J. OPERATOR THEORY
16(1986) 375–385

GROWTH CONDITIONS ON THE RESOLVENT AND MEMBERSHIP IN THE CLASSES A AND $A_{\aleph_0}$

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1. INTRODUCTION

Let $\mathscr{H}$ be a separable, infinite dimensional, complex Hilbert space, and let $\mathscr{L}(\mathscr{H})$ denote the algebra of all bounded linear operators on $\mathscr{H}$. The subsets **A** and $\mathbf{A}_{\aleph_0}$ of $\mathscr{L}(\mathscr{H})$, to be defined below, were introduced in [5] and studied in many papers over the last two years (cf. [7] for an in-depth development of the theory of dual algebras and a bibliography of pertinent articles). These classes, together with the intermediate classes $\mathbf{A}_n$, $n = 1, 2, \ldots$, defined below, have become important in the study of the structure theory of contraction operators. One reason for this is that much is known about the structure of an operator T in $\mathbf{A}_{\aleph_0}$. For example, such a T has a huge invariant subspace lattice, is reflexive, and is a "universal dilation" (cf. [7, Chapters IV, V, VI, IX]). On the other hand, the class **A** figures prominently in the invariant subspace problem for contractions T such that the spectrum $\sigma(T)$ of T contains the unit circle $\mathbf{T}$. For instance, a recent theorem from [13] says that if the weak* and weak operator topologies coincide on the dual algebra generated by an arbitrary operator in **A**, then every contraction T in $\mathscr{L}(\mathscr{H})$ satisfying $\sigma(T) \supset \mathbf{T}$ has nontrivial invariant subspaces. Various sufficient conditions that a contraction belong to **A** or to $\mathbf{A}_{\aleph_0}$ are known (cf. [2], [13]) and, in particular, there is a growth condition on the resolvent that provides a sufficient condition for membership in **A** and one on the essential resolvent that provides a sufficient condition for membership in $\mathbf{A}_{\aleph_0}$ (cf. [1], [6]). The purpose of this note is to "revisit" these theorems. In the case of the first theorem (concerning the class **A**) we are able to improve the theorem and give a rather transparent proof. In the case of the theorem concerning the class $\mathbf{A}_{\aleph_0}$, we give a short proof, which avoids the use of the Sz.-Nagy—Foiaş canonical model of a contraction, of a somewhat different theorem. The hope is that the transparency of these proofs will make it clearer to the interested reader exactly what role these growth conditions play in this circle of ideas.

2. GROWTH OF THE RESOLVENT AND THE CLASS A

The notation and terminology employed herein agree with that in [7]. Nevertheless, for completeness, we begin by reviewing a few pertinent definitions and important results. It is well-known that $\mathscr{L}(\mathscr{H})$ is the dual space of the Banach space (and ideal) $\mathscr{C}_1(\mathscr{H})$ of trace-class operators on $\mathscr{H}$ equipped with the trace norm $\|\ \|_1$. This duality is implemented by the bilinear functional

$$\langle T, L\rangle = \mathrm{tr}(TL), \qquad T \in \mathscr{L}(\mathscr{H}),\ L \in \mathscr{C}_1(\mathscr{H}).$$

A subalgebra $\mathscr{A}$ of $\mathscr{L}(\mathscr{H})$ that contains $1_{\mathscr{H}}$ and is closed in the weak* topology on $\mathscr{L}(\mathscr{H})$ is called a dual algebra. It follows from general principles (cf. [12]) that if $\mathscr{A}$ is a dual algebra, then $\mathscr{A}$ can be identified with the dual space of $Q_{\mathscr{A}} = \mathscr{C}_1(\mathscr{H})/{}^{\perp}\mathscr{A}$, where ${}^{\perp}\mathscr{A}$ is the preannihilator of $\mathscr{A}$ in $\mathscr{C}_1(\mathscr{H})$, under the pairing

$$(1) \qquad \langle T, [L]\rangle = \mathrm{tr}(TL), \qquad T \in \mathscr{A},\ [L] \in Q_{\mathscr{A}}.$$

