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DERIVATIONS AND FREE GROUP ACTIONS ON C^*-ALGEBRAS

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1. MAIN RESULTS, AND NOTATION

Let G be a locally compact group which is abelian or compact with dual $\hat{G}$ and α an action of G on a C^*-algebra $\mathscr{A}$ as a group of $*$-automorphisms. Let $\mathscr{A}^\alpha_{\mathrm{F}} = \bigcup\{\mathscr{A}^\alpha(K) \mid K \subseteq \hat{G},\ K \text{ compact}\}$ denote the dense $*$-subalgebra of G-finite elements, where $\mathscr{A}^\alpha(K)$ is the spectral subspace of $\mathscr{A}$ corresponding to the compact set K, [16], [11], [4]. When G is a Lie group, it is an open problem whether or not all derivations δ defined on $\mathscr{A}^\alpha_{\mathrm{F}}$, and mapping $\mathscr{A}^\alpha_{\mathrm{F}}$ into $\mathscr{A}^\alpha_{\mathrm{F}}$, in short $\delta \in \mathrm{Der}(\mathscr{A}^\alpha_{\mathrm{F}}, \mathscr{A}^\alpha_{\mathrm{F}})$, are pregenerators (i.e. δ is closable and its closure generates a one-parameter group of $*$-automorphisms, see [16]). (Throughout this paper we adopt the convention that derivations commute with the $*$-operation). This is false if G is not Lie, see [12, Example 5.14]. A systematic treatment of positive results in this direction is given in [4]; we will here only mention the following theorems: If G is compact, abelian, and Lie, α is ergodic, $\mathscr{A}$ is simple and $\delta \in \mathrm{Der}(\mathscr{A}_{\mathrm{F}}, \mathscr{A}_{\mathrm{F}})$, then δ has the decomposition $\delta = \delta_0 + \tilde{\delta}$ where δ_0 is the generator of a one-parameter subgroup of $\alpha(G)$ and $\tilde{\delta}$ is inner, so a posteriori δ is a pregenerator, [7, Theorem 2.1]. If $G = \mathbf{T}$ = the circle group, $\mathscr{A}$ is separable, and α fixes any (closed, two-sided) ideal in $\mathscr{A}$ and $\delta \in \mathrm{Der}(\mathscr{A}^\alpha_{\mathrm{F}}, \mathscr{A})$, then δ is a generator, [25, Theorem 2.1] and [26, Theorem]. If G is compact (not necessarily abelian), and there exists a faithful, G-covariant representation of $\mathscr{A}$ with $(\mathscr{A}^\alpha)' \cap \mathscr{A}'' = \mathbf{C}1$, where $\mathscr{A}^\alpha$ is the fixed point algebra for α, and $\delta \in \mathrm{Der}(\mathscr{A}^\alpha_{\mathrm{F}}, \mathscr{A})$ with $\delta(\mathscr{A}^\alpha) \subseteq \mathscr{A}^\alpha_{\mathrm{F}}$, then δ again has the decomposition $\delta = \delta_0 + \tilde{\delta}$ with $\tilde{\delta}$ bounded, and thus δ is a pregenerator, [11, Theorem 2.5]. If G is compact, abelian, Lie and there exists a faithful G-covariant representation on a Hilbert space $\mathscr{H}$ such that the range projections $E(\gamma) = [\mathscr{A}^\alpha(\gamma)\mathscr{H}]$ are all equal to $\mathbf{1}$, and $\delta \in \mathrm{Der}(\mathscr{A}^\alpha_{\mathrm{F}}, \mathscr{A}^\alpha_{\mathrm{F}})$, then δ is a pregenerator, [11, Theorem 3.4].

Our first main result is:

THEOREM 1.1. *Let G be a separable locally compact group which is abelian or compact, α an action of G on a simple, separable, unital C^*-algebra $\mathscr{A}$, and* *assume that there exists an automorphism* τ *of* $\mathscr{A}$ *such that* $\tau\alpha_g = \alpha_g\tau$ *for all* $g \in G$ *and* $\lim_{|n|\to\infty} \|[\tau^n(x), y]\| = 0$ *for all* $x, y \in \mathscr{A}$, *where* [,] *denotes the commutator. Let* δ *be a derivation from* $\mathscr{A}_F^\alpha$ *into* $\mathscr{A}$ *such that* $\delta \mid \mathscr{A}^\alpha(K)$ *is bounded for each compact subset* $K \subseteq \hat{G}$. *(This condition is automatically fulfilled if* G *is abelian and compact and* $\delta(\mathscr{A}) \subseteq \mathscr{A}_F^\alpha$.*) It follows that* δ *has a decomposition*

$$\delta = \delta_0 + \tilde{\delta}$$

on $\mathscr{A}_F$, *where* δ_0 *is the generator of a one-parameter subgroup of the action* α, *and* $\tilde{\delta}$ *is bounded. In particular* δ *is a pregenerator. If the action* α *of* G *is faithful the decomposition is unique.*

Except for the separability assumptions, this theorem generalizes the main results of [27], [34] and [3], where one in addition assumes that δ commute with τ, and reaches the stronger conclusion that $\delta = \delta_0$. The proof of Theorem 1.1 is based on the fact that if G, α, $\mathscr{A}$ and τ are as above, then there exists a pure state ω on $\mathscr{A}$ such that the center of the direct integral representation $\pi = \int_G^\oplus dg\, \pi_\omega \circ \alpha_g$ consists of exactly the diagonal operators of this representation. Here dg denotes Haar measure. This is Theorem 2.1 of the present paper and this result is also discussed in relation to results by Baker and Powers, [1], [2]. Theorem 1.1 will be proved in Section 2.

When $G = \mathbf{T}$, it follows from [25], [26] and [28] that the assumption of the existence of τ in Theorem 1.1 can be replaced by the weaker assumption $\Gamma(\alpha) \neq \{0\}$, where Γ denotes the Connes spectrum, [29, 8.8.2], and the conclusion $\delta = \delta_0 + \tilde{\delta}$ where $\tilde{\delta}$ is bounded can still be reached. However if $G = \mathbf{T}^2$ this conclusion is false in general when $\Gamma(\alpha) = \widehat{\mathbf{T}^2} = \mathbf{Z}^2$. An example is the ergodic actions of $\mathbf{T}^2$ on simple C^*-algebras, and it follows from the results of [7] that the corresponding derivations $\delta \in \mathrm{Der}(\mathscr{A}_F^\alpha, \mathscr{A})$ has a decomposition $\delta = \delta_0 + \tilde{\delta}$, where $\tilde{\delta}$ is approximately inner but not neccessarily inner unless $\delta \in \mathrm{Der}(\mathscr{A}_F^\alpha, \mathscr{A}_F^\alpha)$. We do not know in general if $\delta \in \mathrm{Der}(\mathscr{A}_F^\alpha, \mathscr{A}_F^\alpha)$ has the decomposition $\delta = \delta_0 + \tilde{\delta}$ if G is abelian and $\Gamma(\alpha) = \hat{G}$. The following result is true, however.

THEOREM 1.2. *Let* G *be a compact abelian Lie group, and* α *an action of* G *on a* C^**-algebra* $\mathscr{A}$ *such that* $\Gamma(\alpha) = \hat{G}$ *and* $\mathscr{A}$ *is G-prime* (i.e. *any two nontrivial G-invariant ideals in* $\mathscr{A}$ *has nontrivial intersection). If* δ *is a derivation from* $\mathscr{A}_F^\alpha$ *into* $\mathscr{A}_F^\alpha$, *then* δ *is a pregenerator. Furthermore* $\mathscr{A}_F^\alpha$ *consists of analytic elements for* δ, *and there exists a* $t > 0$ *such that* $\sum_{n=0}^{\infty} \frac{t^n}{n!} \|\delta^n(x)\| < \infty$ *for all* $x \in \mathscr{A}_F^\alpha$.

This theorem follows from [11, Theorem 3.4] and Theorem 3.1 of the present paper. The latter theorem implies that there exists a pure state ω on $\mathscr{A}^\alpha$, such that if π is the cyclic representation on $\mathscr{H}$ defined by the G-invariant state $\omega \circ P_0$ on $\mathscr{A}$, where $P_0 = \int_G dg\, \alpha_g$, then $\pi(\mathscr{A}^\alpha(\gamma))\mathscr{H}$ is dense in $\mathscr{H}$ for all $\gamma \in \hat{G}$.

Our next theorem contains a somewhat unsatisfactory extra assumption. We have already mentioned that the theorem is known in the case that all prime ideals are fixed by the action.

THEOREM 1.3. *Let α be on action of the circle group $\mathbf{T} = \mathbf{R}/\mathbf{Z}$ on a separable C^*-algebra $\mathscr{A}$ with the property that there is an $\varepsilon > 0$ such that for each prime ideal $\mathscr{P}$ in $\mathscr{A}$, $\mathscr{P}$ is either globally fixed by α or $\alpha_t(\mathscr{P}) \neq \mathscr{P}$ for $0 < t < \varepsilon$. It follows that all $\delta \in \mathrm{Der}(\mathscr{A}_F^\alpha, \mathscr{A}_F^\alpha)$ are pregenerators.*

We expect that the condition involving ε is unneccessary, and prove that this is indeed the case if $\mathscr{A}$ is abelian, Theorem 4.7. However, the proof in this case shows that the problems encountered in the $\varepsilon \to 0$ limit are nontrivial, due to a loss of uniform analyticity of $\mathscr{A}_F^\alpha$.

Our last theorem concerns invariant derivations δ, with no other assumptions on the domain $D(\delta)$ than it is globally invariant under the G-action.

THEOREM 1.4. *Let G be a compact abelian group and α an action of G on a C^*-algebra $\mathscr{A}$. Let δ be a closed derivation of $\mathscr{A}$ such that $\delta\alpha_g = \alpha_g\delta$ for all $g \in G$ and $\mathscr{A}^\alpha \subseteq D(\delta)$, where $\mathscr{A}^\alpha$ is the fixed point algebra for the action α.*
It follows that δ is a generator.

In the case that $\delta \mid \mathscr{A}^\alpha = 0$ (or inner) this theorem was shown independently in [13, Theorem 5.1], and by Kishimoto, see [22, Appendix]. The present generalization was also announced in a postscript to [30].

This theorem fails if the condition $\mathscr{A}^\alpha \subseteq D(\delta)$ is replaced by $\delta \mid \mathscr{A}^\alpha$ is a generator, see [13, Example 6.1], but see also [14]. The theorem also fails if the compactness assumption on G is removed, see [13, Example 6.5] and [5, Example 2.4]. It is unknown if the theorem holds in general if the condition that G is abelian is removed, but it still holds for non-abelian G when $\mathscr{A}$ is abelian, [20], or if $\mathscr{A}$ is separable type I and $\delta \mid \mathscr{A}^\alpha = 0$, see [21].

This theorem will be proved in Section 6 by combining techniques from [13], [22] and [11].

2. ACTIONS COMMUTING WITH AN ASYMPTOTICALLY ABELIAN AUTOMORPHISM

THEOREM 2.1. *Let $\mathscr{A}$ be a unital, simple, separable C^*-algebra, G a separable locally compact group which is abelian or compact, and α a faithful action of G on $\mathscr{A}$. Assume that there is an automorphism τ of $\mathscr{A}$ such that*

$$\alpha_g \tau = \tau \alpha_g$$

for all $g \in G$, and

$$\lim_{n\to\infty} \|\tau^n(x)y - y\tau^n(x)\| = 0$$

for all $x, y \in \mathscr{A}$.

It follows that there exists a pure state ω on $\mathscr{A}$ such that the center of the direct integral representation

$$\pi = \int_G^{\oplus} dg\, \pi_\omega \circ \alpha_g$$

consists of just the diagonal operators of this direct integral decomposition. In particular the states $\omega \circ \alpha_g$ are all disjoint for distinct $g \in G$.

REMARK 2.2. As noted implicitly in [1], [2], the conclusion on the center of π does not follow from disjointness of the states $\omega \circ \alpha_g$ alone. In fact if α is the gauge action of $\mathbf{T}$ on the CAR algebra $\mathscr{A}$, there exists a pure state ω on $\mathscr{A}$ such that the states $\omega \circ \alpha_t$, $t \in \mathbf{T}$, are all disjoint, but $\int_{\mathbf{T}} dt\, \omega \circ \alpha_t$ is a factor state, see Example 2.3.

Proof of Theorem 2.1. We first prove the theorem when G is abelian, and thereafter describe the modifications needed when G is compact.

As G is separable, there exists a dense sequence $\{\gamma_n\}$ in $\hat{G}$, and a decreasing sequence $\{\Omega_n\}$ forming a basis of compact neighbourhoods of $0 \in \hat{G}$. If G also is compact, the sequence $\{\gamma_n\}$ should be chosen such that each $\gamma \in \hat{G}$ occurs an infinite number of times in the sequence.

The existence of τ implies that the spectrum of α is a subgroup of $\hat{G}$, and as α is faithful, this subgroup is equal to $\hat{G}$ (actually $\Gamma(\alpha) = \hat{G}$). Thus the spectral subspace $\mathscr{A}^\alpha(\gamma_n + \Omega_n)$ contains an x_n with

$$\|x_n\| = 1.$$

Put $y_n = x_n + x_n^*$. By modifying x_n by a phase factor we may assume

$$\|y_n\| = \sup(\mathrm{Spec}(y_n)),$$

and

$$\|y_n\| > 1$$

(where the last relation gives the optimal general estimate). Define a continuous function f_n on $\mathbf{R}$ by

$$f_n(t) = \begin{cases} 0 \ , & t \leqslant 0, \\ t/\|y_n\|, & t \geqslant 0, \end{cases}$$

and set $e_n = f_n(y_n)$. Then $e_n \geqslant 0$ and $\|e_n\| = 1$.

