta(N))TP(N)\| : N \in \mathcal{N}\} \leqslant \|\theta - \mathrm{id}\| \, \|T\| < \varepsilon + \delta.$$

Thus there is an element V of $\mathrm{Alg}\mathcal{N}$ with $\|T - V\| < \varepsilon + \delta$. Hence

$$\|U - V\| < \varepsilon + 2\delta$$

and so V is invertible.

Next, it will be shown that V^{-1} belongs to $\mathrm{Alg}\mathcal{N}$. This is equivalent to showing that $VN = N$ for every N in $\mathcal{N}$. Otherwise, there is some N in $\mathcal{N}$ for which VN is a proper subspace of N. Pick any unit vector x in $N \ominus VN$, and choose a vector y in the unit ball of $M = \theta(N)$ with $\|x - y\| \leqslant \varepsilon$. Since $T^{-1}M = N$, the vector $z = T^{-1}y$ belongs to N. Thus

$$\|Vz - x\| \leqslant \|V - T\| \, \|z\| + \|y - x\| < 2\varepsilon + \delta < 1.$$

This contradiction establishes that V^{-1} is in $\mathrm{Alg}\mathcal{N}$.

Hence $S = TV^{-1}$ implements θ, and

$$\|S - I\| \leqslant \|T - V\| \, \|V^{-1}\| < \frac{\varepsilon + \delta}{1 - \varepsilon - \delta} < 2\varepsilon. \qquad ▨$$

THEOREM 4.2. *Let $\mathcal{A} = \mathrm{Alg}\mathcal{N}$ be a nest algebra, and let $\mathcal{A}_1$ be a norm closed algebra such that*

$$d(\mathcal{A}, \mathcal{A}_1) = \varepsilon < 0.01\,.$$

Then $\mathcal{A}_1$ is similar to $\mathcal{A}$ via an invertible operator S satisfying

$$\|S - I\| < 8\varepsilon.$$

Proof. By Theorem 2.2, $\mathcal{M} = \mathrm{Lat}\,\mathcal{A}_1$ is a nest and there is an order isomorphism θ of $\mathcal{N}$ onto $\mathcal{M}$ with $\|\theta - \mathrm{id}\| \leqslant 4\varepsilon$. By Theorem 4.1, θ is implemented by an invertible operator S with $\|S - I\| < 8\varepsilon$. Hence

$$d(\mathscr{A}_1, \operatorname{Alg}\mathscr{M}) \leqslant d(\mathscr{A}_1, \mathscr{A}) + d(\mathscr{A}, S\mathscr{A}S^{-1}) < \varepsilon + \frac{16\varepsilon}{1 - 8\varepsilon} < 21\varepsilon < 1.$$

(See the proof of Proposition 2.1 for a similar estimate.) But $\mathscr{A}_1$ is a subspace of $\operatorname{Alg}\mathscr{M}$, so they must be equal. ▨

The following corollary is immediate.

COROLLARY 4.3. *For two nests $\mathscr{N}$ and $\mathscr{M}$ on a separable Hilbert space, the following are equivalent:*

i) *$\mathscr{N}$ and $\mathscr{M}$ are close,*

ii) *$\operatorname{Alg}\mathscr{N}$ and $\operatorname{Alg}\mathscr{M}$ are close,*

iii) *$\mathscr{N}$ and $\mathscr{M}$ are similar via an invertible operator close to I.*

These same methods apply to a family of reflexive algebras, called nest subalgebras of von Neumann algebras, studied by Gilfeather and Larson [9, 10]. These algebras have the form $\mathscr{A} = \mathscr{B} \cap \operatorname{Alg}\mathscr{N}$ where $\mathscr{B}$ is a von Neumann algebra and $\mathscr{N}$ is a nest of projections in $\mathscr{B}$. The usefulness of a distance formula should be apparent from the preceding proof. Provided that $\mathscr{B}$ is approximately finite, there is a distance formula for $\mathscr{A}$ [10]. Namely, there is a universal constant K such that if T is any operator,

$$d(T, \mathscr{A}) \leqslant K \sup\{\|P^\perp TP\| : P = P(L),\ L \in \operatorname{Lat}\mathscr{A}\}.$$

The AF condition on $\mathscr{B}$ guarantees an amenable group G of unitaries, which generate the commutant $\mathscr{B}'$ as a von Neumann algebra. Equivalently, there is an expectation $E_{\mathscr{B}}$ of $\mathscr{B}(\mathscr{H})$ onto $\mathscr{B}$. Averaging over this group will yield the desired extension.

