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C^*-ALGEBRAS ASSOCIATED WITH DENJOY HOMEOMORPHISMS OF THE CIRCLE

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1. INTRODUCTION

Homeomorphisms of the circle $S^1 = \mathbf{R}/\mathbf{Z}$ were studied by Poincaré (1885) [19] in connection with the qualitative investigation of trajectories on the torus. He introduced the notion of rotation number $0 \leqslant \rho(\varphi) < 1$ of a homeomorphism $\varphi\colon S^1 \to S^1$ and showed that homeomorphisms without periodic orbits are characterized by having an irrational rotation number $\rho(\varphi) = \alpha$. He proved the fundamental theorem that for such homeomorphisms the points on any orbit $\{\varphi^n(x) \mid n \in \mathbf{Z}\}$ are placed on S^1 in the same order as $\{n\alpha \,(\mathrm{mod}\, 1) \mid n \in \mathbf{Z}\}$. (Cf. Theorem 3.1.) This immediately yields the fundamental semiconjugation

$$h \circ \varphi = R_\alpha \circ h, \tag{1}$$

where $R_\alpha\colon t \to t + \alpha \,(\mathrm{mod}\, 1)$ denotes the rigid rotation of the circle through the angle α (or $2\pi\alpha$, if one prefers), and h is an (essentially) unique orientation-preserving continuous map of S^1 onto S^1.

In 1932 Denjoy [4] proved the remarkable result that all C^2-diffeomorphisms φ of the circle with no periodic orbits are *conjugate* to R_α, where $\alpha = \rho(\varphi)$; in other words, the map h in (1) is a homeomorphism. He also showed that for any irrational number $0 < \alpha < 1$ there exists a homeomorphism φ with rotation number α such that φ is not conjugate to R_α. Arnold (1961) [1] studied the delicate question of how the diffeomorphism type of the conjugating map h of (1) depends upon the rational approximation properties of the irrational number α. The final step was taken by Herman (1976) [7] proving a conjecture of Arnold. More specifically, he showed that there exists a measurable subset A of the irrational numbers between 0 and 1 with Lebesgue measure 1 such that if φ is a C^n-diffeomorphism, $n \geqslant 2$, with $\rho(\varphi) \in A$, then the conjugating map h of (1) is C^{n-2}.

Building upon Herman's work Katznelson studied in two papers [9], [10] the von Neumann algebras that C^2-diffeomorphisms with no periodic orbits give rise to via the W^*-crossed product construction. To be more specific, any C^2-diffeomorphism φ acts non-singularly on $(S^1, \mathcal{B}, m)$, where $\mathcal{B}$ denotes the Borel sets and m is the Lebesgue measure. The action turns out to be ergodic if the rotation number of φ is irrational and so one gets a factor by the crossed product construction. Katznelson showed that in this way one gets precisely all *injective factors* of *product type*. In fact, they may all be obtained from C^∞-diffeomorphisms. In particular, to get the type III factors the rotation number α must have very strong rational approximation properties, and α will be a so-called Liouville number.

The situation is somewhat different in the C^*-setting. Here one considers the action that a homeomorphism $\varphi: S^1 \to S^1$ induces on $C(S^1)$ and then form the C^*-crossed product. Assuming φ has irrational rotation number $\rho(\varphi) = \alpha$ we have to distinguish between two cases. First case: φ is conjugate to R_α and so the crossed product will be isomorphic to the irrational rotation algebra A_α, which has been analysed by Rieffel [22] and Pimsner-Voiculescu [17]. Second case: φ is a so-called Denjoy homeomorphism with orbits that are non-dense. In this case there is a unique minimal φ-invariant closed set $\Sigma \subsetneq S^1$, which turns out to be totally disconnected. In the present paper we analyse the C^*-algebras that Denjoy homeomorphisms give rise to via the C^*-crossed product construction, both when we consider them as homeomorphisms of S^1 and when we restrict to the invariant sets Σ. These C^*-algebras turn out to have an interesting and rich structure. In fact, all the information of the homeomorphism is retained at the C^*-algebra level, i.e., the complete set of invariants that Markley (1969) [11] found for Denjoy homeomorphisms can be recovered from the C^*-algebras. (Cf. Theorem 3.6.)

2. NOTATION AND TERMINOLOGY

A *discrete flow* (X, φ) is a topological space X together with a homeomorphism $\varphi: X \to X$. If (X, φ) and (Y, ψ) are two discrete flows we say that (X, φ) is *conjugate* (*semiconjugate*) to (Y, ψ) if there exists a homeomorphism (continuous map) h of X onto Y so that $h \circ \varphi = \psi \circ h$. We call h a conjugating (semiconjugating) map.

Let (X, φ) be a discrete flow and let $x \in X$. The *orbit* of x (under φ) is the set $\{\varphi^n(x) \mid n \in \mathbf{Z}\}$, where φ^n denotes the n'th iterate of φ. A subset Σ of X is a *minimal set* of (X, φ) if Σ is the closure of the orbit of x for every x in Σ. So, in particular, a minimal set Σ is a closed *φ-invariant set*, i.e. $\varphi(\Sigma) = \Sigma$. If X itself is a minimal set of (X, φ) we say that the discrete flow (X, φ) is minimal. A point $x \in X$ is a *periodic point* of (X, φ) if there exists a positive integer n such that $\varphi^n(x) = x$. We say (X, φ) has a *periodic orbit* if (X, φ) has a periodic point x; in other words, the orbit of x is finite.

Let t be a real number. Then we can write uniquely $t = [t] + \{t\}$. Here $[t]$ is the greatest integer less than or equal to t, and $\{t\}$ is the *fractional part* of t, so $0 \leqslant \{t\} < 1$. For example, $\{-2.7\} = 0.3$. We will interchangeably write $t \pmod 1$ instead of $\{t\}$.

Let X be a compact Hausdorff space and let $\varphi: X \to X$ be a homeomorphism. Then φ induces an action of the integers $\mathbf{Z}$ on $C(X)$ and by common abuse of notation we will denote this action φ. Let $C(X) \underset{\varphi}{\rtimes} \mathbf{Z}$ denote the C^*-crossed product of the C^*-dynamical system $(C(X), \mathbf{Z}, \varphi)$, cf. [14; 7.6]. For $f \in C(X)$ and $n \in \mathbf{Z}$, the element of $C(X) \underset{\varphi}{\rtimes} \mathbf{Z}$ which, as a function from $\mathbf{Z}$ to $C(X)$, has the value f at n and zero elsewhere, will be denoted $f \otimes \delta_n$. With this notation the algebraic operations in $C(X) \underset{\varphi}{\rtimes} \mathbf{Z}$ become:

$$(f \otimes \delta_n) \cdot (g \otimes \delta_m) = f \cdot \varphi^n(g) \otimes \delta_{n+m}$$

and

$$(f \otimes \delta_n)^* = \overline{\varphi^{-n}(f)} \otimes \delta_{-n}.$$

There is a natural embedding $j: C(X) \to C(X) \underset{\varphi}{\rtimes} \mathbf{Z}$ given by $f \to f \otimes \delta_0$.

We let $\mathscr{K}$ denote the compact operators on a separable, infinite-dimensional Hilbert space.

3. DENJOY HOMEOMORPHISMS

Let $S^1 = \{z \in \mathbf{C} \mid |z| = 1\}$ be the unit circle endowed with the counter-clockwise orientation. We will identify S^1 with $\mathbf{R}/\mathbf{Z}$. By choosing $[0, 1)$ as a representative set for $\mathbf{R}/\mathbf{Z}$ we have the identification $[0, 1) \ni t \leftrightarrow e^{2\pi i t} \in \{z \in \mathbf{C} \mid |z| = 1\}$. We will freely use that representation of S^1 which is most convenient.

Let φ be an orientation-preserving homeomorphism of S^1. (Note: If φ is orientation-reversing, it has at least two fixed points.) Then φ can be "lifted" to a strictly increasing continuous function $\tilde{\varphi}: \mathbf{R} \to \mathbf{R}$ which satisfies $\tilde{\varphi}(x + 1) = \tilde{\varphi}(x) + 1$. By normalizing so that $0 \leqslant \tilde{\varphi}(0) < 1$, $\tilde{\varphi}$ is uniquely determined. The limit

$$\lim_{n \to \infty} \frac{\tilde{\varphi}^n(x)}{n}$$

exists and is independent of $x \in \mathbf{R}$. It will be a number in the interval $[0, 1]$ and is called the *rotation number* of φ and denoted by $\rho(\varphi)$. (More accurately, if the limit is 1 we set by convention $\rho(\varphi) = 0$ and so always $0 \leqslant \rho(\varphi) < 1$.) For example, if φ is equal to the rigid rotation $R_\alpha: t \to t + \alpha \pmod 1$, then $\rho(R_\alpha) = \alpha$. The rotation number $\rho(\varphi)$ measures the average "stretch" along any orbit of φ. We list some elementary properties of $\rho(\varphi)$ and refer to [7], [9], [2; Chapter 3, § 3] for further details:

(i) $\rho(\varphi)$ is rational if and only if φ has a periodic orbit. In particular, $\rho(\varphi) = 0$ if and only if φ has a fixed point.

(ii) $\rho(\varphi) = \rho(h^{-1} \circ \varphi \circ h)$ for h an orientation-preserving homeomorphism of S^1, and $\rho(\varphi) = 1 - \rho(k^{-1} \circ \varphi \circ k)$ for k an orientation-reversing homeomorphism of S^1. More generally, the analogous result is true if (S^1, φ) and (S^1, ψ) are merely semi-conjugate.

(iii) $\rho(\varphi^n) = \{n\rho(\varphi)\}$, $n \in \mathbf{Z}$.

We will henceforth only consider discrete flows (S^1, φ) with no periodic orbits, which by (i) is equivalent to $\rho(\varphi)$ being irrational. We now state the fundamental theorem of Poincaré alluded to in the Introduction.

THEOREM 3.1. (Poincaré [19]). *Let φ be a homeomorphism of the circle S^1 with no periodic orbits, and let $\alpha = \rho(\varphi)$ be the irrational rotation number of φ. Let x be any point of S^1. Then the points $x_n = \varphi^n(x)$ are placed on S^1 in the same order as the points $y_n = n\alpha \pmod 1$, $n \in \mathbf{Z}$.*

COROLLARY 3.2. *Let φ be as in the theorem. There exists an orientation--preserving (rather, non-reversing) map $h: S^1 \to S^1$ so that*

$$h \circ \varphi = R_\alpha \circ h, \tag{1}$$

where $R_\alpha: t \to t + \alpha \pmod 1$. The map h will necessarily be continuous and surjective and so (S^1, φ) is semiconjugate to (S^1, R_α). Moreover, h in (1) is unique up to a rotation, i.e. *the maps of the form $h_1 = R_\beta \circ h$, where $\beta \in [0, 1)$, are precisely the orientation-preserving maps satisfying* (1). *Also, φ is uniquely ergodic,* i.e. *there exists a unique φ-invariant probability measure μ on S^1. In fact, $\mu = \mathrm{d}h$ and so, in particular, $\mu([a, b)) = h(b) - h(a)$, where $0 \leqslant a < b < 1$.*

Proof. For $x \in S^1$ define $h: x_n \to n\alpha \pmod 1$, where $x_n = \varphi^n(x)$, $n \in \mathbf{Z}$. Then h admits a unique orientation-preserving (or rather, non-reversing) extension to a, necessarily continuous, mapping of S^1 onto itself. This follows readily from the theorem and the fact that the points $n\alpha \pmod 1$, $n \in \mathbf{Z}$, are dense in S^1. We denote the extension again by h and observe that (1) is satisfied. The other assertions follows immediately from (1). For details, cf. [9].

DEFINITION 3.3. A *Denjoy homeomorphism* is a homeomorphism φ of S^1 with no periodic orbits such that φ is not conjugate to a rigid rotation. In other words, φ is a Denjoy homeomorphism if $\rho(\varphi) = \alpha$ is irrational and φ is not conjugate to R_α.