Since $\mathscr{A}$ is separable, it follows from asymptotic abelianness of τ that there is an increasing sequence $\{m_k\}$ of natural numbers such that $\{\tau^{n_k}(x_k)\}$ is a central sequence whenever $n_k \geqslant m_k$ for all k. Define

$$z_1 = \tau^{n_1}(e_1),$$

and define a sequence $n_k \in \mathbf{N}$, $z_k \in \mathscr{A}_+$ inductively by

$$n_k \geqslant m_k,$$

$$\|[\tau^{n_k}(e_k),\ z_{k-1}^{1/2}]\| < k^{-1},$$

$$\mu_k \equiv \|z_{k-1}^{1/2}\tau^{n_k}(e_k)\ z_{k-1}^{1/2}\| > 1 - \frac{1}{k^2},$$

and

$$z_k = \mu_k^{-1} z_{k-1}^{1/2}\tau^{n_k}(e_k) z_{k-1}^{1/2},$$

where the next last relation is possible to obtain since the simplicity and unitality of $\mathscr{A}$ together with asymptotic abelianness of τ implies that $\lim_{n\to\infty}\|\tau^n(x)y\| = \|x\|\,\|y\|$ for all $x, y \in \mathscr{A}$, see [27, Lemma 2.2].

Let $\mathscr{S}$ be the set of states ω on $\mathscr{A}$ such that $\lim_{k\to\infty}\omega(z_k) = 1$. As $0 \leqslant z_k \leqslant 1$, the set $\mathscr{S}$ is clearly a face in the state space of $\mathscr{A}$, and $\mathscr{S}$ is non-empty by the ollowing reasoning:

Since $\mu_k z_k \leqslant z_{k-1}$, if we set $\lambda_k = \prod_{n=k+1}^{\infty} \mu_n\ (>0)$, $\{\lambda_k^{-1} z_k\}$ forms a decreasing sequence.

Let ω_k be a state with $\omega_k(z_k) = 1$, and let ω be a weak*-limit point of $\{\omega_k\}$. Then if $n \geqslant k$,

$$\lambda_n^{-1} \leqslant \lambda_k^{-1}\omega_n(z_k).$$

Since $\lambda_n \to 1$, it follows that $\omega(z_k) \geqslant \lambda_k$, and this in turn implies that $\lim_{k\to\infty}\omega(z_k) = 1$. Thus $\omega \in \mathscr{S}$ and $\mathscr{S}$ is non-empty.

Note next that the inequality $\mu_k z_k \leqslant z_{k-1}$ implies $\lambda_k z_n \leqslant z_k$ whenever $n > k$, and hence $\mathscr{S}$ consists of the states ω on $\mathscr{A}$ with the property that $\omega(z_k) \geqslant \lambda_k$ for all k. It follows that $\mathscr{S}$ is closed in the state space of $\mathscr{A}$, and hence $\mathscr{S}$ is a compact nonempty face, and contains an extreme point ω which is a pure state of $\mathscr{A}$.

Since $\|[\tau^{n_k}(e_k),\ z_{k-1}^{1/2}]\| < k^{-1} \to 0$ we obtain

$$\lim_{k\to\infty} \omega(z_{k-1}^{1/2}[\tau^{n_k}(e_k), z_{k-1}^{1/2}]) = 0,$$

and as $\omega(z_k) \to 1 = \omega(1)$ it follows from Schwarz's inequality that

$$\lim_{k\to\infty} \omega((1 - z_k)\tau^{n_k}(e_k)) = 0.$$

Thus

$$\omega(\tau^{n_k}(e_k)) = \omega(z_{k-1}^{1/2}\tau^{n_k}(e_k)z_{k-1}^{1/2}) - \omega(z_{k-1}^{1/2}[\tau^{n_k}(e_k), z_{k-1}^{1/2}]) +$$

$$+ \omega((1 - z_k)\tau^{n_k}(e_k)) \to 1 - 0 + 0 = 1$$

as $k \to \infty$, where we used $z_{k-1}^{1/2}\tau^{n_k}(e_k)z_{k-1}^{1/2} = \mu_k z_k$. But since $\omega\left(\tau^{n_k}\left(\frac{y_{k+} + y_{k-}}{\|y_k\|}\right)\right) \leqslant 1$ and $e_k = \frac{y_{k+}}{\|y_k\|}$ it follows that $\omega(\tau^{n_k}(y_{k-})) \to 0$ and thus

$$\liminf_{k\to\infty} \omega(\tau^{n_k}(y_k)) = \liminf_{k\to\infty} \|y_k\| \geqslant 1.$$

(Here y_{k+} and y_{k-} are the positive and negative parts of y_k.) Hence

$$\liminf_{k\to\infty} \operatorname{Re}\omega(\tau^{n_k}(x_k)) \geqslant \frac{1}{2}.$$

Now, for each $\gamma \in \hat{G}$, there is a subsequence $\{\gamma_{k'}\}$ of $\{\gamma_k\}$ such that $\gamma_{k'} \to \gamma$, and by going to a subsequence we may assume that

$$\lim_{k'\to\infty} \omega(\tau^{n_{k'}}(x_{k'})) = \lambda$$

exists. Then $\operatorname{Re}\lambda > 1/2$, so $\lambda \neq 0$. But as $\tau^{n_{k'}}(x_{k'})$ is a central sequence and π_ω is irreducible, it follows that

$$\lim_{k'\to\infty} \pi_\omega(\tau^{n_{k'}}(x_{k'})) = \lambda 1,$$

where the limit is in the weak operator topology. On the other hand

$$\lim_{k'\to\infty} \|\alpha_g(x_{k'}) - \langle \gamma, g\rangle x_{k'}\| = 0,$$

see [18, Lemme 2.3.5], and hence

$$\lim_{k'\to\infty} \pi(\tau^{n_{k'}}(x_{k'})) = \int_G^{\oplus} dg\, \lambda \langle \gamma, g\rangle.$$

Since $\tau''^{k'}(x_{k'})$ is a central sequence, it follows that $\int_G^{\oplus} dg\, \langle \gamma, g \rangle$ is contained in the center of π for all $\gamma \in \hat{G}$, and thus the set of diagonal operators $1 \otimes L^\infty(G)$ is contained in the center of π. But as the representations $\pi_\omega \circ \alpha_g$ are all irreducible, the center is contained in the set of diagonal operators. This ends the proof of Theorem 2.1 in the case that G is abelian.

Next assume that G is compact, and let $C_F(G)$ be the dense *-subalgebra of $C(G)$ spanned by matrix elements of irreducible unitary representations of G (equivalently $C_F(G)$ are the elements in $C(G)$ which are G-finite under left (or right) translation). The action τ is strongly topologically transitive in the sense of [8], see [27, Proposition 2.1], and it follows from Observation 2 in the proof of Theorem 2.1 in [8] that any $f \in C_F(G)$ has the form

$$f(g) = \sum_{i=1}^{n} \varphi_i(\alpha_g(x_i)),$$

where $x_i \in \mathscr{A}_F^\alpha$, $\varphi_i \in \mathscr{A}^*$ and n is finite (see also [34, Lemma 1.4]).

Now, if $\gamma \in \hat{G}$ and ξ is a unit vector in the representation Hilbert space of γ, choose a matrix representation of γ such that $\gamma_{11}(g) = (\xi, \gamma(g)\xi)$. By the above remark, there is an $x \in \mathscr{A}_F$ such that

$$x_{\gamma,\xi} \equiv x_1 \equiv \int dg\, d(\gamma)\overline{\gamma_{11}(g)}\alpha_g(x) \neq 0,$$

where $d(\gamma)$ is the dimension of γ, and $[\gamma_{ij}(g)]$ is a matrix representative of $\gamma(g)$. If we put

$$x_j = \int_G dg\, d(\gamma)\overline{\gamma_{j1}(g)}\alpha_g(x_1),$$

for $j = 1, \ldots, d$, then we have

$$\alpha_g(x_j) = \sum_k x_k \gamma_{kj}(g).$$

By replacing x_j by λx_j for a suitable $\lambda \in \mathbf{C}$, we may assume that

$$\|x_1\| = 1,$$

and with $y = x_1 + x_1^*$,

$$\|y\| = \sup(\mathrm{Spec}(y)) \geqslant 1.$$

Because of the relation between x_j, and x_1 we then have

$$\|x_j\| \leqslant d(\gamma)\|x_1\| < d(\gamma), \quad j = 1, \ldots, d(\gamma).$$

Now, let (γ_n, ξ_n) be a sequence with $\gamma_n \in \hat{G}$, ξ_n a unit vector in the representation Hilbert space $\mathscr{H}(\gamma_n)$ of γ_n, such that $\{\gamma_n \mid n \in \mathbf{N}\} = \hat{G}$ and $\{\xi_n \mid \xi_n \in \mathscr{H}(\gamma)\}$ is dense in the unit sphere of $\mathscr{H}(\gamma)$ for each $\gamma \in \hat{G}$. For each $n \in \mathbf{N}$, construct a $x(n) = x_1 = x_{\gamma_n, \xi_n}$ as above, and define

$$x_j(n) = \int_G \mathrm{d}g\, d(\gamma)\overline{\gamma_{j1}(g)}\alpha_g(x(n)).$$

Now, repeat the argument, word for word, in the proof for abelian G, with x_n replaced by the present $x(n)$, to construct a pure state ω and a sequence $\{n_k\}$ such that $\{\tau^{n_k}(x(k))\}$ is a central sequence and

$$\liminf_{k\to\infty} \operatorname{Re}\omega(\tau^{n_k}(x(k))) \geqslant \frac{1}{2}\cdot$$

We may construct the sequence $\{(\gamma_n, \xi_n)\}$ such that each (γ, ξ) occuring in the sequence occurs an infinite number of times, and then use the same $x(n)$, $\gamma_{ij}(g)$ and $x_j(n)$ each time this particular (γ, ξ) occurs. Doing this, assume that (γ, ξ) occurs in the sequence, and let $x(k')$ be the subsequence of those $x(k)$ such that $(\gamma_k, \xi_k) = (\gamma, \xi)$. Going to a subsequence again, still denoted by $x(k')$, it follows from the estimate $\|x_j(k')\| \leqslant d(\gamma)$ that we may assume that the limits

$$\lim_{k'\to\infty} \omega(\tau^{n_{k'}}(x_j)) = \lambda_j,$$

exist, where we used the notation $x_j = x_j(k')$ (where $x_j(k')$ is independent of k'). Then

$$\operatorname{Re}(\lambda_1) \geqslant \frac{1}{2}\cdot$$

But as $\tau^{n_{k'}}(x_j)$ is a central sequence and π_ω is irreducible, it follows that

$$\lim_{k'\to\infty} \pi_\omega(\tau^{n_{k'}}(x_j)) = \lambda_j 1$$

where the limit is in the weak operator topology. On the other hand

$$\alpha_g(\tau^{n_{k'}}(x_j)) = \sum_m \tau^{n_{k'}}(x_m)\gamma_{mj}(g)$$

for all k', and thus we get

$$\lim_{k'\to\infty} \pi(\tau^{n_{k'}}(x_j)) = \int_G^{\oplus} \mathrm{d}g \sum_m \lambda_m \gamma_{mj}(g).$$

Since $\tau^{n_k'}(x_j)$ is a central sequence, it follows that $\int_G^{\oplus} dg \sum_m \lambda_m \gamma_{mj}(g)$ is contained in the center of π. But as π_ω is irreducible, the center of π must be a G-invariant subalgebra $\mathscr{C}$ of the algebra $L^\infty(G)$ of diagonal operators in $\pi = \int_G^{\oplus} dg\, \pi_\omega \circ \alpha_g$, and $\mathscr{C}$ contains the functions $\sum_m \lambda_m \gamma_{mj}(g)$ constructed above. By G-invariance of $\mathscr{C}$, to show that the center is all of $L^\infty(G)$ it suffices to show that these functions separate any $g \in G \setminus \{e\}$ from e. But the function $\sum_m \lambda_m \gamma_{m1}(g)$ has the form $(\eta_\xi, \gamma(g)\xi)$, where $\xi = (1, 0, \ldots, 0)$ and $\eta_\xi = (\lambda_1, \lambda_2, \ldots, \lambda_{d(\gamma)})$. But as $\|\eta_\xi\| = 1$ and $\mathrm{Re}(\eta_\xi, \xi) = = \mathrm{Re}\, \lambda_1 \geqslant 1/2$ and $\|\eta_\xi\| \leqslant \sqrt{1 + (d(\gamma) + 1)d(\gamma)^2} \leqslant (d(\gamma))^{3/2}$, and the set of ξ's we are considering is dense in the unit sphere of $\mathscr{H}(\gamma)$, it follows that for any ξ in the unit sphere of $\mathscr{H}(\gamma)$ there exists an $\eta_\xi \in \mathscr{H}(\gamma)$ such that $(\eta_\xi, \xi) \neq 0$ and $(\eta_\xi, \gamma(\cdot)\xi) \in \in \mathscr{C}$. But if $g \in G$, and g is not separated from e by the continuous functions in $\mathscr{C}$, then

$$(\eta_\xi, \gamma(g)\xi) = (\eta_\xi, \xi)$$

for all $\xi \in \mathscr{H}(\gamma)$. Taking ξ to be an eigenvector for the unitary matrix $\gamma(g)$ with eigenvalue μ we get $\mu(\eta_\xi, \xi) = (\eta_\xi, \xi) \neq 0$ and hence $\mu = 1$. It follows that $\gamma(g) = 1$. But as this holds for all $\gamma \in \hat{G}$, it follows that $g = e$, and hence $\mathscr{C} = L^\infty(G)$, indeed.

EXAMPLE 2.3. As a complement to Theorem 2.1, Theorem 3.15 in [2] and Theorem 6.15 in [1], we will construct a pure state ω on the CAR-algebra $\mathscr{A}$, [17], such that $\omega \circ \alpha_t$ is disjoint from ω for all gauge automorphisms α_t, $t \in \mathbf{T}$, but nevertheless $\int_{\mathbf{T}} dt\, \omega \circ \alpha_t$ is a factor state.

