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THE DECOMPOSITION PROPERTY FOR C^*-ALGEBRAS

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1. INTRODUCTION

Associated with each bounded linear map $\varphi: B \to A$ between C^*-algebras is a sequence of bounded maps $\varphi \otimes I_n : B \otimes M_n \to A \otimes M_n$, $n \geqslant 1$. The map φ is said to be completely positive if each $\varphi \otimes I_n$ is positive, and is said to be completely bounded if $\sup_n \|\varphi \otimes I_n\| < \infty$. In case φ is completely bounded the completely bounded norm $\|\varphi\|_{\mathrm{cb}}$ of φ is defined to be this supremum. Each completely positive map is completely bounded, $\|\varphi\| = \|\varphi\|_{\mathrm{cb}}$, and if B is unital then these norms are equal to $\|\varphi(1)\|$. If the range or domain is commutative then positive maps are completely positive [1, 3, 30] while if the range is commutative then all maps are completely bounded and $\|\varphi\|_{\mathrm{cb}} = \|\varphi\|$ [27]. A C^*-algebra A is said to possess the decomposition property if each completely bounded self-adjoint map $\varphi : E \to A$, where E is an operator system, has a completely positive decomposition $\varphi = \varphi^+ - \varphi^-$ satisfying $\|\varphi^+ + \varphi^-\| = \|\varphi\|_{\mathrm{cb}}$. In any decomposition it is automatic that $\|\varphi^+ + \varphi^-\| \geqslant \|\varphi\|_{\mathrm{cb}}$ and so the real restriction in the definition is the reverse inequality.

C^*-algebras with the decomposition property are unital [18] and so, unless explicitly mentioned to the contrary, all C^*-algebras in this paper are assumed to be unital and all linear maps are assumed to be self-adjoint.

In the last five years completely bounded maps have begun to play a prominent role in certain areas of C^*-algebra theory. Wittstock's decomposition theorem [35], Haagerup's characterization of injective von Neumann algebras [15], the Effros-Haagerup work on Ext [11] and Paulsen's investigations of operator similarities [24] should especially be mentioned in this connection. A more complete list of recent work will be found in the references. Briefly stated, the object of this paper is to begin a more detailed study of the relationships between the decomposition property, nuclearity and injectivity for general C^*-algebras.

Section 2 builds on Haagerup's characterization of injective von Neumann algebras to obtain a new characterization of nuclear C^*-algebras in terms of the decomposition and approximation of completely bounded maps (Theorem 2.1).

This is used in the final section to characterize nuclear C^*-algebras with the decomposition property, and in particular to extend Wassermann's results [34] on nuclear and injective von Neumann algebras to the C^*-algebra case. The third section considers a weakened form of the decomposition property, equivalent to the decomposition property for von Neumann algebras but distinct for C^*-algebras (Corollary 3.6). The situation is still unclear but some partial results are given. In particular the norms of decompositions in the commutative case are related to topological properties of the maximal ideal space (Theorem 3.5).

The fourth section is concerned with the case of separable operator systems as the domain, and here a characterization of the decomposition property for commutative ranges is possible in terms of substonean spaces (Theorem 4.6). The fifth section considers the relationship between the decomposition property and injectivity in light of their equivalence for von Neumann algebras [15]. It is shown that algebras possessing the decomposition property are AW*-algebras and that on separable Hilbert spaces this implies injectivity under a mild restriction (Proposition 5.2 and Theorem 5.3).

For the basic definitions and results in the theory of completely bounded maps the reader is referred to [18, 23, 24, 27, 28, 32, 35], and to the forthcoming book by Paulsen [25].

2. NUCLEARITY AND THE DECOMPOSITION PROPERTY

In [15] Haagerup obtained a characterization of injective von Neumann algebras in terms of the decomposition of completely bounded maps from ℓ_n. Given the close connection between nuclearity and injectivity [7, 9, 12, 22], it should be possible to transfer this characterization to nuclear C^*-algebras. The analogues of Haagerup's conditions are properties P_1 and P_1' below, but are insufficient in themselves to determine nuclearity. It is necessary to consider in addition two closely related approximation properties. For Theorem 2.1 the restriction that A must be unital will be lifted.

Consider the following properties which a C^*-algebra A may or may not possess:

P_1 : Given $\varphi : \ell_n \to A$ and $\varepsilon > 0$, there exist completely positive maps $\varphi^{\pm} : \ell_n \to A$ such that $\varphi = \varphi^+ - \varphi^-$ and $\|\varphi^+ + \varphi^-\| \leqslant \|\varphi\|_{\mathrm{cb}} + \varepsilon$.

P_1': There exists a constant $c > 0$ dependent only on A such that, given $\varphi : \ell_n \to A$ there exist completely positive maps $\varphi^{\pm} : \ell_n \to A$ satisfying $\varphi = \varphi^+ - \varphi^-$ and $\|\varphi^+ + \varphi^-\| \leqslant c\|\varphi\|_{\mathrm{cb}}$.

P_2 : Given $\varphi : \ell_n \to A^{**}$ there exists a net of maps $\varphi_\lambda : \ell_n \to A$ such that $\|\varphi_\lambda\|_{\mathrm{cb}} \leqslant \|\varphi\|_{\mathrm{cb}}$ and $\lim\limits_{\lambda} \varphi_\lambda = \varphi$ in the point w*-topology.

P_2': There exists a constant $d > 0$ dependent only on A such that, given $\varphi : \ell_n \to A^{**}$, there exists a net of maps $\varphi_\lambda : \ell_n \to A$ satisfying $\|\varphi_\lambda\|_{cb} \leqslant d\|\varphi\|_{cb}$ and $\lim_\lambda \varphi_\lambda = \varphi$ in the point w*-topology.

It is now possible to prove the following characterization of nuclearity.

THEOREM 2.1. *The following statements concerning a C^*-algebra A are equivalent:*

1) *A is nuclear,*
2) *A satisfies P_1 and P_2,*
3) *A satisfies P_1' and P_2'.*

Proof. 1) $\Rightarrow$ 2). Initially assume that A is unital. Fix an integer n and a map $\varphi : \ell_n \to A^{**}$ with $\|\varphi\|_{cb} = 1$. By assumption A is nuclear and so both A^{**} and $A^{**} \otimes M_2$ are injective [7]. On the subspace

$$\left\{\begin{pmatrix} \lambda & a \\ b & \mu \end{pmatrix} : \lambda, \mu \in \mathbf{C},\ a, b \in \ell_n\right\}$$

of $\ell_n \otimes M_2$ define a map ψ into $A^{**} \otimes M_2$ by

$$\psi\begin{pmatrix} \lambda & a \\ b & \mu \end{pmatrix} = \begin{pmatrix} \lambda & \varphi(a) \\ \varphi(b) & \mu \end{pmatrix}.$$

Since $\|\varphi\|_{cb} = 1$ it follows, by work of Paulsen [23], that ψ is a unital completely positive map which thus extends to a completely positive map $\tilde{\psi} : M_n \otimes M_2 \to A^{**} \otimes M_2$ by injectivity. $M_n \otimes M_2$ is isomorphic to the matrix algebra M_{2n} and so there exists a net $\psi_\lambda : M_n \otimes M_2 \to A \otimes M_2$ of unital completely positive maps converging in the point w*-topology to $\tilde{\psi}$ [27]. Now define $\varphi_\lambda : \ell_n \to A$ by

$$\varphi_\lambda(a) = (1 \quad 0)\,\psi_\lambda(a \otimes E_{12})\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad a \in \ell_n.$$

Then $\|\varphi_\lambda\|_{cb} \leqslant \|\psi_\lambda\| = 1$ and $\lim_\lambda \varphi_\lambda = \varphi$ in the point w*-topology. These maps may not be self-adjoint but can be replaced by $(\varphi_\lambda + \varphi_\lambda^*)/2$. Thus A satisfies P_2.

