J. OPERATOR THEORY
16(1986), 213 – 233

THE CHARACTERIZATION OF DIFFERENTIAL OPERATORS BY LOCALITY: C^*-ALGEBRAS OF TYPE I

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0. INTRODUCTION

Let A be a C*-algebra and let τ be a strongly continuous one-parameter automorphism group of A with infintesimal generator δ. Let H be a linear operator on A defined on the space of smooth elements for τ, and say that H is local with respect to τ if $\omega(H(a)) = 0$ whenever ω is a pure state of A and $\omega(\delta^k(a)) = 0$ for all $k \geqslant 0$. If A an elementary C^*-algebra, and if k is an eigenvalue of τ for each $k \in \mathbf{Z}$, then a local operator with respect to τ must be a polynomial in δ with scalar coefficients. If A is a postliminary C^*-algebra, and if the set X_0 of points of the spectrum of A fixed by τ has no interior, then a local operator with respect to τ must be a polynominal in δ with coefficients functions on the spectrum of A.

In [12], [1], and [7], it was shown that a linear operator H on a commutative C^*-algebra A, defined on the subalgebra A_∞ of smooth elements with respect to a strongly continuous action τ of $\mathbf{R}$ on A, and local in the sense that the support of Hf is contained in the support of f for each function f in the domain of H, must be a polynomial in the infinitesimal generator δ of τ, the coefficients of which are functions on the spectrum of A. The functions which can arise as coefficients of a local operator from A_∞ into A were described in [7] by continuity and growth conditions.

The purpose of this paper is to describe some analogous results for non-commutative C^*-algebras. Results of a somewhat similar nature have already been obtained in [5], [6], [4], and [2]. These papers consider various notions of relative locality of one linear operator on a C^*-algebra with respect to another, which imply in suitable circumstances that the local operator is a polynomial in the reference operator, with coefficients functions on the primitive spectrum of the C^*-algebra. Usually the reference operator is assumed to be a derivation (and in [5] both operators are).

A common feature of these four earlier papers is that the degree of the polynomial, or, rather, a fixed upper bound for the degree, is explicitly built into the definition of relative locality. By contrast, in [7] the definition of locality contains no such explicit information, and hence, as the domain is A_∞, polynomials of arbitrary degree are allowed. In fact, the property of locality in [7] has no relation to the reference derivation; it can be defined in the same way for an operator with any domain. In other words, the only relation of a local operator H defined on A_∞ as in [7] to the reference derivation is through its domain. The reason that H must be a polynomial of finite degree in δ, seen from one point of view, is that any closed operator from the Fréchet space A_∞ into A must be continuous, with respect to one of the C^n-norms. While H was not assumed to be closed as a map from A_∞ to A, locality implies this. (See [3], where the case that δ is an abstract derivation is considered.)

If A is not commutative, it is no longer clear that locality of H can be formulated as an abstract property. For instance, if A is a primitive C^*-algebra and an element a of A is zero on an open set of pure states of A, then by analyticity, $a = 0$, so the obvious formulation of locality is vacuous.

We can however formulate a relative property of locality of H with respect to τ, when H is defined on A_∞, which, with some restriction on τ, seems to force H to be a polynomial in δ without restricting the degree of the polynomial. We shall say that H is local relative to τ if, whenever ω is a pure state of A and $a \in A_\infty$,

$$\omega(\delta^k(a)) = 0 \text{ for all } k \geqslant 0 \Rightarrow \omega(H(a)) = 0.$$

Some restriction is needed on τ; as pointed out in [4] in the case $n = 2$, if A is the C^*-algebra of $n \times n$ matrices, and $\delta = \operatorname{ad} ih$ where $h = \operatorname{diagonal}(\beta_1, \ldots, \beta_n)$ with all $\beta_i - \beta_j$ distinct ($i \neq j$), then the transpose operation is local with respect to δ, but is not a polynomial in δ unless $n = 1$.

THEOREM 1. *Let $(A, \mathbf{R}, \tau)$ be a C^*-dynamical system with A primitive. Let δ denote the infinitesimal generator of τ, and suppose that δ is unbounded. Assume that A contains an elementary closed two-sided ideal I, necessarily invariant under τ. Assume that the spectrum of the restriction of τ to I contains $\mathbf{Z}$, and that furthermore each $k \in \mathbf{Z}$ is an eigenvalue of the restriction $\tau|I$. Let H be a linear operator from $A_\infty = \bigcap_{n \geqslant 1} D(\delta^n)$ into A.*

The following two conditions are equivalent.

1. *H is local with respect to τ.*
2. *There exist $n \geqslant 0$ and $\lambda_0, \lambda_1, \ldots, \lambda_n \in \mathbf{C}$ such that*

$$H = \sum_{m=0}^{n} \lambda_m \delta^m | A_\infty .$$

THEOREM 2. *Let* $(A, \mathbf{R}, \tau)$ *be a* C^**-dynamical system, and let* δ *denote the infinitesimal generator of* τ. *Set* Prim $A = X$, *and denote by* X_0 *the set of points of* X *fixed under the action of* $\mathbf{R}$ *on* X *determined by* τ. *Suppose that* $X \setminus X_0$ *is dense in* X. *Denote by* S *the set of points of* $X \setminus X_0$ *which are open relative to their closures in* X *and closed relative to their orbits. Assume that* S *is dense in* X. *(This is the case if* X *has a dense* τ*-invariant open subset* O *such that every point of* O *is closed relative to* O, *— for example, if* A *is postliminary.) Let* H *be a linear operator from* A_∞ *into* A.

The following two conditions are equivalent.

1. *H is local with respect to* τ.
2. *There exist* $n \geqslant 0$ *and complex-valued functions* $l_0, l_1, \ldots, l_n$ *on* X *such that*

$$H = \sum_{m=0}^{n} l_m \delta^m \mid A_\infty,$$

i.e. *for any* $a \in A_\infty$ *and* $\gamma \in X$,

$$Ha + \gamma = \sum_{m=0}^{n} l_m(\gamma)\delta^m a + \gamma.$$

1. PROOF OF THEOREM 1

We must prove $1 \Rightarrow 2$. (The implication $2 \Rightarrow 1$ is immediate.)

OBSERVATION 1. *Let* h *be an unbounded selfadjoint operator on the Hilbert space of a faithful irreducible representation of* A *such that*

$$\tau_t a = \mathrm{e}^{ith} a\, \mathrm{e}^{-ith}, \quad t \in \mathbf{R},\ a \in A.$$

(See [9], Example 3.2.35.*) For each* $k \in \mathbf{Z}$ *there exist eigenvalues* β_k, β_k' *of* h *such that*

$$\beta_k - \beta_k' = k.$$

Proof. By assumption there exists $0 \neq a \in I$ such that $\tau_t a = \mathrm{e}^{itk} a$ for all $t \in \mathbf{R}$. Therefore a^*a is fixed by τ. Note that I is the algebra of compact operators. Multiplying a on the right, first by a minimal spectral projection of a^*a on which a^*a is not zero, and second by the restriction of $(a^*a)^{-1/2}$ to this projection, we may suppose that a is a partial isometry. Then a^*a and aa^* are projections fixed by τ, i.e. commuting with h. Since a^*a and aa^* are of finite rank there exists, by application of the spectral theorem to a^*ah, a subprojection of a^*a of rank one which commutes with h. Multiplying a on the right by this projection we may suppose that a itself is of rank one. Then there exist $\beta_k, \beta_k' \in \mathbf{R}$ such that

$$haa^* = \beta_k aa^*, \quad ha^*a = \beta_k' a^*a,$$

and so β_k, β'_k are eigenvalues of h. Hence

$$ha = \beta_k a, \quad ah = \beta'_k a,$$

$$\delta a = i(ha - ah) = i(\beta_k - \beta'_k)a = ika,$$

and consequently $\beta_k - \beta'_k = k$.

OBSERVATION 2. *Denote by P the set of pure states $\omega \in P_A$ for which there exists a sequence $(a_k)_{k\geq 0}$ of partial isometries of rank one such that for all $k \geq 0$,*

$$ha_k = \beta_k a_k, \quad a_k h = \beta'_k a_k,$$

$$\omega(a_k) \neq 0, \quad \textit{and} \quad \omega(a_k)^{-1} = O((k+1)^2).$$

P is dense in P_A.

Proof. Choose sequences $(\xi_k)_{k\geq 0}$ and $(\xi'_k)_{k\geq 0}$ of (non zero) eigenvectors of h such that

$$h\xi_k = \beta_k\xi_k, \quad h\xi'_k = \beta'_k\xi'_k,$$

and such that if $\beta_{k_1} = \beta_{k_2}$, $\beta'_{k_1} = \beta'_{k_2}$, or $\beta_{k_1} = \beta'_{k_2}$ then $\xi_{k_1} = \xi_{k_2}$, $\xi'_{k_1} = \xi'_{k_2}$, or $\xi_{k_1} = \xi'_{k_2}$ respectively.