THEOREM 4.4. *Let $\mathscr{B}$ be an approximately finite von Neumann algebra, and let $\mathscr{N}$ and $\mathscr{M}$ be nests of projections in $\mathscr{B}$. Then the following are equivalent:*

i) *$\mathscr{N}$ and $\mathscr{M}$ are close,*

ii) *$\mathscr{B} \cap \operatorname{Alg}\mathscr{N}$ and $\mathscr{B} \cap \operatorname{Alg}\mathscr{M}$ are close,*

iii) *$\mathscr{N}$ and $\mathscr{M}$ are similar via an invertible element of $\mathscr{B}$ close to the identity.*

Proof. Proposition 2.1 gives iii) implies i) immediately. For i) implies ii), let T be a norm one element of $\mathscr{B} \cap \operatorname{Alg}\mathscr{N}$. For M in $\mathscr{M}$, choose N so that $\|P(N) - P(M)\| \leqslant d(\mathscr{N}, \mathscr{M})$. Then

$$\|P(M)^\perp TP(M)\| = \|P(M)^\perp TP(M) - P(N)^\perp TP(N)\| \leqslant$$
$$\leqslant 2\|T\|\,\|P(N) - P(M)\| \leqslant 2d(\mathscr{N}, \mathscr{M}).$$

By the distance formula for nests there is an element A of $\operatorname{Alg}\mathscr{M}$ with

$$\|T - A\| \leqslant 2d(\mathscr{N}, \mathscr{M}).$$

However, $B = E_{\mathscr{B}}A$ belongs to $\mathscr{B} \cap \operatorname{Alg}\mathscr{M}$ since

$$P(M)^{\perp}E_{\mathscr{B}}(A)P(M) = E_{\mathscr{B}}(P(M)^{\perp}AP(M)) = 0$$

for all M in $\mathscr{M}$. And so

$$\|T - B\| = \|E_{\mathscr{B}}(T - A)\| \leqslant 2d(\mathscr{N}, \mathscr{M}).$$

Reversing the role of $\mathscr{N}$ and $\mathscr{M}$ yields ii).

Next, consider ii) implies i). Suppose $\mathscr{A}_1$ is a subalgebra of $\mathscr{B}$ such that

$$d(\mathscr{B} \cap \operatorname{Alg}\mathscr{N}, \mathscr{A}_1) < \varepsilon \leqslant 0.01 .$$

As in the proof of Theorem 2.2, each projection $P(N)$ for N in $\mathscr{N}$ is within 20ε of an idempotent in $\mathscr{A}_1$ with range $\mathscr{M}$ in $\operatorname{Lat}\mathscr{A}_1$. Thus

$$\|P(N) - P(M)\| < 40\varepsilon.$$

Furthermore, M is unique. For suppose Q_1 and Q_2 were two idempotents in $\mathscr{A}_1$ with ranges M_1 and M_2 respectively, and

$$\|P(N) - Q_i\| < 20\varepsilon.$$

So

$$\|P(N) - Q_1Q_2\| = \|(P(N) - Q_1)Q_2 + P(N)(P(N) - Q_2)\| < 20\varepsilon(2 + 20\varepsilon).$$

From this, it follows that $\|Q_1 - Q_1Q_2\| < 1$. But Q_1Q_2 is also an idempotent in $\mathscr{A}_1$ with range $M_1 \wedge M_2$. If $M_1 \nleqslant M_2$, then $\|Q_1 - Q_1Q_2\| \geqslant 1$. Reversing the roles of 1 and 2 yields $M_1 = M_2$. Denote M by $\theta(N)$.

Now if $N_1 < N_2$, it follows as in the proof of Theorem 2.2 that $\theta(N_1) < \theta(N_2)$. Hence

$$\mathscr{M} = \{P(\theta(N)) : N \in \mathscr{N}\}$$

is a nest of projections in $\mathscr{B}$ close to $\mathscr{N}$. Thus, the implication i) implies ii) shows that $\mathscr{A}$ is close to $\mathscr{A}_2 = \mathscr{B} \cap \operatorname{Alg}\mathscr{M}$. But $\mathscr{A}_1$ is contained in $\mathscr{A}_2$, so the triangle inequality allows $d(\mathscr{A}_1, \mathscr{A}_2) < 1$ from which we conclude $\mathscr{A}_1 = \mathscr{B} \cap \operatorname{Alg}\mathscr{M}$.