If A is non-unital let B be the C^*-algebra obtained by adjoining a unit to A, and fix an approximate identity (e_α), $0 \leqslant e_\alpha \leqslant 1$ for A. From above there exists a net $\varphi_\lambda : \ell_n \to B$ of completely contractive maps converging to φ in the point w*-topology with respect to B^{**}. It is easy to see that maps of the form $e_\alpha \varphi_\lambda e_\alpha : \ell_n \to A$ are completely contractive and approximate φ in the point w*-topology with respect to A^{**}. Thus in both cases A satisfies P_2.

In order to establish P_1 first consider an arbitrary map $\theta : \ell_n \to A$, write $(p_i)_{i=1}^n$ for the n minimal projections in ℓ_n and denote by a_i the element $\theta(p_i) \in A$.

Then $\|a_i\| \leqslant \|\theta\|$, and so by the spectral theorem there exist positive elements $a_i^{\pm} \in A$ such that $\|a_i^+ + a_i^-\| \leqslant \|\theta\|$ and $a_i = a_i^+ - a_i^-$. Define two maps $\theta^{\pm}: \ell_n \to A$ by

$$\theta^{\pm}\left(\sum_{i=1}^{n} \lambda_i p_i\right) = \sum_{i=1}^{n} \lambda_i a_i^{\pm}.$$

Each map is positive, hence completely positive since ℓ_n is commutative [30], and $\theta = \theta^+ - \theta^-$. In addition

$$\|\theta^+ + \theta^-\| = \|\theta^+(1) + \theta^-(1)\| = \left\| \sum_{i=1}^{n} (a_i^+ + a_i^-) \right\| \leqslant n\|\theta\|.$$

Now consider $\varphi: \ell_n \to A$ and fix $\varepsilon > 0$. Since ℓ_n is finite dimensional and A is nuclear, there exists a matrix algebra M_r and contractive completely positive maps

$$A \xrightarrow{\tau} M_r \xrightarrow{\sigma} A$$

such that

$$\|\varphi - \sigma\tau\varphi\| \leqslant \varepsilon/n$$

[6, 21]. Since M_r is injective $\tau\varphi$ has a decomposition $\tau\varphi = \psi^+ - \psi^-$, $\psi^{\pm}: \ell_n \to M_r$ and

$$\|\psi^+ + \psi^-\| = \|\tau\varphi\|_{\text{cb}} \leqslant \|\varphi\|_{\text{cb}}.$$

Then $\sigma\tau\varphi: \ell_n \to A$ can be decomposed as $\sigma\tau\varphi = \sigma\psi^+ - \sigma\psi^-$ where

$$\|\sigma\psi^+ + \sigma\psi^-\| \leqslant \|\psi^+ + \psi^-\| \leqslant \|\varphi\|_{\text{cb}}.$$

From above there exist completely positive maps $\theta^{\pm}: \ell_n \to A$ such that

$$\varphi - \sigma\tau\varphi = \theta^+ - \theta^- \quad \text{and} \quad \|\theta^+ + \theta^-\| \leqslant n\|\varphi - \sigma\tau\varphi\| \leqslant \varepsilon.$$

Then φ has a decomposition

$$\varphi = (\sigma\psi^+ + \theta^+) - (\sigma\psi^- + \theta^-)$$

and

$$\|\sigma\psi^+ + \theta^+ + \sigma\psi^- + \theta^-\| \leqslant \|\sigma\psi^+ + \sigma\psi^-\| + \|\theta^+ + \theta^-\| \leqslant \|\varphi\|_{\text{cb}} + \varepsilon.$$

Thus A satisfies P_1.

2) $\Rightarrow$ 3). It is clear from the definitions that c may be chosen to be 2 in P_1' and d to be 1 in P_2'.

3) $\Rightarrow$ 1). Consider a map $\varphi: \ell_n \to A^{**}$. From P_2' there exist a constant $d > 0$ and a net $\varphi_\lambda: \ell_n \to A$ converging to φ in the point w*-topology and satisfying

$\|\varphi_\lambda\|_{cb} \leqslant d\|\varphi\|_{cb}$. By P_1' each φ_λ has a completely positive decomposition $\varphi_\lambda = \varphi_\lambda^+ - \varphi_\lambda^-$ where $\varphi_\lambda^\pm : \ell_n \to A$ and $\|\varphi_\lambda^+ + \varphi_\lambda^-\| \leqslant c\|\varphi_\lambda\|_{cb} \leqslant cd\|\varphi\|_{cb}$. Passing to convergent subnets if necessary, there exist completely positive maps $\varphi^\pm : \ell_n \to A^{**}$ such that $\lim_\lambda \varphi_\lambda^\pm = \varphi^\pm$ in the point w*-topology.

Then

$$\|\varphi^+ + \varphi^-\| \leqslant \limsup_\lambda \|\varphi_\lambda^+ + \varphi_\lambda^-\| \leqslant cd\|\varphi\|_{cb}$$

and $\varphi^+ - \varphi^-$ is a decomposition of φ. It follows from Haagerup's characterization [15] that A^{**} is injective, and thus A is nuclear [7]. This completes the proof of the theorem. ▨

REMARK 2.2. With only minor modifications it is possible to replace ℓ_n by M_n throughout the proof. This emphasizes a basic difference between completely bounded and completely positive maps: the approximations of P_2 and P_2' always hold for completely positive maps [27], but may fail for completely bounded maps as will be seen below.

In order to show that each of these properties alone is insufficient to characterize nuclearity it is first necessary to show that P_2 and P_2' are inherited by subalgebras.

LEMMA 2.3. *Let B be a C^*-algebra satisfying P_2 (respectively P_2'). Then any C^*-subalgebra A also satisfies P_2 (respectively P_2' with the same constant d).*

Proof. Suppose that B satisfies P_2' for a certain constant $d > 0$, and let A be a C^*-subalgebra. Consider a map $\varphi : \ell_n \to A^{**}$. In order to construct point w*-approximations to φ the following is sufficient: given $\varepsilon > 0$ and states $\theta_1, \ldots, \theta_r \in A^*$ there should exist a map $\psi : \ell_n \to A$ such that $\|\psi\|_{cb} \leqslant d\|\varphi\|_{cb} + \varepsilon$ and

$$|\theta_i(\psi(p_j)) - \theta_i(\varphi(p_j))| < \varepsilon, \quad 1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant n$$

where $(p_j)_{j=1}^n$ are the minimal projections in ℓ_n. Fix a positive number δ, to be chosen later.

Since B satisfies P_2' there exists a net of maps $\varphi_\lambda : \ell_n \to B$ such that $\|\varphi_\lambda\|_{cb} \leqslant d\|\varphi\|_{cb}$ and $\lim_\lambda \varphi_\lambda = \varphi$ in the point w*-topology with respect to B^{**}. Choose λ_0 such that

$$|\omega_i(\varphi(p_j)) - \omega_i(\varphi_\lambda(p_j))| < \delta, \quad 1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant n$$

for all $\lambda \geqslant \lambda_0$, where each $\omega_i \in B^*$ is a fixed state extension of θ_i.

Consider now the n-fold direct sum $B \oplus \ldots \oplus B$ of n copies of B and view $A \oplus \ldots \oplus A$ as a subalgebra. Let $S = \text{conv}\{(\varphi_\lambda(p_1), \ldots, \varphi_\lambda(p_n)) : \lambda \geqslant \lambda_0\} \subseteq B \oplus \oplus \ldots \oplus B$. If S and $A \oplus \ldots \oplus A$ were a strictly positive distance η apart then by Hahn-Banach separation there would exist linear functionals $\tau_i \in B^*$ such that

$$\sum_{i=1}^{n} \tau_i(a_i) = 0 \quad \text{for all } a_i \in A$$

but

$$\left| \sum_{i=1}^{n} \tau_i \varphi_\lambda(p_i) \right| \geqslant \eta > 0 \quad \text{for } \lambda \geqslant \lambda_0 .$$

It would thus be impossible for the nets $\{\varphi_\lambda(p_i)\}$ to converge to $\{\varphi(p_i)\}$ in the point w*-topology. Consequently there exists a map $\sigma : \ell_n \to B$ which is a convex combination of φ_λ's for $\lambda \geqslant \lambda_0$, and elements $a_1, \ldots, a_n \in A$ such that

$$\|a_i - \sigma(p_i)\| < \delta, \quad 1 \leqslant i \leqslant n.$$

Note that $\|\sigma\|_{\mathrm{cb}} \leqslant d\|\varphi\|_{\mathrm{cb}}$.