(Here and throughout the note we write $[L]$ or $[L]_{\mathscr{A}}$ for the coset in $Q_{\mathscr{A}}$ containing the operator L in $\mathscr{C}_1(\mathscr{H})$.) If x and y are vectors in $\mathscr{H}$, then the associated rank-one operator $x \otimes y$, defined as usual by $(x \otimes y)(u) = (u, y)x$, $u \in \mathscr{H}$, belongs to $\mathscr{C}_1(\mathscr{H})$ and satisfies $\mathrm{tr}(x \otimes y) = (x, y)$. Thus if $\mathscr{A}$ is a given dual subalgebra of $\mathscr{L}(\mathscr{H})$, $[x \otimes y] \in Q_{\mathscr{A}}$. As is well-known, every operator L in $\mathscr{C}_1(\mathscr{H})$ can be written as $L = \sum_{i=1}^{\infty} x_i \otimes y_i$ for certain square-summable sequences $\{x_i\}$ and $\{y_i\}$ (with convergence in the norm $\|\ \|_1$), and it follows easily that every element of $Q_{\mathscr{A}}$ has the form $[L] = \sum_{i=1}^{\infty} [x_i \otimes y_i]$. A dual algebra $\mathscr{A}$ is said to have property (A_1) if for every element $[L]$ of $Q_{\mathscr{A}}$ there exist vectors x and y in $\mathscr{H}$ satisfying $[L] = [x \otimes y]$. More generally, if n is any nonzero cardinal number not exceeding $\aleph_0$, and if for every doubly indexed family $\{[L_{ij}]\}_{0 \leq i,j < n}$ of elements of $Q_{\mathscr{A}}$, there exist sequences $\{x_i\}_{0 \leq i < n}$ and $\{y_j\}_{0 \leq j < n}$ of vectors from $\mathscr{H}$ such that

$$[L_{ij}] = [x_i \otimes y_j], \qquad 0 \leq i, j < n,$$

then $\mathscr{A}$ is said to have property (A_n). (Note, in particular, that this defines property $(A_{\aleph_0})$.)

Let $\mathbf{N}$ be the set of positive integers, let $\mathbf{D}$ be the open unit disc in $\mathbf{C}$, and let $\mathbf{T} = \partial\mathbf{D}$. A set $\Lambda \subset \mathbf{D}$ is said to be dominating for $\mathbf{T}$ if almost every point of $\mathbf{T}$ is a nontangential limit of a sequence of points from Λ. The spaces $L^p = L^p(\mathbf{T})$ and $H^p = H^p(\mathbf{T})$, $1 \leq p \leq \infty$, are the usual function spaces. If T is an absolutely continuous contraction in $\mathscr{L}(\mathscr{H})$ (i.e., a contraction whose maximal unitary direct summand is either absolutely continuous or acts on the space (0)), we denote by $\mathscr{A}_T$ the dual algebra generated by T, and we write Q_T for the predual $Q_{\mathscr{A}_T}$. For such T, as is well-known (cf. [7, Theorem 4.1]), the Sz.-Nagy—Foiaş functional calculus Φ_T is a weak*-continuous, norm-decreasing, algebra homomorphism of H^∞ onto a weak* dense subalgebra of $\mathscr{A}_T$, and we define the class $\mathbf{A}=\mathbf{A}(\mathscr{H})$ to be the set of all absolutely continuous contractions T in $\mathscr{L}(\mathscr{H})$ for which Φ is an isometry of H^∞ onto $\mathscr{A}_T$. Furthermore, for each cardinal number n satisfying $1 \leqslant n \leqslant \aleph_0$, we define the class $\mathbf{A}_n=\mathbf{A}_n(\mathscr{H})$ to be the set of all T in $\mathbf{A}$, for which $\mathscr{A}_T$ has property $(\mathbf{A}_n)$. The class $\mathbf{A}$ arises naturally when one is studying the invariant subspace problem because of the following theorem of Apostol [1].

THEOREM 2.1. *If T is a contraction in $\mathscr{L}(\mathscr{H})$ such that $\sigma(T) \supset \mathbf{T}$, then either T has a nontrivial hyperinvariant subspace or $T \in \mathbf{A}$.*

To look into how Apostol proved Theorem 2.1 in [1], let us introduce, for each contraction T in $\mathscr{L}(\mathscr{H})$, and for each θ satisfying $0<\theta \leqslant 1$, the following set:

$$\zeta_\theta(T)=(\mathbf{D} \cap \sigma(T)) \cup \{\lambda \in \mathbf{D} \setminus \sigma(T) : \theta\|(T-\lambda)^{-1}\| > 1/(1-|\lambda|)\}. \tag{2}$$

If one makes the remark that any contraction T that has a nonzero unitary direct summand also has nontrivial hyperinvariant subspaces (cf. [16]), then Theorem 2.1 is an immediate consequence of its "two halves".