Let $\mathscr{A}_n$ be the 2×2 complex matrix algebra and let $\alpha^{(n)}$ be the action of $\mathbf{T} = \mathbf{R}/\mathbf{Z}$ on $\mathscr{A}_n$ of the form

$$\alpha_t^{(n)} = \mathrm{Ad} \begin{bmatrix} e^{2\pi i \lambda_n t} & 0 \\ 0 & 1 \end{bmatrix},$$

where $\lambda_n \in \mathbf{Z}$. Let $\mathscr{A} = \bigotimes_{n=1}^{\infty} \mathscr{A}_n$ and $\alpha_t = \bigotimes_{n=1}^{\infty} \alpha_t^{(n)}$.

For $\mu \in (0, 1/2]$, define a pure state on $\mathscr{A}_n$ by

$$\mathrm{Tr}\left\{\begin{bmatrix} \mu & \sqrt{\mu(1-\mu)} \\ \sqrt{\mu(1-\mu)} & 1-\mu \end{bmatrix} \cdot \right\},$$

and let ω_μ be the state on $\mathscr{A}$ defined by the infinite tensor product of these states. Then ω_μ is a pure state.

OBSERVATION 2.3.1. *Let* $\lambda_n = 3^n + \varepsilon_n$, *where* $\varepsilon_{2n} = 0$ *and* $\varepsilon_{2n-1} = 1$ *for* $n = 1, 2, \ldots$. *Then* $\omega_\mu \circ \alpha_t$ *is disjoint from* ω_μ *for any non-zero* $t \in \mathbf{T}$, *but*

$$\overline{\omega_\mu} = \int_{\mathbf{T}} dt\, \omega_\mu \circ \alpha_t$$

is a factor state on $\mathscr{A}$. *(More precisely, if* $\mu = 1/2$, $\overline{\omega_\mu}$ *is the unique tracial state, and if* $\mu \in (0, 1/2)$, $\overline{\omega_\mu}$ *is the Powers type* $\mathrm{III}_{\mu/(1-\mu)}$ *factor state.)*

Proof. Note that $\{\lambda_n\}$ is chosen such that

$$\sum_{k=1}^{n} \lambda_k \leqslant (3^{n+1} - 3)/2 + (n + 1)/2 < \lambda_{n+1}.$$

Hence the fixed point algebra $\mathscr{A}^\alpha$ under α is the maximal abelian C^*-subalgebra generated by the diagonal matrices $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in \mathscr{A}_n$. Also, if $P : \mathscr{A} \to \mathscr{A}^\alpha$ is the canonical projection

$$P = \int_{\mathbf{T}} dt\, \alpha_t$$

then

$$P(xy) = P(x)P(y) \quad \text{if } x \in \mathscr{A}_n,\ y \in \mathscr{A}_m,\ n \neq m$$

because of the special choice of $\{\lambda_n\}$. Thus

$$\overline{\omega}_\mu = \overline{\omega}_\mu \circ P = \int_{\mathbf{T}} dt\, \omega_\mu \circ \alpha_t \circ P = \omega_\mu \circ P$$

becomes the infinite tensor product of the states $\mathrm{Tr}\left\{\begin{pmatrix} \mu & 0 \\ 0 & 1-\mu \end{pmatrix} \cdot \right\}$, and this proves that $\overline{\omega_\mu}$ is a factor state of the asserted type, [18].

Note that $\omega_\mu \circ \alpha_t$ is the infinite tensor product of the states

$$\mathrm{Tr}\left\{\begin{pmatrix} \mu & \sqrt{\mu(1-\mu)}\,e^{-2\pi i \lambda_n t} \\ \sqrt{\mu(1-\mu)}\,e^{2\pi i \lambda_n t} & 1-\mu \end{pmatrix} \cdot \right\}$$

with $n = 1, 2, \ldots$. Hence to prove $\omega_\mu \circ \alpha_t \,\delta\, \omega_\mu$ when $t \neq 0$ it suffices to show that

$$\lambda = \limsup_{n \to \infty} |e^{2\pi i \lambda_n t} - 1| > 0,$$

see e.g. [31, Theorem 2.7]. But if $t \in 3^{-k}\mathbf{Z}/\mathbf{Z}$, then

$$\lambda \geqslant |e^{2\pi i t} - 1| > 0.$$

Otherwise

$$\lambda > \limsup_{n\to\infty} |e^{2\pi i 9^n t} - 1| = 2.$$

The observation follows.

PROPOSITION 2.3.2. *Let α be the action of* $\mathbf{T}$ *on* $\mathscr{A} = \bigotimes_{n=1}^{\infty} \mathscr{A}_n$ *with all* $\lambda_n = 1$. *There exists a pure state ω on $\mathscr{A}$ such that $\omega \circ \alpha_t$ is disjoint from ω for $t \neq 0$, but*

$$\bar{\omega} = \int_{\mathbf{T}} dt\, \omega \circ \alpha_t$$

is a factor state on $\mathscr{A}$.

Proof. Let $\lambda_n = 3^n + \varepsilon_n$ be as in Observation 2.3.1, and define

$$k_n = \sum_{j=1}^{n} \lambda_j, \quad \mathscr{B}_n = \bigotimes_{j=k_{n-1}+1}^{k_n} \mathscr{A}_j.$$

Then there is a two-dimensional projection $p_n \in \mathscr{B}_n$ such that $\alpha_t(p_n) = p_n$ for all t, and

$$\alpha_t|p_n\mathscr{B}_n p_n \simeq \mathrm{Ad}\begin{pmatrix} e^{2\pi i\lambda_n t} & 0 \\ 0 & 1 \end{pmatrix}.$$

Let $e_n = p_1 p_2 \ldots p_n$ and define a completely positive map

$$e_n \mathscr{A} e_n \to e_{n+1} \mathscr{A} e_{n+1}$$

by multiplying p_{n+1} from both sides. The inductive limit of this system of C^*-algebras and completely positive maps is naturally isomorphic to

$$\mathscr{B} = \bigotimes_{n=1}^{\infty} p_n \mathscr{B}_n p_n$$

and the completely positive map $E : \mathscr{A} \to \mathscr{B}$ defined by

$$E(x) = \varinjlim e_n x e_n$$

for $x \in \mathscr{A}$, is a surjection. Now, if β is the automorphism group which is called α in Observation 2.3.1, with $\lambda_n = 3^n + \varepsilon_n$, one clearly has

$$E \circ \alpha_t = \beta_t \circ E$$

for $t \in \mathbf{T}$. Let $\omega = \omega_\mu \circ E$, where ω_μ is defined before Observation 2.3.1. We claim that ω satisfies the desired properties.

Let $e = \lim_n \pi_\omega(e_n)$ is the cyclic representation of $\mathscr{A}$ defined by ω. Then $E(x) \to e\pi_\omega(x)e$ defines the cyclic representation of $\mathscr{B}$ associated to ω_μ on the Hilbert space $e\mathscr{H}_\omega$. This remark also applies to $\bar{\omega}$, $\overline{\omega_\mu}$ instead of ω, ω_μ. Proposition 2.3.2 is now a straightforward consequence of Observation 2.3.1.

After this diversion we now return to

Proof of Theorem 1.1. Adopt the hypotheses of Theorem 1.1 and let

$$\pi = \int_G^\oplus dg\, \pi_\omega \circ \alpha_g$$

be a direct integral representation of $\mathscr{A}$ as described in Theorem 2.1. This representation acts on

$$\mathscr{H} = \int_G^\oplus dg\, \mathscr{H}_\omega = \mathscr{H}_\omega \otimes L^2(G)$$

where $\mathscr{H}_\omega$ is the representation Hilbert space of π_ω. Now, as all the representations $\pi_\omega \circ \alpha_g$ are irreducible, and the center of $\mathscr{M} = \pi(\mathscr{A})''$ consists of just the diagonal operators by Theorem 2.1, it follows that

$$\mathscr{M} = \mathscr{L}(\mathscr{H}_\omega) \otimes L^\infty(G)$$

where $\mathscr{L}(\mathscr{H}_\omega)$ is the algebra of all bounded operators on $\mathscr{H}_\omega$. If R is the right regular representation of G on $\mathscr{H}_\omega \otimes L^2(G)$, i.e.

$$R_h(\xi \otimes \varphi)(g) = \xi\varphi(gh)$$

for $\xi \in \mathscr{H}_\omega$, $\varphi \in L^2(G)$, then $\pi(\alpha_h(x)) = R_h\pi(x)R_h^*$ for all $x \in \mathscr{A}$, $h \in G$, thus the representation π is G-covariant, and the action α extends to the action of right translation on $\mathscr{M} = \mathscr{L}(\mathscr{H}_\omega) \otimes L^\infty(G)$. We denote the extended action also by α, and we have in particular

$$\mathscr{M}^\alpha = \mathscr{L}(\mathscr{H}_\omega) \otimes 1.$$

Since the derivation δ is bounded on each of the spectral subspaces $\mathscr{A}^\alpha(K)$, where K is compact, it follows from [23] that $\delta \mid \mathscr{A}^\alpha(K)$ is σ-weakly continuous in the representation π, and δ extends by continuity to $\mathscr{M}^\alpha(K)$. The extended operator, also denoted by δ, on $\mathcal{M}^{\alpha}_{\mathrm{F}} = \bigcup_K \mathcal{M}^{\alpha}(K)$, is still a derivation. (We remark parenthetically that if G is compact and abelian, and $\delta(\mathcal{A}^{\alpha}) \subseteq \mathcal{A}^{\alpha}_{\mathrm{F}}$, then it follows from the techniques of [11] that δ automatically is bounded on each $\mathcal{A}^{\alpha}(K)$, see [4, Theorem 2.3.8].) Note that since α is the action by right translation, we have

$$\mathcal{M}^{\alpha}_{\mathrm{F}} = \mathcal{L}(\mathcal{H}_{\omega}) \otimes L^{\infty}_{\mathrm{F}}(G)$$

where $L^{\infty}_{\mathrm{F}}(G)$ are the functions with compactly supported Fourier transform, and the tensor product is algebraic if G is compact.

The restriction of δ to $\mathcal{M}^{\alpha} = \mathcal{L}(\mathcal{H}_{\omega}) \otimes 1$ is implemented by a $h = -h^* \in \mathcal{L}(\mathcal{H}_{\omega}) \otimes L^{\infty}(G)$, see e.g. [16, Example 3.2.34].

We now discuss the two cases that G is compact and G is abelian separately. Assume first that G is compact. Following [10], [11], if $\gamma \in \hat{G}$ and $[\gamma_{ij}]$ is a particular matrix representative of γ, then define $\mathcal{A}^{\alpha}_n(\gamma)$ as the set of finite sequences $x = (x_{i1}, \ldots, x_{id})^n_{i=1}$ in $\mathcal{A}^d$, where $d = d(\gamma) = \dim(\gamma)$, such that

$$\alpha_g(x) = x(1 \otimes \gamma(g)).$$

Then the linear span of the matrix elements in $\mathcal{A}^{\alpha}_1(\gamma)$ is the spectral subspace $\mathcal{A}^{\alpha}(\gamma) =$ range of the projection $\int_G dg\, d(\gamma)\, \overline{\mathrm{Tr}(\gamma(g))}\alpha_g$, where dg is normalized and Tr is un-normalized. Now, the action α and the derivation δ on $\mathcal{L}(\mathcal{H}_{\omega}) \otimes L^{\infty}(G)$ has obvious extensions $\bar{\alpha}$ and $\bar{\delta}$ to $\mathcal{L}(\mathcal{H}_{\omega}) \otimes L^{\infty}(G) \otimes M_d$, where M_d is the algebra of complex $d \times d$ matrices, and we have $1 \otimes \gamma \in 1 \otimes L^{\infty}(G) \otimes M_d$ and $x \in \mathcal{L}(\mathcal{H}_{\omega}) \otimes L^{\infty}(G) \otimes M_d$ for $x \in \mathcal{M}^{\alpha}_{d(\gamma)}(\gamma)$. But the transformation law of x under α implies that $x(1 \otimes \gamma^*)$ is contained in the fixed point algebra for the right regular representation of G on $\mathcal{L}(\mathcal{H}_{\omega}) \otimes L^{\infty}(G) \otimes M_d$, i.e.

$$x(1 \otimes \gamma^*) \in \mathcal{L}(\mathcal{H}_{\omega}) \otimes 1 \otimes M_d.$$

Hence

$$\bar{\delta}(x(1 \otimes \gamma^*)) = [h \otimes 1_d, x(1 \otimes \gamma^*)]$$

and thus

$$\bar{\delta}(x)(1 \otimes \gamma^*) + x\bar{\delta}(1 \otimes \gamma^*) = [h \otimes 1_d, x](1 \otimes \gamma^*).$$

Multiplying to the right with $1 \otimes \gamma$ we obtain

$$\bar{\delta}(x) = [h \otimes 1_d, x] - x\bar{\delta}(1 \otimes \gamma^*)(1 \otimes \gamma).$$

But

$$\bar{\delta}(1 \otimes \gamma^*)(1 \otimes \gamma) \in 1 \otimes L^\infty(G) \otimes M_d$$

by the following reasoning: We have $\delta(\text{Centre}\,\mathscr{M} \cap D(\delta)) \subseteq \text{Centre}\,\mathscr{M}$ since if $x \in \text{Centre}\,\mathscr{M} \cap D(\delta)$ and $y \in D(\delta)$, then

$$y\delta(x) = \delta(yx) - \delta(y)x = \delta(xy) - x\delta(y) = \delta(x)y.$$

Thus $\delta(1 \otimes L_F^\infty(G)) \subseteq 1 \otimes L^\infty(G)$ and hence $\bar{\delta}(1 \otimes \gamma^*) \in 1 \otimes L^\infty(G) \otimes M_d$.