REMARK. If $\rho(\varphi) = \alpha$, where α is irrational, then φ is a Denjoy homeomorphism if and only if one (and hence every) φ-orbit is non-dense. This is an immediate consequence of (1). Likewise we see from (1) that φ is a Denjoy homeomorphism if and only if h is not one-to-one.

PROPOSITION 3.4. *Let φ be a Denjoy homeomorphism with irrational rotation number α and with unique invariant probability measure $\mu = \mathrm{d}h$, where h is the map in* (1). *Let $\Sigma =$ support (μ). Then $\Sigma \subsetneq S^1$ is a closed φ-invariant set which is the only minimal set of (S^1, φ). Furthermore, Σ is a Cantor set,* i.e. *Σ is a totally disconnected compact set with no isolated points. Moreover, Σ coincides, for every $x \in S^1$, with the limit points of the orbit $\{\varphi^n(x) \mid n \in \mathbf{Z}\}$.*

Proof. μ is non-atomic since h is continuous, and so Σ is perfect, i.e. closed and without isolated points. Now $h(t) - h(0) = \int_0^t \mathrm{d}\mu$, $t \in [0, 1)$. Hence $h(\Sigma) = S^1$. Also, if $h(F) = S^1$ for a closed subset F of S^1, then $\Sigma \subset F$. Assume now that F is a closed φ-invariant subset of S^1, i.e. $\varphi(F) = F$. Because of

$$(1) \qquad h \circ \varphi = R_\alpha \circ h,$$

we have that $h(F)$ is R_α-invariant. Hence $h(F) = S^1$ and so $\Sigma \subset F$. Hence Σ is the only minimal set of (S^1, φ). In particular, $\Sigma \subsetneq S^1$. Moreover, Σ is a Cantor set since the boundary of Σ is φ-invariant and so must coincide with Σ. To prove the last assertion we first observe that the limit points of the orbit $\{\varphi^n(x) \mid n \in \mathbf{Z}\}$, where $x \in \Sigma$, coincides with Σ. It remains to prove that the limit points of any two orbits coincide. For the proof of this, cf. [2; Chapter 3, § 3, Theorem 4].

We are now in a position to introduce a complete set of invariants for a Denjoy homeomorphism φ up to conjugacy. The first invariant is the rotation number $\rho(\varphi) = \alpha$, where α is an irrational number. Let h be the (essentially) unique map associated to φ satisfying (1) and let $\Sigma = \text{support}(\mu)$ be the invariant Cantor set associated with φ, cf. Proposition 3.4. The second invariant is the value taken by h at the gaps of the Cantor set Σ, i.e. the μ-measure of those parts of Σ that lie between the disjoint open intervals of the complement of Σ. To be more specific, let $\Sigma = S^1 \setminus \bigcup_{n=1}^{\infty} I_n$, where $\bigcup_{n=1}^{\infty} I_n$ is a countable disjoint union of open intervals, the intervals $I_1, I_2, \ldots, I_n, \ldots$ being the components of $S^1 \setminus \Sigma$. The map h in (1) collapses each of the intervals $I_n = (a_n, b_n)$ into a single point. By continuity $h(a_n) = h(b_n)$. We call the countable set $\{a_n, b_n \mid n = 1, 2, \ldots\}$ the *accessible* points of Σ. The accessible points pair naturally two and two by being end-points of disjoint components in $S^1 \setminus \Sigma$. Also, h is one-to-one on the inaccessible points of Σ, i.e. on $\Sigma \setminus \{a_n, b_n \mid n = 1, 2, \ldots\}$.

DEFINITION 3.5. Let φ be a Denjoy homeomorphism with rotation number $\rho(\varphi) = \alpha$ and let h and Σ be as above. Set

$$Q(\varphi) = \{h(x) \mid x \text{ accessible point of } \Sigma\} \ (= \{h(I_n) \mid n \in \mathbf{Z}\}).$$

$Q(\varphi)$ is uniquely determined by φ up to a rigid rotation. The set $Q(\varphi)$ is countable and invariant under R_α, a fact that follows readily from (1). Let $1 \leqslant n(\varphi) \leqslant \aleph_0$ be the number of disjoint orbits of $Q(\varphi)$ under R_α.

REMARK. $Q(\varphi)$ is the complement of the invariant $T(\varphi)$ of [11]. Note that $n(\varphi)$ may be equal to $\aleph_0$.

THEOREM 3.6. (Markley [11]). *Let φ_1 and φ_2 be two Denjoy homeomorphisms of S^1. Then (S^1, φ_1) is conjugate to (S^1, φ_2) via an orientation-preserving conjugating map if and only if $\rho(\varphi_1) = \rho(\varphi_2)$ and $Q(\varphi_1) \equiv Q(\varphi_2)$. Similarly, (S^1, φ_1) is conjugate to (S^1, φ_2) via an orientation-reversing conjugating map if and only if $\rho(\varphi_1) = 1 - \rho(\varphi_2)$ and $Q(\varphi_1) \equiv 1 - Q(\varphi_2)$.*

(Here $C \equiv D$, where $C, D \subset S^1$, means $C = R_\beta(D)$ for some $\beta \in [0, 1)$.)

We are now able to give a simple description and visualisation of a typical Denjoy homeomorphism φ, which clearly brings forth the invariants $\rho(\varphi)$, $Q(\varphi)$ and $n(\varphi)$. Recall that $Q(\varphi)$ may be partitioned into $n(\varphi)$ disjoint orbits under R_α, say $Q(\varphi) = \bigcup_{i=1}^{n(\varphi)} Q_i$, where each Q_i is an orbit under R_α. Here $\alpha = \rho(\varphi)$. This corresponds to the fact that φ "permutes" the disjoint intervals in $S^1 \setminus \Sigma$, i.e. φ maps any component in $S^1 \setminus \Sigma$ homeomorphically onto another, and this "permutation" can be partitioned into $n(\varphi)$ disjoint "orbits". Each Q_i is of the form $\gamma_i + n\alpha \pmod 1$, $n \in \mathbf{Z}$, and $\gamma_i - \gamma_j \notin \{n\alpha \pmod 1) \mid n \in \mathbf{Z}\}$ if $i \neq j$. According to Theorem 3.6 it is not $Q(\varphi)$ itself, but $Q(\varphi)$ modulo rigid rotations that is of interest. Hence we are interested in the "gaps" $\gamma_i - \gamma_1$, $i = 2, \ldots, n(\varphi)$. So we may assume that $\gamma_1 = 0$. (This corresponds to a choice of the map h in (1) such that h takes the value 0 at an accessible point of Σ.) Based upon (1) and the properties of the map h that we have previously outlined we can now say that (S^1, φ) is conjugate to $(S^1_{Q(\varphi)}, \hat{R}_\alpha)$. Here $S^1_{Q(\varphi)}$ is the circle S^1 with the points $Q(\varphi)$ being "doubled", i.e. at each $x \in Q(\varphi)$ the circle is cut open with the two end-points adjoined, and then the pair of end-points are connected by an open arc. The continuous map $\hat{R}_\alpha \colon S^1_{Q(\varphi)} \to S^1_{Q(\varphi)}$ is rotation of $S^1_{Q(\varphi)}$ through α, with the proviso that $\hat{R}_\alpha$ maps the arc at the doubled point $x \in S^1$ homeomorphically onto the arc at the doubled point $x + \alpha \pmod 1$. The conjugating map between (S^1, φ) and $(S^1_{Q(\varphi)}, \hat{R}_\alpha)$ maps the component $I_n =$

$= (a_n, b_n)$ of $S^1 \setminus \Sigma = \bigcup_{n=1}^{\infty} I_n$ onto one of the attached arcs with the pair a_n, b_n corresponding to the relevant doubled point of S^1. The lengths of the attached arcs tend to zero as n increases to ∞.

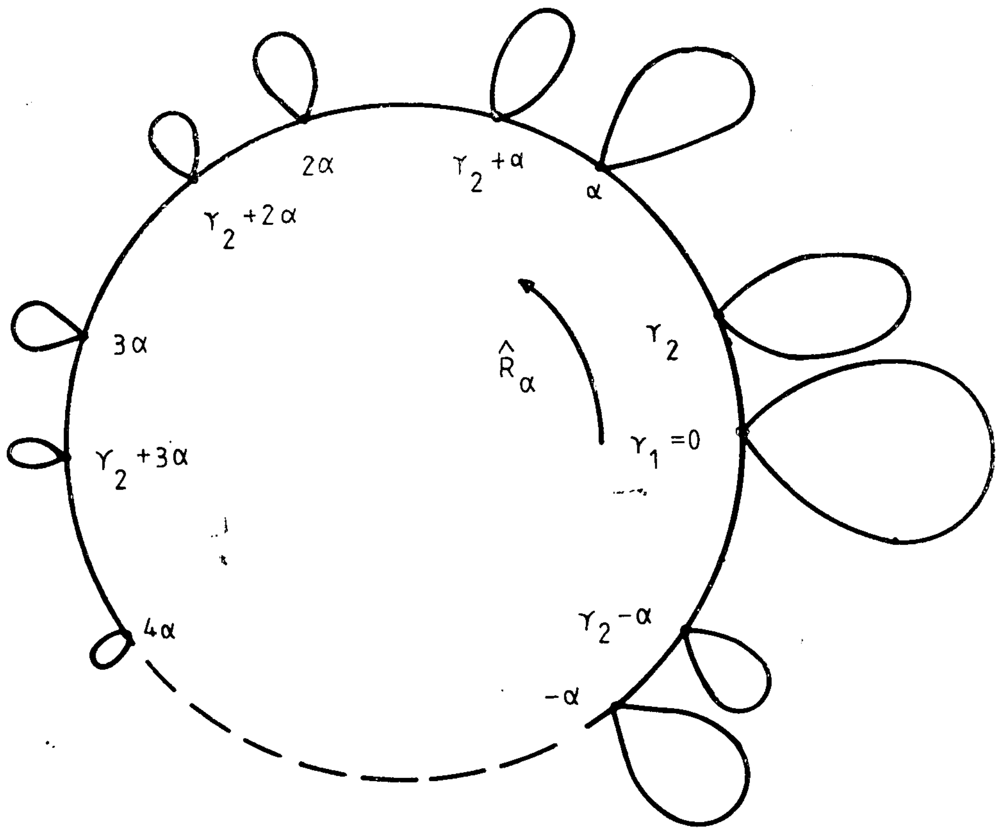


Figure 1.

Figure 1 shows the case where $n(\varphi) = 2$ and $Q(\varphi) = Q_1 \cup Q_2$, where $Q_1 = \{n\alpha \mid n \in \mathbf{Z}\}$ and $Q_2 = \{\gamma_2 + n\alpha \mid n \in \mathbf{Z}\}$, everything being taken modulo 1.

If we remove the attached open arcs of $S^1_{Q(\varphi)}$ altogether we are left with a totally disconnected compact set without isolated points, i.e. a Cantor set. We will denote this set $\hat{S}^1_{Q(\varphi)}$. Observe that $\hat{S}^1_{Q(\varphi)}$ is invariant under $\hat{R}_\alpha$. It follows from the discussion above that $(\hat{S}^1_{Q(\varphi)}, \hat{R}_\alpha)$ is conjugate to (Σ, φ), where we now, by abuse of notation, use $\hat{R}_\alpha$ and φ to denote restriction mappings.

REMARK 1. An alternative way to look at the disconnected circle $\hat{S}^1_{Q(\varphi)}$ is to consider the abelian C^*-algebra $\mathscr{A}$ generated by $C(S^1)$ and all characteristic functions of the form $\chi_{[a,b)}$, where a and b are distinct elements of $Q(\varphi)$, equipped with the supremum norm. Then the maximal ideal space of $\mathscr{A}$ is $\hat{S}^1_{Q(\varphi)}$. Compare this with [3; 2.5]. A relevant reference is also [12].