Now define a map $\psi : \ell_n \to A$ by

$$\psi\left(\sum_{i=1}^{n} \mu_i p_i \right) = \sum_{i=1}^{n} \mu_i a_i .$$

From above $\|\psi - \sigma\| \leqslant n\delta$, and so, as in the proof of Theorem 2.1 $\psi - \sigma$ has a completely positive decomposition

$$\psi - \sigma = \xi^+ - \xi^-$$

where $\xi^\pm : \ell_n \to B$ and $\|\xi^+ + \xi^-\| \leqslant n^2\delta$. It follows that $\|\psi - \sigma\|_{\mathrm{cb}} \leqslant n^2\delta$ and so

$$\|\psi\|_{\mathrm{cb}} \leqslant d\|\varphi\|_{\mathrm{cb}} + n^2\delta.$$

Finally observe that for $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant n$

$$|\theta_i\psi(p_j) - \theta_i\varphi(p_j)| = |\theta_i\psi(p_j) - \omega_i\sigma(p_j) + \omega_i\sigma(p_j) - \theta_i\varphi(p_j)| \leqslant$$

$$\leqslant |\omega_i(a_j) - \omega_i\sigma(p_j)| + |\omega_i\sigma(p_j) - \omega_i\varphi(p_j)| < \delta + \delta = 2\delta.$$

Now choose δ to be $\min\{\varepsilon/n^2, \varepsilon/2\}$. By dividing ψ by $1 + \varepsilon/d\|\varphi\|_{\mathrm{cb}}$ it is evident that ψ can be chosen to have cb-norm at most $d\|\varphi\|_{\mathrm{cb}}$. Thus A satisfies P_2' for the same constant d.

Satisfying property P_2 is clearly equivalent to satisfying P_2' for every number $d > 1$ and so, from above, P_2 also passes to C^*-subalgebras. ▨

EXAMPLES 2.4. Choi [5] has discovered a separable n-nuclear subalgebra A of the nuclear Cuntz algebra O_2. By Theorem 2.1 O_2 satisfies P_2 and P_2' and so A

satisfies P_2 and P_2' by Lemma 2.3. Since A is non-nuclear it thus cannot satisfy P_1 or P_1', and so P_2 and P_2' are insufficient to characterize nuclearity. Failure to satisfy P_1' may be interpreted in the following manner: for each integer r there is an integer n_r and a completely contractive map $\varphi_r : \ell_{n_r} \to A$ so that for any decomposition $\varphi_r^+ - \varphi_r^-$ of φ_r it must be that $\|\varphi_r^+ + \varphi_r^-\| \geqslant r$. This adds one more pathology to the list for A (see [5]).

Now consider $B(H)$ for an infinite dimensional Hilbert space H. This is an injective von Neumann algebra [1] and so satisfies P_1 and P_1' (Wittstock's theorem [35]). However $B(H)$ is known to be non-nuclear [34], and so by Theorem 2.1 properties P_2 and P_2' must fail in this case. Thus P_1 and P_1' are insufficient to characterize nuclearity.

With a little extra work a separable example can be constructed. There exists a finite dimensional operator system E in $B(H)$ which cannot be embedded in any nuclear C^*-algebra [2]. Define inductively an increasing sequence $B_1 \subseteq B_2 \subseteq \ldots \ldots \subseteq B(H)$ of separable C^*-algebras beginning with $B_1 = C^*(E)$. If $B_1, \ldots, B_r$ have been defined, for each integer n choose a dense sequence of maps $\varphi_{nj} : \ell_n \to B_r$, $j \geqslant 1$. Each φ_{nj} has a completely positive decomposition $\varphi_{nj}^{\pm} : \ell_n \to B(H)$ and so define B_{r+1} to be the C^*-algebra generated by B_r and the ranges of the maps $\{\varphi_{nj}^{\pm}\}_{n,j=1}^{\infty}$. Then let B be the norm closure of $\bigcup_{r=1}^{\infty} B_r$. A short approximation argument shows that B satisfies P_1 and P_1'. However B contains E and thus cannot be nuclear. It follows that P_2 and P_2' fail for the separable C^*-algebra B.

The following result will be useful later.

COROLLARY 2.5. *A nuclear C^*-algebra cannot contain an injective non-nuclear C^*-algebra.*

Proof. Let A be nuclear and let B be an injective subalgebra. By Wittstock's theorem [35] B satisfies P_1 and P_1'. By Theorem 2.1 A satisfies P_2 and P_2', and so the subalgebra B also satisfies P_2 and P_2' by Lemma 2.3. One last application of Theorem 2.1 now shows that B is nuclear. ▨

3. THE WEAK DECOMPOSITION PROPERTY

A C^*-algebra A is said to have the weak decomposition property if, for each C^*-algebra B and each completely bounded map $\varphi: B \to A$, there exist completely positive maps $\varphi^{\pm} : B \to A$ such that $\varphi = \varphi^+ - \varphi^-$. Notice that the norm requirement on $\varphi^+ + \varphi^-$ has been dropped. However there is an automatically imposed condition on this quantity.

LEMMA 3.1. *Let A be a C^*-algebra with the weak decomposition property. Then there exists a constant $c > 0$, dependent only on A, such that any completely bounded map $\varphi : B \to A$ decomposes as $\varphi = \varphi^+ - \varphi^-$ with $\|\varphi^+ + \varphi^-\| \leqslant c\|\varphi\|_{cb}$.*

Proof. If the lemma is false then there exist C^*-algebras B_n and completely bounded maps $\varphi_n : B_n \to A$ such that, for any decompositions,

$$\|\varphi_n\|_{cb} = 1, \quad \|\varphi_n^+ + \varphi_n^-\| \geq 4^n.$$

Let B be the direct sum $\bigoplus_{n=1}^{\infty} B_n$ and define $\varphi : B \to A$ by

$$\varphi(b_n) = \sum_{n=1}^{\infty} 2^{-n}\varphi_n(b_n), \qquad n \geq 1.$$

Then $\|\varphi\|_{cb} \leq 1$. Suppose that φ has a decomposition $\varphi = \varphi^+ - \varphi^-$. If ψ_n denotes the natural embedding of B_n as the n^{th} summand of B then $2^n\varphi^+\psi_n - 2^n\varphi^-\psi_n$ is a decomposition of φ_n, and $\|2^n\varphi^+\psi_n + 2^n\varphi^-\psi_n\| \leq 2^n\|\varphi^+ + \varphi^-\|$. For sufficiently large n this contradicts the lower bound of 4^n. ▨

REMARK 3.2. Haagerup's work shows that, in the case of von Neumann algebras, the weak decomposition property implies injectivity which in turn implies the decomposition property [15]. As will be seen below this is not the case for general C^*-algebras (Corollary 3.6).

The weak decomposition property will now be examined for commutative C^*-algebras. This is the simplest case and also the one for which the most information is available. The following example illustrates some of the possibilities.