In the expression for $\bar{\delta}(x)$ above, all the elements are contained in $\mathscr{L}(\mathscr{H}_d) \otimes \otimes L^\infty(G) \otimes M_d$, and can therefore be viewed as function from G into $\mathscr{L}(\mathscr{H}_d) \otimes M_d$. If we consider an $x \in \mathscr{A}^\alpha_{d(\gamma)}(\gamma)$, and evaluate these functions at $g \in G$, we get

$$\alpha_g(\delta(x)) = [h(g) \otimes 1_d, \alpha_g(x)] - \alpha_g(x)(\delta(1 \otimes \gamma^*)(1 \otimes \gamma^*))(g)$$

for almost all $g \in G$, where we have surpressed the π. Since $\mathscr{A}$ and G are separable and thus $\hat{G}$ is countable, there is a $g \in G$ such that the expression above is valid for all $x \in \mathscr{A}^\alpha_{d(\gamma)}(\gamma)$ and all $\gamma \in \hat{G}$. But as $\delta(1 \otimes \gamma^*)(1 \otimes \gamma)(g)$ is a scalar matrix in M_d, it follows that

$$[h(g) \otimes 1_d, \alpha_g(x)] \in \mathscr{A} \otimes M_d$$

for all $x \in \mathscr{A}^\alpha_{d(\gamma)}(\gamma)$ and $\gamma \in \hat{G}$, and hence

$$[h(g), \mathscr{A}^\alpha_F] \subseteq \mathscr{A}.$$

It follows that $\text{ad}(h(g)) = \tilde{\delta}'$ defines a bounded, and thus inner derivation of the simple unital C^*-algebra $\mathscr{A}$, and hence $\tilde{\delta} = \alpha_{-g}\tilde{\delta}'\alpha_g$ is an inner derivation. If we define $\delta_0 = \delta - \tilde{\delta}$, then δ_0 is a derivation from $\mathscr{A}^\alpha_F$ into $\mathscr{A}$ such that $\delta_0 \big| \mathscr{A}^\alpha = 0$, and, furthermore

$$\delta_0(x) = - x(\delta(1 + \gamma^*)(1 \otimes \gamma))(g) = - xR(\gamma)$$

for $x \in \mathscr{A}^\alpha_{d(\gamma)}(\gamma)$, where $R(\gamma)$ is a scalar matrix. To conclude that δ_0 is the generator of a one-parameter subgroup of $\alpha(G)$, one can now either use the argument in the last part of the proof of Theorem 2.5 in [11], or one sees immediately from the above expression for δ_0 that δ_0 commutes with τ, and hence the conclusion follows from [34], Theorem 2.1 or [8], Theorem 3.1. This finishes the proof of Theorem 1.1 in the case that G is compact.

Next, assume that G is locally compact and abelian. As for compact groups, we deduce

$$\delta(x) = [h, x] - x\delta(\gamma)\gamma$$

when $x \in \mathcal{M}^\alpha(\gamma)$, and $\gamma \in \hat{G}$. The map $\hat{G} \to L^\infty(G)$: $\gamma \to -\delta(\bar{\gamma})\gamma = \Phi(\gamma, \cdot)$ is an additive map, i.e. $\Phi(\gamma_1 + \gamma_2, \cdot) = \Phi(\gamma_1, \cdot) + \Phi(\gamma_2, \cdot)$ in $L^\infty(G)$, as a consequence of the derivation property of δ. Furthermore, Φ maps into purely imaginary functions as δ is a *-map. Also, as $\delta \,|\, \mathcal{M}^\alpha(K)$ is σ-weakly continuous when $K \subseteq \hat{G}$ is compact, the map $\gamma \to \Phi(\gamma, \cdot)$ is continuous when $L^\infty(G)$ is equipped with the weak* topology from $L^1(G)$, and as $\Phi(\gamma, \cdot) = -\delta(\bar{\gamma})\gamma$, Φ is uniformly bounded on compacts in $\hat{G}$.

For each pair $\gamma_1, \gamma_2 \in \hat{G}$, we have that

$$\Phi(\gamma_1 + \gamma_2, g) = \Phi(\gamma_1, g) + \Phi(\gamma_2, g)$$

for almost all $g \in G$. Thus, if Γ is a countable, dense subgroup of $\hat{G}$ we have that the relation above holds for all $\gamma_1, \gamma_2 \in \Gamma$ and all g in a subset W of G of full Haar measure. Since Φ is essentially bounded on compacts in $\hat{G}$ and $\hat{G}$ is σ-compact, we may also choose W such that $\gamma \to \Phi(\gamma, g)$ is bounded on compacts for $g \in W$. But then

$$\gamma \in \Gamma \to \Phi(\gamma, g) \in \mathrm{i}\,\mathbf{R}$$

is continuous for $g \in W$ by the following reasoning: If $\gamma_n \in \Gamma$ and $\gamma_n \to 0$, but $\Phi(\gamma_n, g) \nrightarrow 0$, then, as $\Phi(\cdot, g)$ is bounded on compacts we may assume that $\Phi(\gamma_n, g) \to \varepsilon \neq 0$. Since $\gamma_n \to 0$, there is a sequence k_n of integers such that $k_n \to \infty$, but $\{k_n \gamma_n\}$ is contained in a compact subset of $\hat{G}$. Then $\Phi(k_n\gamma_n, g) = k_n\Phi(\gamma_n, g) \to \infty$, and this contradicts that $\Phi(\cdot, g)$ is bounded on compacts. Thus $\gamma \in \Gamma \to \Phi(\gamma, g) \in \mathrm{i}\,\mathbf{R}$ is continuous for $g \in W$. Now, for $g \in W$, let $\Psi(\gamma, g)$ be the function obtained from $\Phi(\cdot, g) \,|\, \Gamma$ by extending the latter function to $\hat{G}$ by continuity. We argue that if $\gamma \in \hat{G}$, then $\Psi(\gamma, g) = \Phi(\gamma, g)$ almost everywhere in G. Let γ_n be a sequence in Γ such that $\gamma_n \to \gamma$. Then $\Psi(\gamma_n, g) \to \Psi(\gamma, g)$ for $g \in W$. Thus $\Psi(\gamma_n, \cdot) \to \Psi(\gamma, \cdot)$ weakly in $L^\infty(G)$. But since $\Psi(\gamma_n, \cdot) = \Phi(\gamma_n, \cdot) \to \Phi(\gamma, \cdot)$ weakly, we have $\Psi(\gamma, \cdot) = \Phi(\gamma, \cdot)$ in $L^\infty(G)$, and hence we may replace Φ by ψ, i.e. we may assume that

$$\Phi(\gamma_1 + \gamma_2, g) = \Phi(\gamma_1, g) + \Phi(\gamma_2, g)$$

pointwise for all $g \in W$, $\gamma_1, \gamma_2 \in \hat{G}$, and hence for all $g \in G$.

Then, for each $g \in G$, $t \in \mathbf{R}$, the map

$$\gamma \in \hat{G} \to \mathrm{e}^{t\Phi(\gamma, g)}$$

is a continuous character, and by Pontryagin's duality theorem there is an element $h(g, t) \in G$ such that

$$\langle \gamma, h(g, t) \rangle = \mathrm{e}^{t\Phi(\gamma, g)}.$$

Then $t \to h(g, t)$ is a continuous one-parameter subgroup of G. Let δ_g be the generator of the corresponding one-parameter group of automorphisms of $\mathcal{M}$ by right translations by $h(g, t)$ and define

$$\delta_0(x) = \delta(x) - [h, x], \quad x \in \mathcal{L}(\mathcal{H}_\omega) \otimes L^\infty_{\mathrm{F}}(G)$$

$$\delta_0'(x)(g) = (\delta_g x)(g), \quad x \in \mathcal{L}(\mathcal{H}_\omega) \otimes L^\infty_{\mathrm{F}}(G),\ g \in G.$$

We will argue that $\delta_0 = \delta_0'$. Firstly, as δ_g is the generator of the group $1 \otimes R_{h(g,t)}$ on $\mathcal{M} = \mathcal{L}(\mathcal{H}_\omega) \otimes L^\infty(G)$, it follows that $\delta_g \,|\mathcal{L}(\mathcal{H}_\omega) = 0$, and thus $\delta_0'|\mathcal{L}(\mathcal{H}_\omega) = 0 = \delta_0|\mathcal{L}(\mathcal{H}_\omega)$. Secondly, if $x \in \mathcal{M}^\alpha(\gamma)$ for $\gamma \in \hat{G}$, then x has the form $x = y \otimes \gamma$, where $y \in \mathcal{L}(\mathcal{H}_\omega)$, and we have

$$\delta_0(x) = -\, x\delta(\bar{\gamma})\gamma = \Phi(\gamma, \cdot)x$$

by the definition of Φ, i.e.

$$\delta_0(x)(g) = \Phi(\gamma, g)x(g) = \Phi(\gamma, g)y\gamma(g) =$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} y\gamma(gh(g, t)) = \delta_g(y \otimes \gamma)(g) = \delta_0'(x)(g)$$

where we used

$$\Phi(\gamma, g) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \mathrm{e}^{t\Phi(\gamma,g)} = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \langle \gamma, h(g, t)\rangle.$$

Thus

$$\delta_0 \,|\, \mathcal{M}^\alpha(\gamma) = \delta_0' |\, \mathcal{M}^\alpha(\gamma)$$

for all $\gamma \in \hat{G}$. But as $\delta_0 \,|\, \mathcal{M}^\alpha(K)$ is σ-weakly continuous for each compact subset $K \subseteq \hat{G}$, and any function in $L^\infty(G)$ with α-spectrum in K can be approximated in the weak*-topology by linear combinations of characters $\gamma \in K'$ where K' is a compact subset of $\hat{G}$ containing K in its interior, the formula

$$\delta_0 \,|\, \mathcal{M}^\alpha(K) = \delta_0' \,|\, \mathcal{M}^\alpha(K)$$

follows once we can show that δ_0' is a well defined derivation such that $\delta_0' \,|\, \mathcal{M}^\alpha(K)$ is σ-weakly continuous for each compact $K \subseteq \hat{G}$. For well-definedness, we have to verify that

$$g \to (\delta_g x)(g)$$

is in $\mathscr{L}(\mathscr{H}_\omega) \otimes L^\infty(G)$ for all $x \in \mathscr{L}(\mathscr{H}_\omega) \otimes L^\infty_{\mathrm{F}}(G)$. It suffices to do this for $x \in 1 \otimes \otimes L^\infty_{\mathrm{F}}(G)$ i.e. we must prove that the δ' defined on $L^\infty_{\mathrm{F}}(G)$ by

$$(\delta' f)(g) = (\delta_g f)(g)$$

has the asserted properties.

Fix a compact set K in $\hat{G}$ and another compact set K' in $\hat{G}$ with K and 0 in its interior. We may view Φ as a mapping from G into $\check{\hat{G}} =$ the space of continuous additive characters from $\hat{G}$ into $\mathrm{i}\,\mathbf{R}$. Equip $\check{\hat{G}}$ with the topology defined by the seminorms

$$\|\varphi\| = \sup_{\gamma \in K'} |\varphi(\gamma)|.$$

Then $\check{\hat{G}}$ is a Banach space. (Actually all these norms are equivalent since K' contains a neighbourhood of 0.)

As mentioned above, any $\varphi \in \check{\hat{G}}$ defines a derivation δ_φ on $L^\infty(G)$, where δ_φ is the generator of the one-parameter subgroup of right translations by $h(\varphi, t)$, where $h(\varphi, t)$ is defined by the requirement

$$\langle \gamma, h(\varphi, t) \rangle = \mathrm{e}^{t\varphi(\gamma)}$$

for all $\gamma \in \hat{G}$. We now have that the map

$$\varphi \in \check{\hat{G}} \to \delta_\varphi \,|\, \mathscr{M}^\alpha(K)$$

is bounded in the sense that

$$\|\delta_\varphi \,|\, L^\alpha_\infty(K)\| \leqslant C \|\varphi\|$$

for a constant C. To see this, note first that

$$\mathrm{Spec}(\delta_\varphi \,|\, L^\alpha_\infty(K)) = \{\varphi(\gamma) \,|\, \gamma \in K\}$$

(as $\gamma \in L_\infty(K)$ when $\gamma \in K$, and $\delta_\varphi(\gamma) = \varphi(\gamma)\gamma$, one obviously has $\supseteq$, but as $L_\infty(K)$ is contained in the σ-weakly closed linear hull of $\gamma \in K + \Omega$ for any neighborhood Ω of 0 in $\hat{G}$, one has $\mathrm{Spec}(\delta_\varphi | L^\alpha_\infty(K)) \subseteq \bigcap_\Omega \{\varphi(\gamma) \,|\, \gamma \in K + \Omega\} = \{\varphi(\gamma) \,|\, \gamma \in K\}$.)

But as $K \subseteq K'$, we have

$$\sup\{|\varphi(\gamma)| \,|\, \gamma \in K\} \leqslant \|\varphi\|$$

and [28, Lemma 2.4] implies that

$$\|\delta_\varphi \,|\, \mathcal{M}^\alpha_\infty(K)\| \leqslant C\|\varphi\|$$

for all $\varphi \in \hat{G}$, and some constant $C > 0$.