Finally, i) implies iii). By Theorem 4.1, there is an invertible operator S in $\mathscr{B}(\mathscr{H})$ close to the identity which implements the similarity of $\mathscr{N}$ and $\mathscr{M}$. Let $T = E_{\mathscr{B}}(S)$. Clearly,

$$\|T - I\| = \|E_{\mathscr{B}}(S - I)\| \leqslant \|S - I\|.$$

So T^{-1} exists, $\|T - S\| \leqslant 2\|S - I\|$, and TN is closed for all N in $\mathscr{N}$. It remains to show that $TN = \theta(N) = M$. Now

$$P(M)^{\perp}TP(N) = E_{\mathscr{B}}(P(M)^{\perp}SP(N)) = 0.$$

Thus TN is contained in M. Suppose there is a unit vector y in M orthogonal to TN. Then $x = S^{-1}y$ belongs to N, and

$$1 \leqslant \|Tx - y\| \leqslant \|T - S\|\|S^{-1}y\| \leqslant 2\|S - I\|(1 - \|S - I\|)^{-1} < 1$$

for $\|S - I\| < 1/3$. This contradiction establishes the claim. ▨

5. GENERAL CONSIDERATIONS

For studying perturbation questions for reflexive algebras, it seems reasonable to distinguish certain related but different perturbation properties that such algebras might have. First of all, there is the distance formula which proves to be so useful for nests and von Neumann algebras. This property has been called hyper reflexivity [3, 14]. Beyond this, there are the properties that closeness of the algebras, or even the lattices implies similarity.

DEFINITION 5.1. Let $\mathscr{C}$ be a class of reflexive algebras which is invariant under similarity.

Property (D) *(Distance Formula):* $\mathscr{C}$ is said to have property (D) if for each $\mathscr{A}$ in $\mathscr{C}$, there is a constant $K < \infty$ such that

$$d(T, \mathscr{A}) \leqslant K \sup\{\|P(L)^{\perp} T P(L)\| : L \in \operatorname{Lat} \mathscr{A}\}$$

for all T in $\mathscr{B}(\mathscr{H})$.

Property (A) *(Algebra Perturbation):* $\mathscr{C}$ has property (A) if for each $\mathscr{A}$ in $\mathscr{C}$, there are constants $K < \infty$ and $\varepsilon_0 > 0$ such that if $\mathscr{A}_1$ is another algebra in $\mathscr{C}$ with $d(\mathscr{A}, \mathscr{A}_1) = \varepsilon < \varepsilon_0$, then there is an invertible operator S with $\|S - I\| \leqslant K\varepsilon$ and $S\mathscr{A}S^{-1} = \mathscr{A}_1$.

Property (L) *(Lattice Perturbation):* $\mathscr{C}$ has property (L) if for each $\mathscr{A}$ in $\mathscr{C}$, there are constants $K < \infty$ and $\varepsilon_0 > 0$ such that if θ is a lattice isomorphism of $\mathscr{L} = \operatorname{Lat} \mathscr{A}$ onto $\mathscr{L}_1 = \operatorname{Lat} \mathscr{A}_1$ for some $\mathscr{A}_1$ in $\mathscr{C}$ with $\|\theta - \mathrm{id}\| = \varepsilon < \varepsilon_0$, then there is an invertible operator S implementing θ such that $\|S - I\| \leqslant K\varepsilon$.

The class of nest algebras has been seen to have all these properties. The class of algebras spanned by a bounded σ-complete Boolean algebras of idempotents has this property. This can be seen by Proposition 2.1 and the results of Christensen for type I von Neumann algebras [5]. The class of all sub-Boolean algebras fails to have property (D) [8] and thus fails to have property (L) as well. This makes it seem unlikely that it has property (A), but this is presently unresolved. It is to be hoped that perhaps certain tractable subclasses (maybe width two algebras) have some of these properties. What follows are a few simple propositions relating these ideas.

PROPOSITION 5.2. *Property* (L) *implies property* (D).

Proof. Let $\mathscr{A}$ belong to $\mathscr{C}$ with constants K and ε_0 provided by property (L). Let T be an operator on $\mathscr{H}$ of unit norm. Let

$$\varepsilon = \min\{\varepsilon_0/2, (3K)^{-1}, 1/3\}.$$

Set $V = I + \varepsilon T$ and $\mathscr{M} = V\mathscr{L}$, where $\mathscr{L} = \operatorname{Lat} \mathscr{A}$. Let

$$\Delta = \sup\{\|P(L)^{\perp} T P(L)\| : L \in \mathscr{L}\}.$$

For any subspace L in $\mathscr{L}$, decompose $\mathscr{H} = L \oplus L^\perp$. With respect to this decomposition, T is a 2×2 operator matrix $T = (T_{ij})$, and $V = (V_{ij})$ where $V_{ii} = I + \varepsilon T_{ii}$ and $V_{ij} = \varepsilon T_{ij}$ for $i \neq j$.