EXAMPLE 3.3. Let A be the C^*-subalgebra of $\ell_\infty \oplus \ell_\infty$ defined by

$$A = \{(f, g) : f - g \in c_0\}.$$

Any completely bounded map $\varphi : B \to A$ may be viewed as $\theta \oplus (\theta + \psi)$ where $\theta : B \to \ell_\infty$ and $\psi : B \to c_0$ are completely bounded maps. Since ℓ_∞ is injective and thus has the decomposition property (the weak decomposition property would suffice) there are completely positive maps $\theta^\pm : B \to \ell_\infty$, $\psi^\pm : B \to \ell_\infty$ such that

$$\theta = \theta^+ - \theta^-, \qquad \psi = \psi^+ - \psi^-.$$

Then φ has a completely positive decomposition

$$\varphi = (\theta^+ + \psi^-) \oplus (\theta^+ + \psi^+) - (\theta^- + \psi^-) \oplus (\theta^- + \psi^-).$$

It is easy to check that these maps have their ranges in A. Thus A has the weak decomposition property.

Now define $\psi : c \to c_0$ by

$$\psi(\lambda_n) = (\lambda_n - \lim_n \lambda_n), \quad (\lambda_n) \in c,$$

and define $\varphi : c \oplus c \to A$ by

$$\varphi(a, b) = (\psi(a), \psi(b)), \quad a, b \in c.$$

Then $\|\varphi\|_{cb} = 2$, and from above φ has a completely positive decomposition

$$\varphi = \varphi^+ - \varphi^-, \quad \varphi^\pm : c \oplus c \to A.$$

Write $\varphi^+(0, 1) = (f, g)$ and $\varphi^+(1, 0) = (h, k)$. If $a \in c_0^+$, $\|a\| \leqslant 1$ then

$$(f, g) = \varphi^+(0, 1) \geqslant \varphi^+(0, a) \geqslant \varphi(0, a) = (0, a)$$

and so $g \geqslant 1$. Similarly $h \geqslant 1$. Since $\varphi(0, 1) = \varphi(1, 0) = 0$, it follows that

$$(\varphi^+ + \varphi^-)(1, 1) = (\varphi^+ + \varphi^-)((1, 0) + (0, 1)) =$$

$$= 2\varphi^+(1, 0) + 2\varphi^+(0, 1) = (2f + 2h, 2g + 2k).$$

In ℓ_∞ / c_0, $\dot{f} = \dot{g}$ and so

$$2\dot{f} + 2\dot{h} \geqslant 4.$$

Thus, $\|\varphi^+ + \varphi^-\| \geqslant 4 = 2\|\varphi\|_{cb}$. The computations show that A has the weak decomposition property, but not the decomposition property.

Similar calculations establish the following result, in which commutativity is not assumed.

PROPOSITION 3.4. *Let A have the weak decomposition property and let $J_1, \ldots J_n$ be closed two-sided ideals in A. Define B to be the C^*-subalgebra of $\bigoplus_{i=0}^{n} A$*

$$B = \{(a_0, a_1, \ldots, a_n) : a_0 - a_i \in J_i\}.$$

Then B has the weak decomposition property.

It may be that, starting from an algebra A with the decomposition property, this construction accounts for all algebras B with the weak decomposition property; all known examples are of this type.

Let N denote the positive integers, βN the Stone-Čech compactification of N, and N* the corona set βN — N [33]. If the construction of Proposition 3.4 is specialized to $A = \ell_\infty$ and $J_1 = J_2 = \ldots = J_n = c_0$ then the maximal ideal space of the resulting algebra B is readily seen to be n copies of βN with the corona sets N* identified pointwise. The calculations of Example 3.3 then suggest the following result, which has also been obtained by Huruya [17].

THEOREM 3.5. *Let X be a compact Hausdorff space containing a point x_0 which is in the closures of n disjoint open subsets of X. Then there exists a C^*-algebra B and a completely bounded map $\varphi : B \to C(X)$ such that any decomposition $\varphi^{\pm} : B \to C(X)$ satisfies*

$$\|\varphi^+ + \varphi^-\| \geqslant n\|\varphi\|_{cb}.$$

Proof. Let $U_1, \ldots, U_n$ denote the n disjoint open sets whose closures contain a common point x_0. Let X_1 be the compact Hausdorff space obtained by identifying $(U_1 \cup \ldots \cup U_n)^c$ to the point x_0 and let Z be two copies of X_1 identified at x_0. The second copy of U_i will be denoted V_i, and the point in V_i corresponding to $x \in U_i$ will be written $\tilde{x}$. Finally define Y to be $Z \times \{1, 2, \ldots, n\}$, n topologically disjoint copies of Z.

Now define $\theta : X \to C(Y)^*$ by

$$\theta(x) = \begin{cases} \delta(x, i) - \delta(\tilde{x}, i) & x \in U_i \\ 0 & \text{otherwise}, \end{cases}$$

where $\delta(x, i)$ denotes the point measure at x in the i^{th} copy of Z. It is clear b construction that $\|\theta(x)\| \leqslant 2$ for all $x \in X$ and that θ is continuous in the w*-topology of $C(Y)^*$. It thus corresponds to a completely bounded map $\varphi : C(Y) \to C(X)$ for which $\|\varphi\|_{cb} = 2$. Any decomposition of φ induces a decomposition $\theta = \theta^+ - \theta^-$ of θ, where $\theta^{\pm}$ are w*-continuous selections of measures.

If $x \in U_i$ then clearly

$$\theta^+(x) \geqslant \delta(x, i) \quad \text{and} \quad \theta^-(x) \geqslant \delta(\tilde{x}, i).$$

It follows that

$$\theta^+(x) + \theta^-(x) \geqslant \delta(x, i) + \delta(\tilde{x}, i), \quad x \in U_i.$$

By continuity

$$\theta^+(x_0) + \theta^-(x_0) \geqslant 2\delta(x_0, i)$$

and since this is true for each i, $1 \leqslant i \leqslant n$

$$\theta^+(x_0) + \theta^-(x_0) \geqslant 2 \sum_{i=1}^{n} \delta(x_0, i).$$

Thus $\|\theta^+(x_0) + \theta^-(x_0)\| \geqslant 2n$ and so

$$\|\varphi^+ + \varphi^-\| \geqslant 2n = n\|\varphi\|_{cb}.$$

This completes the proof. ▨

COROLLARY 3.6. *For each integer n there exists a C^*-algebra A_n with the weak decomposition property, a C^*-algebra B_n and a completely bounded map $\varphi_n : B_n \to A_n$ such that any decomposition $\varphi_n^\pm : B_n \to A_n$ satisfies*

$$\|\varphi_n^+ + \varphi_n^-\| \geqslant n\|\varphi_n\|_{cb}.$$

Proof. Let A_n be the subalgebra of $\overset{n}{\underset{0}{\oplus}} \ell_\infty$ defined by

$$A_n = \{(f_0, \ldots, f_n) : f_i \in \ell_\infty,\ \ f_0 - f_i \in c_0\}.$$

This algebra is commutative and so may be represented as $C(X_n)$ where X_n is the maximal ideal space. As noted above, X_n contains n open sets (copies of $\mathbf{N}$) whose closures intersect in the corona set $\mathbf{N}^*$.

By Proposition 3.4 A_n has the weak decomposition property. By Theorem 3.5 there exists a C^*-algebra B_n and a completely bounded map $\varphi_n : B_n \to A_n$ such that any decomposition satisfies

$$\|\varphi_n^+ + \varphi_n^-\| \geqslant n\|\varphi\|_{cb}, \qquad ▨$$

completing the proof.

It is trivial to see that any direct sum of algebras with the decomposition property again has the decomposition property. The above theorem, in conjunction with Lemma 3.1, shows that this is not the case for the weak decomposition property. Consider, for example, $\overset{\infty}{\underset{n=1}{\oplus}} A_n$.

The following characterization of the decomposition property for commutative C^*-algebras has also been obtained recently by Huruya [17].

COROLLARY 3.7. *A commutative C^*-algebra $C(X)$ has the decomposition property if and only if X is stonean.*

Proof. If X is stonean then $C(X)$ is injective and has the decomposition property, by Wittstock's theorem [35].