Next consider the map

$$\Phi : G \to \check{\hat{G}} \subset C(K').$$

This map is weakly measurable in the sense that if μ is any measure on K', then

$$g \to \int_{K'} \mathrm{d}\mu(\gamma)\Phi(\gamma, g)$$

is measurable by separability of K', and as $C(K')$ is separable it follows from Pettis's theorem, [36, Section V.4], that the map is strongly measurable in the sense that there exists a sequence $\Phi_n : G \to \check{\hat{G}}$ of measurable functions such that $\Phi_n(\cdot, G)$ is a finite subset of $\check{\hat{G}}$ for each n, i.e. for each n

$$\Phi_n(\cdot, g) = \sum_m \chi_m(g)\varphi_m(\gamma)$$

where χ_m is a projection in $L^\infty(G)$, $\varphi_m \in \check{\hat{G}}$, the sum is finite and $\|\Phi_n(\cdot, g) - \Phi(\cdot, g)\| \to 0$ for almost all g. But if δ'_n is the derivation of $L^\infty(G)$ corresponding to Φ_n, then

$$(\delta_n f)(g) = \sum_m \chi_m(g)(\delta_{\varphi_m} f)(g)$$

and this function of g is obviously in $L^\infty(G)$. But

$$\|(\delta_{\Phi_n(\cdot g)} - \delta_{\Phi(\cdot g)}) \,|\, L^\alpha_\infty(K)\| \leqslant C\|\Phi_n(\cdot g) - \Phi(\cdot g)\| \to 0$$

so if $f \in L^\alpha_\infty(K)$, then

$$(\delta_{\Phi_n(\cdot g)} f - \delta_g f)(h) \to 0$$

for almost all $g, h \in G$. But since these functions are equicontinuous as functions of h, we deduce that

$$(\delta_g f)(g) = \lim_{n \to \infty} (\delta_n f)(g)$$

for almost all $g \in G$, and hence $g \to (\delta_g f)(g) \in L^\infty(G)$. Thus δ'_0 is well defined, and then $\delta'_0 \,|\, L^\alpha_\infty(K)$ is σ-weakly continuous, either by [23] or by limit arguments. It follows that $\delta_0 = \delta_0'$, and we have proved that δ has the decomposition

$$\delta(x)(g) = (\delta_g x)(g) + [h(g), x(g)]$$

for $x \in \mathcal{M}_{\mathrm{F}}^{\alpha}$ and almost all $g \in G$. Now it follows from separability of $\mathcal{A}$, that there is a $g \in G$ such that the above relation is true for all $x \in \mathcal{A}_{\mathrm{F}}^{\alpha}$, i.e.

$$\alpha_g(\delta(x)) = \delta_g(\alpha_g x) + [h(g), \alpha_g(x)].$$

It follows that $\mathrm{ad}(h(g))(\mathcal{A}_{\mathrm{F}}^{\alpha}) \subseteq \mathcal{A}$, and thus $\tilde{\delta}' = \mathrm{ad}(h(g))$ defines a bounded, and thus inner derivation of the simple C^*-algebra $\mathcal{A}$. Applying $\alpha_{g^{-1}}$ to the relation above, and setting $\delta_0 = \delta_g$, $\tilde{\delta} = \alpha_{-g}\tilde{\delta}'\alpha_g$, we obtain the desired decomposition

$$\delta = \delta_0 + \tilde{\delta}.$$

Finally, we note that the decomposition is unique both when G is compact and abelian since the existence of the representation π in Theorem 2.1 prevents all the nonzero generators δ_0 of one-parameter subgroups of $\alpha(G)$ from being inner.

REMARK 2.4. In a certain sense, the representation π of Theorem 2.1 is not a good one to prove Theorem 1.1; it would be more convenient to use a G-covariant representation π such that $\pi(\mathcal{A}^{\alpha})' \cap \pi(\mathcal{A})'' = \mathbf{C}\,1$. If such a representation π exists, Theorem 1.1 is an immediate corollary of Theorem 2.5 in [11] in the case that G is compact, at least if $\delta(\mathcal{A}^{\alpha}) \subseteq \mathcal{A}_{\mathrm{F}}^{\alpha}$. However, when G is not compact, examples of quasifree ergodic actions show that one may have $(\mathcal{A}^{\alpha})' \cap \mathcal{A} \neq \mathbf{C}\,1$, so a representation π of the above type does not exist. If G is compact a simple argument using $\mathcal{A}_1^{\alpha}(\gamma)\mathcal{A}_1^{\alpha}(\gamma)^* \subseteq \mathcal{A}^{\alpha}$ (see [10, Theorem 3.2]) shows that $(\mathcal{A}^{\alpha})' \cap \mathcal{A} = \mathbf{C}\,1$, but nevertheless it may be hard to find representations π such that $\pi(\mathcal{A}^{\alpha})' \cap \pi(\mathcal{A})'' = \mathbf{C}\,1$. An example illustrating the problems in $G = \mathbf{T}$ acting as gauge automorphisms on the Cuntz algebra $\mathcal{O}_{\infty}$, and τ a quasi-free automorphism of $\mathcal{O}_{\infty}$ induces by a unitary operator U with absolutely continuous spectrum. Then, by [24, Proposition 5.3], we have $\lim_{N\to\infty} (2N+1)^{-1} \sum_{n=-N}^{N} \tau^n(x) = \omega(x)1$, where the convergence is in norm, and ω is the Fock state. But $\pi(\mathcal{A}^{\alpha})' \cap \pi(\mathcal{A})'' \approx L^{\infty}(N)$ if π is the Fock representation.

3. FREE ACTIONS

THEOREM 3.1. *Let G be a compact, abelian, separable group and α an action of G on a C^*-algebra $\mathcal{A}$ such that $\Gamma(\alpha) = \hat{G}$ and $\mathcal{A}$ is G-prime. It follows that*

1. *There exists a pure state* φ *on the fixed point algebra* $\mathscr{A}^\alpha$ *such that*

$$\|\varphi|\mathscr{A}^\alpha(\gamma_1)\dots\mathscr{A}^\alpha(\gamma_n)\mathscr{A}^\alpha(\gamma_n)^*\dots\mathscr{A}^\alpha(\gamma_1)^*\|=1$$

for all finite sequences $\gamma_1,\dots,\gamma_n$ *in* $\hat{G}$.

2. *If* φ *is a pure state on* $\mathscr{A}^\alpha$ *satisfying property* 1 *for all finite sequences* $\gamma_1,\dots,\gamma_n\in\hat{G}$ *with* $n\leqslant 2$, *and* $(\mathscr{H},\pi,\Omega)$ *is the cyclic representation of* $\mathscr{A}$ *associated to the G-invariant state* $\omega=\varphi\circ P_0$, *where* $P_0=\int_G \mathrm{dg}\,\alpha_g$, *then*

$$[\pi(\mathscr{A}^\alpha(\gamma))\mathscr{H}]=\mathscr{H}$$

for all $\gamma\in\hat{G}$.

Proof. The conditions on α imply that $\mathscr{A}^\alpha$ is prime, [29, Theorem 8.10.4]. It follows by induction on n that all the ideals $\mathscr{A}^\alpha(\gamma_1)\dots\mathscr{A}^\alpha(\gamma_n)\mathscr{A}^\alpha(\gamma_n)^*\dots\mathscr{A}^\alpha(\gamma_1)^*$ in $\mathscr{A}^\alpha$ are non-zero. (If $\mathscr{A}^\alpha(\gamma_2)\dots\mathscr{A}^\alpha(\gamma_2)^*\neq 0$, then $(\mathscr{A}^\alpha(\gamma_1)^*\mathscr{A}^\alpha(\gamma_1))\mathscr{A}^\alpha(\gamma_2)\dots\dots\mathscr{A}^\alpha(\gamma_2)^*(\mathscr{A}^\alpha(\gamma_1)^*\mathscr{A}^\alpha(\gamma_1))\neq 0$ by primeness of $\mathscr{A}^\alpha$.) Since $\hat{G}$ is countable these ideals can be enumerated. Let $\mathscr{I}_n$ be the intersection of the n first ideals in this enumeration. By primeness of $\mathscr{A}^\alpha$, $\mathscr{I}_n\neq 0$ and $\mathscr{I}_n$ is a decreasing sequence of essential ideals. We will inductively construct a decreasing sequence of positive elements $e_n\in\mathscr{I}_n$ such that

$$\|e_n\|=1$$

$$\mathscr{D}_n\equiv\{a\in\mathscr{A}\mid e_nae_n=a\}\neq 0$$

$$e_{n+1}\in\mathscr{D}_n\cap\mathscr{I}_{n+1}.$$

First let e, be any positive element in $\mathscr{I}_1$ such that $\mathscr{D}_1\neq 0$ (possible by spectral theory). When e_n has been constructed, note that $\mathscr{D}_n$ is a hereditary C^*-subalgebra of $\mathscr{A}$, and thus $\mathscr{D}_n\cap\mathscr{I}_{n+1}\neq 0$ since $\mathscr{I}_{n+1}$ is an essential ideal. Thus $\mathscr{D}_n\cap\mathscr{I}_{n+1}$ contains a positive element e'_{n+1} of norm 1, and replacing e'_{n+1} by $e_{n+1}=f(e'_{n+1})$ where

$$f(t)=\begin{cases}2t, & 0\leqslant t\leqslant\frac{1}{2}\\[2ex] 1, & \frac{1}{2}\leqslant t\leqslant 1\end{cases}$$

we have $c_{r+1}\in\mathscr{D}_n\cap\mathscr{I}_{n+1}$ and $\mathscr{D}_{n+1}\neq 0$.

For each n, let φ_n be a state of $\mathscr{A}^\alpha$ such that $\varphi_n(e_n) = 1$, and let φ be a weak*-limit of φ_n as $n \to \infty$. As $\varphi_n(e_n) = 1$ for $m \geqslant n$, it follows that $\varphi(e_n) = 1$ for all n. Thus $\{\varphi \in E_{\mathscr{A}^\alpha} \mid \varphi(e_n) = 1, \ \forall n\}$ is a nonempty compact face in the state space $E_{\mathscr{A}^\alpha}$ of $\mathscr{A}^\alpha$. If φ is an extremal point in this face, φ has the properties in 1..

We now prove 2.. $\mathscr{H}$ carries a representation of G defined by

$$U(g)\pi(x)\Omega = \pi(\alpha_g(x))\Omega, \quad x \in \mathscr{A},$$

and if $\mathscr{H}^U(\gamma)$ is the spectral subspace of $\mathscr{H}$ corresponding to $\gamma \in \hat{G}$, then

$$\mathscr{H}^U(\gamma) = \overline{\mathscr{A}^\alpha(\gamma)\Omega} \quad \text{(suppressing } \pi).$$

Then $(\pi|\mathscr{A}^\alpha)|\mathscr{H}^U(0)$ is the cyclic representation of $\mathscr{A}^\alpha$ corresponding to φ, and this representation is irreducible by purity of φ.

The spaces $\mathscr{H}^U(\gamma)$ are also invariant under $\mathscr{A}^\alpha$, and we next argue that $\mathscr{A}^\alpha$ acts irreducible on each $\mathscr{H}^U(\gamma)$. We denote the extension of $\alpha = \mathrm{Ad}(U)$ to the σ-weak closure $\mathscr{M}$ of $\mathscr{A}$ also by α, and let $\mathscr{M}^\alpha(\gamma)$ denote the corresponding spectral subspaces in $\mathscr{M}$. Then $\mathscr{M}^\alpha(\gamma)$ is the σ-weak closure of $\mathscr{A}^\alpha(\gamma)$ for each $\gamma \in \hat{G}$. We will show that if $\xi_1, \xi_2 \in \mathscr{M}^\alpha(\gamma)\mathscr{H}^U(0)$ and $\|\xi_1\| = \|\xi_2\|$, there then is a $x \in \mathscr{M}^\alpha$ such that $\|x\| = 1$ and $\xi_2 = x\xi_1$. It is then clear that any vector in $\mathscr{H}^U(\gamma)$ is cyclic for $\mathscr{M}^\alpha$, thus $\mathscr{A}^\alpha$ acts irreducibly.

Consider first the case that $\xi_1 = x_1\eta_1$, $\xi_2 = x_2\eta_2$, $x_1, x_2 \in \mathscr{M}^\alpha(\gamma)$, $\eta_1, \eta_2 \in \mathscr{H}^U(0)$. Using polar decomposition of x_1, x_2 it suffices to assume that x_1, x_2 are partial isometries in $\mathscr{M}^\alpha(\gamma)$ with $\eta_i = x_i^* x_i \eta_i$ for $i = 1, 2$, see e.g. Lemma 4.1. But by Kadison's transitivity theorem there is a unitary $u \in \mathscr{M}^\alpha$ such that $u\eta_1 = \eta_2$, [35, Theorem 1.21.16]. If $x = x_2 u x_1^*$ then $x \in \mathscr{M}^\alpha$, $\|x\| \leqslant 1$, and $x\xi_1 = x_2 u x_1^* \xi_1 = x_2 u \eta_1 = x_2\eta_2 = \xi_2$.

Consider next the general case that $\xi_1 = \sum_{i=1}^n x_i\eta_i$, $\xi_2 = \sum_{i=1}^n y_i\psi_i$, $x_i, y_i \in \mathscr{M}^\alpha(\gamma)$ $\eta_i, \psi_i \in \mathscr{H}^U(0)$.

These relations may be written

$$\begin{bmatrix} \xi_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 & \dots & x_n \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_n \end{bmatrix}$$

with a similar relation for ξ_2. Thus, using the argument in the previous paragraph on $\mathscr{M} \otimes M_n$, $\alpha \otimes \iota$, where M_n is the algebra of $n \times n$ matrices, we find an

$x = [x_{ij}] \in \mathcal{M}^\alpha \otimes M_n$ such that $\|x\| \leqslant 1$ and

$$\begin{bmatrix} \xi_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} x_{11} & \dots & x_{1n} \\ x_{21} & \dots & x_{2n} \\ \vdots & & \vdots \\ x_{n1} & \dots & x_{nn} \end{bmatrix} \begin{bmatrix} \xi_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

But then $x_{11} \in \mathcal{M}^\alpha$, $\|x_{11}\| \leqslant 1$ and $\xi_2 = x_{11}\xi_1$.

We have established that $\mathcal{H}^U(\gamma)$ is a minimal invariant subspace for $\mathcal{M}^\alpha$, thus the projection $P(\gamma): \mathcal{H} \mapsto \mathcal{H}^U(\gamma)$ is a minimal projection in the commutant $(\mathcal{M}^\alpha)'$. But if $E(\gamma)$ is the range projection of the ideal $\mathcal{M}^\alpha(\gamma)\mathcal{M}^\alpha(\gamma)^*$ in $\mathcal{M}^\alpha$, then $E(\gamma) \in \mathcal{M}^\alpha \cap (\mathcal{M}^\alpha)'$. But $P(\xi)E(\gamma) \neq 0$ for all $\xi, \gamma \in \hat{G}$ since $\mathcal{A}^\alpha(\gamma)^*\mathcal{A}^\alpha(\xi)\,\Omega \neq 0$ by property 1.. As $P(\xi)$ is minimal in $(\mathcal{M}^\alpha)'$ it follows that $P(\xi) \leqslant E(\gamma)$ for all $\xi \in \hat{G}$, and hence $E(\gamma) = \mathbf{1}$. But

$$E(\gamma) = [\mathcal{A}^\alpha(\gamma)\mathcal{H}],$$

so this establishes 2..