REMARK 2. The description we have given of a Denjoy homeomorphism also shows how one can conversely construct a Denjoy homeomorphism φ with a given minimal Cantor set $\Sigma \subsetneq S^1$, a given irrational rotation number $\rho(\varphi) = \alpha$ and a given $Q(\varphi) = Q$, where Q is a countable subset of S^1 invariant under R_α. In fact, according to a theorem of Cantor, cf. [2; Chapter 3, § 3, p. 81], the components of $S^1 \setminus \Sigma$ and the set Q can be put in a one-to-one correspondence which preserves the induced orientation from S^1. Using this fact it is straightforward to construct a Denjoy homeomorphism φ with minimal Cantor set Σ such that (S^1, φ) is conjugate to $(S^1_Q, \hat{R}_\alpha)$.

REMARK 3. If φ_1 and φ_2 are two Denjoy homeomorphisms with invariant Cantor sets Σ_1 and Σ_2, respectively, then (Σ_1, φ_1) is conjugate to (Σ_2, φ_2) if and only if (S^1, φ_1) is conjugate to (S^1, φ_2). This can be shown by using the description we have given above of Denjoy homeomorphisms. (Cf. also [11; Theorem 3.3].) Hence the restriction of a Denjoy homeomorphism φ to its unique minimal Cantor set Σ has the same conjugacy invariants as (S^1, φ).

4. THE C^*-ALGEBRAS A_φ AND D_φ AND THE FUNDAMENTAL EXTENSION

Let φ be a Denjoy homeomorphism with irrational rotation number $\rho(\varphi) = \alpha$ and let $\Sigma \subsetneq S^1$ be its unique minimal Cantor set. Set $Y = S^1 \setminus \Sigma = \bigcup_{n=1}^{\infty} I_n$, where $\{I_n\}$ are the components of Y. We have the short exact sequence

$$0 \to C_0(Y) \xrightarrow{i_1} C(S^1) \xrightarrow{q_1} C(\Sigma) \to 0,$$

where i_1 is the natural inclusion and q_1 is the restriction mapping, i.e. if $g \in C_0(Y)$ then $i_1(g)$ is g on Y and 0 on $S^1 \setminus Y = \Sigma$, and $q_1(f) = f \mid \Sigma$ for $f \in C(S^1)$. Since Σ and Y are φ-invariant, φ gives rise to $\mathbf{Z}$-actions on each of the abelian C^*-algebras in the above short exact sequence. Let us use φ to denote each of these actions. Taking C^*-crossed products we get the following short exact sequence (cf. [24]):

$$0 \to \underbrace{C_0(Y) \rtimes_\varphi \mathbf{Z}}_{\mathscr{I}_\varphi} \xrightarrow{i} \underbrace{C(S^1) \rtimes_\varphi \mathbf{Z}}_{A_\varphi} \xrightarrow{q} \underbrace{C(\Sigma) \rtimes_\varphi \mathbf{Z}}_{D_\varphi} \to 0. \tag{2}$$

Here i is the natural inclusion and $q(f \otimes \delta_n) = q_1(f) \otimes \delta_n$, where $f \in C(S^1)$, $n \in \mathbf{Z}$. So A_φ is an extension of D_φ by $\mathscr{I}_\varphi$, where

$$A_\varphi = C(S^1) \underset{\varphi}{\rtimes} \mathbf{Z}, \quad D_\varphi = C(\Sigma) \underset{\varphi}{\rtimes} \mathbf{Z}, \quad \mathscr{I}_\varphi = C_0(Y) \underset{\varphi}{\rtimes} \mathbf{Z}.$$

The extension (2) is easy to describe. In fact, the extension stems from the underlying fact that a continuous function on Σ can be extended to a continuous function on S^1, the non-uniqueness of the extension being measured by an element of $C_0(Y)$.

The following three propositions give the general properties of the C^*-algebras A_φ and D_φ. Note that Proposition 4.1 contains implicitly the complete ideal structure of A_φ.

PROPOSITION 4.1. *A_φ has a unique normalized trace* $\mathrm{Tr} = \mathrm{Tr}_\varphi$. *Moreover,*

$$\mathrm{Tr}(f \otimes \delta_n) = \begin{cases} \int_{S^1} f \, d\mu & \text{if } n = 0 \\ 0 & \text{otherwise,} \end{cases}$$

where μ is the unique φ-invariant probability measure on S^1. $\mathscr{I}_\varphi$ is the unique maxima two-sided ideal in A_φ and $\mathscr{I}_\varphi$ is equal to $\{x \in A_\varphi \mid \mathrm{Tr}(x^*x) = 0\}$. Furthermore,

$$\mathscr{I}_\varphi \cong \oplus \sum_{1}^{n(\varphi)} C_0(\mathbf{R}) \otimes \mathscr{K},$$

where $n(\varphi)$ is defined in Definition 3.5 *and $\mathscr{K}$ is the algebra of compacts.*

PROPOSITION 4.2. *D_φ is a simple C^*-algebra with a unique normalized (faithful) trace* $\hat{\mathrm{Tr}} = \hat{\mathrm{Tr}}_\varphi$. *In fact,* $\mathrm{Tr} = \hat{\mathrm{Tr}} \circ q$, *where q is the map in* (2).

PROPOSITION 4.3. *A_φ and D_φ can be embedded into* AF-*algebras.*

Proof of 4.1 and 4.2. D_φ is simple since (Σ, φ) is minimal, cf. [20]. By [5], [24] every closed two-sided ideal in $A_\varphi = C(S^1) \underset{\varphi}{\rtimes} \mathbf{Z}$ is of the form $C_0(U) \underset{\varphi}{\rtimes} \mathbf{Z}$, where U is an open φ-invariant subset of S^1. Since Y according to Proposition 3.4 is the unique maximal, open, φ-invariant (proper) subset of S^1 we conclude that $\mathscr{I}_\varphi = C_0(Y) \underset{\varphi}{\rtimes} \mathbf{Z}$ is the unique maximal two-sided ideal in A_φ. Since φ has a unique invariant probability measure μ it follows by [5], [24] that A_φ has a unique normalized trace Tr as stated. The assertion about the trace $\hat{\mathrm{Tr}}$ on D_φ is then straightforward. Set $\mathscr{J} = \{x \in A_\varphi \mid \mathrm{Tr}(x^*x) = 0\}$. Then $\mathscr{J}$ is a closed two-sided ideal in A_φ and so $\mathscr{J} \subset \mathscr{I}_\varphi$. Conversely, since $\Sigma = \mathrm{support}(\mu)$ we get that $x \in \mathscr{I}_\varphi$

implies $\mathrm{Tr}(x^*x) = 0$, and so $\mathscr{I}_\varphi \subset \mathscr{J}$. In fact, if $f \in C_0(Y)$, then $(f \otimes \delta_n)^*(f \otimes \delta_n) = |\varphi^{-n}(f)|^2 \otimes \delta_0$, and so

$$\mathrm{Tr}((f \otimes \delta_n)^* f \otimes \delta_n) = \int_{S^1} \varphi^{-n}(|f|^2)\mathrm{d}\mu = \int_{S^1} |f|^2\mathrm{d}\mu = 0.$$

Now elements of the form $f \otimes \delta_n$, $f \in C_0(Y)$, $n \in \mathbf{Z}$, span a dense sub-algebra of $\mathscr{I}_\varphi$, and so $\mathrm{Tr}(x^*x) = 0$ for every x in $\mathscr{I}_\varphi$.

From the description we have given in Section 3 we conclude that (Y, φ) is conjugate to (X, τ), where $X = \{(x, y) \in \mathbf{R}^2 \mid y = k,\ k \in \mathbf{Z}\}$ and τ is a "shift" map, i.e. τ is a map of the form $(x, k) \to (x, k + m)$ for $x \in \mathbf{R}$, $k \in \mathbf{Z}$, and appropriate $m \in \mathbf{Z}$. The number of disjoint "orbits" is $n(\varphi)$. Figure 2 illustrates this for $n(\varphi) = 2$.

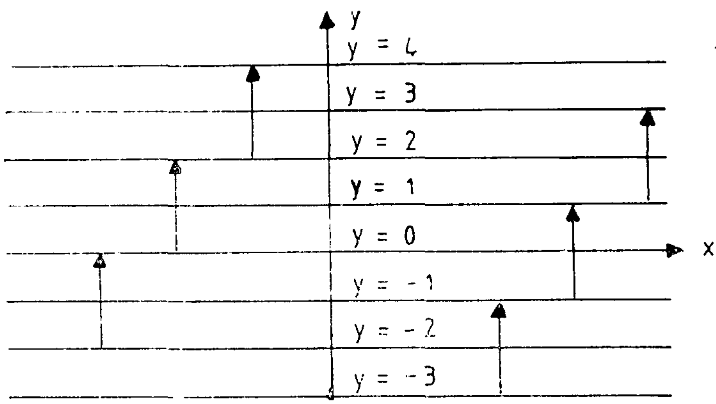


Figure 2.

Hence we get that $\mathscr{I}_\varphi = C_0(Y) \times_\varphi \mathbf{Z}$ is isomorphic to $\oplus \sum_1^{n(\varphi)} C_0(\mathbf{R}) \otimes \mathscr{K}$, cf. [6]. Cf. also remark below.

Proof of 4.3. Pimsner [15] has given a necessary and sufficient dynamic condition for when the C^*-crossed product associated to a discrete flow (X, θ) is embeddable into an AF-algebra. The condition is that every point should be "pseudo-non-wandering". For metric spaces this property is the same as "chain recurrence":

Let X be a metric space with metric d, and let θ be a homeomorphism of X. A point x in X is chain-recurrent if, for any $\varepsilon > 0$, there are points $x_1, x_2, \ldots, x_n$ in X ($n \geqslant 2$) such that $x_1 = x_n = x$ and $d(\theta(x_i), x_{i+1}) < \varepsilon$ for all $i = 1, 2, \ldots, n-1$.

Pimsner's result is that $C(X) \times_\theta \mathbf{Z}$ is embeddable into an AF-algebra if and only if every point of X is chain recurrent. Now we observe that if (X, θ) is minimal then every point in X is chain recurrent. Hence D_φ is embeddable into an AF-algebra since (Σ, φ) is minimal. To prove the same for A_φ we first note that clearly every point in Σ is chain recurrent for (S^1, φ). So let $x \in Y = S^1 \setminus \Sigma$. Then $x \in I$, where I is a component of Y. Given $\varepsilon > 0$ there is an $n \geqslant 1$ such that the length of $\varphi^n(I)$ is less than ε. Let y be one of the end-points of $\varphi^n(I)$. Then $y \in \Sigma$ and $d(\varphi^n(x), y) < \varepsilon$. Our ε-chain from x back to x will consist of three parts. The first part is: $x, \varphi(x), \ldots, \varphi^{n-1}(x), y$. Now we can find an $m \leqslant -1$ such that $\varphi^m(I)$ has length less than $\varepsilon/2$. Choose z to be an end-point of $\varphi^m(I)$. Then $z \in \Sigma$ and $d(\varphi^m(x), z) < \varepsilon/2$. The third part of the chain will be: $\varphi^m(x), \varphi^{m+1}(x), \ldots, \varphi^{-1}(x), x$. Now y and z are both in the minimal set Σ and so we can find $k \geqslant 0$ with $d(\varphi^k(y), z) < \varepsilon/2$. The middle part of the chain will be: $y, \varphi(y), \ldots, \varphi^{k-1}(y), \varphi^m(x)$. Now

$$d(\varphi^k(y), \varphi^m(x)) \leqslant d(\varphi^k(y), z) + d(z, \varphi^m(x)) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

The ε-chain is: $x, \varphi(x), \ldots, \varphi^{n-1}(x), y, \varphi(y), \ldots, \varphi^{k-1}(y), \varphi^m(x), \varphi^{m+1}(x), \ldots, \varphi^{-1}(x), x$.