Next choose the norms of the vectors in these two sequences in the following manner. Set $\|\xi_0\| = \|\xi'_0\| = 1$. If β_1 (β'_1) coincides with β_0 $(= \beta'_0)$ then $\|\xi_1\|$ $(\|\xi'_1\|)$ is fixed. If not, set $\|\xi_1\| = 2^{-1}$ $(\|\xi'_1\| = 2^{-1})$. If β_2 (β'_2) coincides with one of the earlier eigenvalues $\beta_0, \beta_1, \beta'_1$ then $\|\xi_2\|$ $(\|\xi'_2\|)$ is fixed. If not, set $\|\xi_2\| = 3^{-1}$ $(\|\xi'_2\| = 3^{-1})$. Continue in this way to fix the norms of all the ξ_k and ξ'_k. Thus

$$\|\xi_k\| \geq (k+1)^{-1}, \quad \|\xi'_k\| \geq (k+1)^{-1}, \quad k \geq 0,$$

by construction. Finally note that if repetitions are eliminated then the sequences $(\|\xi_k\|)$ and $(\|\xi'_k\|)$ are square summable.

Next let η be a unit vector such that for some $\gamma > 0$,

$$\|\eta_k\| \geq \gamma(k+1)^{-1}, \quad \|\eta'_k\| \geq \gamma(k+1)^{-1}, \quad k \geq 0,$$

where η_k (η'_k) denotes the component of η in the eigenspace corresponding to β_k (β'_k). It follows that the vector pure state $\omega = \omega_\eta$ determined by η belongs to P: if a_k is a partial isometry of rank one which is isometric on η'_k and takes η'_k into a positive multiple of η_k then

$$\omega(a_k) = \|\eta_k\| \, \|\eta'_k\| \geq \gamma^2(k+1)^{-2},$$

so $\omega(a_k) \neq 0$ and $\omega(a_k)^{-1} = O((k+1)^2)$, $k \geq 0$.

Next let η be a unit vector such that all components η_k and η'_k are non zero and all except finitely many of them (excluding repetitions) are equal respectively to ξ_k and ξ'_k constructed above. It follows that η satisfies the hypothesis of the preceding paragraph, whence $\omega_\eta \in P$.

Let ζ be any unit vector. It is clear that ζ can be approximated arbitrarily closely by a vector η satisfying the hypotheses of the immediately preceding paragraph, and therefore such that $\omega_\eta \in P$. This shows that P is dense in P_I (even in the norm topology). But P_I is dense in P_A (in the weak* topology), since I is essential in A, so P is dense in P_A.

OBSERVATION 3. *For every* $\omega \in P$ *there exist* $n(\omega) \geqslant 0$ *and* $l_0(\omega), l_1(\omega), \ldots$ $\ldots, l_{n(\omega)}(\omega) \in \mathbf{C}$ *such that*

$$\omega H = \sum_{m=0}^{n(\omega)} l_m(\omega)\omega\delta^m \,|\, A_\infty .$$

Proof. Fix $\omega \in P$. In order to apply Peetre's theorem, [13], to deduce the assertion, it is sufficient to show that every function in $C_{00}^\infty(]-\pi, \pi[)$ (i.e. every smooth function on $]-\pi, \pi[$ with compact support) can be expressed as

$$\omega(\tau a) :]-\pi, \pi[\ni t \mapsto \omega(\tau_t a)$$

for some $a \in A_\infty$. For if $a \in A_\infty$ and $\omega(\tau a)$ is zero on an open subset O of $]-\pi, \pi[$, then differentiation yields that $\omega(\tau\delta^m a)$ is zero on O for all m, whence by Condition 1 $\omega(\tau Ha)$ is zero on O. In other words, if $\omega(\tau a) \in C_{00}^\infty(]-\pi, \pi[)$ then also $\omega(\tau Ha) \in C_{00}^\infty(]-\pi, \pi[)$ and the map $\omega(\tau a) \mapsto \omega(\tau Ha)$ is local. By [13] a local operator on $C_{00}^\infty(]-\pi, \pi[)$ is a differential operator (locally of finite order). Recalling that the derivative of $\omega(\tau a)$ is $\omega(\tau\delta a)$, and evaluating at 0, we obtain

$$(*) \qquad \omega(Ha) = \sum_{m=0}^{n(\omega)} l_m(\omega)\omega(\delta^m a),$$

with $l_m(\omega) \in \mathbf{C}$, for all $a \in A_\infty$ with $\omega(\tau a) \in C_{00}^\infty(]-\pi, \pi[)$. Since also the map $\omega(\tau a) \mapsto$ $\mapsto \omega(\tau Ha)$ is a local operator on the space of all functions in $C^\infty(]-\pi, \pi[)$ which can be expressed as $\omega(\tau a)$ for some $a \in A_\infty$, it must by locality be given by the same differential operator as on the smaller subspace $C_{00}^\infty(]-\pi, \pi[)$. Hence $(*)$ above holds for all $a \in A_\infty$.

If $f \in C_{00}^\infty(]-\pi, \pi[)$, then f can be considered as a C^∞-function on $\mathbf{T}$ (equal to zero in a neighbourhood of $-1 \in \mathbf{T}$). Therefore

$$f(t) = \sum_{k \in \mathbf{Z}} \gamma_k e^{ikt}, \qquad t \in]-\pi, \pi[,$$

where $\gamma_k = O(|k|^{-p})$ for all $p > 0$. Then with a_k as in the statement of Observation 2 for $k \geqslant 0$, and a_k defined as a^*_{-k} for $k < 0$, $\omega(a_k)^{-1}\gamma_k = O(|k|^{-p})$ for all $p > 0$, so the series

$$\sum_{k \in \mathbf{Z}} \omega(a_k)^{-1}\gamma_k a_k$$

converges in A and defines an element a of A_∞. Since

$$\omega(\tau_t a_k) = \omega(a_k)e^{ikt}$$

it follows that, for all $t \in]-\pi, \pi[$,

$$\omega(\tau_t a) = \sum_{k \in \mathbf{Z}} \omega(a_k)^{-1}\gamma_k \omega(\tau_t a_k) = \sum_{k \in \mathbf{Z}} \gamma_k e^{ikt} = f(t).$$

OBSERVATION 4. *The functions l_m and n on P are constant.*

Proof. It is sufficient to prove this in restriction to a dense subset of P: if $\omega H = \sum_{m=0}^{n} \gamma_m \omega\delta^m | A_\infty$ for fixed $\lambda_0, \lambda_1, \ldots, \lambda_n$ and a dense set of $\omega \in P_A$, then this holds for all $\omega \in P_A$.

In order to select a suitable dense subset of P, we shall first modify the choice of the eigenvalues β_k and β'_k of h with $\beta_k - \beta'_k = k$, $k \in \mathbf{Z}^+$. This may change P, but as pointed out above this is immaterial.

The additional property of the sequences (β_k) and (β'_k) that we require is that there should exist a sequence (k_r) in $\mathbf{Z}^+$ such that all β_{k_r} are distinct and all β'_{k_r} are distinct. Note that with the initial choice of β_k and β'_k, all β_k might have been identical. To choose β_k and β'_k with this property, we introduce an equivalence relation on the set of all eigenvalues of h as follows: β is equivalent to β' if $\beta - \beta'$ is an integer. If all equivalence classes are finite, then any choice of β_k and β'_k will do; since there must be infinitely many equivalence classes which for at least one k contain the chosen pair $\{\beta_k, \beta'_k\}$, we can choose one k_r for each one, and then all β_{k_r} and all β'_{k_r} are distinct (even taken together, if no k_r is zero). Suppose that there exists an infinite equivalence class of eigenvalues of h. Then for any k, there is an arbitrarily large pair of eigenvalues with difference at least k. This allows us to choose a sequence (k_r) and sequences (β_{k_r}), (β'_{k_r}) of distinct eigenvalues with $\beta_{k_r} - \beta'_{k_r} = k_r$, and we then as before make any choice of β_k and β'_k with $\beta_k - \beta'_k = k$ for values of k not among the k_r.

Now fix $\omega_\zeta \in P$, and define a subset Q of P as follows: $\omega_\xi \in Q$ if $\omega_\xi \in P$ and for almost all r the components of ξ in the eigenspaces of h corresponding to β_{k_r} and β'_{k_r} are equal to the components of ζ. It is not difficult to see that Q is dense in P.

Let $\omega_1 = \omega_\xi$ and $\omega_2 = \omega_\eta$ be distinct pure states in Q and denote by ξ_k, ξ'_k and η_k, η'_k the components of ξ and η in the eigenspaces of h corresponding to the eigenvalues β_k and β'_k, $k \in \mathbf{Z}^+$. (Recall that these components are non zero.) Then for almost all r, $\xi_{k_r} = \eta_{k_r}$ and $\xi'_{k_r} = \eta'_{k_r}$.