Proof of Theorem 1.2. Theorem 1.2 is a corollary of Theorem 3.1 and [11, Theorem 3.4]. It is essential here that G is a Lie group, i.e. $G = \mathbf{T}^d \times F$ where $\mathbf{T}$ is the torus, $d \in \{0, 1, 2, \dots\}$ and F is a finite abelian group.

4. THE CIRCLE GROUP

In this section we will prove Theorem 1.3. To that end we first need the following general lemma.

LEMMA 4.1. *Let α be an action of a compact abelian group G on a C^*-algebra $\mathcal{A}$. If $x \in \mathcal{A}^\alpha(\gamma)$ for some $\gamma \in \hat{G}$, then x has the form $x = ay$ where $a \in \overline{\mathcal{A}^\alpha(\gamma)\mathcal{A}^\alpha(\gamma)^*}^{\|\,\|}$ and $y \in \mathcal{A}^\alpha(\gamma)$.*

Proof. Assume that $\mathcal{A}$ is faithfully and covariantly represented on a Hilbert space $\mathcal{H}$ (such a representation exists since $P_0 = \int_G \mathrm{d}g\,\alpha_g$ is faithful), let $\mathcal{M} = \mathcal{A}''$ be the weak closure of $\mathcal{A}$, and let α also denote the extension of α to $\mathcal{M}$. If $x = (xx^*)^{1/2}u$ is the polar decomposition of $x \in \mathcal{A}^\alpha(\gamma)$, then $(xx^*)^{1/2} \in \mathcal{A}^\alpha$ and $u \in \mathcal{M}^\alpha(\gamma)$, see [11, Observation 3 in the proof of Theorem 2.5]. Put $a = (xx^*)^{1/4}$, $y \in (xx^*)^{1/4}\,u$. Then $x = ay$, $a \in \overline{\mathcal{A}^\alpha(\gamma)\mathcal{A}^\alpha(\gamma)^*}^{\|\,\|}$, $y \in \mathcal{M}^\alpha(\gamma)$ and if f_m is the real positive function defined by

$$f_m(t) = \begin{cases} t^{-1/4} & \text{for } t \geqslant \dfrac{1}{m} \\[2ex] tm^{5/4} & \text{for } 0 \leqslant t \leqslant \dfrac{1}{m} \end{cases}$$

and $a_m = f_m(xx^*)$, then $a_m \in \mathscr{A}^\alpha$, thus $a_m x \in \mathscr{A}^\alpha(\gamma)$, but $\|a_m x - y\| = \|f_m(xx^*)(xx^*)^{1/2} - (xx^*)^{1/4}\| \leqslant (1/m)^{1/4} \to 0$ as $m \to \infty$, thus $y \in \mathscr{A}^\alpha(\gamma)$.

LEMMA 4.2. *Let α be an action of a compact abelian group G on a C^*-algebra $\mathscr{A}$. Let $\mathscr{I}$ be an α-invariant (closed two-sided) ideal in $\mathscr{A}$, and let $\delta\colon \mathscr{A}^\alpha_{\mathrm{F}} \mapsto \mathscr{A}$ be a derivation. Then $\mathscr{I} \cap \mathscr{A}^\alpha_{\mathrm{F}} = \mathscr{I}^\alpha_{\mathrm{F}}$ is dense in $\mathscr{I}$ and $\delta(\mathscr{I}^\alpha_{\mathrm{F}}) \subseteq \mathscr{I}$, thus δ induces *-derivations of $\mathscr{I}$ and $\mathscr{A}/\mathscr{I}$ in a natural fashion.*

Proof. This result is related to Lemma 1 in [21]. Since trivially $\mathscr{I}^\alpha_{\mathrm{F}} = \mathscr{I} \cap \mathscr{A}^\alpha_{\mathrm{F}}$, the latter *-algebra is dense in $\mathscr{I}$. But by Lemma 4.1, $\mathscr{I}^\alpha \mathscr{I}^\alpha_{\mathrm{F}} = \mathscr{I}^\alpha_{\mathrm{F}}$, thus $\mathscr{I}^\alpha_{\mathrm{F}} \mathscr{I}^\alpha_{\mathrm{F}} = \mathscr{I}^\alpha_{\mathrm{F}}$ and it follows from the derivation property of δ and the ideal property of $\mathscr{I}$ that $\delta(\mathscr{I}^\alpha_{\mathrm{F}}) \subseteq \delta(\mathscr{I}^\alpha_{\mathrm{F}})\mathscr{I}^\alpha_{\mathrm{F}} + \mathscr{I}^\alpha_{\mathrm{F}}\delta(\mathscr{I}^\alpha_{\mathrm{F}}) \subseteq \mathscr{I}$.

LEMMA 4.3. *Adopt the same hypotheses on α, G, $\mathscr{A}$ and δ as in Lemma 4.2, and let $\mathscr{J} \subset \mathscr{I}$ be α-invariant ideals in $\mathscr{A}$. Assume that the derivations induced by δ on $\mathscr{A}/\mathscr{I}$ and on $\mathscr{I}/\mathscr{J}$ are pregenerators. It follows that the derivation induced by δ on $\mathscr{A}/\mathscr{J}$ is a pregenerator.*

Proof. This result is similar to Lemma 2 in [21]. It is clearly enough to assume $\mathscr{J} = 0$. Let $\delta_{\mathscr{I}}$, resp. $\delta^{\mathscr{I}}$ be the derivations induced by δ on $\mathscr{I}$, resp. $\mathscr{A}/\mathscr{I}$ and let $\rho\colon \mathscr{A} \to \mathscr{A}/\mathscr{I}$ be the quotient map. We first show that $(\delta \pm 1)(\mathscr{A}^\alpha_{\mathrm{F}})$ is dense in $\mathscr{A}$. Let $x \in \mathscr{A}$ and let $\varepsilon > 0$. As $\delta^{\mathscr{I}}$ is a pregenerator and $\rho(\mathscr{A}^\alpha_{\mathrm{F}}) = (\mathscr{A}/\mathscr{I})_{\mathrm{F}}$, there are a $y \in \mathscr{A}$ and an $a \in \mathscr{A}^\alpha_{\mathrm{F}}$ such that $\|\rho(x-y)\| < \varepsilon/2$ and $y = (\delta + 1)(a)$. But then there is a $z \in \mathscr{I}$ such that $\|x - y - z\| < \varepsilon/2$ and as $\delta_{\mathscr{I}}$ is a pregenerator there are a $v \in \mathscr{I}$ and a $b \in \mathscr{I}_{\mathrm{F}}$ such that $\|z - v\| < \varepsilon/2$ and $v = (\delta + 1)(b)$. But then $y + v = (\delta + 1)(a + b)$ and $\|x - (y + v)\| < \varepsilon$. Thus $(\delta + 1)(\mathscr{A}^\alpha_{\mathrm{F}})$ is dense in $\mathscr{A}$. Similarly $(\delta - 1)(\mathscr{A}^\alpha_{\mathrm{F}})$ is dense in $\mathscr{A}$.

To show that δ is a pregenerator, it now suffices to show that

$$\|(1 + \lambda\delta)(x)\| \geqslant \|x\|$$

for all $\lambda \in \mathbf{R}$, $x \in \mathscr{A}_{\mathrm{F}}$, [16, Theorem 3.1.10]. Let $\pi_{\mathscr{I}}$, resp $\pi^{\mathscr{I}}$ be a faithful representation of $\mathscr{I}$, resp. $\mathscr{A}/\mathscr{I}$, such that the automorphism group generated by δ on $\mathscr{I}$, resp. $\mathscr{A}/\mathscr{I}$, is covariant. $\pi_{\mathscr{I}}$ has a canonical extension to $\mathscr{A}$, [19, Proposition 2.11.2]. Then $\delta_{\mathscr{I}}$ and $\delta^{\mathscr{I}}$ extend to generators $\bar{\delta}_{\mathscr{I}}$, $\bar{\delta}^{\mathscr{I}}$ respectively of σ-weakly continuous one-parameter groups of automorphisms of $\pi_{\mathscr{I}}(\mathscr{I})''$ and $\pi^{\mathscr{I}}(\mathscr{A}/\mathscr{I})''$. Let $\pi = \pi_{\mathscr{I}} \oplus \pi^{\mathscr{I}}$ and $E = 1 \oplus 0$. Then E is the central projection in $\pi(\mathscr{A})''$ corresponding to the ideal $\mathscr{I}$, thus $E \in D(\bar{\delta}_{\mathscr{I}})$ and $\bar{\delta}_{\mathscr{I}}(E) = 0$. If $x \in \mathscr{A}^\alpha_{\mathrm{F}}$, then

$$\pi(x) = \pi(x)E + \pi(x)(1 - E) = \pi(x)E + \pi^{\mathscr{I}}(\rho(x))(1 - E).$$

Now, if e_τ is an approximate identity for $\mathscr{I}^\alpha$, then e_τ is an approximate identity for $\mathscr{I}$, [9, Lemma 4.1]. But $\delta|\mathscr{I}^\alpha$ is bounded, [32], and since $\delta|\mathscr{I}^\alpha$ is a restriction of $\delta_{\mathscr{I}}$ which is σ-weakly closed, it follows from compactness of the unit sphere of $\pi_{\mathscr{I}}(\mathscr{I})''$

that $\lim_{\tau} \delta(e_\tau) = \bar{\delta}_{\mathscr{J}}(E) = 0$ in the σ-weak topology, since $\lim_{\tau} \pi_{\mathscr{J}}(e_\tau) = E$ in this topology. Now, if $x \in \mathscr{A}_{\mathrm{F}}$, then $xe_\tau \in \mathscr{J}_{\mathrm{F}}$ and

$$(1 + \lambda\delta)(xe_\tau) = (1 + \lambda\delta)(x)e_\tau + \lambda x\delta(e_\tau)$$

by the derivation property of δ. Taking the σ-weak limit $\tau \mapsto \infty$, it follows that $\pi(x)E \in D(\bar{\delta}_{\mathscr{J}})$ and

$$(1 + \lambda\bar{\delta}_{\mathscr{J}})(\pi(x)E) = \pi((1 + \lambda\delta)(x))E.$$

It follows from the generator property of $\bar{\delta}_{\mathscr{J}}$ that

$$\|\pi((1 + \lambda\delta)(x))E\| = \|(1 + \lambda\bar{\delta}_{\mathscr{J}})(\pi(x)E)\| \geqslant \|\pi(x)E\|.$$

On the other hand, as $\delta^{\mathscr{J}}$ extends to a generator we have

$$\|\rho((1 + \lambda\delta)(x))\| > \|\rho(x)\|$$

and as E is the range projection of the kernel $\mathscr{J}$ of ρ we have $\|\rho(y)\| = \|\pi(y)(1 - E)\|$ for all $y \in \mathscr{A}$, and hence

$$\|\pi((1 + \lambda\delta)(x))(1 - E)\| \geqslant \|\pi(x)(1 - E)\|.$$

But as E is a central projection in $\pi(\mathscr{A})''$, it follows that

$$\|\pi(y)\| = \|\pi(y)E\| \vee \|\pi(y)(1 - E)\|$$

for all $y \in \mathscr{A}$, and hence

$$\|(1 + \lambda\delta)(x)\| = \|\pi((1 + \lambda\delta)(x))\| =$$

$$= \|\pi((1 + \lambda\delta)(x))E\| \vee \|\pi((1 + \delta)(x))(1 - E)\| \geqslant$$

$$\geqslant \|\pi(x)E\| \vee \|\pi(x)(1 - E)\| = \|\pi(x)\| = \|x\|.$$

This finishes the proof that δ is a pregenerator.

Next, in order to prove Theorem 3.1, define

$$\mathscr{J} = \bigcap \{\mathscr{P} \in \mathrm{Prim}(\mathscr{A}) \mid \exists\, t \in \mathbb{T}:\ \alpha_t(\mathscr{P}) \neq \mathscr{P}\},$$

where $\mathrm{Prim}(\mathscr{A})$ is the space of primitive ideals in $\mathscr{A}$, and

$$\mathscr{B} = \mathscr{A}/\mathscr{J}.$$

Note that if $\mathscr{A}$ is abelian with spectrum Ω, $\mathscr{J}$ is the ideal corresponding to the interior of the set of α-fixed points in Ω.

LEMMA 4.4. *$\mathscr{J}$ is α-invariant and any ideal of $\mathscr{J}$ is α-invariant.*

Proof. Define $\mathrm{Prim}^\alpha(\mathscr{A}) = \{\mathscr{P} \in \mathrm{Prim}(\mathscr{A}) \mid \alpha_t(\mathscr{P}) = \mathscr{P}$ for all $t \in \mathbf{T}\}$. Then $\mathrm{Prim}^\alpha(\mathscr{A})$ is α-invariant, hence $\mathrm{Prim}(\mathscr{A}) \setminus \mathrm{Prim}^\alpha(\mathscr{A})$ is α-invariant, and then

$$\mathscr{J} = \bigcap \{\mathscr{P} \in \mathrm{Prim}(\mathscr{A}) \setminus \mathrm{Prim}^\alpha(\mathscr{A})\}$$

is α-invariant.

If $\mathscr{I}$ is an ideal in $\mathscr{J}$, then $\mathscr{I}$ is an ideal in $\mathscr{A}$ and

$$\mathscr{I} = \bigcap \{\mathscr{P} \cap \mathscr{J} \mid \mathscr{P} \in \mathrm{Prim}(\mathscr{A}),\ \mathscr{P} \supseteq \mathscr{I},\ \mathscr{J} \setminus \mathscr{P} \neq \emptyset\}$$

see [19, Proposition 2.11.5]. Thus if $\mathscr{I}$ is not α-invariant, there is a $\mathscr{P} \in \mathrm{Prim}(\mathscr{A})$ which is not α-invariant such that $\mathscr{J} \setminus \mathscr{P} \neq \emptyset$. But this contradicts the definition of $\mathscr{J}$.