THEOREM 2.1a. *Suppose T is a completely nonunitary contraction in $\mathscr{L}(\mathscr{H})$ satisfying $\sigma(T) \supset \mathbf{T}$ and there exists some $0<\theta<1$ such that $\zeta_\theta(T)$ is not dominating for $\mathbf{T}$. Then the bicommutant of T is not an integral domain, and consequently T has nontrivial hyperinvariant subspaces.*

THEOREM 2.1b. *Suppose T is a completely nonunitary contraction in $\mathscr{L}(\mathscr{H})$ such that for every θ satisfying $0<\theta<1$, $\zeta_\theta(T)$ is dominating for $\mathbf{T}$. Then $T \in \mathbf{A}$.*

The proof of Theorem 2.1a is a (by now classical) argument in which one uses the hypothesis on T to construct two disjoint Jordan loops J_1 and J_2 situated as illustrated in Figure 1 with the properties that $(J_1 \cup J_2)\setminus\mathbf{T} \subset \mathbf{C}\setminus\sigma(T)$ and the

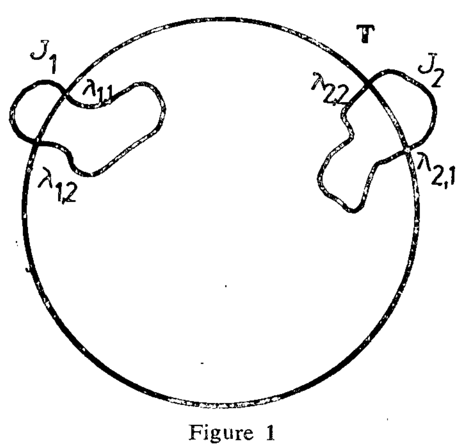

Figure 1

inequality

$$\|(T-\lambda I)^{-1}\| \leqslant 1/\theta(1-|\lambda|)$$

is satisfied on $(J_1 \cup J_2)\setminus\mathbf{T}$. It follows easily that the function

$$\|(\lambda-\lambda_{i,1})(\lambda-\lambda_{i,2})(T-\lambda)^{-1}\|$$

is bounded on $J_i\setminus\mathbf{T}$, $i=1,2$, and thus one may employ the Cauchy integral formula to produce two nonzero operators defined by

$$f_i(T)=\frac{1}{2\pi i}\int_{J_i}(\zeta-\lambda_{i,1})(\zeta-\lambda_{i,2})(\zeta-T)^{-1}d\zeta, \quad i=1,2,$$

belonging to the rational algebra generated by T such that $f_1(T)f_2(T)=0$. One of these operators must have a nonzero kernel, which is clearly a hyperinvarant subspace for T. (See [1] or [11] for more details.)

Concerning Theorem 2.1b, there is a stronger result which has a fairly transparent proof.

THEOREM 2.1b'. *Suppose T is a completely nonunitary contraction in $\mathcal{L}(\mathcal{H})$ such that for some θ satisfying $0<\theta<1/2$, $\zeta_0(T)$ is dominating for* $\mathbf{T}$. *Then $T \in \mathbf{A}$.*

Proof. We must show that $\|h\|_\infty=\|h(T)\|$ for every h in H^∞. Since the functional calculus is norm decreasing, one always has $\|h(T)\| \leqslant \|h\|_\infty$, so it suffices to prove the reverse inequality

$$\|h(T)\| \geqslant \|h\|_\infty, \quad h \in H^\infty. \tag{3}$$

To this end, let $h \in H^\infty$ be arbitrary, and choose θ satisfying $0<\theta<1/2$ such that $\zeta_\theta(T)$ is dominating for $\mathbf{T}$. We will show that

$$\|h(T)\| \geqslant \|h\|_\infty(1-2\theta), \tag{4}$$

which is enough to imply (3). Indeed, h being arbitrary in H^∞, it follows from (4) that

$$\|h(T)\|^n \geqslant \|h^n(T)\| \geqslant \|h^n\|_\infty(1-2\theta)=\|h\|_\infty^n(1-2\theta), \quad n\in\mathbf{N}, \tag{5}$$

and taking n^{th} roots in (5) and letting n tend to infinity yields (3). To establish (4), it suffices, in view of the fact that

$$\sup_{\lambda\in\zeta_\theta(T)}|h(\lambda)|=\|h\|_\infty, \quad h\in H^\infty,$$