We shall now use an idea from [4]. Let ρ and σ be non zero complex numbers such that $\rho + \sigma \neq 0$ and $\|\rho\zeta + \sigma\eta\| = 1$. Thus the pure state $\omega_3 = \omega_{\rho\zeta+\sigma\eta}$ is distinct from ω_1 and ω_2. Assume that $\omega_3 \in P$. Recall that the set of pure states of $M_2(\mathrm{C})$ is a two sphere (see, for example, [9], Example 4.2.7), and so the real affine span of three distinct pure states of $M_2(\mathrm{C})$ intersects the set of pure states of $M_2(\mathrm{C})$ in a circle. Let μ_1, μ_2, and μ_3 be real numbers with sum one such that $\omega = \sum_{j=1}^{3} \mu_j\omega_j$ is a pure state. Assume that ω belongs to P. By Observation 3,

$$\omega_j H = \sum_{m\geq 0} l_m(\omega_j)\omega_j\delta^m|A_\infty,$$

$$\omega H = \sum_{m\geq 0} l_m(\omega)\omega\delta^m|A_\infty.$$

Substituting $\omega = \sum_{j=1}^{3} \mu_j\omega_j$ yields

$$\sum_{m\geq 0}\left(\sum_{j=1}^{3}(l_m(\omega) - l_m(\omega_j))\mu_j\omega_j\right)\delta^m = 0.$$

Applying this to a_k such that $\delta a_k = \mathrm{i}ka_k$ then yields

$$\sum_{m\geq 0}\left(\sum_{j=1}^{3}(l_m(\omega) - l_m(\omega_j))\mu_j\omega_j(a_k)\right)(\mathrm{i}k)^m = 0.$$

By the choice of ω_1, ω_2, and ω_3, for almost any r there exist a_{k_r} with $\delta(a_{k_r}) = \mathrm{i}k_r a_{k_r}$ and $\gamma_{k_r} > 0$ such that

$$\omega_j(a_{k_r}) = \alpha_j\gamma_{k_r}, \quad j = 1, 2, 3,$$

where $\alpha_1 = \alpha_2 = 1$ and $\alpha_3 = |\rho + \sigma|^2 > 0$. (With r such that $\xi_{k_r} = \eta_{k_r}$ and $\xi'_{k_r} = \eta'_{k_r}$ we may take $a_{k_r} = (\cdot\,|\xi'_{k_r})\xi_{k_r}$ and $\gamma_{k_r} = \|\xi'_{k_r}\|^2\|\xi_{k_r}\|^2$.) Substituting this into the preceding equation yields

$$\sum_{m\geq 0}\left(\sum_{j=1}^{3}(l_m(\omega) - l_m(\omega_j))\,\mu_j\alpha_j\right)(\mathrm{i}k_r)^m = 0,$$

for almost all r. Choosing a number of r for which this holds equal to the number of m appearing (which is finite — at most $\max(n(\omega_j), n(\omega))$), and noting that the coefficient matrix $((ik_r)^m)_{r,m}$ is a Vandermonde matrix and therefore invertible, we have

$$\sum_{j=1}^{3}(l_m(\omega)-l_m(\omega_j))\mu_j\alpha_j=0, \quad m=0,1,2,\ldots .$$

Of course, we still have to ensure that $\omega_3=\omega_{\rho\xi+\sigma\eta}$ and $\omega=\sum_{j=1}^{3}\mu_j\omega_j$ belong to P, so that $l_m(\omega_3)$ and $l_m(\omega)$ exist.

The strategy of our proof that $l_m(\omega_1)=l_m(\omega_2)$ for all m can be briefly described as follows. Fix $\omega_1=\omega_\xi\in Q$. We shall show that for a dense set of $\omega_2=\omega_\eta$ in Q it is possible to choose ρ and σ, and μ_1,μ_2,μ_3, so that ω_3 and ω belong to P, and so that, furthermore, the vector $(\mu_j\alpha_j)$ in $\mathbf{R}^3$ is orthogonal to $(1,1,1)$, but not to $(l_m(\omega_j))$ unless $l_m(\omega_1)=l_m(\omega_2)$. This together with the result of the preceding paragraph, that $(\mu_j\alpha_j)$ is orthogonal to the vectors

$$l_m(\omega)(1,1,1)-(l_m(\omega_j)), \quad m=0,1,2,\ldots,$$

yields $l_m(\omega_1)=l_m(\omega_2)$ for all m. This shows that the functions l_m are constant on a dense subset of Q, and in particular on a dense subset of P, as desired.

Fix $\omega_\xi\in Q$. By writing $\omega_\eta\in Q$ for a unit vector η, let us understand that η itself, not just up to a phase, has the property that $\eta_{k_r}=\zeta_{k_r}$ and $\eta'_{k_r}=\zeta'_{k_r}$ for almost all r. Let ω_0 be an arbitrary vector state in P, distinct from ω_ξ, and choose a unit vector φ such that $\omega_0=\omega_\varphi$ and $(\varphi\mid\xi)\geqslant 0$. ($\varphi$ is unique if $(\varphi\mid\xi)\neq 0$.) Approximate φ by a unit vector η, having the same projection as φ orthogonal to the eigenspaces of h, such that $\omega_2=\omega_\eta\in Q$. Changing the components of η in finitely many eigenspaces of h, we may suppose that $(\eta\mid\xi)=(\varphi\mid\xi)$, and that still η is a unit vector close to φ and $\omega_\eta\in Q$.

Next, note that if a finite number of pairs of complex numbers $(\rho_1,\sigma_1),\ldots$ $\ldots,(\rho_p,\sigma_p)$ are given, such that $\rho_q+\sigma_q\neq 0$ and $\|\rho_q\xi+\sigma_q\eta\|=1$, a property depending only on $(\eta\mid\xi)$ which is now fixed, then we can choose η so that, in addition, each of the pure states $\omega_{\rho_q\xi+\sigma_q\eta}$ belongs to P. To make this simpler, we shall suppose that the reference state ω_ζ for the definition of Q is such that $\|\zeta_k\|^{-1}=O(k+1)$, $\|\zeta'_k\|^{-1}=O(k+1)$. (See the proof of Observation 2. In general, $\omega_\zeta\in P$ just means that $\|\zeta_k\|^{-1}\|\zeta'_k\|^{-1}=O((k+1)^2)$.) We shall also suppose that $\|\xi_k\|^{-1}=O(k+1)$, $\|\xi'_k\|^{-1}=O(k+1)$, and that $\|\eta_k\|^{-1}=O(k+1)$, $\|\eta'_k\|^{-1}=O(k+1)$, as we can do without loss of generality. To ensure that $\omega_{\rho_q\xi+\sigma_q\eta}\in P$, first change η by a small amount, as described below, so that the components $(\rho_q\xi+\sigma_q\eta)_k$ and $(\rho_q\xi+\sigma_q\eta)'_k$ are non zero, and the inverses of their norms are $O(k+1)$. Second, make slight changes to finitely many components of η, such that this property is not destroyed, and so that again $\|\eta\|=1$ and $(\eta\mid\xi)=(\varphi\mid\xi)$ as before. Since $\rho_q+\sigma_q\neq 0$ it is not necessary to change the components η_{k_r}, η'_{k_r}

which are equal to ξ_{k_r}, ξ'_{k_r}, and preserving these components one still has $\omega_\eta \in Q$. The way to change η in the first step is to break the components η_k, η'_k with $k \notin \{k_r\,; r = 1, 2, \ldots\}$ into finitely many groups in an appropriate way and then multiply each group by an appropriate positive scalar close to 1. This is done as follows.

First let $\varepsilon_1 > 0$ and break the components $(\rho_1\xi + \sigma_1\eta)_k$, $(\rho_1\xi + \sigma_1\eta)'_k$ into two groups according as they are greater than, or strictly less than the corresponding components of $\varepsilon_1\eta$ in norm. Multiplying the components of η in the latter group by any positive number larger than $1 + 2\varepsilon_1|\sigma_1|^{-1}$ makes this group disappear; that is, one then has the first inequality ($\geqslant$) everywhere. Next let $\varepsilon_2 > 0$ and perform a similar operation to render all (relevant) components of $\rho_2\xi + \sigma_2\eta$ greater in norm than the corresponding components of $\varepsilon_2\eta$. Since this involves multiplying certain components of η by $1 + 2\varepsilon_2|\sigma_2|^{-1}$, or by a larger number γ, it follows that if ε_2 is sufficiently small relative to ε_1, and if γ is not much larger than $1 + 2\varepsilon_2|\sigma_2|^{-1}$, then the components of $\rho_1\xi + \sigma_1\eta$ are still greater in norm than those of $\varepsilon_1\eta/2$. Continuing in this way, after performing a similar operation p times, we have ensured that all relevant components of all $\rho_q\xi + \sigma_q\eta$ are greater in norm than the corresponding components of $\gamma\eta$ for some $\gamma > 0$. Since $\|\eta_k\|^{-1} = O(k + 1)$ and $\|\eta'_k\|^{-1} = O(k + 1)$, the vectors $\rho_q\xi + \sigma_q\eta$ now have the same property.