Conversely if $C(X)$ has the decomposition property then any completely bounded map $\varphi : B \to C(X)$ has a decomposition

$$\varphi = \varphi^+ - \varphi^- \quad \text{with} \quad \|\varphi^+ + \varphi^-\| = \|\varphi\|_{cb}.$$

By Theorem 3.5 no point in X can be in the closures of two or more disjoint open sets. This is just a reformulation of the definition of stonean spaces. ▨

COROLLARY 3.8. *ℓ_∞ has the decomposition property but ℓ_∞/c_0 does not.*

Proof. It is clear that ℓ_∞ has the decomposition property. The maximal ideal space $\mathbf{N}^*$ of ℓ_∞/c_0 contains two disjoint open subsets whose closures are not disjoint [33]. Thus $\mathbf{N}^*$ is not stonean and so ℓ_∞/c_0 does not have the decomposition property, by Corollary 3.7. ▨

4. THE SEPARABLE DECOMPOSITION PROPERTY

As was noted in the previous section, the decomposition property may hold for a C^*-algebra but fail for one of its quotients. The object of this section is to consider a weaker property which, in certain circumstances, does extend from a C^*-algebra to its quotient by a two-sided ideal.

DEFINITION 4.1. A C^*-algebra A is said to have the *separable decomposition property* if every completely bounded map $\varphi : E \to A$ from a separable operator system into A has a decomposition $\varphi = \varphi^+ - \varphi^-$ where $\varphi^\pm : E \to A$ are completely positive, and $\|\varphi^+ + \varphi^-\| = \|\varphi\|_{cb}$.

LEMMA 4.2. *Let A be a nuclear C^*-algebra, and let S be a separable subset. Then there is a separable, unital, nuclear subalgebra of A containing S.*

Proof. An increasing sequence of separable subalgebras of A will be defined in the following way: B_1 is the C^*-algebra generated by S and the identity. Assuming that $B_1, \ldots, B_j$ have been defined, choose a countable dense sequence $\{b_i\}_{i=1}^\infty$ in B_j and for any integer r, choose a matrix algebra M_{n_r} and completely positive maps

$$A \xrightarrow{\tau_r} M_{n_r} \xrightarrow{\sigma_r} A$$

such that

$$\|\sigma_r\tau_r(b_i) - b_i\| < \frac{1}{r}, \quad 1 \leqslant i \leqslant r.$$

Define B_{j+1} to be the C^*-algebra generated by B_j and its images under the maps $\sigma_r\tau_r$. Let B be the norm closure of $\bigcup_{j=1}^{\infty} B_j$. Then B is separable, contains S, and has approximate point norm factorizations through matrix algebras by construction. B is thus nuclear. ▨

PROPOSITION 4.3. *Let A be a nuclear C^*-algebra with a closed two-sided ideal J. If A has the separable decomposition property then so does A/J.*

Proof. Let E be a separable operator system and consider a completely bounded map $\varphi : E \to A/J$. The image of E is a separable subspace of A/J. Since A is nuclear, so too is A/J [7] and so there is a separable, unital, nuclear subalgebra B of A/J containing $\varphi(E)$ by Lemma 4.2. From the Choi-Effros lifting theorem [8] there exists a completely positive unital map $\psi : B \to A$ such that $\rho\psi = \text{id}$ on B where $\rho : A \to A/J$ is the quotient map. Then $\psi\varphi : E \to A$ is completely bounded with $\|\psi\varphi\|_{\text{cb}} = \|\varphi\|_{\text{cb}}$ and so by hypothesis there exist completely positive maps $\theta^{\pm} : E \to A$ such that $\psi\varphi = \theta^+ - \theta^-$ and

$$\|\theta^+ - \theta^-\| = \|\psi\varphi\|_{\text{cb}} = \|\varphi\|_{\text{cb}}.$$

The map φ then has a decomposition $\varphi = \rho\theta^+ - \rho\theta^-$, $\rho\theta^{\pm} : E \to A/J$ and clearly

$$\|\rho\theta^+ + \rho\theta^-\| = \|\varphi\|_{\text{cb}}.$$

Thus A/J has the separable decomposition property. ▨

REMARK 4.4. This proposition applies to ℓ_∞/c_0 since ℓ_∞ is nuclear and being injective, also enjoys the decomposition property.

For the remainder of the section interest will be focused on commutative algebras since in this case a complete characterization of the separable decomposition property can be given. Some preliminary definitions are necessary before the theorem can be stated.

DEFINITION 4.5. 1) A compact Hausdorff space is said to be *substonean* if any two disjoint co-zero sets have disjoint closures.

2) A C^*-algebra A has the *countable Riesz separation property* if, given an increasing sequence $\{a_i\}_{i=1}^{\infty}$ and a decreasing sequence $\{b_j\}_{j=1}^{\infty}$ from A_{sa} satisfying $a_i \leqslant b_j$ for all $i, j \geqslant 1$, there exists $c \in A_{\text{sa}}$ such that

$$a_i \leqslant c \leqslant b_j \quad \text{for} \quad i, j \geqslant 1.$$

3) A C^*-algebra A is *separably injective* if, given a separable operator system F, a subspace E containing the identity, and a completely positive map $\varphi : E \to A$, there exists a completely positive extension $\psi : F \to A$ of φ.

The characterization of the separable decomposition property for commutative C^*-algebras $C(X)$ may now be stated.

THEOREM 4.6. *The following are equivalent for a commutative C^*-algebra $C(X)$:*
1) *$C(X)$ has the separable decomposition property,*
2) *X is substonean,*
3) *$C(X)$ has the countable Riesz separation property,*
4) *$C(X)$ is separably injective.*

Proof. 1) $\Rightarrow$ 2). Assume that $C(X)$ has the separable decomposition property, but that X is not substonean. Then there exist disjoint co-zero sets U and V with at least one point x_0 in the intersection of their closures. Choose two functions $f, g \in C(X)$ such that $0 \leqslant f, g \leqslant 1$ and

$$U = \{x : f(x) \neq 0\}, \quad V = \{x : g(x) \neq 0\}.$$

Define a w*-continuous map $\theta : X \to C[0, 3]^*$ by

$$\theta(x) = \begin{cases} \delta_{1+f(x)} - \delta_{1-f(x)} & x \in U, \\ \delta_{2+g(x)} - \delta_{2-g(x)} & x \in V, \\ 0 & \text{otherwise,} \end{cases}$$

where δ_y denotes the point measure at $y \in [0, 3]$. The map θ arises as the dual of a completely bounded map $\varphi : C[0, 3] \to C(X)$ of cb-norm 2. $C[0, 3]$ is separable and so a decomposition of φ exists which induces a w*-continuous decomposition of θ, by duality. The required contradiction will be obtained by showing that no suitable decomposition of θ exists.

Let $\theta^{\pm} : X \to C[0, 3]^*$ be w*-continuous positive maps satisfying

$$\theta = \theta^+ - \theta^- \quad \text{and} \quad \|\theta^+(x) + \theta^-(x)\| \leqslant 2$$

for all $x \in X$. From the definition of θ it is clear that

$$\theta^+(x) \geqslant \delta_{1+f(x)}, \quad \theta^-(x) \geqslant \delta_{1-f(x)} \quad \text{for } x \in U$$

and

$$\theta^+(x) \geqslant \delta_{2+g(x)}, \quad \theta^-(x) \geqslant \delta_{2-g(x)} \quad \text{for } x \in V.$$

Since x_0 is the closure of both U and V, by w*-continuity

$$\theta^+(x_0) \geqslant \delta_1 + \delta_2 \quad \text{and} \quad \theta^-(x_0) \geqslant \delta_1 + \delta_2.$$

Thus

$$\|\theta^+(x_0) + \theta^-(x_0)\| \geqslant 4$$

and a contradiction has been reached.