(which is equivalent to $\zeta_\theta(T)$ being dominating) to show that for every $\lambda\in\zeta_\theta(T)$,

$$\|h(T)\| \geqslant |h(\lambda)|-2\theta\|h\|_\infty. \tag{6}$$

If $\lambda \in \sigma(T)$, one knows from [17] that $h(\lambda) \in \sigma(h(T))$, so (6) is certainly satisfied. Thus we may suppose that $\lambda \in \zeta_\theta(T) \setminus \sigma(T)$, and from (2) we obtain the existence of a unit vector v in $\mathscr{H}$ such that

$$\|(T-\lambda)^{-1}v\| > 1/\theta(1-|\lambda|). \tag{7}$$

Upon setting

$$u = \frac{(T-\lambda)^{-1}v}{\|(T-\lambda)^{-1}v\|}, \tag{8}$$

we get

$$\|(T-\lambda)u\| \leqslant \theta(1-|\lambda|). \tag{9}$$

Now write

$$h(\zeta) = h(\lambda) + g(\zeta)(\zeta-\lambda), \quad \zeta \in \mathbf{D}. \tag{10}$$

It is clear that $g \in H^\infty$ and satisfies

$$\|g\|_\infty \leqslant 2\|h\|_\infty(1-|\lambda|)^{-1}. \tag{11}$$

Therefore, from (8), (9), (10), and (11), we get

$$\|h(T)\| \geqslant \|h(T)u\| = \|h(\lambda)u + g(T)(T-\lambda)u\| \geqslant$$

$$\geqslant |h(\lambda)| - \|g(T)(T-\lambda)u\| \geqslant$$

$$\geqslant |h(\lambda)| - (2\|h\|_\infty(1-|\lambda|)^{-1}(\theta(1-|\lambda|)) = |h(\lambda)| - 2\theta\|h\|_\infty,$$

so (6) is established and the proof is complete.

It is easy to see that the Hilbert-space structure is never used in the proofs of Theorems 2.1a and 2.1b′, so the following somewhat more general result is true.

THEOREM 2.2. *Let $\mathscr{X}$ be a complex Banach space, and suppose Φ is a representation (of norm one) of $H^\infty(\mathbf{T})$ into $\mathscr{L}(\mathscr{X})$ such that $\Phi(1) = 1_{\mathscr{X}}$ and $\sigma(\Phi(e^{it})) \supset \mathbf{T}$. Then either $\Phi(e^{it})$ has a nontrivial hyperinvariant subspace or Φ is an isometry.*

To see how (2) compares with conditions on the growth of the resolvent satisfied by an arbitrary contraction T with $\sigma(T) \supset \mathbf{T}$, we close this section with the following elementary result.

PROPOSITION 2.3. *If T is a contraction in $\mathscr{L}(\mathscr{H})$ satisfying $\sigma(T) \supset \mathbf{T}$, then*

$$\|(T-\lambda)^{-1}\| = 1/(|\lambda|-1), \quad |\lambda| > 1,$$

and

$$\|(T-\lambda)^{-1}\| \geqslant 1/(1-|\lambda|), \quad \lambda \in \mathbf{D} \setminus \sigma(T).$$

Proof. The formula $r((T-\lambda)^{-1}) \leqslant \|(T-\lambda)^{-1}\|$, together with the spectral mapping theorem, implies that

$$\|(T-\lambda)^{-1}\| \geqslant 1/|1-|\lambda||, \quad \lambda \in \mathbf{C}\setminus\sigma(T),$$

and we have from von Neumann's theorem that

$$\|(T-\lambda)^{-1}\| \leqslant 1/(|\lambda|-1), \quad |\lambda| > 1.$$

The proposition follows.

3. GROWTH OF THE ESSENTIAL RESOLVENT AND THE CLASS $\mathbf{A}_{\aleph_0}$

We denote by $\mathbf{K} = \mathbf{K}(\mathscr{H})$ the (closed) ideal of compact operators in $\mathscr{L}(\mathscr{H})$, and recall that the quotient algebra $\mathscr{C}(\mathscr{H}) = \mathscr{L}(\mathscr{H})/\mathbf{K}$ is the Calkin algebra, which is a C^*-algebra. The projection of $\mathscr{L}(\mathscr{H})$ onto $\mathscr{C}(\mathscr{H})$ is denoted by π, and the essential spectrum of an operator T in $\mathscr{L}(\mathscr{H})$ (i.e., the spectrum of $\pi(T)$ in $\mathscr{C}(\mathscr{H})$) is denoted by $\sigma_e(T)$. The function $\lambda \to (\pi(T)-\lambda)^{-1}$ $(= (\pi(T)-\lambda 1_{\mathscr{C}(\mathscr{H})})^{-1})$ defined on $\mathbf{C}\setminus\sigma_e(T)$ is called the essential resolvent of T. For each contraction T in $\mathscr{L}(\mathscr{H})$, and for each θ satisfying $0<\theta<1$, the following set was introduced by Apostol in [1]:

$$\zeta_\theta(T) = \{\sigma_e(T) \cap \mathbf{D}\} \cup \{\lambda \in \mathbf{D}\setminus\sigma_e(T) : \theta\|(\pi(T)-\lambda)^{-1}\| > 1/(1-|\lambda|)\}. \tag{12}$$

We now define the somewhat bigger set

$$\tilde{\zeta}_\theta(T) = \{\sigma_e(T) \cap \mathbf{D}\} \cup \mathscr{F}(T) \cup \{\lambda \in \mathbf{D}\setminus\sigma_e(T) : \theta\|(\pi(T)-\lambda)^{-1}\| > 1/(1-|\lambda|)\},$$

where $\mathscr{F}(T)$ is the open set consisting of the union of those holes H in $\sigma_e(T)$ such that $H \subset \sigma(T)$.

Apostol showed that if T is a contraction in $C_{00}(\mathscr{H})$ (i.e., a contraction in $\mathscr{L}(\mathscr{H})$ such that both sequences $\{T^n\}$ and $\{T^{*n}\}$ converge to zero in the strong operator topology), and for some $0<\theta<1$, $\zeta_\theta(T)$ is dominating for $\mathbf{T}$, then T has a non-trivial invariant subspace. Later it was shown in [6] that if T is an absolutely continuous contraction in $\mathscr{L}(\mathscr{H})$ such that $\zeta_\theta(T)$ is dominating for $\mathbf{T}$ for some $0<\theta<1$, then $T \in \mathbf{A}_{\aleph_0}$. A somewhat shorter proof was given in [7, Chapter VIII], but both of these arguments employ the Sz.-Nagy—Foiaş functional model of a contraction, and are not transparent. The purpose of this section is to give a short proof of the following related result, which generalizes [7, Theorem 6.8].

THEOREM 3.1. *Suppose that T is a contraction in $C_{00}(\mathscr{H})$ and there exists some θ satisfying $0<\theta<1/2$ such that $\tilde{\zeta}_\theta(T)$ is dominating for* $\mathbf{T}$. *Then $T \in \mathbf{A}_{\aleph_0}$.*

The principal tool needed for the proof of this theorem is a result from [7] which uses the following concept. If T is an absolutely continuous contraction in $\mathscr{L}(\mathscr{H})$, $\lambda \in \mathbf{D}$, and there exists an element $[L]$ of Q_T such that

$$\langle h(T), [L] \rangle = h(\lambda), \quad h \in H^\infty,$$

then we write $[L] = [C_\lambda]$ and call this (necessarily unique) element of Q_T *evaluation at* λ. (It is easy to see that if $T \in \mathbf{A}$ then $[C_\lambda]$ exists in Q_T for every $\lambda \in \mathbf{D}$.) The result from [7] that we need is as follows (cf. Definition 2.7 and Theorem 3.7).

PROPOSITION 3.2. *Suppose* $T \in \mathbf{A}(\mathscr{H}) \cap C_{00}$, $0 < \gamma < 1$, $\Lambda \subset \mathbf{D}$ *is dominating for* $\mathbf{T}$, *and for each* $\lambda \in \Lambda$, *there exists a sequence* $\{x_n\} = \{x_n^\lambda\}$ *of unit vectors converging weakly to zero such that*

$$\|[C_\lambda] - [x_n \otimes x_n]\|_{Q_T} \leqslant \gamma, \quad n \in \mathbf{N}. \tag{13}$$

Then $T \in \mathbf{A}_{\aleph_0}$.

A link between Theorem 3.1 and Proposition 3.2 is supplied by the following lemma, which is useful in many situations.

LEMMA 3.3. *Suppose* $A \in \mathscr{L}(\mathscr{H})$, $\pi(A)$ *is invertible in* $\mathscr{C}(\mathscr{H})$, *and* $\alpha > 0$. *Then*

(i) *if* $\|\pi(A)^{-1}\| > \alpha$, *then there exists a sequence* $\{x_n\}$ *of unit vectors in* $\mathscr{H}$, *converging weakly to zero, such that* $\|Ax_n\| < 1/\alpha$ *for* $n \in \mathbf{N}$.