REMARK. In Section 6 it will be useful to have an explicit isomorphism between $\oplus \sum_{1}^{n(\varphi)} C_0(\mathbf{R}) \otimes \mathscr{K}$ and $\mathscr{I}_\varphi$, cf. Theorem 4.1. For our purpose the most convenient way is to do the following: Let $Y = S^1 \setminus \Sigma = \bigcup_{n-1}^{\infty} I_n$, where as before the open intervals $\{I_n\}$ are the components of Y. Let us assume we have enumerated $\{I_n\}$ so that $I_1, I_2, \ldots, I_{n(\varphi)}$ are representatives from each of the $n(\varphi)$ disjoint "orbits" of φ "permuting" the I_n's, cf. Section 3. Set $Y_0 = \bigcup_{i=1}^{n(\varphi)} Y_i$. Let $\mathscr{K}(\ell^2(\mathbf{Z}))$ be the compact operators on $\ell^2(\mathbf{Z})$. Then clearly $C_0(Y_0) \otimes \mathscr{K}(\ell^2(\mathbf{Z}))$ is isomorphic to $\oplus \sum_{1}^{n(\varphi)} C_0(\mathbf{R}) \otimes \mathscr{K}$, and by the choice of Y_0 it is easy to write down an explicit isomorphism $\beta\colon C_0(Y_0) \otimes \mathscr{K}(\ell^2(\mathbf{Z})) \to \mathscr{I}_\varphi$. In fact, let $\{e_i\}$ be the usual basis for $\ell^2(\mathbf{Z})$ and let k_{ij} be the rank one operator taking e_j to e_i, i.e. for $\xi \in \ell^2(\mathbf{Z})$ let $k_{ij}(\xi) = \langle \xi, e_j \rangle e_i$. Then for $g \in C_0(Y_0)$, $\beta(g \otimes k_{ij}) = \varphi^{-i}(g) \otimes \delta_{j-i}$. (The main part of the calculation is the isomorphism between $C_0(\mathbf{Z}) \rtimes_{\tau_1} \mathbf{Z}$ and $\mathscr{K}(\ell^2(\mathbf{Z}))$, where $\tau_1\colon \mathbf{Z} \to \mathbf{Z}$ is the translation $m \to m+1$, as done in [6]. In fact, $\eta\colon Y_0 \times \mathbf{Z} \to Y$ defined by $\eta\colon (y, m) \to \varphi^m(y)$ is a conjugation between $(Y_0 \times \mathbf{Z}, \mathrm{id} \otimes \tau_1)$ and (Y, φ). So

$$\mathscr{I}_\varphi = C_0(Y) \rtimes_{\varphi} \mathbf{Z} \underset{\eta}{\cong} C_0(Y_0 \times \mathbf{Z}) \rtimes_{\mathrm{id}\otimes\tau_1} \mathbf{Z} \cong (C_0(Y_0) \otimes C_0(\mathbf{Z})) \rtimes_{\mathrm{id}\otimes\tau_1} \mathbf{Z} \cong$$

$$\cong C_0(Y_0) \otimes (C_0(\mathbf{Z}) \rtimes_{\tau_1} \mathbf{Z}) \cong C_0(Y_0) \otimes \mathscr{K}(\ell^2(\mathbf{Z})).)$$

Retaining the notation introduced in the above remark we state the following fact about the primitive ideal space $\mathrm{Prim}(A_\varphi)$ of A_φ endowed with the Jacobson topology.

PROPOSITION 4.4. *$\mathrm{Prim}(A_\varphi)$ is homeomorphic to $Y_0 \cup \{\mathscr{I}_\varphi\}$, with the usual topology on Y_0, and the single point $\mathscr{I}_\varphi$ in the closure of every non-empty set.*

Proof. Immediate consequence of Proposition 4.1 and the above remark. (Recall that every closed two-sided ideal in $C_0(Y_0) \otimes \mathscr{K}(\ell^2(\mathbf{Z}))$ is of the form $C_0(U) \otimes \mathscr{K}(\ell^2(\mathbf{Z}))$, where U is an open set in Y_0.)

REMARK. Just as Y_0 has $n(\varphi)$ open components $I_1, I_2, \dots, I_{n(\varphi)}$, we have a natural decomposition $\mathscr{I}_\varphi = \oplus \sum_{k=1}^{n(\varphi)} \mathscr{I}_k$, where

$$\mathscr{I}_k = \beta(C_0(I_k) \otimes \mathscr{K}(\ell^2(\mathbf{Z}))),$$

and where $\beta\colon C_0(Y_0) \otimes \mathscr{K}(\ell^2(\mathbf{Z})) \to \mathscr{I}_\varphi$ is the isomorphism introduced in the previous remark. Notice that the $\mathscr{I}_k$'s depend only on $\mathscr{I}_\varphi$ (and its primitive ideal space) and not on the choice of the I_k's or β, except for permutation of the indices. So the decomposition $\mathscr{I}_\varphi = \oplus \sum_{k=1}^{n(\varphi)} \mathscr{I}_k$ is an isomorphism invariant of $\mathscr{I}_\varphi$, and hence of A_φ.

5. MAIN RESULTS

THEOREM 5.1. *Let φ and ψ be two Denjoy homeomorphisms of the circle S^1 and let $A_\varphi = C(S^1) \rtimes_\varphi \mathbf{Z}$, $A_\psi = C(S^1) \rtimes_\psi \mathbf{Z}$. Then A_φ is isomorphic to A_ψ if and only if φ is conjugate to ψ or to ψ^{-1}.*

In order to prove Theorem 5.1 we shall need to recover from A_φ the complete set of invariants $\rho(\varphi)$ and $Q(\varphi)$ of a Denjoy homeomorphism φ, cf. Theorem 3.6. In particular, we shall need a result on the K-theory of A_φ, which we state as a separate theorem. (For the ideal structure of A_φ, cf. Proposition 4.4.)

THEOREM 5.2. *Let φ be a Denjoy homeomorphism with $\rho(\varphi) = \alpha$. Then $\mathrm{K}_0(A_\varphi) \cong \mathbf{Z} \oplus \mathbf{Z}$ and $\mathrm{K}_1(A_\varphi) \cong \mathbf{Z} \oplus \mathbf{Z}$. Furthermore, $\mathrm{Tr}_*\colon \mathrm{K}_0(A_\varphi) \to \mathbf{Z} + \mathbf{Z}\alpha$ is an isomorphism of ordered groups, where $\mathbf{Z} + \mathbf{Z}\alpha$ inherits the order structure from $\mathbf{R}$ and Tr_* is the homomorphism induced by the unique normalized trace Tr on A_φ. Moreover, the irrational rotation algebra $A_\alpha = C(S^1) \rtimes_{R_\alpha} \mathbf{Z}$ is a C^*-subalgebra of A_φ.*

Before we state the next theorem recall (Definition 3.5) that for φ a Denjoy homeomorphism with $\rho(\varphi) = \alpha$ we have $Q(\varphi) = \bigcup_{i=1}^{n(\varphi)} Q_i$, where $Q_1, \dots, Q_{n(\varphi)}$ are the $n(\varphi)$ disjoint R_α-orbits of $Q(\varphi)$. So we have

$$Q_i = \{\gamma_i + n\alpha \pmod 1 \mid n \in \mathbf{Z}\}; \quad i = 1, \dots, n(\varphi),$$

where $\gamma_i - \gamma_j \notin \{n\alpha \,(\mathrm{mod}\, 1) \mid n \in \mathbf{Z}\}$ when $i \neq j$. As pointed out in Section 3 we may assume without loss of generality that $\gamma_1 = 0$. We will do so henceforth.

THEOREM 5.3. *Let φ be a Denjoy homeomorphism with $\rho(\varphi) = \alpha$ and let Σ be the unique minimal Cantor set. Let D_φ be the simple C^*-algabra $C(\Sigma) \times_\varphi \mathbf{Z}$ with unique (faithful) normalized trace $\hat{\mathrm{Tr}}$. Then $\mathrm{K}_0(D_\varphi) \cong \bigoplus_1^{n(\varphi)+1} \mathbf{Z}$ and $\mathrm{K}_1(D_\varphi) \cong \mathbf{Z}$. Moreover, the range of $\hat{\mathrm{Tr}}$ on the projections in D_φ is $(\mathbf{Z} + \mathbf{Z}\alpha + \mathbf{Z}\gamma_2 + \ldots + \mathbf{Z}\gamma_{n(\varphi)}) \cap [0, 1]$. In particular, if $1, \alpha, \gamma_2, \ldots, \gamma_{n(\varphi)}$ are linearly independent over the rational numbers, then*

$$\hat{\mathrm{Tr}}_*: \mathrm{K}_0(D_\varphi) \to \mathbf{Z} + \mathbf{Z}\alpha + \mathbf{Z}\gamma_2 + \ldots + \mathbf{Z}\gamma_{n(\varphi)}$$

is an order-isomorphism of ordered groups, where $\mathbf{Z} + \mathbf{Z}\alpha + \mathbf{Z}\gamma_2 + \ldots + \mathbf{Z}\gamma_{n(\varphi)}$ inherits the order from $\mathbf{R}$ and $\hat{\mathrm{Tr}}_*$ is the induced homomorphism.

As above let $Q(\varphi) = \bigcup_{i=1}^{n(\varphi)} Q_i$. Set $C_1 = Q_1$, $C_2 = Q_1 \cup Q_2, \ldots, C_k = \bigcup_{i=1}^{k} Q_i, \ldots$ $\ldots, C_{n(\varphi)} = \bigcup_{i=1}^{n(\varphi)} Q_i = Q(\varphi)$. Set $B_i = C(\hat{S}^1_{C_i}) \times_{\hat{R}_\alpha} \mathbf{Z}$, $i = 1, 2, \ldots, n(\varphi)$. (Confer Section 3 for notation.) Then $B_{n(\varphi)} \cong D_\varphi$ and B_i, for $1 \leqslant i \leqslant n(\varphi)$, is isomorphic to D_{ψ_i} where ψ_i is a Denjoy homeomorphism with the invariants $\rho(\psi_i) = \rho(\varphi) = \alpha$ and $Q(\psi_i) = C_i$, cf. Section 3. As before let A_α be the irrational rotation algebra $C(S^1) \times_{R_\alpha} \mathbf{Z}$. Then we can state the following proposition:

PROPOSITION 5.4. *Let φ be as in Theorem 5.3. Then we have a natural hierarchical embedding:*

$$A_\alpha \subsetneq B_1 \subsetneq \ldots \subsetneq B_{n(\varphi)-1} \subsetneq D_\varphi .$$

6. PROOFS

We shall make extensive use of K-theory for C^*-algebras and the theorems on exact sequences for K-groups, cf. [23] and [18]. Let us introduce some notation. For p a (self-adjoint) projection in the C^*-algebra $\mathscr{A}$ let $[p]_0$ denote its class in $\mathrm{K}_0(\mathscr{A})$. Similarly, for u a unitary in $\mathscr{A}^+$ (where $^+$ denotes adjunction of a unit) let $[u]_1$ denote its class in $\mathrm{K}_1(\mathscr{A})$.

We will organize the proofs so that we first prove Theorem 5.3 and Proposition 5.4, arguing directly with D_φ and not invoking A_φ at all. This gives an indication that it might be possible to recover the invariants $\rho(\varphi)$ and $Q(\varphi)$ of a Denjoy homeomorphism φ just from D_φ. We raise this question in Section 7.

In the sequel we make the obvious modifications if $n(\varphi) = \aleph_0$. Recall that $Q(\varphi) = \bigcup_{i=1}^{n(\varphi)} Q_i$, where $Q_i = \gamma_i + n\alpha \pmod 1$, $n \in \mathbf{Z}$. As before we may assume $\gamma_1 = 0$. Recall also that (Σ, φ) is conjugate to $(\hat{S}^1_{Q(\varphi)}, \hat{R}_\alpha)$, where $\hat{S}^1_{Q(\varphi)}$ is the disconnected ("doubled") circle of Section 3.