2) $\Rightarrow$ 3). Assume that X is substonean, and consider an increasing sequence $\{f_n\}_{n=1}^{\infty}$ and a decreasing sequence $\{g_m\}_{m=1}^{\infty}$ from $C(X)$ such that $f_n \leqslant g_m$ for $m, n \geqslant 1$. Without loss of generality assume that $0 \leqslant f_n, g_m \leqslant 1$ for $m, n \geqslant 1$. For each

real $r \in [0, 1]$ define

$$U_r = \bigcup_{n=1}^{\infty} \{x : f_n(x) > r\}, \quad V_r = \bigcup_{m=1}^{\infty} \{x : g_m(x) < r\}.$$

Observe that $V_0 = \emptyset$ and $U_1 = \emptyset$. In addition each U_r and V_r is a co-zero set, $U_r \cap V_r = \emptyset$ and, if $r < s$ then $V_r \subseteq V_s$ and $U_s \subseteq U_r$. For each diadic rational $r = q/2^p$ in $[0, 1]$ an open co-zero set W_r will be constructed by induction satisfying

a) $\bar{V}_r \subseteq W_r \subseteq \bar{W}_r \subseteq \bar{U}_r^c$,

b) if $r < s$ then $\bar{W}_r \subseteq W_s$.

To begin the induction define $W_0 = \emptyset$ and $W_1 = X$. Now suppose that W_r has been defined for $r = q/2^p$, $q = 0, 1, \ldots 2^p$ so that a) and b) hold. Consider now a rational $r = (2q + 1)/2^{p+1}$. For convenience of notation write $\alpha = q/2^p$, $\beta = (q + 1)/2^p$.

Since U_r and V_r are disjoint, the hypothesis implies the disjointness of $\bar{U}_r$ and $\bar{V}_r$. Consequently $\bar{V}_r \subseteq \bar{U}_r^c$. By construction $\bar{V}_r \subseteq \bar{V}_\beta \subseteq W_\beta$, and so $\bar{V}_r \subseteq W_\beta \cap \bar{U}_r^c$. It follows from the inequality $\alpha < r < \beta$ that $\bar{W}_\alpha \subseteq W_\beta$ and that $\bar{W}_\alpha \subseteq \bar{U}_\alpha^c \subseteq \bar{U}_r^c$. Thus

$$\bar{V}_r \cup \bar{W}_\alpha \subseteq W_\beta \cap \bar{U}_r^c.$$

By Urysohn's lemma there exists a function $f \in C(X)$ such that $0 \leqslant f \leqslant 1$, $f \equiv 1$ on $\bar{V}_r \cup \bar{W}_\alpha$ and vanishes outside $W_\beta \cap \bar{U}_r^c$. Define $W_r = \{x : f(x) > 1/2\}$ and observe that

$$\bar{V}_r \cup \bar{W}_\alpha \subseteq W_r \subseteq \bar{W}_r \subseteq \{x : f(x) \geqslant 1/2\} \subseteq W_\beta \cap \bar{U}_r^c.$$

The induction step is complete.

Now define a function $h(x)$ on X by $h(x) = \inf\{r : x \in W_r\}$. It remains to show that $h \in C(X)$ and that h separates $\{f_n\}_{n=1}^{\infty}$ and $\{g_m\}_{m=1}^{\infty}$. If h were not continuous then there would exist a point x and a net $x_\lambda \to x$ such that $h(x_\lambda) \to L \neq h(x)$. It is clear from the definition of h that $L < h(x)$. Choose two diadic rationals satisfying

$$L < r < s < h(x).$$

Then, for sufficiently large λ, $x_\lambda \in W_r$ and so $x \in \bar{W}_r$. Thus, by construction $x \in W_s$ and so $h(x) \leqslant s$, a contradiction. It follows that h is continuous.

For a given point $x \in X$ suppose that $g_m(x) < h(x)$ for some integer m, and choose a diadic rational r such that $g_m(x) < r < h(x)$. Then $x \notin W_r$ and $x \notin V_r$. From the definition of V_r this implies that $g_m(x) \geqslant r$, a contradiction. Thus $h \leqslant g_m$ for $m \geqslant 1$, and the verification of $f_n \leqslant h$ for $n \geqslant 1$ is similar. It follows that $C(X)$ has the countable Riesz separation property. This part of the proof follows the section on Urysohn's lemma in [10].

3) $\Rightarrow$ 4). Suppose that $C(X)$ has the countable Riesz separation property. Let F be a separable operator system, let E be a separable subspace containing

the identity, and let $\varphi : E \to C(X)$ be a completely positive map. Choose a self-adjoint element $a \in F \setminus E$ and consider the following two sets:

$$S = \{b \in E : b \leqslant a\}, \quad T = \{c \in E : a \leqslant c\}.$$

Since E contains the identity both sets are non-empty and separable. Choose countable dense subsets $\{k_n\}_{n=1}^{\infty}$ and $\{l_m\}_{m=1}^{\infty}$ of $\varphi(S)$ and $\varphi(T)$ respectively and observe that

$$k_n \leqslant l_m \quad \text{for } m, n \geqslant 1$$

by positivity of φ.

The hypothesis may be applied to the sequences $\{f_n = \sup(k_1, \ldots, k_n)\}_{n=1}^{\infty}$ and $\{g_m = \inf(l_1, \ldots, l_m)\}_{m=1}^{\infty}$ to obtain $h \in C(X)$ satisfying

$$k_n \leqslant h \leqslant l_m \quad \text{for} \quad m, n \geqslant 1.$$

From this it is easy to see that φ may be extended to a positive map on $\operatorname{span}\{E, a\}$ by defining $\varphi(a)$ to be h. A countable repetition of this argument leads to a positive extension $\psi : F \to C(X)$. However when the range is commutative a positive map is automatically completely positive [30].

4) $\Rightarrow$ 1). There is a duality between completely positive maps $\varphi : E \to A \otimes M_2$ and completely positive maps $\psi : E \otimes M_2 \to A$ where A is a C^*-algebra and E is an operator system. Given $\varphi : E \to A \otimes M_2$ consider $\varphi \otimes I_2 : E \otimes M_2 \to A \otimes M_4$ which is also completely positive. Let $V = (1, 0, 0, 1)$ and define $\tilde{\varphi} : E \otimes M_2 \to A$ by $\tilde{\varphi} = V\varphi \otimes I_2 V^*$. Then $\tilde{\varphi}$ is completely positive. Conversely given $\psi : E \otimes M_2 \to A$, let $P \in M_4^+$ be the matrix $E_{11} + E_{14} + E_{41} + E_{44}$ and define $\tilde{\psi} : E \to A \otimes M_2$ by

$$\tilde{\psi}(a) = \psi \otimes I_2(a \otimes P).$$

It is not difficult to see that these operations are mutually inverse to one another. This is sufficient for present purposes, but the duality is valid when M_2 is replaced by any matrix algebra M_n (see [29] for the case $A = \mathbf{C}$).

Now consider a completely bounded map $\varphi : E \to C(X)$ where E is separable. Assume without loss of generality that $\|\varphi\|_{cb} = 1$. Let S be the self-adjoint separable subspace of $E \otimes M_2$ of the form

$$\left\{\begin{pmatrix} \lambda & b \\ c & \mu \end{pmatrix} : \lambda, \mu \in \mathbf{C}, \quad b, c \in E\right\}$$

and define $\theta : S \to C(X) \otimes M_2$ by

$$\theta\begin{pmatrix} \lambda & b \\ c & \mu \end{pmatrix} = \begin{pmatrix} \lambda & \varphi(b) \\ \varphi(c) & \mu \end{pmatrix}.$$

Following Paulsen [23], θ is completely positive and so $\tilde{\theta}: S \otimes M_2 \to C(X)$ is completely positive. By hypothesis there is a completely positive extension $\psi : E \otimes \otimes M_4 \to C(X)$ of $\tilde{\theta}$ and then $\tilde{\psi} : E \otimes M_2 \to C(X) \otimes M_2$ is a completely positive extension of θ. The completely positive decomposition of φ is now obtained by following the proof of [23]. This completes the proof of these equivalences. ▨

Two recent preprints by Grove and Pedersen [13, 14] contain many results on substonean spaces in connection with the problem of diagonalizing matrices over $C(X)$. Some of this information can be quickly recovered from Proposition 4.3 and Theorem 4.6. If X is substonean and Y is a closed subspace then Y is the maximal ideal space of a quotient $C(X)/J$. $C(X)/J$ has the separable decomposition property and so Y is substonean. If X is a σ-compact space and $C^b(X)$ is the algebra of bounded continuous functions on X, then it is not difficult to show that $C^b(X)/C_0(X)$ has the countable Riesz separation property. Thus the maximal ideal space is substonean.