(ii) *if there exists a sequence of unit vectors* $\{y_n\}$ *in* $\mathscr{H}$ *converging weakly to zero such that* $\|Ay_n\| < 1/\alpha$ *for* $n \in \mathbf{N}$, *then* $\|\pi(A)^{-1}\| \geqslant \alpha$.

Proof. Write $A = UP$, the polar decomposition of A. Since, by hypothesis, A is a Fredholm operator, $\pi(U)$ must be unitary, and, of course, $\pi(P)$ is positive with P. Hence

$$\|\pi(A)^{-1}\| = \|\pi(P)^{-1}\pi(U)^{-1}\| = \|\pi(P)^{-1}\| = 1/\min \sigma_e(P). \tag{14}$$

To prove (i), suppose $\|\pi(P)^{-1}\| > \alpha$. Then, according to (14), $\lambda_0 = \min\{\lambda : : \lambda \in \sigma_e(P)\}$ satisfies $\lambda_0 < 1/\alpha$. Since λ_0 is either an eigenvalue of infinite multiplicity for P or an accumulation point of $\sigma(P)$, it follows easily from the spectral theorem that there exists an orthogonal sequence $\{x_n\}_{n=1}^\infty$ in $\mathscr{H}$ such that $\|Px_n\| < 1/\alpha$, $n \in \mathbf{N}$, and since

$$\|Ax\| = \|Px\|, \quad x \in \mathscr{H}, \tag{15}$$

we have proved (i).

To prove (ii), suppose $\{y_n\}$ is a sequence of unit vectors converging weakly to zero such that $\|Ay_n\| < 1/\alpha$, $n \in \mathbf{N}$. Then, by (15), we have $\|Py_n\| < 1/\alpha$, $n \in \mathbf{N}$. On the other hand, by virtue of (14), if we set, as before, $\lambda_0 = \min \sigma_e(P)$, then it suffices to prove that $\lambda_0 \leqslant 1/\alpha$. Suppose, to the contrary, that $\lambda_0 > 1/\alpha$. Then there exist at most finitely many points λ in $\sigma(P)$ satisfying $\lambda \leqslant (\lambda_0 + 1/\alpha)/2$, and thus there exists a finite-rank positive operator $F \in \mathbf{K}(\mathscr{H})$ such that $\min \sigma(P + F) \geqslant \geqslant (\lambda_0 + 1/\alpha)/2$. Thus $\|(P + F)x\| \geqslant (\lambda_0 + 1/\alpha)/2$ for every unit vector x in $\mathscr{H}$, contradicting

$$\liminf_n \|(P + F)y_n\| = \liminf_n \|Py_n\| \leqslant 1/\alpha,$$

so (ii) is proved.

Proof of Theorem 3.1. We are given some $0 < \theta < 1/2$ such that $\tilde{\zeta}'_\theta(T)$ is dominating for T. Since it is obvious that $\tilde{\zeta}'_\theta(T) \subset \zeta_\theta(T)$, it follows from Theorem 2.1b′ that $T \in \mathbf{A}$. Thus, according to Proposition 3.2, it suffices to construct, for each $\lambda \in \tilde{\zeta}'_\theta(T)$, a sequence $\{x_n\}$ of unit vectors converging weakly to zero and satisfying (13) with $\gamma = 2\theta$. Suppose first that $\lambda \in \sigma_{\mathrm{le}}(T)$. The argument in this case is by now standard; for completeness we give it. One knows that there exists an orthonormal sequence $\{x_n\}$ in $\mathscr{H}$ satisfying $\|(T - \lambda)x_n\| \to 0$. Since $T \in \mathbf{A}$, by the Hahn-Banach theorem, there exists, for each $n \in \mathbf{N}$, a function $h_n \in H^\infty$ of norm one such that

$$\begin{aligned}\|[C_\lambda] - [x_n \otimes x_n]\|_{Q_T} &= \langle h_n(T), [C_\lambda] - [x_n \otimes x_n]\rangle = \\ = h_n(\lambda) - \langle h_n(T), [x_n \otimes x_n]\rangle &= h_n(\lambda) - \mathrm{tr}(h_n(T)(x_n \otimes x_n)) = h_n(\lambda) - (h_n(T)x_n, x_n).\end{aligned} \tag{16}$$