LEMMA 6.1. *$K_1(D_\varphi) \cong \mathbf{Z}$ and is generated by $[1 \otimes \delta_1]_1$. $K_0(D_\varphi) \cong \bigoplus_1^{n(\varphi)+1} \mathbf{Z}$ and is generated by $[\chi_{[0,\alpha)} \otimes \delta_0]_0, [\chi_{[0,\gamma_2)} \otimes \delta_0]_0, \ldots, [\chi_{[0,\gamma_{n(\varphi)})} \otimes \delta_0]_0$ and $[1 \otimes \delta_0]_0$. Here $\chi_{[0,\alpha)}, \chi_{[0,\gamma_2)}, \ldots, \chi_{[0,\gamma_{n(\varphi)})}$ are characteristic functions that lie in $C(\hat{S}^1_{Q(\varphi)})$.*

Proof. Set $A = C(\hat{S}^1_{Q(\varphi)})$ and note that $K_1(A) = 0$ since A is a (commutative) AF-algebra. Now $D_\varphi \cong A \rtimes_{\hat{R}_\alpha} \mathbf{Z}$. From [18] we have the following six-term exact sequence:

$$
\begin{array}{ccccc}
K_0(A) & \xrightarrow{\mathrm{id}-(\hat{R}_\alpha)_*} & K_0(A) & \xrightarrow{j_*} & K_0(A \rtimes_{\hat{R}_\alpha} \mathbf{Z}) \\
\uparrow \nu & & & & \downarrow \rho \\
K_1(A \rtimes_{\hat{R}_\alpha} \mathbf{Z}) & \xleftarrow{j_*} & K_1(A) & \xleftarrow{\mathrm{id}-(\hat{R}_\alpha)_*} & K_1(A) \\
 & & \| & & \| \\
 & & 0 & & 0
\end{array} \tag{3}
$$

Here $j\colon A \to A \rtimes_{\hat{R}_\alpha} \mathbf{Z}$ is the natural embedding and ρ and ν are boundary maps described in [18] (cf. also [13]). Observe that

$$K_0(A) \cong F_0 \oplus F_1 \oplus \ldots \oplus F_{n(\varphi)},$$

where F_i, $2 \leqslant i \leqslant n(\varphi)$, is the free abelian group generated by the projections $\{\chi_{[m\alpha, \gamma_i + m\alpha)} \mid m \in \mathbf{Z}\}$, F_1 is the free abelian group generated by the projections $\{\chi_{[m\alpha, (m+1)\alpha)} \mid m \in \mathbf{Z}\}$ and, finally, F_0 is the free abelian group generated by the identity 1, hence $F_0 \cong \mathbf{Z}$. (In fact, open-closed intervals $[a, b)$, where a and b are in $Q(\varphi)$, form a basis for $\hat{S}^1_{Q(\varphi)}$.)

Now $\mathrm{id} - (\hat{R}_\alpha)_*$ maps each F_i into itself. For $1 \leqslant i \leqslant n(\varphi)$, let $a_m = \chi_{[m\alpha, \gamma_i + m\alpha)}$ (respectively, $a_m = [m\alpha, (m+1)\alpha)$ for $i = 1$). Define the homomorphism $\Gamma_i\colon F_i \to \mathbf{Z}$ by

$$\Gamma_i\colon \sum_{m \in \mathbf{Z}} r_m a_m \to \sum_{m \in \mathbf{Z}} r_m.$$

A simple computation shows that $\operatorname{Ker}\Gamma_i = \operatorname{Im}\Lambda_i$, where $\Lambda_i\colon F_i \to F_i$ is the restriction of $\mathrm{id} - (\hat{R}_\alpha)_*$ to F_i. So

$$\mathbf{Z} \cong F_i/\operatorname{Ker}\Gamma_i = F_i/\operatorname{Im}\Lambda_i.$$

For $i = 0$ the restriction Λ_0 of $\mathrm{id} - (\hat{R}_\alpha)_*$ to F_0 is the null map. Hence $\mathbf{Z} \cong F_0 = F_0/\operatorname{Im}\Lambda_0$. From the exact sequence (3) we get

$$\mathrm{K}_0(A \underset{\hat{R}_\alpha}{\times\!\!|} \mathbf{Z}) \cong \mathrm{K}_0(A)/\operatorname{Im}(\mathrm{id} - (\hat{R}_\alpha)_*),$$

and so $\mathrm{K}_0(D_\varphi) = \mathrm{K}_0(A \underset{\hat{R}_\alpha}{\times\!\!|} \mathbf{Z}) \cong \bigoplus_1^{n(\varphi)+1} \mathbf{Z}$, with generators as asserted.

Clearly the kernel of $\mathrm{id}-(\hat{R}_\alpha)_*\colon \mathrm{K}_0(A) \to \mathrm{K}_0(A)$ is $F_0 \cong \mathbf{Z}$. By (3) ν is injective and so

$$\mathrm{K}_1(D_\varphi) = \mathrm{K}_1(A \underset{\hat{R}_\alpha}{\times\!\!|} \mathbf{Z}) \cong \operatorname{Im}(\nu) = \operatorname{Ker}(\mathrm{id} - (\hat{R}_\alpha)_*) \cong \mathbf{Z}.$$

From [18] (or [13]) we get that ν maps $[1 \otimes \delta_1]_1$ onto the generator $[1]_0$ of F_0.

This completes the proof.

LEMMA 6.2. *The range of* $\hat{\mathrm{Tr}}$ *on the projections in* D_φ *is:*

$$(\mathbf{Z} + \mathbf{Z}\alpha + \mathbf{Z}\gamma_2 + \ldots + \mathbf{Z}\gamma_{n(\varphi)}) \cap [0, 1].$$

Proof. If p is a projection in D_φ then $[p]_0$ can be expressed as a finite linear combination over $\mathbf{Z}$ of the generators of $\mathrm{K}_0(D_\varphi)$ exhibited in Lemma 6.1. The values of the induced homomorphism $\hat{\mathrm{Tr}}_*$ on these generators are $1, \alpha, \gamma_2, \ldots, \gamma_{n(\varphi)}$, respectively. Hence $\hat{\mathrm{Tr}}(p) = \hat{\mathrm{Tr}}_*([p]_0)$ is a number in the given set.

To prove the converse it will be convenient to make an ad hoc notational change and set $\gamma_1 = \alpha$. Let $m_1, m_2, \ldots, m_{n(\varphi)}$ be integers. We must find a projection p in $D_\varphi \cong C(\hat{S}^1_{Q(\varphi)}) \underset{\hat{R}_\alpha}{\times\!\!|} \mathbf{Z}$ so that $\hat{\mathrm{Tr}}(p) = \theta$, where

$$\theta = m_1\gamma_1 + m_2\gamma_2 + \ldots m_{n(\varphi)}\gamma_{n(\varphi)} \pmod 1.$$

(We may assume $0 < \theta < 1$). In fact, we will construct a projection p of the form $p = \chi_V \otimes \delta_0$, where V is a finite union of disjoint open-closed intervals $[a, b)$, a and b in $Q(\varphi)$, of total (Lebesgue-) length θ. Let $\delta > 0$ be the smallest distance between the points $\{s\alpha\}$ ($= s\alpha \pmod 1$), $s = 1, 2, \ldots, \sum_{i=1}^{n(\varphi)} |m_i|$, on $\hat{S}^1_{Q(\varphi)}$. For each $1 \leqslant i \leqslant n(\varphi)$ choose $n_i \in \mathbf{Z}$ so that $\{\gamma_i + n_i\alpha\}$ has distance from 0 less than $\delta/2$. If $m_i \geqslant 0$ ($m_i < 0$), we choose n_i so that $\{\gamma_i + n_i\alpha\}$ lies to the "right" of 0, i.e. in the interval (0, 1/2) (to the "left" of 0, i.e. in the interval (1/2, 0)). For $m_i \geq 0$ the length of the open-closed interval $[0, \{\gamma_i + n_i\alpha\})$ is $\gamma_i + \{n_i\alpha\} - 1$. For $m_i < 0$ the length of the open-closed interval $[\{\gamma_i + n_i\alpha\}, 0)$ is $1 - \gamma_i - \{n_i\alpha\}$. We now rotate m_i of the intervals $[0, \{\gamma_i + n_i\alpha\})$ (respectively, $|m_i|$ of the intervals $[\{\gamma_i + n_i\alpha\}, 0)$ if $m_i < 0$) for each $i = 1, 2, \ldots, n(\varphi)$, so that we get altogether $\sum_{i=1}^{n(\varphi)} |m_i|$ disjoint open-closed intervals in $\hat{S}^1_{Q(\varphi)}$, each with one end-point in the set $\{s\alpha\}$, $s = 1,2, \ldots, \sum_{i=1}^{n(\varphi)} |m_i|$. If $m_i \geq 0$, $\{s\alpha\}$ shall be a "left" end-point of the corresponding rotated intervals, while if $m_i < 0$, $\{s\alpha\}$ shall be a "right" end-point of the corresponding rotated intervals. The total length of these intervals is

$$\text{(i)} \qquad \beta = \sum_{i \in \mathscr{S}} m_i(\gamma_i + \{n_i\alpha\} - 1) + \sum_{i \in \mathscr{T}} |m_i|(1 - \gamma_i - \{n_i\alpha\}),$$

where $\mathscr{S} = \{i \mid m_i \geq 0\}$, $\mathscr{T} = \{i \mid m_i < 0\}$. Observe that $\beta = \left\{\sum_{i=1}^{n(\varphi)} m_i\gamma_i + \left\{\sum_{i=1}^{n(\varphi)} m_in_i\alpha\right\}\right\}$. This is an easy consequence of the elementary relations: $\{-a\} = 1 - \{a\}$, $\{a + b\} = \{\{a\} + \{b\}\}$ and $\{a + l\} = \{a\}$ if $l \in \mathbf{Z}$. So if we can show

$$\text{(ii)} \qquad \left\{-\sum_{i=1}^{n(\varphi)} m_in_i\alpha\right\} < 1 - \beta,$$

then we may place a finite set of disjoint open-closed intervals, with both end-points at $\{n\alpha\}$'s, $n \in \mathbf{Z}$, in the complement of the original $\sum_{i=1}^{n(\varphi)} |m_i|$ intervals and with total length $\left\{-\sum_{i=1}^{n(\varphi)} m_in_i\alpha\right\}$. Then the open-closed set V that will do the job will be the union of the intervals considered. Now let $k \in \mathbf{Z}$ so that

$$\text{(iii)} \qquad k < \sum_{i \in \mathscr{S}} m_i\gamma_i + \sum_{i \in \mathscr{T}} |m_i|(1 - \gamma_i) < k + 1.$$

(Note that we will have strict inequality in (iii) since $0 < \theta < 1$.) By choosing the previous n_i's so that $\{n_i\alpha\}$ is sufficiently close to $1 - \gamma_i$ for every i, we may assume that

$$\text{(iv)} \qquad k < \sum_{i \in \mathscr{S}} m_i(1 - \{n_i\alpha\}) + \sum_{i \in \mathscr{T}} |m_i|\{n_i\alpha\} < k + 1.$$

Now (iv) implies that

$$\text{(v)} \qquad \left\{-\sum_{i=0}^{n(\varphi)} m_i n_i \alpha\right\} = \sum_{i\in\mathscr{S}} m_i(1-\{n_i\alpha\}) + \sum_{i\in\mathscr{T}} |m_i|\{n_i\alpha\} - k.$$

By (i) and (v), to prove the inequality (ii) we must show

$$\text{(vi)} \qquad 1 - (\sum_{i\in\mathscr{S}} m_i\gamma_i + \sum_{i\in\mathscr{T}} |m_i|(1-\gamma_i)) + k > 0.$$

However, (vi) is implied by (iii).