Finally if X is an infinite compact metric space, take an infinite convergent sequence of distinct points, $x_n \to x$. Take disjoint open sets W_n containing x_n and set $U = \bigcup_{n \text{ even}} W_n$, $V = \bigcup_{n \text{ odd}} W_n$. These are disjoint open co-zero sets which have a common point x in the closures. Thus X is not substonean. It follows that no infinite dimensional separable $C(X)$ can have the separable decomposition property.

5. INJECTIVITY AND THE DECOMPOSITION PROPERTY

For von Neumann algebras injectivity is equivalent to possessing the decomposition property [15, 35]. The situation for C^*-algebras is less well understood, although the decomposition property is a consequence of injectivity without restriction [35]. The remaining question is whether the decomposition property implies injectivity for C^*-algebras, and this will be investigated below. It should be noted that Hamana [16] has discovered an example of an injective non-nuclear C^*-algebra which fails to be a von Neumann algebra (in any faithful representation). In the commutative case, the algebra of bounded Borel functions on [0, 1] modulo the ideal of functions supported on sets of first category is an old example of this phenomenon, due to Dixmier.

It is now necessary to establish some preliminary results before the main theorem can be stated.

LEMMA 5.1. *Let B, represented as $C(X)$, be a maximal abelian subalgebra of a unital C^*-algebra A. Let x_0 be a non-isolated point in X and let J denote the ideal in B consisting of functions vanishing at x_0. If $\varphi : B \to A$ is a completely positive unital map satisfying $\varphi(j) \geqslant j$ for $j \in J^+$, then φ is the identity.*

Proof. Define $\psi : J \to A$ by

$$\psi(j) = \varphi(j) - j \quad \text{for} \quad j \in J.$$

By hypothesis ψ is positive and thus completely positive since the domain is commutative [30]. Let p denote the identity of J^{**} (corresponding to the characteristic function of $X \setminus \{x_0\}$) and consider $\varphi^{**} : B^{**} \to A^{**}$. Note that φ^{**} is completely positive, unital, and $\|\varphi^{**}\| = 1$. Then $\varphi^{**}(p) = p + \psi^{**}(p)$. Since p is a projection it follows that $\psi^{**}(p)$ is orthogonal to p in order that $\|\varphi^{**}(p)\| \leqslant 1$.

If $j_1, j_2 \in J^+$ and $\|j_1\|, \|j_2\| \leqslant 1$ then $0 \leqslant j_1 \leqslant p$ and $0 \leqslant \psi^{**}(j_2) \leqslant \psi^{**}(p)$. Thus

$$j_1\psi(j_2) = \psi(j_2)j_1 = 0.$$

By linearity this orthogonality condition can be extended to all $j_1, j_2 \in J$. Since J is a maximal ideal in B it is clear that each $\psi(j)$ commutes with every element of B and so $\psi(j) \in B$ for all $j \in J$, by hypothesis.

For each $x \in X \setminus \{x_0\}$ choose $j_x \in J$ such that $j_x(x) = 1$. The orthogonality conditions applied to j_x and $\psi(j)$ imply that

$$\psi(j)\big| X \setminus \{x_0\} = 0 \quad \text{for } j \in J.$$

Since x_0 is not an isolated point in X it follows that $\psi = 0$, and so φ is the identity on J. Thus φ is the identity on B. ▨

Recall that a C^*-algebra A is said to be monotone complete if any increasing bounded net from A has a least upper bound in A. In the case of a commutative algebra $C(X)$ this is equivalent to the topological property that the closure of any open subset of X is again open [33]. Of course, this is equivalent to the definition of X being stonean given earlier.

PROPOSITION 5.2. *Suppose that a C^*-algebra A has the decomposition property. Then every maximal abelian subalgebra of A is monotone complete.*

Proof. Assume the contrary. Then there is a maximal subalgebra $B = C(X)$ for which X is not stonean. Let U be an open subset of X whose closure is not open, and set V equal to the complement of $\bar{U}$. Let $E = \bar{U} \cap \bar{V}$, which is non-empty by construction, and fix an element $x_0 \in E$. Also observe that $X = \bar{U} \cup \bar{V}$. Form a new compact Hausdorff space Y as the disjoint topological union of $\bar{U}$ and $\bar{V}$. E embeds into $\bar{U}$ and $\bar{V}$ as E_1 and E_2 respectively and the point x_0 may be identified with two points $x_1 \in E_1$ and $x_2 \in E_2$. Now define two operator systems in $C(Y)$ by

$$T = \{f \in C(Y) : f|E_1 \text{ and } f|E_2 \text{ differ by a multiple of the identity}\}$$

and

$$S = \{f \in C(Y) : f|E_1 = f|E_2\}.$$

S is clearly completely order isomorphic to $C(X)$ and T may be obtained from S by adjoining the projection $p \in C(Y)$ which is the characteristic function of $\bar{U}$.

Define $\varphi : T \to C(X)$ by

$$\varphi(f) = \begin{cases} f|\bar{U} - f(x_1) & \text{on } \bar{U} \\ f|\bar{V} - f(x_2) & \text{on } \bar{V} \end{cases} \quad (f \in T).$$

The definition of T ensures that $\varphi(f)$ is well defined as an element of $C(X)$. Moreover $\|\varphi\| \leqslant 2$ and, by choosing an element $f \in T$ of unit norm which takes the value 1 at x_1 and the value -1 at some other point of $\bar{U}$, it is clear that $\|\varphi\| = 2$. Additionally the range of φ is commutative and so $\|\varphi\|_{cb} = \|\varphi\| = 2$. By assumption φ has a completely positive decomposition $\varphi = \varphi^+ - \varphi^-$ where $\varphi^\pm : T \to A$ and $\|\varphi^+ + \varphi^-\| = 2$. Now $\varphi(1) = 0$ and so $\varphi^+(1) = \varphi^-(1)$. It follows that

$$2 = \|\varphi^+(1) + \varphi^-(1)\| = 2\|\varphi^+(1)\|$$

and so $\|\varphi^+(1)\| = 1$. If $\varphi^+(1) \neq 1$ then choose a state ω on T and define new completely positive maps $\psi^\pm : T \to A$ by

$$\psi^\pm(f) = \varphi^\pm(f) + \omega(f)(1 - \varphi^\pm(1)).$$

Then $\varphi = \psi^+ - \psi^-$ and

$$\|\psi^+ + \psi^-\| = \|\psi^+(1) + \psi^-(1)\| = 2.$$

Thus it may be assumed from the outset that φ can be decomposed by unital completely positive maps $\varphi^\pm : T \to A$.

Set $J = \{f \in C(X) : f(x_0) = 0\}$ and remember that S is identified with $C(X)$. Then $\varphi : C(X) \to C(X)$ and φ is the identity on J. Then $\varphi^+ : C(X) \to A$ and, for $j \in J^+$,

$$\varphi^+(j) = \varphi(j) + \varphi^-(j) = j + \varphi^-(j) \geqslant j.$$

Clearly x_0 is not an isolated point of X and so Lemma 5.1 may be applied to obtain that φ^+ is the identity on $C(X)$.