We shall now apply the preceding remark to the three pairs (ρ_1, σ_1), (ρ_2, σ_2), and (ρ_3, σ_3) of complex numbers defined as follows. First,

$$\rho_1 = \sigma_1 = \|\xi + \eta\|^{-1}.$$

Note that $\omega_{\rho_1\xi+\sigma_1\eta}$, ω_ξ, and ω_η lie on a circle of pure states — a circle on the two sphere $\{\omega_\psi\,; \psi \in \mathbf{C}\xi + \mathbf{C}\eta\}$. Since $(\eta\,|\,\xi)$ is real, this circle is equal to $\{\omega_\psi\,; \psi \in \mathbf{R}\xi + \mathbf{R}\eta\}$, which, again since $(\eta\,|\,\xi)$ is real (so that the orthogonal complement of ξ in the real Hilbert space $\mathbf{R}\xi + \mathbf{R}\eta$ is also orthogonal to ξ in $\mathbf{C}\xi + \mathbf{C}\eta$), is a great circle on the two sphere $\{\omega_\psi\,; \psi \in \mathbf{C}\xi + \mathbf{C}\eta\}$. Furthermore, the angle between ω_ξ and ω_η on this circle is bisected by $\omega_{\rho_1\xi+\sigma_1\eta}$: the cosine of the angle between $\omega_{\rho_1\xi+\sigma_1\eta}$ and ω_ξ or ω_η is $(\eta\,|\,\xi)$, which is positive. Choose ρ_2, σ_2 and ρ_3, σ_3 so that $\omega_{\rho_2\xi+\sigma_2\eta}$ and $\omega_{\rho_3\xi+\sigma_3\eta}$ are the points on the circle through ω_ξ, ω_η, and $\omega_{\rho_1\xi+\sigma_1\eta}$ that lie on the line parallel to the line through ω_ξ and ω_η and at distance $2\sqrt{2}/3$ from $\omega_{\rho_1\xi+\sigma_1\eta}$. (Recall that the diameter of the great circle is two.) The pairs (ρ_j, σ_j), $j = 1, 2, 3$, depend only on $(\eta\,|\,\xi)$, which is fixed. Trivially, $\rho_1 + \sigma_1 \neq 0$, and it is not difficult to check that $\rho_2 + \sigma_2 \neq 0$ and $\rho_3 + \sigma_3 \neq 0$.

Now consider the plane in $\mathbf{R}^3$ through the points $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, \alpha_3)$, where $\alpha_3 = |\rho_1 + \sigma_1|^2$. In this plane consider the ellipse

$$\{(\mu_1\alpha_1, \mu_2\alpha_2, \mu_3\alpha_3)\,; \mu_1\omega_1 + \mu_2\omega_2 + \mu_3\omega_3 \in P_A\},$$

where $\alpha_1 = \alpha_2 = 1$, $\omega_1 = \omega_\xi$, $\omega_2 = \omega_\eta$, and $\omega_3 = \omega_{\rho_1\xi+\sigma_1\eta}$. The intersection of this ellipse with the orthogonal complement of $(1, 1, 1)$ consists of two points which are $\sqrt{2/3}$ of the way along the length of the ellipse from $(0, 0, \alpha_3)$, i.e. the points corresponding to $\omega_{\rho_2\xi+\sigma_2\eta}$ and $\omega_{\rho_3\xi+\sigma_3\eta}$, in the affine correspondence which takes ω_1, ω_2, and ω_3 to $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, \alpha_3)$. The line through these two points is the intersection of the plane of the ellipse with the orthogonal complement of $(1, 1, 1)$. Fix $m = 0, 1, 2, \ldots$. If $l_m(\omega_1) \neq l_m(\omega_2)$ then the orthogonal complement of $(l_m(\omega_j))$ is a plane through $(0, 0, 0)$ which is different from the orthogonal complement of $(1, 1, 1)$, and therefore the intersection of this plane with the plane of the ellipse is a line different from the above. Since these two lines can therefore intersect in at most one point, not both the points in the intersection of the first line and the ellipse can lie on the second line. Thus, either the point corresponding to $\omega_{\rho_2\xi+\sigma_2\eta}$ or that corresponding to $\omega_{\rho_3\xi+\sigma_3\eta}$ is not orthogonal to $(l_m(\omega_j))$, assuming that $l_m(\omega_1) \neq l_m(\omega_2)$. Choosing

$$\omega = \sum_{j=1}^{3} \mu_j \omega_j$$

to be one of these states, then we have $(\mu_j \alpha_j)$ orthogonal to $(1, 1, 1)$, but not to $(l_m(\omega_j))$, unless $l_m(\omega_1) = l_m(\omega_2)$.

Condition 2 of the theorem follows immediately from Observation 4 and the density of P.

2. PROOF OF THEOREM 2

2.1. LEMMA. *Let (A, G, τ) be a C^*-dynamical system. Then* (Prim A, G, τ) *is a topological dynamical system.*

Proof. We must show that the map

$$G \times \text{Prim}\, A \ni (g, \gamma) \mapsto \tau_g \gamma \in \text{Prim}\, A$$

is continuous. This follows from 3.4.11 of [10] and the fact that the map

$$G \times P_A \ni (g, \omega) \mapsto \omega\tau_g \in P_A$$

is continuous.

2.2. LEMMA. *Let X be a topological space (not necessarily Hausdorff) and let $(X, \mathbf{R}, \tau)$ be a topological dynamical system (*i.e. *$X \times \mathbf{R} \ni (\gamma, t) \mapsto \tau_t\gamma \in X$ is continuous, τ_0 is the identity, and $\tau_{s+t} = \tau_s\tau_t$). Let γ be a point of X which is open (*i.e. *isolated) in its closure, and let $\varepsilon > 0$.*

If γ is relatively closed in its orbit in X, or if ε is sufficiently small, then the compact subset $\tau_{[-\varepsilon,\varepsilon]}\gamma$ of X is relatively open in its closure.

Proof. Let us first show that if K is a compact subset of $\mathbf{R}$ then the closure of $\tau_K\gamma$ is the union of the closures of its points. Let $\gamma' \in X$ be such that there is a net (γ_i) in X converging to γ' with $\gamma_i \in \{\tau_{t_i}\gamma\}^-$ where $t_i \in K$. Passing to a subnet of (γ_i), we may suppose that the net (t_i) is convergent in K, say to t. Since τ_{t_i} is a homeomorphism, $\tau_{-t_i}\gamma_i \in \{\gamma\}^-$. By continuity, $\tau_{-t_i}\gamma_i$ converges to $\tau_{-t}\gamma'$. Hence $\tau_{-t}\gamma' \in \{\gamma\}^-$, i.e. $\gamma' \in \{\tau_t\gamma\}^-$.

Now choose an open neighbourhood O of γ in X such that $O \cap \{\gamma\}^- = \{\gamma\}$. If $\{\gamma\}$ is not relatively closed in the orbit of γ, restrict ε to be small enough that $\tau_{[-2\varepsilon,2\varepsilon]}\gamma \subseteq O$. Let us show that $\tau_{[-\varepsilon,\varepsilon]}\gamma$ is open in its closure. In view of the above description of the closure, what we must show is that if $\gamma' \in X$ and (γ_i) is a net in $X \setminus \tau_{[-\varepsilon,\varepsilon]}\gamma$ converging to γ' with $\gamma_i \in \{\tau_{t_i}\gamma\}^-$, where $t_i \in [-\varepsilon, \varepsilon]$, then $\gamma' \in X \setminus \tau_{[-\varepsilon,\varepsilon]}\gamma$.

As above, we may suppose that the net (t_i) converges in $[-\varepsilon, \varepsilon]$, say to t. We have $\tau_{-t_i}\gamma_i \in \{\gamma\}^-$, and $\tau_{[-\varepsilon,\varepsilon]}\gamma_i \subseteq X \setminus \{\gamma\}$, whence $\tau_{-t_i}\gamma_i \in \{\gamma\}^- \setminus \{\gamma\}$. By continuity, $\tau_{-t_i}\gamma_i$ converges to $\tau_{-t}\gamma'$. Since $\{\gamma\}^- \setminus \{\gamma\}$ is closed in X, it follows that

$$\tau_{-t}\gamma' \in \{\gamma\}^- \setminus \{\gamma\}.$$

In the case that $\{\gamma\}$ is relatively closed in the orbit of γ, it follows that $\tau_{-t}\gamma'$ does not belong to the orbit of γ; equivalently, γ' does not belong to the orbit of γ. In the opposite case, by the choice of ε, $\tau_{[-2\varepsilon,2\varepsilon]}\gamma \subseteq O$. As $t \in [-\varepsilon, \varepsilon]$, we have $\tau_{-t}\tau_{[-\varepsilon,\varepsilon]}\gamma \subseteq O$, and since $\tau_{-t}\gamma' \in \{\gamma\}^- \setminus \{\gamma\} \subseteq X \setminus O$ it follows that, also in this case, $\gamma' \in X \setminus \tau_{[-\varepsilon,\varepsilon]}\gamma$, as desired.