This completes the proof.

Proof of Theorem 5.3. By Lemma 6.1 the value of $\hat{\mathrm{Tr}}_*$ on the generators of $\mathrm{K}_0(D_\varphi) \cong \bigoplus_1^{n(\varphi)+1} \mathbf{Z}$ is 1, α, $\gamma_2, \ldots, \gamma_{n(\varphi)}$, respectively. So the range of $\hat{\mathrm{Tr}}_*$ is $\mathbf{Z} + \mathbf{Z}\alpha + \mathbf{Z}\gamma_2 \ldots + \mathbf{Z}\gamma_{\varphi(n)}$. If 1, α, $\gamma_2, \ldots, \gamma_{\varphi(n)}$ are linearly independent over the rational numbers then $\hat{\mathrm{Tr}}_*$ is a group isomorphism between $\mathrm{K}_0(D_\varphi)$ and $\mathbf{Z} + \mathbf{Z}\alpha + \mathbf{Z}\gamma_2 + \ldots + \mathbf{Z}\gamma_{\varphi(n)}$. By Lemma 6.2 we conclude that $\hat{\mathrm{Tr}}_*$ also is an order-isomorphism.

Proof of Proposition 5.4. Recall that for $1 \leqslant k \leqslant n(\varphi) - 1$ we have $B_k = C(\hat{S}^1_{C_k}) \underset{\hat{R}_\alpha}{\rtimes} \mathbf{Z}$, where $C_k = \bigcup_{i=1}^{k} Q_i$ and $Q(\varphi) = \bigcup_{i=1}^{n(\varphi)} Q_i$. There is a natural embedding

$$C(\hat{S}^1) \subsetneqq C^1(\hat{S}^1_{B_1}) \subsetneqq \cdots \subsetneqq C^1(\hat{S}^1_{B_{n(\varphi)-1}}) \subsetneqq C(\hat{S}^1_{Q(\varphi)}).$$

All the C^*-subalgebras of $C(\hat{S}^1_{Q(\varphi)})$ above are $\hat{R}_\alpha$-invariant. Hence we have a natural embedding of the B_k's as claimed, according to [14; 7.7.9].

We are now going to calculate the K-groups of $A_\varphi = C(S^1) \underset{\varphi}{\rtimes} \mathbf{Z}$, thereby proving Theorem 5.2. We will do this working with A_φ directly and not using the results on D_φ we have already obtained. We state two elementary K-theory facts that we shall need in the sequel:

$$\mathrm{K}_0(C(S^1)) \cong \mathbf{Z} \quad \text{and is generated by } [1]_0.$$

$$\mathrm{K}_1(C(S^1)) \cong \mathbf{Z} \quad \text{and is generated by } [f]_1,$$

where $f(t) = e^{2\pi i t}$, $t \in [0, 1)$.

N.B. For the rest of this section f will always denote the function $f(t) = e^{2\pi i}$, $t \in [0, 1)$.

We are again going to invoke the six-term exact sequence of [18]:

$$\text{(4)}\qquad \begin{array}{ccccc} K_0(C(S^1)) & \xrightarrow{\mathrm{id}-\varphi_*} & K_0(C(S^1)) & \xrightarrow{j_*} & K_0(C(S^1) \rtimes_\varphi \mathbf{Z}) \\ \uparrow \nu & & & & \downarrow \rho \\ K_1(C(S^1) \rtimes_\varphi \mathbf{Z}) & \xleftarrow{j_*} & K_1(C(S^1)) & \xleftarrow{\mathrm{id}-\varphi_*} & K_1(C(S^1)). \end{array}$$

Since $\varphi(1) = 1$, and using our description of $K_0(C(S^1))$, id $- \varphi_* = 0$ on the top row. On the bottom row, id $- \varphi_* = 0$ since φ is orientation-preserving. Thus (4) splits into two short exact sequences:

$$\text{(4)}' \qquad 0 \longrightarrow K_0(C(S^1)) \xrightarrow{j_*} K_0(A_\varphi) \xrightarrow{\rho} K_1(C(S^1)) \longrightarrow 0$$

$$\text{(4)}'' \qquad 0 \longrightarrow K_1(C(S^1)) \xrightarrow{j_*} K_1(A_\varphi) \xrightarrow{\nu} K_0(C(S^1)) \longrightarrow 0.$$

LEMMA 6.3. *$K_1(A_\varphi) \cong \mathbf{Z} \oplus \mathbf{Z}$ and is generated by $[f \otimes \delta_0]_1$ and $[1 \otimes \delta_1]_1$.*

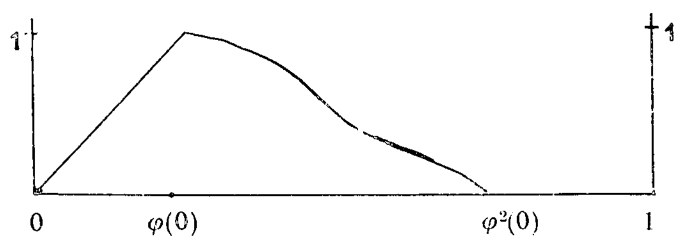
Proof. Immediate consequence of (4)'' by noting that $j_*([f]_1) = [f \otimes \delta_0]_1$ and $\nu([1 \otimes \delta_1]_1) = [1]_0$.

LEMMA 6.4. *$K_0(A_\varphi) \cong \mathbf{Z} \oplus \mathbf{Z}$ and (assuming $\alpha \in (0, 1/2)$) is generated by $[1 \otimes \delta_0]_0$ and $[p]_0$, where p is a (self-adjoint) projection in A_φ of the form $p = \varphi^{-1}(f_1) \otimes \delta_{-1} + f_0 \otimes \delta_0 + f_1 \otimes \delta_1$ (p is a so-called Rieffel projection) such that $\mathrm{Tr}(p) = \alpha$. Here f_0 and f_1 are particular functions in $C(S^1)$ to be described in the proof.*

Proof. We may assume that $\rho(\varphi) = \alpha \in (0, 1/2)$. (In fact, if $\rho(\varphi) \in (1/2, 1)$, replace φ by φ^{-1} and notice that $A_\varphi \cong A_{\varphi^{-1}}$ and $\rho(\varphi^{-1}) = 1 - \rho(\varphi)$.) By Theorem 3.1 we conclude that $\varphi^2(0)$ does not lie between 0 and $\varphi(0)$. Define $f_0 \in C(S^1)$ as follows:

$$f_0(t) = \begin{cases} t/\varphi(0) & \text{for } 0 \leqslant t \leqslant \varphi(0) \\ 1 - \varphi^{-1}(t)/\varphi(0) & \text{for } \varphi(0) \leqslant t \leqslant \varphi^2(0) \\ 0 & \text{otherwise} \end{cases}$$

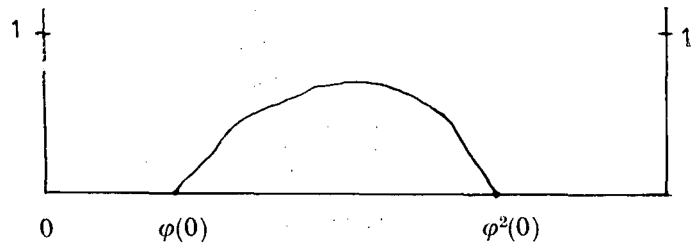
i.e.



Define $f_1 \in C(S^1)$ as follows:

$$f_1(t) = \begin{cases} \sqrt{f_0(t) - f_0(t^2)} & \text{for } \varphi(0) \leqslant t \leqslant \varphi^2(0) \\ 0 & \text{otherwise} \end{cases}$$

i.e.



It is easily seen that:

(i) $$f_0 = f_0^2 + \varphi^{-1}(\bar{f}_1 \cdot f_1) + f_1 \cdot f_1$$

(ii) $$f_1 = f_0 \cdot f_1 + \varphi(f_0) \cdot f_1$$

(iii) $$0 = f_1 \cdot \varphi^{-1}(f_1).$$

(i) — (iii) imply that $p = \varphi^{-1}(f_1) \otimes \delta_{-1} + f_0 \otimes \delta_0 + f_1 \otimes \delta_1$ is a self-adjoint projection in A_φ. As in the appendix of [18], $\rho([p]_0) = [f]_1$. Clearly, $j_*([1]_0) = [1 \otimes \delta_0]_0$. So by (4)′ we get that $K_0(A_\varphi) \cong \mathbf{Z} \oplus \mathbf{Z}$ with generators $[1 \otimes \delta_0]_0$ and $[p]_0$. It remains to prove that $\mathrm{Tr}(p) = \alpha$. But:

$$\mathrm{Tr}(p) = \int_{S^1} f_0 \mathrm{d}\mu = \int_{[0,\, \varphi(0)]} t/\varphi(0)\, \mathrm{d}\mu(t) + \int_{[\varphi(0),\, \varphi^2(0)]} (1 - \varphi^{-1}(t)/\varphi(0))\, \mathrm{d}\mu(t) =$$

$$= \int_{[0,\, \varphi(0)]} t/\varphi(0) \mathrm{d}\mu(t) + \int_{[0,\, \varphi(0)]} (1 - t/\varphi(0))\, \mathrm{d}\mu(t) =$$

$$= \int_{[0,\, \varphi(0)]} 1 \, \mathrm{d}\mu = \mu([0,\, \varphi(0)]) = h(\varphi(0)) - h(0) = \alpha.$$

We have used the φ-invariance of $\mu = \mathrm{d}h$ and the properties of h as outlined in Corollary 3.2. This completes the proof.

REMARK. We could alternatively have constructed the generator p in the above lemma by using the natural embedding of A_α into A_φ (cf. proof of Theorem 5.2 below) and then find a Rieffel projection in A_α with the desired properties. This would correspond to choosing the functions f_0 and f_1 above to be constant on each of the intervals of the complement of the invariant Cantor set of φ. However, the above proof is direct and shows the freedom we have in choosing f_0.

Proof of Theorem 5.2. Let D be the functions in $C(S^1)$ that are constant on each of the intervals $\{I_n\}$ of the complement of the invariant Cantor set Σ of φ. Alternative description: D consists of the continuous functions on $S^1_{Q(\varphi)}$ that are constant on each of the attached arcs of the "doubled" circle, cf. Figure 1. Then D is a C^*-subalgebra of $C(S^1_{Q(\varphi)})$ that is invariant under the action $\hat{R}_\alpha$. There is a natural isomorphism of $D \times_{\hat{R}_\alpha} \mathbf{Z}$ with $C(S^1) \times_{R_\alpha} \mathbf{Z} = A_\alpha$. By [14; 7.7.9] we have $D \times_{\hat{R}_\alpha} \mathbf{Z} \subsetneq C(S^1_{Q(\varphi)}) \times_{\hat{R}_\alpha} \mathbf{Z} \cong A_\varphi$. So A_α is embedded in a natural way as a C^*-subalgebra of A_φ. By Lemma 6.3 and Lemma 6.4 we know $K_0(A_\varphi)$ and $K_1(A_\varphi)$ as abstract groups. What remains to show is that $\mathrm{Tr}_*\colon K_0(A_\varphi) \to \mathbf{Z} + \mathbf{Z}\alpha$ is an order-isomorphism between $K_0(A_\varphi)$ as an ordered group, and $\mathbf{Z} + \mathbf{Z}\alpha$, with the inherited ordering from $\mathbf{R}$. By Lemma 6.4 we know that the range of Tr_* is $\mathbf{Z} + \mathbf{Z}\alpha$. It will be sufficient to show that if $0 < m + n\alpha < 1$ for some $m, n \in \mathbf{Z}$, then there is a projection q in A_φ so that $\mathrm{Tr}(q) = m + n\alpha$. However, we know that A_α is a C^*-subalgebra of A_φ. By [22] we can find a projection q in A_α with the desired property. So the proof is complete.