Now the C^*-algebra A may be viewed as a subalgebra of a suitably chosen $B(H)$. By the injectivity of $B(H)$ [1], $\varphi^+ : T \to A$ has a completely positive unital extension $\theta : C(Y) \to B(H)$, and it has already been shown that if $f \in S$ then

$$\theta(f^*f) = \varphi^+(f^*f) = \varphi^+(f^*)\varphi^+(f) = \theta(f^*)\theta(f)$$

and

$$\theta(ff^*) = \varphi^+(ff^*) = \varphi^+(f)\varphi^+(f^*) = \theta(f^*)\theta(f).$$

Thus S is contained in the multiplicative domain of θ [4] and so for $f \in S$ and $g \in C(Y)$

$$\theta(fg) = \theta(f)\theta(g) \quad \text{and} \quad \theta(gf) = \theta(g)\theta(f)$$

[4]. Obviously $fg = gf$ and so $\theta(g)$ commutes with each $\theta(f)$ for $f \in S$. However the image of φ^+ acting on S is $C(X)$ and so $\theta(g)$ commutes with every element of B. Since B is a maximal abelian subalgebra of A, it follows in particular that $\varphi^+(p) \in B$.

Take an increasing positive net $(f_\alpha) \in S$ converging pointwise to the characteristic function of U and a decreasing net $(g_\beta) \in S$ converging pointwise to the characteristic function of $\bar{U} \cup E_2$. Then for each α and β

$$\varphi^+(f_\alpha) \leqslant \varphi^+(p) \leqslant \varphi^+(g_\beta)$$

and so

$$\varphi^+(p)|U = 1, \quad \varphi^+(p)|V = 0.$$

This would imply that U and V have disjoint closures, and this contradiction proves the proposition. ▨

The following theorem constitutes the main result of the section.

THEOREM 5.3. *Let A be a C^*-algebra faithfully represented on a separable Hilbert space and assume that the centre of A is a von Neumann algebra. Then A is injective if and only if A has the decomposition property.*

Proof. As observed several times above, injectivity implies the decomposition property [35].

Conversely if A has the decomposition property then A is an AW*-algebra by Proposition 5.2, (see [26]). Since the centre is a von Neumann algebra, A is also a von Neumann algebra [36] and it follows from [15] that A is injective. ▨

The theorem applies, of course, to all simple C^*-algebras and to those with a finite dimensional centre.

6. NUCLEAR AND INJECTIVE C^*-ALGEBRAS

In this section the results of the second and fifth sections will be applied to determine the structure of those nuclear C^*-algebras which possess the decomposition property. As a corollary those C^*-algebras which are both nuclear and injective will be characterized, thus extending the work of Wassermann in the von Neumann algebra case. The methods employed owe much to [34].

It was shown in the previous section that algebras with the decomposition property are AW*-algebras. It is thus necessary to obtain some basic facts for such algebras. Lemma 6.1 is trivial, and Lemmas 6.2 and 6.4 are immediate deductions from Kaplansky's work [19, 20].

LEMMA 6.1. *Let A be a C^*-algebra containing n orthogonal equivalent projections with sum 1. Then there is a C^*-algebra B such that A is isomorphic to $B \otimes M_n$.*

LEMMA 6.2. *Let A be an* AW**-algebra and suppose that there exists an integer n such that any collection of orthogonal equivalent projections has at most n elements. Then there exist integers $n_1 < n_2 < \ldots < n_r \leqslant n$ and stonean spaces $X_1, \ldots, X_r$ such that A is isomorphic to $\bigoplus_{i=1}^{r} C(X_i) \otimes M_{n_i}$.*

LEMMA 6.3. *Let $\{p_i\}_{i=1}^{\infty}$ be a sequence of orthogonal projections in an* AW*-*-algebra A. Then A contains a subalgebra isomorphic to $\bigoplus_{i=1}^{\infty} p_i A p_i$.*

Proof. Let $p = \sum_{i=1}^{\infty} p_i$. By restricting to the AW*-algebra pAp it may be assumed from the outset that $\sum_{i=1}^{\infty} p_i = 1$.

If each p_i were a central projection then the result is true [19, Lemma 2.5]. In general consider $B = \{x : xp_i = p_i x \text{ for all } i \geqslant 1\}$. This is an AW*-algebra [19], contains each projection p_i, and it is clear from the definition that each p_i is central in B. Thus B is isomorphic to $\bigoplus_{i=1}^{\infty} Bp_i$ [19], and since B contains $p_i A p_i$ it follows that B is isomorphic to $\bigoplus_{i=1}^{\infty} p_i A p_i$. ▨

Recall that an AW*-algebra has a unique decomposition as a direct sum of type I, type II and type III AW*-algebras, exactly as in the von Neumann algebra case [19]. Let M_∞ denote the von Neumann algebra $\bigoplus_{n=1}^{\infty} M_n$.

LEMMA 6.4. *Let A be a type* I AW**-algebra. Then either A contains a subalgebra isomorphic to M_∞ or A is isomorphic to a finite direct sum $\bigoplus_{i=1}^{r} C(X_i) \otimes M_{n_i}$ where each X_i is stonean.*

LEMMA 6.5. *Let A be an* AW**-algebra containing no abelian projections. Then A contains a subalgebra isomorphic to M_∞.*

Proof. From [31, p. 302] any projection $p \in A$ may be decomposed as $p = p_1 + p_2$ where $p_1 \sim p_2$ (the proof given for von Neumann algebras is also valid for AW*-algebras). The following procedure can clearly be continued indefinitely: Write $1 = p_1 + p_2$ and define p_{11} to be p_1. Now write $p_2 = p_{21} + p_{22} +$

$+ p_{23} + p_{24}$ where $p_{2i} \sim p_{2j}$ and set aside p_{21} and p_{22}. Divide p_{24} into eight equivalent subprojections

$$p_{24} = \sum_{i=1}^{8} p_{3i}$$

and set aside p_{31}, p_{32} and p_{33}.

In this way an infinite collection (p_{ij}), $1 \leqslant j \leqslant i < \infty$, of orthogonal projections is defined and for each $n \geqslant 1$

$$p_{n1} \sim p_{n2} \sim \dots \sim p_{nn}.$$

The concluding argument of the previous lemma now shows that A contains a subalgebra isomorphic to M_∞.

PROPOSITION 6.6. *An* AW*-*algebra A either contains a subalgebra isomorphic to M_∞ or it is of the form $\bigoplus_{i=1}^{r} C(X_i) \otimes M_{n_i}$ where each X_i is stonean.*

Proof. A may be expressed as the direct sum $A_1 \oplus A_2 \oplus A_3$ of algebras of types I, II and III. If A_2 or A_3 is non-zero then Lemma 6.5 shows that A contains a subalgebra isomorphic to M_∞. On the other hand if both A_2 and A_3 are zero then A is type I and Lemma 6.4 can be applied. ▨

This completes the necessary preliminary work and the main results can be stated.

THEOREM 6.7. *A nuclear C^*-algebra has the decomposition property if and only if it has the form $\bigoplus_{i=1}^{r} C(X_i) \otimes M_{n_i}$ where each X_i is stonean.*

Proof. Algebras of the stated form are nuclear and have the decomposition property. In the opposite direction, if A has the decomposition property then it is an AW*-algebra by Proposition 5.2. Then by Proposition 6.6 there are two possibilities: either A contains M_∞ or A has the desired form. Now M_∞ is a non--nuclear injective von Neumann algebra [34]. The nuclearity of A and Corollary 2.5 then combine to rule out the first possibility, completing the proof. ▨

COROLLARY 6.8. *C^*-algebras which are both injective and nuclear are of the form $\bigoplus_{i=1}^{r} C(X_i) \otimes M_{n_i}$ where each X_i is stonean.*

Proof. In the von Neumann algebra case this is due to Wassermann [34]. In general injective C^*-algebras have the decomposition property and the result then follows from Theorem 6.7. ▨

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