If we write, as in (10),

$$h_n(\zeta) = h_n(\lambda) + g_n(\zeta)(\zeta - \lambda), \quad n \in \mathbf{N},\ \zeta \in \mathbf{D},$$

then each $g_n \in H^\infty$ and satisfies

$$\|g_n\| \leqslant 2\|h_n\|(1 - |\lambda|)^{-1} = 2(1 - |\lambda|)^{-1}. \tag{17}$$

Therefore from (16) and (17) we obtain

$$\begin{aligned}\|[C_\lambda] - [x_n \otimes x_n]\| &= h_n(\lambda) - \{h_n(\lambda) + (g_n(T)(T - \lambda)x_n, x_n)\} \leqslant \\ &\leqslant |(g_n(T)(T - \lambda)x_n, x_n)| \leqslant \|g_n\|\,\|(T - \lambda)x_n\| \leqslant \\ &\leqslant 2(1 - |\lambda|)^{-1}\|(T - \lambda)x_n\| \to 0,\end{aligned} \tag{18}$$

so in this case, using some tail of the sequence $\{x_n\}$, we have the desired conclusion.

Suppose next that $\lambda \in \sigma_{\mathrm{re}}(T)$. Then $\bar{\lambda}_{\mathrm{le}}(T^*)$, and thus there exists an orthonormal sequence $\{x_n\}$ in $\mathscr{H}$ such that $\|(T^* - \bar{\lambda})x_n\| \to 0$. With the h_n and g_n defined as before, we obtain from (18):

$$\|[C_\lambda] - [x_n \otimes x_n]\| \leqslant |(g_n(T)x_n, (T^* - \bar{\lambda})x_n)| \leqslant$$

$$\leqslant \|g_n\| \, \|(T^* - \bar{\lambda})x_n\| \leqslant 2(1 - |\lambda|)^{-1}\|(T^* - \bar{\lambda})x_n\| \to 0,$$

so once again we have the desired conclusion.

We consider now the case in which $\lambda \in \mathscr{F}(T)$, and we observe that $(T - \lambda)$ is a Fredholm operator. We treat the case $\mathrm{i}(T - \lambda) \geqslant 0$; the other case follows by taking adjoints. Elementary properties of the index, together with [13, Lemma 2.2] when $\mathrm{i}(T - \lambda) = 0$, implies that the sequence $\{\mathrm{Ker}(T - \lambda)^n\}$ is strictly increasing. For each $n \in \mathbf{N}$, let x_n be a unit vector in $\mathrm{Ker}(T - \lambda)^{n+1} \ominus \mathrm{Ker}(T - \lambda)^n$. Via (10) we obtain, for any h in H^∞,

$$\langle h(T), [x_n \otimes x_n]\rangle = (h(T)x_n, x_n) = (g(T)(T - \lambda)x_n, x_n) + h(\lambda) =$$

$$= h(\lambda) = \langle h(T), [C_\lambda]\rangle.$$

Thus $[C_\lambda] = [x_n \otimes x_n]$ for $n \in \mathbf{N}$, and since the sequence $\{x_n\}$ is orthonormal, the proof in this case is complete.

Finally, suppose that $\lambda \in \tilde{\zeta}'_\theta(T) \setminus (\mathscr{F}(T) \cup \sigma_e(T))$, so

$$\|(\pi(T) - \lambda)^{-1}\| > 1/\theta(1 - |\lambda|).$$

Then, according to Lemma 3.3 with $A = T - \lambda$, there exists a sequence $\{x_n\}$ of unit vectors in $\mathscr{H}$, converging weakly to zero, such that

$$\|(T - \lambda)x_n\| < \theta(1 - |\lambda|), \quad n \in \mathbf{N}.$$

Hence, with the h_n and g_n defined as in the first two cases, we have from (18) that

$$\|[C_\lambda] - [x_n \otimes x_n]\| \leqslant 2(1 - |\lambda|)^{-1}\|(T - \lambda)x_n\| \leqslant 2\theta = \gamma < 1, \quad n \in \mathbf{N}.$$

Thus, by Proposition 3.2, the proof is complete.

We conclude this note by mentioning the following problem, which is of some importance in the theory of dual algebras, and whose solution might be approached by the methods discussed above.

PROBLEM 3.4. Suppose that for each $n \in \mathbf{N}$, T_n is an operator in the class $\mathbf{A}$. Does it follow that $\sum_{n \in \mathbf{N}}^{\oplus} T_n$ belongs to $\mathbf{A}_{\aleph_0}$?