REMARK. Pimsner [16; Proposition 6] has by a different approach showed that the range of Tr_* is $\mathbf{Z} + \mathbf{Z}\alpha$.

We now turn to the proof of Theorem 5.1. Let us fix φ throughout to be a Denjoy homeomorphism with rotation number $\rho(\varphi) = \alpha$ and invariant Cantor set Σ. Set $Y = S^1 \setminus \Sigma = \bigcup_{n=1}^{\infty} I_n$, where $\{I_n\}$ are the components of Y. As in Proposition 4.4 we assume that we have enumerated $\{I_n\}$ so that $I_1, I_2, \ldots, I_{n(\varphi)}$ are representatives from each of the $n(\varphi)$ disjoint "orbits" of φ "permuting" the I_n's, and we set $Y_0 = \bigcup_{i=1}^{n(\varphi)} I_i$. Recall the fundamental semiconjugation (1):

$$h \circ \varphi = R_\alpha \circ h. \tag{1}$$

Set $\gamma_i = h(I_i)$, $i = 1, 2, \ldots, n(\varphi)$. As before we assume we have chosen an h so that $\gamma_1 = 0$. Now the invariant $Q(\varphi)$ of φ is $Q(\varphi) = \bigcup_{i=1}^{n(\varphi)} Q_i$, where $Q_i = \{\gamma_i + n\alpha \pmod 1 \mid n \in \mathbf{Z}\}$. By Theorem 5.2 we can recover $\rho(\varphi) = \alpha$ from

A_φ. To prove Theorem 5.1 we need to recover $Q(\varphi)$ from A_φ. Before we start the rigorous development, let us give a rough description of our procedure to recover $Q(\varphi)$, thereby pointing out the basic idea.

Recall the fundamental extension (2):

$$0 \longrightarrow \mathscr{I}_\varphi \xrightarrow{i} A_\varphi \xrightarrow{q} D_\varphi \longrightarrow 0. \quad (2)$$

Consider the following function g_k $(1 < k \leqslant n(\varphi))$:

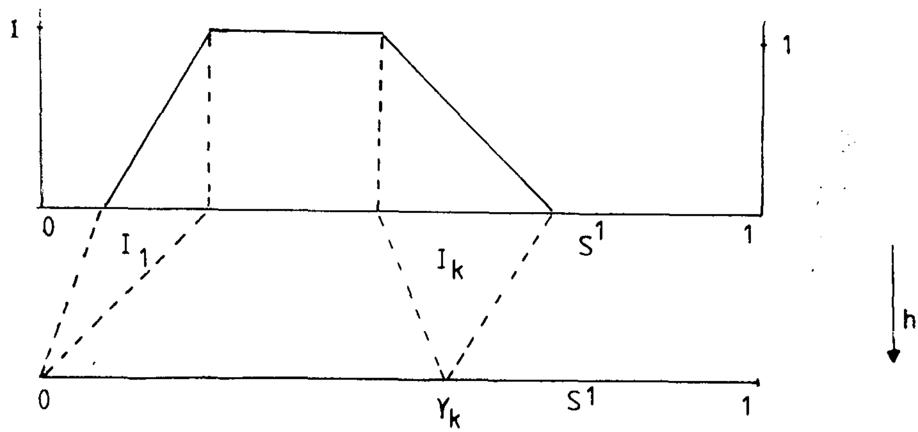


Figure 3.

Observe the following:

(i) g_k restricted to Σ is always zero or one. That is, $q(g_k \otimes \delta_0) = \chi_{[0,\gamma_k)} \otimes \delta_0$ is a projection in D_φ.

(ii) $\exp(2\pi i g_k)$ is identically 1, except on I_1, where it winds once, and on I_k, where it winds once in the opposite direction.

(iii) $\mathrm{Tr}(g_k \otimes \delta_0) = \hat{\mathrm{T}}\mathrm{r}(\chi_{[0,\gamma_k)} \otimes \delta_0) = \gamma_k$.

$Q(\varphi)$ will appear as the trace of certain elements of $\mathrm{K}_0(D_\varphi)$, whose images under exp are specific elements in $\mathrm{K}_1(\mathscr{I}_\varphi)$, cf. (5) below.

We now begin a rigorous development, starting with the unique decomposition $\mathscr{I}_\varphi = \oplus \sum_{k=1}^{n(\varphi)} \mathscr{I}_k$ of $\mathscr{I}_\varphi$, where $\mathscr{I}_k = \beta(C_0(I_k) \otimes \mathscr{K}(\ell^2(\mathbf{Z})))$, β being the isomorphism

$$\beta\colon C_0(Y_0) \otimes \mathscr{K}(\ell^2(\mathbf{Z})) \to \mathscr{I}_\varphi$$

introduced in Section 4. We refer to Section 4 for the details. Let i and i_k denote the natural inclusion maps $i\colon \mathscr{I}_\varphi \to A_\varphi$ and $i_k\colon \mathscr{I}_k \to A_\varphi$. As before, f will denote the function $f(t) = e^{2\pi i t}$, $t \in [0, 1)$.

LEMMA 6.5. *$K_1(\mathcal{I}_k) \cong \mathbf{Z}$ for each $k = 1, 2, \ldots, n(\varphi)$. If x_k is a generator of $K_1(\mathcal{I}_k)$, then $(i_k)_*(x_k) = [f \otimes \delta_0]_1$ or $(i_k)_*(x_k) = -[f \otimes \delta_0]_1$.*

Proof. $K_1(\mathcal{I}_k) \cong K_1(C_0(I_k) \otimes \mathcal{K}(\ell^2(\mathbf{Z}))) \cong K_1(C_0(I_k)) \cong \mathbf{Z}$, since I_k is an open interval.

Now choose $g \in C(S^1)$ such that $[g]_1 = [f]_1$ in $K_1(C(S^1))$, i.e. g has winding number 1, and such that g is identically 1 off of I_k, i.e. $g - 1 \in C_0(I_k)$. Then $y = [1 \otimes I + (g-1) \otimes k_{00}]_1$ is a generator for $K_1(C_0(I_k) \otimes \mathcal{K}(\ell^2(\mathbf{Z})))$. Hence $\beta_*(y)$ is a generator for $K_1(\mathcal{I}_k)$ and so it must be either x_k or $-x_k$. Now

$$\beta(1 \otimes I + (g-1) \otimes k_{00}) = 1 \otimes \delta_0 + \beta((g-1) \otimes k_{00}) =$$

$$= 1 \otimes \delta_0 + (g-1) \otimes \delta_0,$$

and so

$$(i_k)_*(x_k) = \pm (i_k)_* \circ \beta_*(y) = \pm [i_k(1 \otimes \delta_0 + (g-1) \otimes \delta_0)]_1 =$$

$$= \pm [g \otimes \delta_0]_1 = \pm [f \otimes \delta_0]_1$$

by the choice of g. So the proof is complete.

Let us assess the situation: $\mathcal{I}_\varphi$ decomposes naturally into $n(\varphi)$ summands $\mathcal{I}_k$, each corresponding to an orbit of a Cantor gap I_k. By (ii) of the general discussion above we need to know when functions over different I_k's are winding in the same directions or opposite ones. By Lemma 6.5 $K_1(\mathcal{I}_k)$ measures the winding number, and we may control the orientation as follows:

(†) Choose a generator x_1 for $K_1(\mathcal{I}_1)$ arbitrarily. Then for each $k \geq 2$, choose a generator x_k for $K_1(\mathcal{I}_k)$ such that $(i_k)_*(x_k) = (i_1)_*(x_1)$ in $K_1(A_\varphi)$.

REMARK 1. We have a choice of a plus or a minus in selecting x_1. This really corresponds to the fact that we do not recover $Q(\varphi)$ but either $Q(\varphi)$ or $1 - Q(\varphi)$.

REMARK 2. We will also denote by $x_1, \ldots, x_{n(\varphi)}$ the classes in $K_1(\mathcal{I}_\varphi)$ under the natural maps $\mathcal{I}_k \hookrightarrow \mathcal{I}_\varphi$. Clearly $K_1(\mathcal{I}_\varphi) \cong \bigoplus_{1}^{n(\varphi)} \mathbf{Z}$, with generators $x_1, \ldots, x_{n(\varphi)}$. Also $i_*(x_k) = (i_k)_*(x_k)$.

By (2) we get the following fundamental exact sequence of K-theory (cf. [23]):

$$(5) \qquad \begin{array}{ccccc} K_0(\mathcal{I}_\varphi) & \xrightarrow{i_*} & K_0(A_\varphi) & \xrightarrow{q_*} & K_0(D_\varphi) \\ \partial \uparrow & & & & \downarrow \exp \\ K_1(D_\varphi) & \xleftarrow{q_*} & K_1(A_\varphi) & \xleftarrow{i_*} & K_1(\mathcal{I}_\varphi). \end{array}$$

Define, for each $k = 1, 2, \ldots, n(\varphi)$, a subset of $K_0(D_\varphi)$ by

$$X_k = \{y \in K_0(D_\varphi) \mid \exp(y) = x_1 - x_k\}. \tag{††}$$

Our goal is to show that $Q(\varphi) = \hat{\mathrm{Tr}}_*\left(\bigcup_{k=1}^{n(\varphi)} X_k\right)$ up to a sign, everything taken mod 1. (If we had not normalized so that $\gamma_1 = h(I_1) = 0$, we would get $Q(\varphi) \equiv \equiv \hat{\mathrm{Tr}}_*\left(\bigcup_{k=1}^{n(\varphi)} X_k\right)$. (Cf. Theorem 3.6 for notation.).)

LEMMA 6.6. *Let g_k $(k = 2, \ldots, n(\varphi))$ be the functions with graphs shown in Figure 3, and set $g_1 = 0$. Then*

$$[q(g_k \otimes \delta_0)]_0 \in X_k \quad \textit{for all } k,$$

or else

$$-[q(g_k \otimes \delta_0)]_0 \in X_k \quad \textit{for all } k.$$

Proof. If $k = 1$ we get $[q(g_1 \otimes \delta_0)]_0 = -[q(g_1 \otimes \delta_0)]_0 = 0 \in X_1 = \mathrm{Ker}(\exp)$. So we only have to show the result for $k \geqslant 2$. Now

$$e^{2\pi i g_k} = 1 + h_1 + h_k,$$

where $h_1 \in C_0(I_1)$, $h_k \in C_0(I_k)$ and $h_1 \cdot h_k = 0$. (h_1 is the same function for all k.) h_1+1 and h_k+1 have winding number 1 and -1, respectively. Thus $[(h_1+1)\otimes\delta_0]_1$ is a generator for $K_1(\mathscr{I}_1)$, so it must be either x_1 or $-x_1$. Assume, for now, that it is x_1. Analogously, $[(h_k + 1) \otimes \delta_0]_1$ is either x_k or $-x_k$. Now

$$\exp[q(g_k \otimes \delta_0)]_0 = [e^{2\pi i g_k} \otimes \delta_0]_1 = [(1 + h_1 + h_k) \otimes \delta_0]_1 =$$

$$= [(1 + h_1)(1 + h_k) \otimes \delta_0]_1 = [(1 + h_1) \otimes \delta_0]_1 + [(1 + h_k) \otimes \delta_0]_1 =$$

$$= x_1 + [(1 + h_k) \otimes \delta_0]_1 .$$

Since this is in the image of exp, it is in the kernel of i_* according to (5). But

$$i_*(x_1 + [(1 + h_k) \otimes \delta_0]_1) = i_*(x_1 \pm x_k) = i_*(x_1) \pm i_*(x_k) = i_*(x_1) \pm i_*(x_1),$$

by the choice of x_k (cf. (†)). Thus, in order to get 0, we must have a minus sign, so that $[(1 + h_k) \otimes \delta_0]_1 = -x_k$. This is true for each $k \geqslant 2$. So $[q(g_k \otimes \delta_0)]_0 \in X_k$ for all k (cf. (††)). If $[(1 + h_1) \otimes \delta_0]_1 = -x_1$, the same argument shows $[(1 + h_k) \otimes \otimes \delta_0]_1 = x_k$ for $k \geqslant 2$. So in this case we have $-[q(g_k \otimes \delta_0)]_0 \in X_k$ for all k. This completes the proof.