2.3. LEMMA. *Let $(X, \mathbf{R}, \tau)$ be a topological dynamical system. Suppose that τ is transitive,* i.e. *X consists of a single orbit. Suppose that X is not a point, and that distinct points have distinct closures.*

1. *τ is periodic if, and only if, X is a circle.*

2. *If each point of X is closed then $\tau_{[-\varepsilon,\varepsilon]}\gamma$ is closed, compact, and Hausdorff for any $\gamma \in X$ and $\varepsilon > 0$.*

Proof. Ad 1. If τ is periodic, with period p, then by Lemma 17.2 of [12], with G the compact group $\mathbf{R}/\mathbf{Z}p$ and K the subgroup $\{0\}$, X is a circle.

Let X be a circle. To show that τ is periodic we must show that the map $t \mapsto \tau_t$ is not injective, i.e., for fixed $\gamma \in X$ the map $t \mapsto \tau_t\gamma$ is not injective. Note that $X = \bigcup_{n \geq 1} \tau_{[-n,n]}\gamma$, and each $\tau_{[-n,n]}\gamma$ is connected. Since the circle is not the union of an increasing sequence of connected proper subsets, $X = \tau_{[-n,n]}\gamma$ for some n. For such an n, the map $t \mapsto \tau_t\gamma$ is not injective on $[-n, n]$.

Ad 2. By continuity of $t \mapsto \tau_t\gamma$, the set $\tau_{[-\varepsilon,\varepsilon]}\gamma$ is compact in X. Let us show that it is Hausdorff. Suppose that $(\tau_{t_i}\gamma)$ converges in $\tau_{[-\varepsilon,\varepsilon]}\gamma$, say to $\tau_t\gamma$, and let us show that the limit is unique. We use the argument of Lemma 17.2 of [12].

We may assume that (t_i) converges in $[-\varepsilon, \varepsilon]$, say to t'. Then by continuity,

$$\gamma = \tau_{t_i}^{-1}\tau_{t_i}\gamma \mapsto \tau_{t'}^{-1}\tau_t\gamma.$$

Since $\{\gamma\}$ is closed, $\gamma = \tau_{t'}^{-1}\tau_t\gamma$, i.e. $\tau_t\gamma = \tau_{t'}\gamma$.

It remains to show that $\tau_{[-\varepsilon,\varepsilon]}\gamma$ is closed. Let $\delta \in (\tau_{[-\varepsilon,\varepsilon]}\gamma)^-$. Choose $\varepsilon' > \varepsilon$ such that $\delta \in \tau_{[-\varepsilon',\varepsilon']}\gamma$. Since $\tau_{[-\varepsilon',\varepsilon']}\gamma$ is Hausdorff and $\tau_{[-\varepsilon,\varepsilon]}\gamma$ is a compact subset of $\tau_{[-\varepsilon',\varepsilon']}\gamma$, $\tau_{[-\varepsilon,\varepsilon]}\gamma$ is relatively closed in $\tau_{[-\varepsilon',\varepsilon']}\gamma$. In particular, $\delta \in \tau_{[-\varepsilon,\varepsilon]}\gamma$.

2.4. *Proof of Theorem 2.* We must prove $1 \Rightarrow 2$. (The implication $2 \Rightarrow 1$ is immediate.)

OBSERVATION 1. *For every $\gamma \in S$ there exist unique $l_0(\gamma), l_1(\gamma), \ldots, l_{n(\gamma)}(\gamma) \in \mathbf{C}$ such that*

$$Ha + \gamma = \sum_{m=0}^{n(\gamma)} l_m(\gamma)\delta^m a + \gamma, \quad a \in A_\infty.$$

Proof. Fix $\gamma \in S$. Since $\{\gamma\}$ is open in its closure, there is a simple subquotient B of A with primitive spectrum $\{\gamma\}$. We shall show, using Condition 1 and Peetre's theorem, [13], that for each $\omega \in P_B$ (i.e. for each pure state ω of A/γ not zero on the ideal B of A/γ) there exist $l_0(\omega), l_1(\omega), \ldots, l_{n(\omega)}(\omega) \in \mathbf{C}$ such that

$$\omega(Ha) = \sum_{m=0}^{n(\omega)} l_m(\omega)\omega(\delta^m a), \quad a \in A_\infty.$$

Hence by the proof of Theorem 3.1 of [4] (that part which uses Lemma 3.2 of [4]), $l_m(\omega)$ is independent of $\omega \in P_B$. In other words, there exist $n(\gamma) \geqslant 0$ and $l_0(\gamma), l_1(\gamma), \ldots$ $\ldots, l_{n(\gamma)}(\gamma) \in \mathbf{C}$ such that

$$Ha + \gamma = \sum_{m=0}^{n(\gamma)} l_m(\gamma)\delta^m a + \gamma, \quad a \in A_\infty.$$

Fix $\omega \in P_B$. In order to apply Peetre's theorem to deduce that $\omega H = \sum_{m=0}^{n(\omega)} l_m(\omega)\omega\delta^m | A_\infty$, it is sufficient to show that every function in $C_{00}^\infty(]-1, 1[)$ (i.e. every smooth function on $]-1, 1[$ with compact support) can be expressed as

$$\omega(\tau a) :]-1, 1[\ni t \mapsto \omega(\tau_t a)$$

for some $a \in A_\infty$. For if $a \in A_\infty$ and $\omega(\tau a)$ is zero on an open subset O of $]-1, 1[$, then differentiating yields that $\omega(\tau\delta^m a)$ is zero on O for all m, whence by Condition 1 $\omega(\tau Ha)$ is zero on O. In other words, if $\omega(\tau a) \in C_{00}^\infty(]-1, 1[)$ then also $\omega(\tau Ha) \in C_{00}^\infty(]-1, 1[)$ and the map $\omega(\tau a) \mapsto \omega(\tau Ha)$ is local. By [13] a local operator on $C_{00}^\infty(]-1, 1[)$ is a differential operator (locally, of finite order). Recalling that the derivative of $\omega(\tau a)$ is $\omega(\tau\delta a)$, and evaluating at 0, we obtain

$$(*) \qquad \omega(Ha) = \sum_{m=0}^{n(\omega)} l_m(\omega)\omega(\delta^m a),$$

with $l_m(\omega) \in \mathbf{C}$, for all $a \in A_\infty$ with $\omega(\tau a) \in C_{00}^\infty(]-1, 1[)$. Since also the map $\omega(\tau a) \mapsto \mapsto \omega(\tau Ha)$ is a local operator on the space of all functions in $C^\infty(]-1, 1[)$ which can be expressed as $\omega(\tau a)$ for some $a \in A_\infty$, it must by locality be given by the same differential operator as on the smaller subspace $C_{00}^\infty(]-1, 1[)$. Hence $(*)$ above holds for all $a \in A_\infty$.

Of course, the orbit of γ in X may be periodic, but as $\gamma \in S \subseteq X \setminus X_0$, the period p of γ is not zero, so we may change the scale so that $1 < p \leqslant +\infty$. Fix $t \geqslant 0$ with $1 + t < p$, and let us show that the subset $\tau_{]-1,1-t[}\gamma$ of the closure of $\tau_{[-1-t,1]}\gamma$ in X is open relative to this closed set. By Lemma 2.2, combined with Lemma 2.1, $\tau_{[-1-t,1]}\gamma$ is open relative to its closure (as $\gamma \in S$). By Lemma 2.3 applied to the orbit of γ (each point of which is relatively closed, as $\gamma \in S$), $\tau_{[-1-t,1]}\gamma$ is Hausdorff; hence the map

$$[-1-t,1] \ni s \mapsto \tau_s\gamma \in \tau_{[-1-t,1]}\gamma$$

is a homeomorphism. It follows first that the subset $\tau_{]-1,1-t[}\gamma$ is open relative to the set $\tau_{[-1-t,1]}\gamma$, and hence that it is open relative to the closure of this set.