REFERENCES

1. APOSTOL, C., Ultraweakly closed operator algebras, *J. Operator Theory*, **2**(1979), 49–61.
2. APOSTOL, C.; BERCOVICI, H; FOIAŞ, C.; PEARCY, C., Invariant subspaces, dilation theory, and the structure of the predual of a dual algebra. I, *J. Funct. Anal.*, **63**(1985), 369–404.
3. APOSTOL, C.; BERCOVICI, H.; FOIAŞ, C.; PEARCY, C., Invariant subspaces, dilation theory, and the structure of the predual of a dual algebra. II, *Indiana Univ. Math. J.*, **34**(1985), 845–855.
4. BERCOVICI, H.; CHEVREAU, B.; FOIAŞ, C.; PEARCY, C., Dilation theory and systems of simultaneous equations in the predual of an operator algebra. II, *Math. Z.*, **187**(1984), 97–103.
5. BERCOVICI, H.; FOIAŞ, C.; PEARCY, C., Dilation theory and systems of simultaneous equations in the predual of an operator algebra. I, *Michigan Math. J.*, **30**(1983), 335–354.
6. BERCOVICI, H.; FOIAŞ, C.; PEARCY, C., Factoring trace-class operator-valued functions with applications to the class $A_{\aleph_0}$, *J. Operator Theory*, **14**(1985), 351–389.
7. BERCOVICI, H.; FOIAŞ, C.; PEARCY, C., *Dual algebras with applications to invariant subspaces and dilation theory*, CBMS Regional Conference Series in Math., No. **56**, A.M.S., Providence, 1985.
8. BERCOVICI, H.; FOIAŞ, C.; PEARCY, C., On the reflexivity of algebras and linear spaces of operators, *Michigan Math. J.*, **33**(1986), 119–126.
9. BERCOVICI, H.; FOIAŞ, C.; PEARCY, C.; SZ.-NAGY, B., Functional models and generalized spectral dominance, *Acta Sci. Math. (Szeged)*, **43**(1981), 243–254.
10. BERCOVICI, H.; FOIAŞ, C.; PEARCY, C.; SZ.-NAGY, B., Factoring compact operator-valued functions, *Acta Sci. Math. (Szeged)*, **48**(1985), 25–36.
11. BROWN, S., Some invariant subspaces for subnormal operators, *Integral Equations Operator Theory*, **1**(1978), 310–333.
12. BROWN, S.; CHEVREAU, B.; PEARCY, C., Contractions with rich spectrum have invariant subspaces, *J. Operator Theory*, **1**(1979), 123–136.
13. CHEVREAU, B.; PEARCY, C., On the structure of contraction operators with applications to invariant subspaces, *J. Funct. Anal.*, **67**(1986), 360–379.
14. CHEVREAU, B.; PEARCY, C., Sur le problème de sous-espace invariant pour les contractions, *C.R. Acad. Sci. Paris Sér. I Math.*, **301**(1985), 735–738.
15. CHEVREAU, B.; PEARCY, C.; SHIELDS, A., Finitely connected domains G, representations of $H^{\infty}(G)$, and invariant subspaces, *J. Operator Theory*, **6**(1981), 375–405.
16. DOUGLAS, R. G.; PEARCY, C., Hyperinvariant subspaces and transitive algebras, *Michigan Math. J.*, **19**(1972), 1–12.
17. FOIAŞ, C.; MLAK, W., The extended spectrum of completely non-unitary contractions and the spectral mapping theorem, *Studia Math.*, **26**(1966), 239–245.
18. MARSALLI, M., Ph. D. Thesis, University of Michigan, 1985.
19. PEARCY, C., *Some recent results in operator theory*, CBMS-Regional Conference Series in Math., No. **36**, A.M.S., Providence, 1978.
20. ROBEL, G., On the structure of (BCP)-operators and related algebras. I, *J. Operator Theory*, **12**(1984), 23–45.
21. SHIELDS, A., Weighted shift operators and analytic function theory, in *Topics in Operator Theory*, A.M.S., Providence, 1974, pp. 49–128.

22. SZ.-NAGY, B.; FOIAŞ, C., *Harmonic analysis of operators on Hilbert space*, North Holland, Amsterdam, 1970.

23. WESTWOOD, D., Weak operator and weak* topologies on singly generated algebras, *J. Operator Theory*, **15**(1986), 267–280.

24. WESTWOOD, D., On C_{00}-contractions with dominating spectrum, *J. Funct. Anal.*, 66(1986), 96–104.

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Received April 7, 1986.