By Lemma 6.6 and (5) we get

$$X_k = [q(g_k \otimes \delta_0)]_0 + \mathrm{Ker}(\exp) = [q(g_k \otimes \delta_0)]_0 + \mathrm{Im}(q_*),$$

or alternatively,

$$X_k = -[q(g_k \otimes \delta_0)]_0 + \mathrm{Im}(q_*).$$

We are now in a position to prove the crucial lemma:

LEMMA 6.7. *$Q(\varphi) = \bigcup_{k=1}^{n(\varphi)} \hat{\mathrm{Tr}}_*(X_k)$ or $Q(\varphi) = 1 - \bigcup_{k=1}^{n(\varphi)} \hat{\mathrm{Tr}}_*(X_k)$, everything taken* mod 1.

Proof. Case 1: $[q(g_k \otimes \delta_0)]_0 \in X_k$ for all k. Recall that

$$Q(\varphi) = h(Y) = \bigcup_{k=1}^{n(\varphi)} \{h(I_k) + n\alpha \pmod 1 \mid n \in \mathbf{Z}\} =$$

$$= \bigcup_{k=1}^{n(\varphi)} \{\gamma_k + n\alpha \pmod 1 \mid n \in \mathbf{Z}\}.$$

Now

$$\hat{\mathrm{Tr}}(q(g_k \otimes \delta_0)) = \mathrm{Tr}(g_k \otimes \delta_0) = \int_{S^1} g_k \, \mathrm{d}\mu =$$

$$= h(I_k) - h(I_1) = \gamma_k - \gamma_1 = \gamma_k.$$

Hence we get:

$$\hat{\mathrm{Tr}}_*(X_k) = \hat{\mathrm{Tr}}_*([q(g_k \otimes \delta_0)]_0 + q_*(\mathrm{K}_0(A_\varphi))) =$$

$$= \hat{\mathrm{Tr}}(q(g_k \otimes \delta_0)) + (\hat{\mathrm{Tr}} \circ q)_*(\mathrm{K}_0(A_\varphi)) = \gamma_k + \mathrm{Tr}_*(\mathrm{K}_0(A_\varphi)) =$$

$$= \gamma_k + \mathbf{Z} + \mathbf{Z}\alpha,$$

by Theorem 5.2. Thus $Q(\varphi) = \bigcup_{k=1}^{n(\varphi)} \hat{\mathrm{Tr}}_*(X_k)$, everything taken mod 1.

Case 2: $-[q(g_k \otimes \delta_0)]_0 \in X_k$ for all k. By the same calculation we get:

$$\hat{\mathrm{Tr}}_*(X_k) = -\gamma_k + \mathbf{Z} + \mathbf{Z}\alpha.$$

Thus $Q(\varphi) = 1 - \bigcup_{k=1}^{n(\varphi)} \hat{\mathrm{Tr}}_*(X_k)$, everything taken mod 1. So the proof is complete.

REMARK. We emphasize again that the $\mathscr{I}_k$'s, the x_k's and the X_k's have all been defined independently of the choice of β, the I_k's, etc., and without using the embedding $j: C(S^1) \to A_\varphi$. So we may conclude by Lemma 6.7 that $Q(\varphi)$ (or $1 - Q(\varphi)$) is, up to a rigid rotation, an isomorphism invariant of A_φ.

Proof of Theorem 5.1. By Theorem 5.2 we get that $A_\varphi \cong A_\psi$ implies $\rho(\varphi) = \rho(\psi)$ or $\rho(\varphi) = 1 - \rho(\psi)$. By Lemma 6.7 we get that $A_\varphi \cong A_\psi$ implies $Q(\varphi) \equiv Q(\psi)$ or $Q(\varphi) \equiv 1 - Q(\psi)$. In general, if θ is a Denjoy homeomorphism we note that $\rho(\theta^{-1}) = 1 - \rho(\theta)$ and $Q(\theta^{-1}) \equiv Q(\theta)$. (This is a simple consequence of the fundamental semiconjugation (1).) The proof of the theorem is now an immediate consequence of Theorem 3.6.

7. CONCLUDING REMARKS AND OPEN PROBLEMS

As pointed out in Section 6 we proved Theorem 5.3 and Proposition 5.4 directly from D_φ without invoking A_φ. We conjecture it is possible to recover $\rho(\varphi)$ and $Q(\varphi)$ from D_φ directly. Let us put this conjecture in its right perspective:

As pointed out in Remark 3 of Section 3, (Σ, φ) has the same conjugacy invariants as (S^1, φ), where φ is a Denjoy homeomorphism with minimal Cantor set Σ. So if φ_1 and φ_2 are two Denjoy homeomorphism with invariant Cantor sets Σ_1 and Σ_2, respectively, the conjecture would entail the following theorem:

$D_{\varphi_1} = C(\Sigma_1) \underset{\varphi_1}{\rtimes} \mathbf{Z}$ *is isomorphic to* $D_{\varphi_2} = C(\Sigma_2) \underset{\varphi_2}{\rtimes} \mathbf{Z}$ *if and only if* $\varphi_1 \mid \Sigma_1$ *is conjugate to* $\varphi_2 \mid \Sigma_2$ *or to* $\varphi_2^{-1} \mid \Sigma_2$.

This leads to the more general conjecture:

Let $\theta_i: X_i \to X_i$ ($i = 1, 2$) be a minimal homeomorphism of the Cantor set X_i (in other words, X_i is a compact Hausdorff space without isolated points and with a countable basis of open-closed sets) and assume θ_i is uniquely ergodic, i.e. there exists a unique θ_i-invariant probability measure μ_i on X_i. So $B_{\theta_i} = C(X_i) \underset{\theta_i}{\rtimes} \mathbf{Z}$ is a simple C^*-algebra with a unique normalized (and faithful) trace Tr_i. Then $B_1 \cong B_2$ if and only if θ_1 is conjugate to θ_2 or to θ_2^{-1}.

Let us exhibit a concrete example to support the conjecture:

Let (X_i, θ_i), for $i=1,2$, be an odometer (also known as "adding-machines") with (natural number) parameters $d_0^i, d_1^i, \ldots, d_n^i, \ldots$. That is, $X_i = \prod_{k=0}^{\infty} \{0, 1, \ldots, d_k^i - 1\}$ with product topology and $\theta_i: X_i \to X_i$ is "addition of 1 with carry-over" (cf. [10]). Now X_i is naturally organized to a compact abelian group which is monothetic, i.e. it has a dense subgroup which is the homomorphic image of $\mathbf{Z}$. (X_i becomes the group $\Delta_{\mathbf{a}}$ of $\mathbf{a}$-adic integers, where $\mathbf{a} = (d_0^i, d_1^i, \ldots, d_n^i, \ldots)$, cf. [8; § 10 and § 25]. In fact, these are precisely the 0-dimensional monothetic compact groups [8].

Then (X_i, θ_i), $i = 1, 2$, satisfies the conditions of the above conjecture, the unique θ_i-invariant measure being the Haar-measure. Also, it is known that $B_1 = C(X_1) \underset{\theta_1}{\rtimes} \mathbf{Z}$ is isomorphic to $B_2 = C(X_2) \underset{\theta_2}{\rtimes} \mathbf{Z}$ if and only if θ_1 is conjugate to θ_2 or to θ_2^{-1}. In fact, θ_i ($i = 1, 2$) is a minimal rotation R_{ρ_i} of the compact abelian group X_i, where $\rho_i = (1, 0, 0, \ldots, 0, \ldots)$. So the result follows from [21]. It turns out that in this case the range of the trace on projections in the crossed-product is a complete conjugacy invariant, the range being $\left\{\frac{s}{d_0^i d_1^i \ldots d_n^i} \,\middle|\, 0 \leqslant s \leqslant d_0^i d_1^i \ldots d_n^i,\ n = 0, 1, \ldots\right\}$, $i = 1, 2$. (We mention as an aside that $K_0(B_i)$ and $K_1(B_i)$ can be computed using the six-term exact sequence (3) and proceeding as in the proof of Lemma 6.1. We can also construct an embedding of B_i in the UHF-algebra A_∞^i of rank $\{d_k^i \mid k = 0, 1, 2, \ldots\}$. By Glimm's theorem $B_1 \cong B_2$ if and only if $A_\infty^1 \cong A_\infty^2$, cf. [11; 6.4].)

Finally, let us return to the first conjecture about the D_φ's. By Proposition 5.4 (respectively, Theorem 5.2) we know that the irrational rotation algebra A_α is a C^*-subalgebra of D_φ (respectively, A_φ), where $\alpha = \rho(\varphi)$. We conjecture that if A_β is a C^*-subalgebra of D_φ (respectively, A_φ) then $\beta = \{n\alpha\}$ for some $n \in \mathbf{Z}$. This result would imply that we can recover the rotation number $\rho(\varphi)$ from D_φ. However, a word of warning is in order: If φ_1 and φ_2 are two Denjoy homeomorphisms with the same rotation number and $K_0(D_{\varphi_1})$ is order-isomorphic to $K_0(D_{\varphi_2})$, this in itself is not enough to conclude that φ_1 is conjugate to φ_2 or to φ_2^{-1}.

EXAMPLE. Let $1, \alpha, \gamma_2, \gamma_3$ lie in [0,1] and be linearly independent over the rational numbers. Let φ_1 be a Denjoy homeomorphism such that:

$$\rho(\varphi_1) = \alpha \quad \text{and} \quad Q(\varphi_1) = \{n\alpha \ (\mathrm{mod}\ 1) \mid n \in \mathbf{Z}\} \cup \{\gamma_2 + n\alpha \ (\mathrm{mod}\ 1) \mid n \in \mathbf{Z}\} \cup \{\gamma_3 + n\alpha \ (\mathrm{mod}\ 1) \mid n \in \mathbf{Z}\}.$$

Let φ_2 be a Denjoy homeomorphism such that:

$$\rho(\varphi_2) = \alpha \quad \text{and} \quad Q(\varphi_2) = \{n\alpha \ (\mathrm{mod}\ 1) \mid n \in \mathbf{Z}\} \cup \{-\gamma_2 + n\alpha \ (\mathrm{mod}\ 1) \mid n \in \mathbf{Z}\} \cup \{\gamma_3 + n\alpha \ (\mathrm{mod}\ 1) \mid n \in \mathbf{Z}\}.$$

So $n(\varphi_1) = n(\varphi_2) = 3$, and by Theorem 5.3 we know that $K_0(D_{\varphi_1}) \cong K_0(D_{\varphi_2}) \cong \mathbf{Z} + \mathbf{Z}\alpha + \mathbf{Z}\gamma_2 + \mathbf{Z}\gamma_3$ as ordered groups. (Moreover, by Lemma 6.2 the ranges of the traces on the projections in D_{φ_1} and D_{φ_2} coincide.) However, $Q(\varphi_1) \not\equiv Q(\varphi_2)$ and $Q(\varphi_1) \not\equiv 1 - Q(\varphi_2)$, i.e. there is no β_1 so that $Q(\varphi_2) = R_{\beta_1}(Q(\varphi_1))$, or β_2 so that $1 - Q(\varphi_2) = R_{\beta_2}(Q(\varphi_1))$. This is shown by a simple computation. So by Theorem 3.6 (and Remark 3 in Section 3) φ_1 is not conjugate to φ_2 or to φ_2^{-1}.

So to prove the conjecture about the D_φ's, just a recovery of the rotation number $\rho(\varphi)$ is not sufficient.

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