Now V_{11} takes L onto L, and is bounded below by 2/3. Also

$$\|V_{21}\| = \varepsilon \|P(L)^\perp T P(L)\| \leqslant \varepsilon \Delta.$$

If v is a unit vector in $\theta(L) = VL$, there is a vector x in L with $\|x\| \leqslant 3/2$ so that $v = Vx$. Thus

$$d(v, L) = \|P(L)^\perp v\| = \|P(L)^\perp V P(L) x\| \leqslant \frac{3}{2} \varepsilon \Delta.$$

Conversely, if x is a unit vector in L, there is a vector y in L with $\|y\| \leqslant 3/2$ and $V_{11} y = x$. So

$$d(x, \theta(L)) \leqslant \|x - Vy\| = \|P(L)^\perp V P(L) y\| \leqslant \frac{3}{2} \varepsilon \Delta.$$

Hence

$$\|\theta - \mathrm{id}\| \leqslant (3/2)\varepsilon \Delta < \varepsilon_0.$$

By property (L), there is an invertible operator S with

$$\|S - I\| \leqslant K \cdot \frac{3}{2} \cdot \varepsilon \Delta$$

which implements θ. Hence $S^{-1}V$ belongs to $\mathscr{A}$. Whence

$$d(T, \mathscr{A}) = \varepsilon^{-1} d(V, \mathscr{A}) \leqslant \varepsilon^{-1} \|V - S^{-1}V\| \leqslant \varepsilon^{-1} \|S^{-1}\| \, \|S - I\| \, \|V\| \leqslant$$

$$\leqslant \varepsilon^{-1} \frac{K \cdot \frac{3}{2} \cdot \varepsilon \Delta}{1 - K \cdot \frac{3}{2} \cdot \varepsilon \Delta} \cdot \frac{4}{3} \leqslant \frac{2K}{1 - K \cdot \frac{3}{2} \cdot \frac{1}{3K} \cdot 1} = 4K\Delta.$$

So property (D) holds with constant $4K$. ▨

REMARK 5.3. Property (L) implies a stronger version of (D) which we denote as *property* (D*): For each $\mathscr{A}$ in $\mathscr{C}$, there are constants $\varepsilon_0 > 0$ and $K < \infty$ so that if $\mathscr{A}_1$ belongs to $\mathscr{C}$ and θ is a lattice isomorphism of Lat $\mathscr{A}$ onto Lat $\mathscr{A}_1$ with $\|\theta - \mathrm{id}\| < \varepsilon_0$, then $\mathscr{A}_1$ has property (D) with constant K. This follows easily since similar algebras have related constants for property D depending only on $\|S - I\|$ where S implements the similarity.

PROPOSITION 5.4. *Suppose $\mathscr{C}$ is a class of sub-Boolean operator algebras with property* (L). *Then it has property* (A).

Proof. This is an immediate consequence of Theorem 2.2.

THEOREM 5.5. *Let $\mathscr{C}$ be a class of sub-Boolean operator algebras. Then property* (L) *holds if and only if properties* (A) *and* (D*) *hold.*

Proof. Remark 5.3 and Proposition 5.4 provide one direction. Conversely, suppose $\mathscr{C}$ has properties (A) and (D*). It will be shown using (D*) that close lattices implies close algebras. Then property (A) will suffice.

Let $\mathscr{A}$ and $\mathscr{A}_1$ belong to $\mathscr{C}$, and let θ be a lattice isomorphism of $\mathscr{L} = \mathrm{Lat}\,\mathscr{A}$ onto $\mathscr{L}_1 = \mathrm{Lat}\,\mathscr{A}_1$ with $\|\theta - \mathrm{id}\|$ very small. Let K be the distance constant valid for all $\mathscr{A}_1$ such that θ is sufficiently close to the identity provided by (D*). Let $Q(L) = P(\theta(L))$ for L in $\mathscr{L}$. Then for A of norm one in $\mathscr{A}_1$,

$$d(A, \mathscr{A}) \leqslant K \sup\{\|P(L)^{\perp}AP(L)\| : L \in \mathscr{L}\} =$$

$$= K \sup\{\|P(L)^{\perp}AP(L) - Q(L)^{\perp}AQ(L)\| : L \in \mathscr{L}\} < 2K\|\theta - \mathrm{id}\|.$$

Similarly, one obtains for A of norm one in $\mathscr{A}$,

$$d(A, \mathscr{A}_1) < 2K\|\theta - \mathrm{id}\|.$$

So $d(\mathscr{A}, \mathscr{A}_1) < 2K\|\theta - \mathrm{id}\|$.

Now provided that $\|\theta - \mathrm{id}\|$ was sufficiently small, property (A) gives the desired similarity. ▨

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Added in proof. Harrison and Longstaff [20] have some related result. In particular, they have Proposition 1.2 independently.