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STABLE RANGE FOR TENSOR PRODUCTS OF EXTENSIONS OF $\mathscr{K}$ BY $C(X)$

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INTRODUCTION

In [8] G. Nagy studied the stable rank of the algebras $\mathscr{T}_n$ of n-dimensional Toeplitz operators [3, 4]. For n even his results are exact, while for odd n the stable rank is only determined up to 1. In this paper we obtain general results for the stable rank of tensor products of Brown-Douglas-Fillmore extensions [1] of $\mathscr{K}$ (the algebra of compact operators) by commutative C^*-algebras. Our results are sharp under some additional hypothesis which are fulfilled if the spectra of the commutative C^*-algebras involved are manifolds. In particular we complete Nagy's results by obtaining the exact value of the stable rank of $C(X) \otimes \mathscr{T}_n$ if X is a compact manifold. We obtain also some estimates for M. A. Rieffel's connected stable rank csr(A) and introduce an absolute connected stable rank, acsr(A), which turns out to be quite useful and show that it equals the stable rank of $A \otimes C([0, 1])$.

The first section deals with preliminary material due to M. A. Rieffel and G. Nagy. The second section contains some general results concerning the stable rank, the connected stable rank and the absolute connected stable rank. In the third section we generalize G. Nagy's estimates. The last section contains some technical lemmas and the proof of the main theorem.

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I.

We begin by recalling some definitions and results from [9].

For a unital C^*-algebra A and a natural number n we consider $\mathrm{Lg}_n(A)$, the set of n-tuples of elements of A which generate A as a left ideal: the *topological stable rank* of A is the least integer n (if it does not exist it will be taken by definition to be ∞) such that $\mathrm{Lg}_n(A)$ is dense in A^n. According to [6] this number coincides with the usual Bass *stable rank* of A, denoted $\mathrm{sr}(A)$, which is the least integer $n \geqslant 1$ such that for any $(a_1, \ldots, a_n, a_{n+1}) \in \mathrm{Lg}_{n+1}(A)$ there exists $b_1, \ldots, b_n \in A$ such that $(a_1 + b_1 a_{n+1}, \ldots, a_n + b_n a_{n+1}) \in \mathrm{Lg}_n(A)$ (if no such n exists we take $\mathrm{sr}(A)$ to be ∞). We denote by $M_{m,n}(A)$ the set of matrices with entries in A with m rows and n columns; $M_{m,m}(A)$ is also denoted by $M_m(A)$. $\mathrm{GL}(n, A)$ is the group of invertible elements of $M_n(A)$ and $\mathrm{GL}^0(n, A)$ the connected component of 1 in this group. We shall denote by $\mathrm{csr}(A)$ the connected stable rank of A which is the least integer $n \geqslant 1$ such that the action of $\mathrm{GL}^0(m, A)$ by left multiplication on $\mathrm{Lg}_m(A)$ is transitive for any $m \geqslant n$ (if no such integer exists we take $\mathrm{csr}(A) = \infty$). According to Corollary 8.5 of [9] $\mathrm{csr}(A)$ is also the least integer n such that $\mathrm{Lg}_m(A)$ is connected for any $m \geqslant n$. If A is not unital we take $\mathrm{sr}(A)$ ($\mathrm{crs}(A)$) to be $\mathrm{sr}(\tilde{A})$ ($\mathrm{csr}(\tilde{A})$) where $\tilde{A}$ is the algebra obtained from A by adjoining a unit.

We shall use the following results proved in [9] (here J is a two-sided ideal in the C^*-algebra A):

(1.1) $$\mathrm{sr}(J) \leqslant \mathrm{sr}(A)$$ (Theorem 4.4)

(1.2) $$\mathrm{sr}(A/J) \leqslant \mathrm{sr}(A)$$ (Theorem 4.3)

(1.3) $$\mathrm{sr}(A) \leqslant \max\{\mathrm{sr}(J), \mathrm{sr}(A/J), \mathrm{csr}(A/J)\}$$ (Theorem 4.11)

(1.4) $$\mathrm{sr}(A \otimes \mathcal{K}) = \begin{cases} 1 & \text{if } \mathrm{sr}(A) = 1 \\ 2 & \text{if } \mathrm{sr}(A) \geqslant 2, \end{cases}$$ (Theorem 6.4)

($\mathcal{K}$ is standing for the algebra of compact operators on a separable Hilbert space).