LEMMA 4.5. *The derivation defined by δ on $\mathscr{J}$ is a pregenerator.*

Proof. By [25, Theorem 2.1], there exists an α-invariant pure state on $\mathscr{J}$ (at this point the separability of $\mathscr{J}$ is used), and also, for any nonzero ideal $\mathscr{I}$ in $\mathscr{J}$ we can find an α-invariant pure state on $\mathscr{I}$. Thus, by Zorn's lemma, there exists a faithful family of α-covariant irreducible representations of $\mathscr{J}$. But by [11], Lemma 2.3 and 3.1, the restriction of δ to each spectral subspace $\mathscr{A}^\alpha(\gamma)$, $\gamma \in \hat{G}$, is bounded. (The extra assumption $E(\gamma)\delta_0(x) = \delta_0(x)$ for $x \in \mathscr{A}^\alpha(\gamma)$ used in Lemma 3.1 of [11] is unnecessary; this follows from our Lemma 4.1 and the other assumptions of Lemma 3.1 in [11].) It now follows from the main theorem in [26] that δ on $\mathscr{J}$ is a pregenerator.

LEMMA 4.6. *The derivation defined by δ on $\mathscr{B} = \mathscr{A}/\mathscr{J}$ is a pregenerator.*

Proof. By definition of $\mathscr{J}$ we have that $\bigcap \{\mathscr{P} \in \mathrm{Prim}(\mathscr{B}) \mid \alpha_t(\mathscr{P}) \neq \mathscr{P}$ for some $t \in \mathbf{T}\} = \{0\}$. Also, by the general assumptions of Theorem 1.3, there exists an $\varepsilon > 0$ such that if $\mathscr{P} \in \mathrm{Prim}(\mathscr{B})$, and $\alpha_t(\mathscr{P}) \neq \mathscr{P}$ for some t, then $\alpha_t(\mathscr{P}) \neq \mathscr{P}$ for $0 < t < \varepsilon$. Thus if $\gamma(\mathscr{P})$ is the period of $\mathscr{P}$ under α, then $\gamma(\mathscr{P}) > \varepsilon$, i.e. $\gamma(\mathscr{P}) \in \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{N}\right\}$ where N is the largest integer with $\frac{1}{N} \geqslant \varepsilon$.

For each non-invariant $\mathscr{P} \in \mathrm{Prim}(\mathscr{B})$, pick a pure state ω on $\mathscr{B}$ with $\ker(\pi_\omega) = \mathscr{P}$, where π_ω is the irreducible representation of $\mathscr{B}$ defined by ω. The representation

$$\pi_{\mathscr{P}} = \int_{\mathbf{T}}^{\oplus} \mathrm{d}t\, \pi_\omega \circ \alpha_t$$

of $\mathscr{A}$ on $\mathscr{H}_\omega \otimes L^2(\mathbf{T})$ is then α-covariant. As π_ω is irreducible, the center of this representation is clearly contained in $1 \otimes L^\infty(\mathbf{T})$, and it is invariant under translation by $\mathbf{T}$, i.e. it is either equal to $L^\infty(\mathbf{T})$ or $\mathbf{C}1$ or the set $\mathscr{N}_p$ of periodic functions in

$L^\infty(\mathbf{T})$ of period $p = \frac{1}{n}$ for some $n = 2, 3, \ldots$. Put $\mathcal{N}_1 = L^\infty(\mathbf{T})$ and $N_0 = \mathbf{C}1$. We now argue that the assumption that $\alpha_t(\mathscr{P}) \neq \mathscr{P}$ for $0 < t < \varepsilon$ if $\mathscr{P} \in \mathrm{Prim}(\mathscr{B})$ is non-invariant implies that the center is $\mathcal{N}_p$ for some $p \geqslant \varepsilon$. Assume ad absurdum that this was not the case, i.e. that the center was $\mathcal{N}_p$ with $p < \varepsilon$. Then, if $p > 0$, the representations

$$\int_0^{\varepsilon'}{}^{\oplus} \mathrm{d}t\, \pi_\omega \circ \alpha_t \quad \text{and} \quad \int_p^{p+\varepsilon'}{}^{\oplus} \mathrm{d}t\, \pi_\omega \circ \alpha_t$$

would be quasi-equivalent for all $\varepsilon' > 0$. But then the kernel of the two representations are equal, i.e.

$$\bigcap_{0 \leqslant t \leqslant \varepsilon'} \alpha_{-t}(\mathscr{P}) = \bigcap_{0 \leqslant t \leqslant \varepsilon} \alpha_{-t-p}(\mathscr{P}).$$

But

$$\mathscr{P} = \Big(\bigcup_{\varepsilon' > 0} \bigcap_{0 \leqslant t \leqslant \varepsilon'} \alpha_{-t}(\mathscr{P})\Big)^-, \quad \alpha_{-p}(\mathscr{P}) = \Big(\bigcup_{\varepsilon' > 0} \bigcap_{0 \leqslant t \leqslant \varepsilon'} \alpha_{-t-p}(\mathscr{P})\Big)^-$$

by the argument in the first part of the proof of Lemma 8.11.7 in [29]. It follows that $\alpha_{-p}(\mathscr{P}) = \mathscr{P}$, which contradicts the ε-assumption. If $p = 0$, the above argument works with p replaced by any positive number.

Now, assume that the center of the representation $\pi_{\mathscr{P}} = \int_{\mathbf{T}}^{\oplus} \mathrm{d}t\, \pi_\omega \circ \alpha_t$ **is** $1 \otimes \mathcal{N}_p$, where $p \geqslant \varepsilon$ by the above reasoning. The function $t \mapsto e^{2\pi i \cdot \frac{t}{p}}$ then defines a unitary operator U in $\mathcal{N}_p$ and thus in $1 \otimes \mathcal{N}_p$, contained in the spectral subspace $\mathcal{M}^\alpha\left(\frac{1}{p}\right)$ for the extension of α to $\mathcal{M} = \pi_{\mathscr{P}}(\mathscr{A})''$. Thus the spectral subspace $\mathcal{M}^\alpha\left(\frac{m}{p}\right)$ contains the unitary operator U^m for $m \in \mathbf{Z}$. It follows that if N is the least common multiple of all natural numbers n such that $\frac{1}{n} \geqslant \varepsilon$, then $\mathcal{M}^\alpha(N)$ contains a unitary operator for all $\mathscr{P} \in \mathrm{Prim}(\mathscr{B})$ such that $\alpha_t(\mathscr{P}) \neq \mathscr{P}$ for some t. Thus, if π is the direct sum of $\pi_{\mathscr{P}}$ for all non-invariant $\mathscr{P} \in \mathrm{Prim}(\mathscr{B})$, then $(\pi(\mathscr{A})'')^\alpha(N)$ contains a unitary operator $U(N)$. But then, using the technique in the proof of Proposition 3.7 in [11], one verifies that $\mathscr{A}_F^\alpha$ consists of analytic elements for δ, with a uniform radius of convergence for the series $\sum_{n \geqslant 0} \frac{t^n}{n!} \|\delta^n(x)\|$, and δ is a pregenerator. Note that the assumption $E(n)\delta(x) = \delta(x)$ for $x \in \mathscr{A}^\alpha(n)$ in [11, Proposition 3.7] is automatically fulfilled because of our Lemma 4.1.

Proof of Theorem 1.3. The theorem is an immediate consequence of Lemmas 4.3, 4.5 and 4.6.

Presently we do not know how to remove the maximum frequency condition of the primitive ideals from Theorem 1.3. One possible method is to use the ideals $\mathscr{I}_n$ in $\mathscr{B}$ defined as

$$\mathscr{I}_n = \bigcap \left\{ \mathscr{P} \in \operatorname{Prim}(\mathscr{B}) \,\middle|\, \alpha_t(\mathscr{P}) \neq \mathscr{P} \text{ for } 0 < t < \frac{1}{n} \right\}.$$

Then the derivation induced by δ on $\mathscr{B}/\mathscr{I}_n$ is a generator by the argument in Lemma 4.6, and thus δ defines a one-parameter group of automorphisms of the projective limit

$$\mathscr{B}/\mathscr{I}_1 \leftarrow \mathscr{B}/\mathscr{I}_2 \leftarrow \mathscr{B}/\mathscr{I}_3 \leftarrow \ldots .$$

Since $\bigcap_n \mathscr{I}_n = 0$, $\mathscr{B}$ is contained as a subalgebra of this projective limit. But it is not immediately clear that the group of automorphisms defined by δ leaves $\mathscr{B}$ invariant, or that this group is strongly continuous. An example showing the kind of problems which may occur is the C^*-algebra $\mathscr{A}$ of continuous functions on the annulus in the plane given by $1 \leqslant r \leqslant 2$ in polar coordinates. Let T be the map given by $T(r, \varphi) = \left(r, \varphi + \dfrac{1}{r-1}\right)$ for $1 < r \leqslant 2$ and let $\mathscr{I}_n$ be the ideal of functions f which are zero for $1 + \dfrac{1}{n} \leqslant r \leqslant 2$. Then T clearly defines an automorphism of the projective limit, but not of the C^*-algebra $\mathscr{A}$ itself since the circle $r = 1$ is rotated by an infinite angle. Nevertheless, Theorem 1.3 is true for abelian C^*-algebras without the ε-assumption:

PROPOSITION 4.7. *Let α be an action of the circle group $\mathbf{T}$ on an abelian C^*-algebra $\mathscr{A}$. It follows that all $\delta \in \operatorname{Der}(\mathscr{A}_F^\alpha, \mathscr{A}_F^\alpha)$ are pregenerators.*

Proof. The ideal $\mathscr{J}$ is now the ideal corresponding to the interior of the set of α-fixed points in $\Omega = \operatorname{Spectrum}(\mathscr{A})$, and hence $\mathscr{J} \subseteq \mathscr{A}^\alpha \subseteq D(\delta)$. As $\mathscr{J}$ is abelian, it follows that $\delta \mid \mathscr{J} = 0$, and Lemma 4.5 follows trivially without assuming that $\mathscr{A}$ is separable. Again putting $\mathscr{B} = \mathscr{A}/\mathscr{J}$, let $\Omega_{\mathscr{B}} = \operatorname{Spectrum}(\mathscr{B})$. For each $n \in \mathbf{N}$, let $\mathscr{B}_n$ consist of the functions $f \in C_0(\Omega_{\mathscr{B}})$ such that f is constant on orbits of frequency larger than n, i.e. of period less than $\dfrac{1}{n}$. Then $\mathscr{B}_n$ is a closed, α-invariant subalgebra of $\mathscr{B}$, and as the orbits in the spectrum of $\mathscr{B}_n$ are either fixed points, or orbits of period longer than or equal to $\dfrac{1}{n}$, it follows from Lemma 4.6 (where separability was not used) that $\delta \mid (\mathscr{B}_n)_F^\alpha$ is a pregenerator on $\mathscr{B}_n$. For this we just have to check that $\delta((\mathscr{B}_n)^{\alpha}_{\mathrm{F}}) \subseteq \mathscr{B}_n$. But since $\delta \mid \mathscr{B}^{\alpha}(K)$ is bounded for each compact K, [4, Theorem 2.3.8], it follows from the argument in the beginning of the proof of Theorem 5.1 in [6] that the $\mathbf{T}$ orbits in $\Omega_{\mathscr{B}}$ are restriction sets, i.e. if $f \in \mathscr{B}^{\alpha}_{\mathrm{F}}$ and f is constant on some orbit, then $\delta f = 0$ on the same orbit. Thus $\delta((\mathscr{B}_n)^{\alpha}_{\mathrm{F}}) \subseteq (\mathscr{B}_n)^{\alpha}_{\mathrm{F}}$. But $\bigcup_n \mathscr{B}_n$ is dense in $\mathscr{B}$, [6, Lemma 2.7], and it follows that δ on $\mathscr{B}$ is a pregenerator. By Lemma 4.3, δ on $\mathscr{A}$ is a pregenerator.

If in the context of Proposition 4.7, δ_0 is the generator of α, it follows from the analysis in [6] and [28] that $\delta = l\delta_0$, where l is a continuous function on $\Omega \setminus \Omega_0$ (Ω_0 is the fixed points of Ω) which is bounded on sets of bounded frequency, so in particular δ is dominated by a constant multiple of δ_0 on each $\mathscr{B}_n$ (or $\mathscr{A}_n$, defined analogously). The function l may blow up as the frequency tends to ∞, however, and this illustrates the problem of removing the condition involving ε in Theorem 3.1. Proposition 4.7 can be extended to cases with weaker assumptions on the range of δ, e.g. δ maps into Lipschitz continuous elements, [33].

5. INVARIANT DERIVATIONS

In this section we will prove Theorem 1.4, so adopt the notation and assumptions of that theorem.

The algebra $\mathscr{A}$ may be faithfully and covariantly represented on a Hilbert space $\mathscr{H}$. Let $\mathscr{M} = \mathscr{A}''$, and let α also denote the extension of α to $\mathscr{M}$. We first prove that δ is σ-weakly closable in this representation, and its σ-weak closure $\bar{\delta}$ generates a σ-weakly continuous one-parameter group of automorphisms of $\mathscr{M}$.