Note next that an automorphism α of a C^*-algebra R, even if it does not take a closed two-sided ideal I of R onto itself, or even into itself, does induce an isomorphism between two closed two-sided ideals of R/I, which may be different from zero — namely, the images in R/I of the ideals J and αJ where

$$J = \{a \in R\,;\, a(\alpha^{-1}I + I) \subseteq \alpha^{-1}I \cap I\}.$$

To check that the isomorphism $\alpha \,|\, J : J \to \alpha J$ induces a homomorphism between $(J + I)/I$ and $(\alpha J + I)/I$, suppose that $a \in J$ and $a \in I$. From $\alpha a \in \alpha I$ we deduce that αI contains a right approximate unit for αa. From $\alpha a \in \alpha J$ we deduce that $(\alpha a)(\alpha I) \subseteq I$. It follows from these two properties of αa that $\alpha a \in I$, and so one has a homomorphism. A similar argument shows that this homomorphism is injective. It is clearly surjective.

Note now that each subset of X which is open relative to its closure is the primitive spectrum of a subquotient of A, i.e. of an ideal of a quotient of A. Apply the preceding result with $R = A$, $\alpha = \tau_t$ where $0 < t < p - 1$, and I the closed two-sided ideal of A with primitive spectrum the complement of the closure of $\tau_{[-1,1]}\gamma$ in X, to conclude that τ_t induces an isomorphism between the subquotients of A with primitive spectra $\tau_{]-1,1-t[}\gamma$ and $\tau_{]-1+t,1[}\gamma$. The action of τ_t on these primitive spectra may be identified with translation of the interval $]-1, 1-t[$ onto $]-1+t, 1[$ inside $]-1, 1[$.

If follows that the subquotient of A with primitive spectrum $\tau_{]-1,1[}\gamma$ is isomorphic to $C_0(]-1,1[)\otimes B$, and in such a way that for each $0 < t < p-1$ the isomorphism induced by τ_t is just translation from the ideal $C_0(]-1,1-t[)\otimes B$ onto the ideal $C_0(]-1+t,1[)\otimes B$. We deduce from this that, with $\omega\in P_B$ as above, for any $f\in C_{00}(]-1,1[)$ there exists $a\in A$ such that $\omega(\tau a)=f$. Let us now show that if $f\in C_{00}^\infty(]-1,1[)$ then a can be chosen in A_∞. Extend f to a function in $C_{00}^\infty(\mathbf{R})$ with support contained in $[-p,p]$. By [11], Théorème 3.2, f is a finite sum $\sum g_i * h_i$ where $g_i, h_i\in C_{00}^\infty(\mathbf{R})$ and $\operatorname{supp} g_i$ is sufficiently small that

$$[-1,1]\setminus \operatorname{supp} g_i \subseteq [-(1+p)/2,(1+p)/2].$$

Choose $b_i\in A$ as before such that $\omega(\tau_t b_i)=h_i(t)$ for all $t\in]-(1+p)/2,(1+p)/2[$. (Work with this interval instead of $]-1,1[$.) Set $\sum\int \mathrm{d}s\, g_i(s)\tau_{-s}b_i = a$. It follows that $a\in A_\infty$ and for all $t\in]-1,1[$,

$$\omega(\tau_t)a = \sum\int \mathrm{d}s\, g_i(s)\omega(\tau_{t-s}b_i) = \sum\int \mathrm{d}s\, g_i(s)h_i(t-s) = f(t),$$

as desired.

Uniqueness of the numbers $l_0(\gamma), l_1(\gamma),\ldots, l_{n(\gamma)}(\gamma)$ (or, rather, uniqueness of the numbers $l_0(\omega), l_1(\omega),\ldots, l_{n(\omega)}(\omega)$ for any $\omega\in P_B$) holds for example by Lemma 3.2 of [4].

OBSERVATION 2. *There exists $n\geqslant 0$ such that $H: A_\infty\to A$ is bounded with respect to the norm $a\mapsto \|a\|_n=\sup_{0\leqslant k\leqslant n}\|\delta^k a\|$ on A_∞ (and the norm $\|\cdot\|$ on A).*

Proof. It is sufficient to show that $H: A_\infty\to A$ is closed. This follows from Observation 1 and the hypothesis that S is dense in X. (If $a_k\to 0$ in A_∞ and $Ha_k\to b$ in A then by Observation 1, for each $\gamma\in S$, $Ha_k+\gamma=\sum l_m(\gamma)\delta^m a_k + \gamma\to 0$ and hence $b+\gamma=0$; since S is dense this implies $b=0$.)

OBSERVATION 3. *$n(\gamma)\leqslant n$ for all $\gamma\in S$.*

Proof. Fix $\gamma\in S$. By Observation 1, $Ha+\gamma=\sum_{m=0}^{n(\gamma)} l_m(\gamma)\delta^m a+\gamma$, $a\in A_\infty$. We must show that $l_m(\gamma)=0$ if $m>n$, where n is as in Observation 2, i.e. such that $H: A_\infty\to A$ is bounded with respect to the norm $\|\cdot\|_n$ on A_∞.

Denote by I the largest τ-invariant closed two-sided ideal of A contained in $\gamma: I=\bigcap_{t\in\mathbf{R}}\tau_t\gamma$. Note that the image of A_∞ in A/I is equal to all of $(A/I)_\infty$ (by [11], Théorème 3.2, any $x\in(A/I)_\infty$ is a finite sum $\sum\int \mathrm{d}s\, f_i(s)\tau_s y_i$ where $f_i\in C_{00}^\infty(\mathbf{R})$ and $y_i\in A/I$). Let us show that H induces an operator from $(A/I)_\infty$ to A/I. We must show that if $a\in A_\infty\cap I$ then $Ha\in I$. If $a\in A_\infty\cap I$ then $\delta^k a\in I$ for all $k=$

$= 0, 1, 2, \ldots$, so if $\omega \in P_{A/I}$ then $\omega(\delta^k a) = 0$ for all $k = 0, 1, 2, \ldots$, whence by locality of H with respect to τ, $\omega(Ha) = 0$. This proves that $Ha \in I$ if $a \in A_\infty \cap I$. Clearly the induced operator $H : (A/I)_\infty \to A/I$ is local with respect to τ. (In this paragraph we have used only that I is τ-invariant.)

Since $\gamma \in X \setminus X_0$, δ is not bounded in A/I. Therefore, there exists arbitrarily large $\beta \in \mathbf{R}$ such that, for all $\varepsilon > 0$, $(A/I)^\tau([\beta - \varepsilon, \beta + \varepsilon]) \neq 0$. Hence, for arbitrarily large $\beta \in \mathbf{R}$, there exists $a_\beta \in A_\infty$ such that $\|a_\beta + I\| = 1$ and $\|\delta^m a_\beta - (i\beta)^m a_\beta + I\|$ is arbitrarily small for $0 \leqslant m \leqslant \max(n, n(\gamma))$. (See [8], Proposition 1.1.) Replacing a_β by $\tau_t a_\beta$ for suitable $t \in \mathbf{R}$ (note that $\tau_{\mathbf{R}}\gamma$ is dense in the primitive spectrum of A/I), we may suppose that $\|a_\beta + \gamma\|$ is arbitrarily close to $\|a_\beta + I\| = 1$. We now have

(i) $\|a_\beta + I\|_n = O(\beta^n)$,

(ii) $\|Ha_\beta + \gamma\| = |l_{n(\gamma)}(\gamma)|\, \beta^{n(\gamma)} + O(\beta^{n(\gamma)-1})$,

for arbitrarily large β (assuming that β is at least one, say).

If we knew that H is continuous with respect to the norm $\|\cdot\|_n$ in A/I, then the desired inequality $n(\gamma) \leqslant n$ would follow immediately from these estimates. It is not clear, however, how to deduce this from continuity of H with respect to the norm $\|\cdot\|_n$ in A, as it is not clear how to lift an element a from $(A/I)_\infty$ to A_∞ without substantially increasing the norm $\|a\|_n$. Nevertheless, the estimate (i) was obtained in a rather special way, namely, by choosing $a_\beta + I$ of norm one in $(A/I)^\tau([\beta - \varepsilon, \beta + \varepsilon])$ for ε sufficiently small, and it is possible, as we shall now show, to choose a_β in such a way that, in addition,

(i)′ $\|a_\beta\|_n = O(\beta^n)$.

The desired inequality $n(\gamma) \leqslant n$ follows from (i)′ and (ii).