As δ commutes with α, we have $\delta(\mathscr{A}^{\alpha}) \subseteq \mathscr{A}^{\alpha}$, and it follows from Sakai-Kadison's derivation theorem that there exists a $h = -h^* \in \mathscr{M}^{\alpha} = (\mathscr{A}^{\alpha})''$ such that $\delta(x) = hx - xh$ for all $x \in \mathscr{A}^{\alpha}$. Define derivations $\tilde{\delta}$, $\delta_0 : D(\delta) \to \mathscr{M}$ by

$$\tilde{\delta}(x) = hx - xh, \quad \delta_0(x) = \delta(x) - \tilde{\delta}(x)$$

for $x \in D(\delta)$. As $h \in \mathscr{M}^{\alpha}$, it follows that $\tilde{\delta}$ and δ_0 commutes with α. As $\delta_0 \mid \mathscr{A}^{\alpha} = 0$, it follows from the proof of Theorem 5.1 in [13] (see also [4, Lemma 2.7.11]), that for each $\gamma \in \hat{G}$ there exists a (possibly unbounded) closed, skewadjoint operator $L(\gamma)$ affiliated with the abelian von Neumann algebra $(\mathscr{M}^{\alpha} \cap (\mathscr{M}^{\alpha})')E(\gamma)$, where $E(\gamma)$ is the projection onto $[\mathscr{M}^{\alpha}(\gamma)\mathscr{H}]$, such that

$$\delta_0(x) = L(\gamma)x$$

for each $x \in D(\delta) \cap \mathscr{A}^{\alpha}(\gamma)$. Then $e^{tL(\gamma)}$ is a unitary operator in $(\mathscr{M}^{\alpha} \cap (\mathscr{M}^{\alpha})')E(\gamma)$ for each $t \in \mathbf{R}$, and hence

$$\tau_t(x) = e^{tL(\gamma)}x, \quad x \in \mathscr{M}^{\alpha}(\gamma)$$

defines a σ-weakly continuous group of isometries of $\mathcal{M}^{\alpha}(\gamma)$ for each $\gamma \in \hat{G}$. Define a one-parameter group τ_t of $\mathcal{M}^{\alpha}_{\mathrm{F}}$ by

$$\tau_t(\sum_{\gamma} x_{\gamma}) = \sum_{\gamma} e^{tL(\gamma)} x_{\gamma}$$

for $x_{\gamma} \in \mathcal{M}^{\alpha}(\gamma)$.

By [15, Lemma 1.5] there exists a unique *-isomorphism β_{γ} from $(\mathcal{M}^{\alpha} \cap (\mathcal{M}^{\alpha})')E(-\gamma)$ into $(\mathcal{M}^{\alpha} \cap (\mathcal{M}^{\alpha})')E(\gamma)$ such that $\beta_{\gamma}(a)x = xa$ for all $x \in \mathcal{M}^{\alpha}(\gamma)$, $a \in (\mathcal{M}^{\alpha} \cap (\mathcal{M}^{\alpha})')E(-\gamma)$. The isomorphism β_{γ} extends uniquely to unbounded closed operators affiliated to $(\mathcal{M}^{\alpha} \cap (\mathcal{M}^{\alpha})')E(-\gamma)$. It then follows from the derivation property of δ that L satisfies the partial cocycle relation

$$L(\gamma_1 + \gamma_2)E(\gamma_1, \gamma_2) =$$

$$= L(\gamma_1)E(\gamma_1, \gamma_2) + \beta_{\gamma_1}(E(-\gamma_1)L(\gamma_2))E(\gamma_1, \gamma_2)$$

where

$$E(\gamma_1, \gamma_2) = E(\gamma_1)\beta_{\gamma_1}(E(-\gamma_1)E(\gamma_2)),$$

see [11, Lemma 3.1] or [4, Lemma 2.7.5]. But all operators in this expression are affiliated to the abelian von Neumann algebra $\mathcal{M}^{\alpha} \cap (\mathcal{M}^{\alpha})'$, hence after exponentiation we get

$$e^{tL(\gamma_1+\gamma_2)}E(\gamma_1, \gamma_2) = e^{tL(\gamma_1)}\beta_{\gamma_1}(E(-\gamma_1)e^{tL(\gamma_2)})E(\gamma_1, \gamma_2),$$

where we have extended the operators $L(\gamma)$ to $\mathcal{H}$ by defining $L(\gamma)(1 - E(\gamma)) = 0$. Thus

$$e^{tL(\gamma_1+\gamma_2)}x_1x_2 = e^{tL(\gamma_1)}\beta_{\gamma_1}(E(-\gamma_1)\, e^{tL(\gamma_2)})\, x_1x_2 = e^{tL(\gamma_1)}x_1e^{tL(\gamma_2)}x_2$$

for $x_i \in \mathcal{M}^{\alpha}(\gamma_i)$, $i = 1, 2$. This means that

$$\tau_t(x_1x_2) = \tau_t(x_1)\tau_t(x_2)$$

for $x_i \in \mathcal{M}^{\alpha}_{\mathrm{F}}$, and τ is a group of automorphisms of $\mathcal{M}^{\alpha}_{\mathrm{F}}$. Analogously, as δ is a *-map, we get

$$L(-\gamma) = \beta_{-\gamma}(L(\gamma)^*)$$

and then

$$\tau_t(x^*) = \tau_t(x)^*$$

for $x \in \mathcal{M}^{\alpha}_{\mathrm{F}}$. Thus τ is a one-parameter group of *-automorphisms of $\mathcal{M}^{\alpha}_{\mathrm{F}}$. It now follows from [15, Lemma 1.8] that τ is a group of isometries.

The projection $P_0 = \int_G dg\, \alpha_g$ from $\mathcal{M}$ onto $\mathcal{M}^\alpha$ is faithful and normal, and if ω is a normal state on $\mathcal{M}^\alpha$, $\omega \circ P_0$ defines a normal state on $\mathcal{M}$. As $L(0) = 0$ the latter state is τ-invariant, and hence τ is unitarily implemented in the corresponding representation. Since the direct sum of all these representations forms a faithful normal representation of $\mathcal{M}$, and $\mathcal{M}^\alpha_F$ is σ-weakly dense in $\mathcal{M}$, it follows that τ extends by σ-weak continuity to a σ-weakly continuous one-parameter group of *-automorphisms of $\mathcal{M}$. Denote the σ-weak generator ([16, Definition 3.1.5]) of this group by δ_τ. We next argue that δ_τ is the σ-weak closure of δ_0.

We first show that δ_τ extends δ_0. But as δ_0 commutes with α, $D(\delta) \cap \mathcal{A}^\alpha_F$ is a core for δ_0, [4, Lemma 2.5.8]. Hence it suffices to show that $D(\delta) \cap \mathcal{A}^\alpha(\gamma) \subseteq D(\delta_\tau)$ and $\delta_\tau(x) = L(\gamma)x$ for $x \in D(\delta) \cap \mathcal{A}^\alpha(\gamma)$. But this follows from the formula

$$\tau_t(x) - x = e^{tL(\gamma)}x - x = \int_0^t ds\, e^{sL(\gamma)} L(\gamma)x.$$

To show that $\delta_\tau = \bar{\delta}_0$, first note that a simple spectral theoretic argument shows that the set of $x \in D(\delta) \cap \mathcal{A}^\alpha(\gamma)$ such that there exists an $y \in D(\delta) \cap \mathcal{A}^\alpha(\gamma)$ with $x = yy^*x$ is a norm dense subset of $\mathcal{A}^\alpha(\gamma)$. But this subset consists of entire analytic elements for $\bar{\delta}_0$ by the following reasoning: As $y^*x \in \mathcal{A}^\alpha$, we have

$$\delta_0(x) = \delta_0(yy^*x) = \delta_0(y)y^*x.$$

But $\delta_0(y)y^* \in \mathcal{M}^\alpha(\gamma)\mathcal{M}^\alpha(\gamma)^* \subseteq \mathcal{M}^\alpha$, and the module property $\delta_0(ax) = a\delta_0(x)$ for $a \in \mathcal{A}^\alpha$, $x \in \mathcal{A}^\alpha(\gamma)$ implies that $\mathcal{M}^\alpha(\mathcal{A}^\alpha(\gamma) \cap D(\delta)) \subseteq D(\bar{\delta}_0)$ and the module property extends by closure to $a \in \mathcal{M}^\alpha$. Thus $\delta_0(y)y^*x \in D(\bar{\delta}_0)$ and

$$\bar{\delta}_0(\delta_0(y)y^*x) = \delta_0(y)y^*\delta_0(x) = (\delta_0(y)y^*)^2x.$$

Proceeding by induction, we deduce that $x \in D(\bar{\delta}_0^n)$ and

$$\bar{\delta}_0^n(x) = (\delta_0(y)y^*)^n x$$

for $n = 1, 2, \ldots$. This shows that x is entire analytic for $\bar{\delta}_0$.

Since $D(\bar{\delta}_0)$ contains a dense set of entire analytic elements, $D(\bar{\delta}_0)$ is a core for δ_τ [16, Proposition 3.2.58]. Thus $\delta_\tau = \bar{\delta}_0$. But this establishes that $\delta = \delta_0 + \tilde{\delta} = \delta_0 + \mathrm{ad}(h)$ is σ-weakly closable with closure $\bar{\delta} = \delta_\tau + \tilde{\delta}$, $D(\bar{\delta}) = D(\delta_\tau)$, and this closure is a generator of a σ-weakly continuous group of *-automorphisms of $\mathcal{M}$ for any faithful G-covariant representation of $\mathcal{A}$.

Theorem 1.4 now follows from the next lemma.

LEMMA 5.1. *Let G be a locally compact abelian group and α an action of G on a C^*-algebra $\mathcal{A}$. Let δ be a closed derivation on $\mathcal{A}$ commuting with α.*

*If the σ-weak closure $\bar{\delta}$ of δ exists and is a generator of a σ-weakly continuous one-parameter group of *-automorphisms of $\mathscr{A}''$ in each G-covariant representation, then δ is a generator.*

Proof. By the Hille-Yosida theorem, [16, Theorem 3.1.10], we have to show that

$$\|(1 + \lambda\delta)(x)\| \geq \|x\|$$

for all $x \in D(\delta)$ and all real λ, and

$$\overline{(1 + \lambda\delta)(D(\delta))}^{\|\ \|} = \mathscr{A}$$

for all real λ. But the first estimate follows immediately from the generator property of δ in a faithful representation of $\mathscr{A}$, and hence we concentrate on the density of the range of $1 + \lambda\delta$.

Assume ad absurdum that the range of $1 + \lambda\delta$ is not dense. Then there exists a non-zero functional $\eta \in \mathscr{A}^*$ such that

$$\eta((1 + \lambda\delta)(x)) = 0$$

for all $x \in D(\delta)$. But as δ commutes with α, it follows that

$$\eta_f((1 + \lambda\delta)(x)) = 0$$

for all $x \in D(\delta)$, and all $f \in L^1(G)$, where

$$\eta_f = \int_G dg\, f(g)\, \eta \circ \alpha_g .$$

But as $\eta \neq 0$, $\eta_f \neq 0$ for at least one $f \in L^1(G)$ such that $\hat{f}$ has compact support. We will prove in a moment, Lemma 5.2, that this η is a σ-weakly continuous functional in some faithful G-covariant representation of $\mathscr{A}$. But then

$$\eta_f((1 + \lambda\bar{\delta})(D(\bar{\delta}))) = 0$$

where $\bar{\delta}$ denotes the σ-weak closure of δ in this representation. But since $\bar{\delta}$ is a generator, $(1 + \lambda\bar{\delta})(D(\bar{\delta})) = \mathscr{A}''$, and hence $\eta_f = 0$. This contradiction establishes Lemma 5.1 as soon as we have proved:

LEMMA 5.2. *Let G be a locally compact abelian group, and α an action of G on a C*-algebra $\mathscr{A}$. Let $\eta \in \mathscr{A}^*$ be a functional such that the spectrum of η with respect to the action induced by α on $\mathscr{A}^*$ is compact. It follows that there exists a G-covariant representation of $\mathscr{A}$ such that η is σ-weakly continuous in this representation.*

Proof. Let $|\eta|$ be the absolute value of the functional η, [35, Theorem 1.14.4], let $\pi_{|\eta|}$ be the associated cyclic representation on the Hilbert space $\mathscr{H}_{|\eta|}$ and let π be the induced covariant representation of $\mathscr{A}$ on $L^2(\mathscr{H}_{|\eta|}, G)$ defined by $\pi_{|\eta|}$, i.e.

$$(\pi(x)\xi)(g) = \pi_{|\eta|}(\alpha_g(x))\xi(g)$$

$$(U_h\xi)(g) = \xi(g + h)$$

for $x \in \mathscr{A}$, $\xi \in L^2(\mathscr{H}_{|\eta|}, G)$, $g, h \in G$. The polar decomposition theorem, [35, Theorem 1.14.4], implies that η is a vector functional in $\pi_{|\eta|}$, i.e.

$$\eta(x) = (\psi_1, \pi_{|\eta|}(x)\psi_2)$$

for suitable vectors $\psi_1, \psi_2 \in \mathscr{H}_{|\eta|}$. Let now $\varphi \in L^1(G)$ be a function such that the Fourier transform $\hat{\varphi} = 1$ on the α-spectrum of η, and pick functions φ_1, $\varphi_2 \in L^2(G)$ such that the pointwise product $\overline{\varphi}_1 \cdot \varphi_2 = \varphi$. Put

$$\xi_i = \varphi_i \otimes \psi_i \in L^2(G) \otimes \mathscr{H}_{|\eta|} = L^2(\mathscr{H}_{|\eta|}, G).$$

Then

$$(\xi_1, \pi(x)\xi_2) = \int_G \mathrm{d}g\ \varphi(g)\,(\psi_1, \pi_{|\eta|}(\alpha_g(x))\psi_2) =$$

$$= \int_G \mathrm{d}g\ \varphi(g)\,\eta(\alpha_g(x)) = \eta(x).$$

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Note added in proofs. One of the conditions in the hypoteses of Theorem 1.1 is that $\delta|\mathscr{A}^\alpha(K)$ is bounded for each compact subset $K \subseteq \hat{G}$. After this paper was completed, it has been established that this condition is automatically fulfilled if G is compact, and the condition follows from the seemingly weaker condition that $\delta|\mathscr{A}^\alpha(K)$, is bounded for some compact neighbourhood K of 0 in $\hat{G}$ when G is non-compact, [37].

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