Choose $f \in L^1(\mathbf{R})$ such that $\hat{f}$ is equal to 1 on $[-1, 1]$ and to 0 outside $[-2, 2]$. For each $\beta \in \mathbf{R}$ and $\varepsilon > 0$, define $f_{\beta,\varepsilon} \in L^1(\mathbf{R})$ by

$$f_{\beta,\varepsilon}(t) = e^{i\beta t}\varepsilon f(\varepsilon t).$$

Then $\int |f_{\beta,\varepsilon}| = \int |f|$, and $(f_{\beta,\varepsilon})^\wedge$ is equal to 1 on $[\beta - \varepsilon, \beta + \varepsilon]$ and to 0 outside $[\beta - 2\varepsilon, \beta + 2\varepsilon]$. Let $b_\beta \in A$ be such that $\|b_\beta\| = \|b_\beta + I\| = 1$ and $b_\beta + I \in (A/I)^\tau([\beta - \varepsilon, \beta + \varepsilon])$, where $\varepsilon > 0$ will be specified later. Set

$$\int \mathrm{d}s\, f_{\beta,\varepsilon}(s)\tau_s b_\beta = a_\beta .$$

Then $a_\beta \in A^\tau([\beta - 2\varepsilon, \beta + 2\varepsilon])$, $\|a_\beta\| \leqslant \int |f|$, and $a_\beta + I = b_\beta + I$. Hence, if ε is sufficiently small, not only do we have as before the estimates (i) and (ii), but we also have (i)′. (See [8], Proposition 1.1.)

OBSERVATION 4. *For every $\gamma \in X$ such that the spectrum of τ in the quotient $A/\bigcap_{t\in\mathbf{R}} \tau_t\gamma$ consists of at least $n+1$ points, there exists a neighbourhood O of γ in X such that the functions $l_0, l_1, \ldots, l_n$ defined on S are bounded on $S \cap O$.*

Proof. Choose distinct points $\beta_0, \beta_1, \ldots, \beta_n$ in the spectrum of τ in the quotient A/I, where $I = \bigcap_{t\in\mathbf{R}} \tau_t\gamma$. Thus, $(A/I)^\tau([\beta_k - \varepsilon, \beta_k + \varepsilon]) \neq 0$, for all $\varepsilon > 0$. Hence as shown in the proof of Observation 3, $A^\tau([\beta_k - 2\varepsilon, \beta_k + 2\varepsilon]) \nsubseteq \gamma$, for all $\varepsilon > 0$. Choose $0 < \varepsilon < 1$ such that the intervals $[\beta_k - 3\varepsilon, \beta_k + 3\varepsilon]$ are mutually disjoint. (The distance between $[\beta_j - 2\varepsilon, \beta_j + 2\varepsilon]$ and $[\beta_k - 2\varepsilon, \beta_k + 2\varepsilon]$ is then at least $(1/3)|\beta_j - \beta_k|$.)

Denote by O the set of $\gamma' \in X$ such that $A^\tau([\beta_k - 2\varepsilon, \beta_k + 2\varepsilon]) \nsubseteq \gamma'$, for all k. O is open and $\gamma \in O$.

Fix $\gamma' \in O \cap S$. Set $\bigcap_{t\in\mathbf{R}} \tau_t\gamma' = I'$. Since the image of $A^\tau([\beta_k - 2\varepsilon, \beta_k + 2\varepsilon])$ in A/I' is contained in $(A/I')^\tau([\beta_k - 2\varepsilon, \beta_k + 2\varepsilon])$, this subspace of A/I' is non zero. Hence there exists $\beta_k' \in [\beta_k - 2\varepsilon, \beta_k + 2\varepsilon]$ such that

$$(A/I')^\tau([\beta_k' - \varepsilon', \beta_k' + \varepsilon']) \neq 0 \quad \text{for all } \varepsilon' > 0.$$

Note that by the choice of ε, $|\beta_j' - \beta_k'| \geqslant (1/3)|\beta_j - \beta_k|$. Choose $a_k \in A$ such that

$$\|a_k + I'\| = 1, \quad a_k + I' \in (A/I')^\tau([\beta_k' - \varepsilon', \beta_k' + \varepsilon']),$$

where $\varepsilon' > 0$ is to be specified. If ε' is small, then $\delta^m(a_k + I')$ is close in norm to $(i\beta_k')^m(a_k + I')$, $m = 0, 1, \ldots, n$ (see [8], Proposition 1.1). Replace a_k by $\tau_t a_k$ for a suitable t in $\mathbf{R}$ so that $\|a_k + \gamma'\|$ is close to $\|a_k + I'\| = 1$.

As in the proof of Observation 3, change a_k, without changing $a_k + I'$, so that

$$a_k \in A^\tau([\beta' - 2\varepsilon', \beta' + 2\varepsilon']), \quad \|a_k\| \leqslant \int |f|$$

where f is as in the proof of Observation 3 (and independent of all data). If ε' is sufficiently small, $\|a_k\|_n$ is then approximately less than $\max(1, |\beta_k'|^n) \int |f|$.

By Observation 2, there exists $C > 0$ such that $\|Ha\| \leqslant C\|a\|_n$ for all $a \in A_\infty$. Since $\gamma' \in S$, $Ha_k + \gamma'$ is equal to $\sum_{m=0}^{n} l_m(\gamma')\delta^m a_k + \gamma'$ and is therefore close in norm to $(\sum l_m(\gamma')(i\beta_k')^m)(a_k+\gamma')$ (provided ε' is small -- see [8], Proposition 1.1). Since $\|a_k+\gamma'\|$ is close to 1, $\left|\sum_{m=0}^{n} l_m(\gamma')(i\beta_k')^m\right|$ is close to $\|Ha_k+\gamma'\|$ and therefore approximately less than $C\max(1,|\beta_k'|^n)\int|f|$. Since $\varepsilon'>0$ is arbitrary, this proves

$$\left|\sum_{m=0}^{n} l_m(\gamma')(i\beta_k')^m\right| \leqslant C\max(1,|\beta_k'|^n)\int|f|.$$

Since $|\beta_k'|\leqslant \max(|\beta_k+2|,|\beta_k-2|)$, and the entries of the inverse of the $(n+1)\times(n+1)$ matrix $((i\beta_k')^m)$ are bounded in absolute value by $\prod_{j<k}|\beta_j-\beta_k|^{-1}$ times a polynomial in $\beta_0,\beta_1,\ldots,\beta_n$ (independently of $\gamma'\in S\cap O$), it follows from these inequalities for $k=0,1,\ldots,n$ that $l_0(\gamma'),l_1(\gamma'),\ldots,l_n(\gamma')$ are majorized by a function of $\beta_0,\beta_1,\ldots,\beta_n$ which is independent of $\gamma'\in S\cap O$.

OBSERVATION 5. *For every* $\gamma\in X$ *there exist* $l_0(\gamma),l_1(\gamma),\ldots,l_n(\gamma)\in\mathbf{C}$ *such that*

$$Ha+\gamma=\sum_{m=0}^{n} l_m(\gamma)\delta^m a+\gamma,\quad a\in A_\infty.$$

Proof. Let us prove first that for each $a\in A_\infty$ and each $\omega\in P_{A/\gamma}$,

$$\omega(a)=0,\quad \omega(\delta a)=0,\ldots,\omega(\delta^n a)=0\Rightarrow\omega(Ha)=0.$$

Let $a\in A_\infty$ and $\omega\in P_{A/\gamma}$ be such that $\omega(\delta^m a)=0$ for $m=0,1,\ldots,n$. Consider first the case that the spectrum of τ in the quotient $A/\bigcap_{t\in\mathbf{R}}\tau_t\gamma$ consists of at most n points. Then for all $m=0,1,2,\ldots,$ $\delta^m a+\gamma$ is a linear combination of $\delta^0 a+\gamma$, $\delta^1 a+\gamma,\ldots,\delta^n a+\gamma$, whence $\omega(\delta^m a)=0$. Hence in this case, by locality of H with respect to τ, $\omega(Ha)=0$.

Now consider the case that the spectrum of τ in the quotient $A/\bigcap_{t\in\mathbf{R}}\tau_t\gamma$ consists of at least $n+1$ points. By 3.4.11 of [10], the canonical map $P_A\to\operatorname{Prim}A$ is both continuous and open. Because it is open, and S is dense in X, $\bigcup_{\gamma'\in S}P_{A/\gamma'}$ is dense in P_A. Because it is continuous, there is by Observation 4 a neighbourhood U of ω in P_A such that $l_0,l_1,\ldots,l_n$ are bounded on $\bigcup_{\gamma'\in S}P_{A/\gamma'}\cap U$, say by $M>0$. Fix $\varepsilon>0$. Shrink U so that, for each $\omega'\in U$,

$$|\omega'(\delta^m a)|<\varepsilon/(n+1)M,\quad m=0,1,\ldots,n,$$

$$|(\omega-\omega')(Ha)|<\varepsilon.$$

It follows first that for each $\omega' \in \bigcup_{\gamma' \in S} P_{A/\gamma'} \cap U$,

$$|\omega'(Ha)| < \varepsilon.$$

By density of $\bigcup_{\gamma' \in S} P_{A/\gamma'}$ in P_A, such an ω' exists. It follows hence that $|\omega(Ha)| < 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, $\omega(Ha) = 0$.

We shall now deduce the conclusion of the theorem using Theorems 2.1 and 3.1 of [4]. Suppose first that the spectrum of τ in the quotient $A/\bigcap_{t \in \mathbf{R}} \tau_t \gamma$ is infinite, in other words, that the compositions of $\delta^0, \delta^1, \delta^2, \ldots$ with the canonical quotient map $A \to A/\gamma$ are independent. The existence of suitable scalars $l_0(\gamma), l_1(\gamma), \ldots, l_n(\gamma)$ follows in this case immediately from Theorem 3.1 of [4].

Suppose, finally, that the spectrum of τ in the quotient $A/\bigcap_{t \in \mathbf{R}} \tau_t \gamma$ is finite. In particular it follows that γ is τ-invariant, i.e. $\bigcap_{t \in \mathbf{R}} \tau_t \gamma = \gamma$. Furthermore, $(A/\gamma)_\infty = A/\gamma$, so the local operator induced by H in the quotient A/γ, as in the proof of Observation 3, is everywhere defined and continuous in the C^*-algebra norm.

Let us show that H leaves spectral subspaces of A with respect to τ invariant. Suppose that $a \in A_\infty$, $f \in L^1(\mathbf{R})$, and $\int ds\, f(s) \tau_s a = 0$, and let us show that $\int ds\, f(s) \tau_s Ha = 0$. From $\int ds\, f(s) \tau_s a = 0$ of course follows $\int ds\, f(s) \tau_s \delta^m a = 0$, $m = 0, 1, 2, \ldots$. Hence by Observation 1, $\int ds\, f(s) \tau_s Ha + \gamma = 0$ for every $\gamma \in S$. Since S is dense in X, $\int ds\, f(s) \tau_s Ha = 0$. It follows from the definition that H leaves spectral subspaces invariant.

It follows that H leaves spectral subspaces of the quotient A/γ with respect to τ invariant. Here we shall use that the spectrum of τ in A/γ is finite. Let $a \in A$ be such that $a + \gamma \in (A/\gamma)^\tau(\{\beta\})$, and let $\varepsilon > 0$ be such that $(A/\gamma)^\tau([\beta - 2\varepsilon, \beta + 2\varepsilon]) = (A/\gamma)^\tau(\{\beta\})$. As in the proof of Observation 3, we may change a, without changing $a + \gamma$, so that $a \in A^\tau([\beta - 2\varepsilon, \beta + 2\varepsilon])$. Hence by the result of the preceding paragraph, $Ha \in A^\tau([\beta - 2\varepsilon, \beta + 2\varepsilon])$, from which follows

$$Ha + \gamma \in (A/\gamma)^\tau([\beta - 2\varepsilon, \beta + 2\varepsilon]) = (A/\gamma)^\tau(\{\beta\}).$$

Now let us pass to the quotient A/γ. We have that H is a bounded operator $A/\gamma \to A/\gamma$ leaving the (finitely many) spectral subspaces of δ invariant, and if $\omega \in P_{A/\gamma}$ and $a \in A/\gamma$ are such that $\omega(a) = \omega(\delta a) = \ldots = \omega(\delta^n a)$, then $\omega(Ha) = 0$. To establish the existence of scalars $l_0(\gamma), l_1(\gamma), \ldots, l_n(\gamma)$ as required in the conclusion of the theorem we shall first show that the restriction of H to each spectral subspace of δ in A/γ is a scalar multiple of the identity. If the spectral subspace is of dimension one this follows from its invariance under H. If the spectral subspace is of dimension two or more then this follows from the locality of H with respect to δ, by Theorem 2.1 of [4] applied to the restrictions of H and of the identity map to the image of the spectral subspace in a faithful irreducible representation of A/γ. (Note that in the case that the spectral subspace is of dimension two or more we do not need to use that it is invariant under H.)

We have shown that H is a polynomial in δ in the quotient A/γ. We must now show that this polynomial is of degree at most n, modulo the minimal polynomial of δ in A/γ. Since ωH is a linear combination of $\omega, \omega\delta, \ldots, \omega\delta^n$ for each $\omega \in P_{A/\gamma}$, it is sufficient to find $\omega \in P_{A/\gamma}$ such that $\omega, \omega\delta, \ldots, \omega\delta^{r-1}$ are independent where r is the degree of the minimal polynomial of δ in A/γ. (Of course, if $r \leqslant n+1$ we don't have to do anything.) For ω we may take the vector state in any faithful irreducible representation of A/γ defined by the sum of one eigenvector from each eigenspace of a selfadjoint operator h such that $\delta = \operatorname{ad} ih$ in A/γ.

Observation 5 affirms Condition 2.

2.5. Remark. The question of characterizing the finite sequences of functions $l_0, l_1, \ldots, l_n$ on X that determine a linear operator $H : A_\infty \to A$ as in Condition 2 of Theorem 2, in terms of continuity and growth conditions, as was done in the commutative case in [7], — or even the question of whether such a characterization is possible — would seem to be quite interesting.

Analysis of the arguments above reveals the following necessary conditions. (Here A may be any C^*-algebra.) The functions $l_0, l_1, \ldots, l_n$ must be continuous at each $\gamma \in X$ such that the spectrum of τ in $A/\bigcap_{t \in \mathbf{R}} \tau_t \gamma$ has at least $n+1$ points. The scalars $l_0(\gamma), l_1(\gamma), \ldots, l_n(\gamma)$ must have absolute values less than the product of the bound of H with respect to the norm $\|\cdot\|_n$ on A_∞ and a certain number which depends only on the spectrum of τ in the quotient $A/\bigcap_{t \in \mathbf{R}} \tau_t \gamma$. Furthermore, the function l_0 must be bounded, by a (universal) constant times the above bound of H. Finally, if γ is τ-invariant and the spectrum of τ in A/γ has at most n points, then the coefficients of the remainder after the polynomial $l_0 + l_1 x + \ldots + l_n x^n$ is divided by the minimal polynomial of δ in A/γ must be bounded on a neighbourhood of γ in X. It is a natural question whether the coefficients of this remainder must in fact be continuous at γ. This is the case if the functions $l_1, \ldots, l_n$ (as well as l_0) are bounded on a neighbourhood of γ, or on a net in X converging to γ, — for example if the spectrum of τ in $A/\bigcap_{t \in \mathbf{R}} \tau_t \gamma'$ is all of $\mathbf{R}$, or not too sparse, for a dense set of γ' in X.

Whether or not for each $\gamma \in X$ the coefficients of the above remainder must always be continuous at γ, if this is the case for given functions $l_0, l_1, \ldots, l_n$, and if these functions also satisfy the continuity and growth conditions which are known to be necessary, it seems reasonable to ask whether the expression $\sum_{m=0}^{n} l_m \delta^m$ defines an operator from A_∞ to A.

2.6. REMARK. If γ is not fixed in X, and if $\gamma \in S$, as defined in Theorem 2, then it is not difficult to compute the spectrum of τ in the quotient $A/\bigcap_{t\in\mathbf{R}} \tau_t\gamma$, using the techniques of the proof of Observation 1. If γ is periodic, so that for some minimal $t_0 > 0$, $\tau_{t_0}\gamma = \gamma$, then the spectrum of τ in the quotient $A/\bigcap_{t\in\mathbf{R}} \tau_t\gamma$ is determined by the spectrum of τ_{t_0} in the quotient A/γ — it is the set of all $\beta \in \mathbf{R}$ such that $e^{i\beta t_0}$ belongs to the spectrum of τ_{t_0} in A/γ. If γ is not periodic, the spectrum of τ in $A/\bigcap_{t\in\mathbf{R}} \tau_t\gamma$ is equal to $\mathbf{R}$.

Presumably this description of the spectrum of τ at a point $\gamma \in X$ not fixed by τ also holds without the assumption $\gamma \in S$. We note that for such a point γ, either the orbit of γ is periodic or $\tau_t\gamma \neq \gamma$ for all $t \neq 0$. (The stability group of γ is closed: if $\tau_{t_i}\gamma = \gamma$ and $t_i \to t$ then $\tau_t\gamma \supseteq \gamma$, and similarly $\tau_{-t}\gamma \supseteq \gamma$, so $\tau_t\gamma = \gamma$.)

Acknowledgements. This work was begun whilst the first two authors were visiting the Australian National University, at the invitation of the third author, with the support of the Mathematical Sciences Research Centre. Parts of the work were carried out whilst the first and second authors were at the University of Toronto, with the support of the Natural Sciences and Engineering Research Council of Canada, and whilst the first and third authors were at the Mathematical Sciences Research Institute, Berkeley, with the support of that institute. The work was completed whilst the first author was guest professor at the Research Institute for Mathematical Sciences, Kyoto University, and the second author was chercheur associé at the Centre National de la Recherche Scientifique, Université d'Aix-Marseille II, Luminy. The first and second authors wish to acknowledge the support of the Norwegian and Danish Natural Science Research Councils, respectively.

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Received July 27, 1985.