Let $A = \lim\limits_{\iota \in I} A_\iota$ be an inductive limit of C^*-algebras over the directed set I; the following results are proved analogously as Theorem 5.1 of [9]:

(1.5) $$\mathrm{sr}(A) \leqslant \liminf_{\iota \in I} \mathrm{sr}(A_\iota)$$

(1.6) $$\mathrm{csr}(A) \leqslant \liminf_{\iota \in I} \mathrm{csr}(A_\iota).$$

The following results are simple consequences of definitions (A, B are C^*-algebras):

(1.7) $$\mathrm{sr}(A \oplus B) = \max\{\mathrm{sr}(A), \mathrm{sr}(B)\}$$

(1.8) $$\mathrm{csr}(A \oplus B) = \max\{\mathrm{csr}(A), \mathrm{csr}(B)\}.$$

Lemma 2 of [8] reads:

$$\text{csr}(A) \leqslant \max\{\text{csr}(J), \text{csr}(A/J)\}. \tag{1.9}$$

We shall use also the following consequence of a classical theorem in dimension theory (see [9], Theorem 1.1 and Proposition 1.7)

$$\text{sr}(C(X)) = [\dim(X)/2] + 1; \tag{1.10}$$

here X is a compact space.

It is a simple observation that $x = (x_1, \ldots, x_s) \in \text{Lg}_s(A)$ if and only if there exists $\varepsilon > 0$ such that $x_1^* x_1 + \ldots + x_s^* x_s \geqslant \varepsilon$, if and only if $\varphi(x_g^* x_g) > 0$ for any $g \in \{1, \ldots, s\}$ and any pure state φ of the C^*-algebra A; this also shows that $f \in C(X, A^s)$ belongs to $\text{Lg}_s(C(X, A))$ if and only if $f(x) \in \text{Lg}_s(A)$ for any point x in the compact space X.

If A is a C^*-algebra and $x, y \in A^s$ we shall denote by $\|x - y\| = \max_g \|x_g - y_g\|$ for $x = (x_1, \ldots, x_s)$, $y = (y_1, \ldots, y_s)$.

II.

In this section we prove some general results concerning the stable rank and the connected stable rank for C^*-algebras. We also introduce the notion of absolute connected stable rank of A and prove that it is equal to $\text{sr}(C[0, 1]) \otimes A)$.

2.1. LEMMA. *Let J be a two-sided ideal in the C^*-algebra A, $\pi : A \to A/J$ the quotient map, and suppose that $s \geqslant \text{sr}(A)$. Let $\varepsilon > 0$ and $x = (x_1, \ldots, x_s) \in A^s$ be such that $\pi(x) = (\pi(x_1), \ldots, \pi(x_s)) \in \text{Lg}_s(A/J)$. Then there exists $x' \in \text{Lg}_s(A)$ such that $\|x' - x\| < \varepsilon$ and $\pi(x') = \pi(x)$. Moreover, if $y = (y_1, \ldots, y_s) \in A^s$ is such that $\pi(y_1)\pi(x_1) + \ldots + \pi(y_n)\pi(x_n) = 1$ then there exists $y' \in A^s$ such that $y_1' x_1' + \ldots + y_s' x_s' = 1$ and $\pi(y_g') = \pi(y_g)$ for $1 \leqslant g \leqslant s$.*

Proof. Denote also by π the map $M_n(A) \to M_n(A/J)$. The set $V_\delta = \{(\exp\pi(a))\pi(x) \mid \|a\| < \delta, a \in M_s(A)\}$ is a neighbourhood of $\pi(x)$ in $\text{Lg}_s(A/J)$. Choose $x'' \in \pi^{-1}(V) \cap \text{Lg}_s(A)$ such that $\|x'' - x\| < \varepsilon/2$. Then $\pi(x'') = (\exp\pi(a))\pi(x)$ for a suitable $a \in M_s(A)$ with $\|a\| < \delta$. Let $x' = \exp(-a)x'' \in \text{Lg}_s(A)$. Then $\|x' - x\| \leqslant \|x' - x''\| + \|x'' - x'\| \leqslant \|\exp(-a) - 1\|\|x''\| + \varepsilon/2 < (e^\delta - 1)(\|x\| + \varepsilon/2) + \varepsilon/2 < \varepsilon$ if δ is small enough.

Since $\pi(y_1 x_1 + \ldots + y_s x_s) = 1$ we get $y_1 x_1' + \ldots + y_s x_s' = 1 - a$ with $a \in J$. Let $\{u_i\}$ be an approximate unit for J and $y_1'', \ldots, y_s'' \in A^s$ such that $y_1'' x_1' + \ldots + y_s'' x_s' = 1$. Let $y_g' = (1 - u_c)y_g + u_c y_g''$ for $1 \leqslant g \leqslant s$. Then $\pi(y_g') = \pi(y_g)$ for $1 \leqslant g \leqslant s$ and $y_1' x_1' + \ldots + y_s' x_s' = 1 - (1 - u_c)a$. Choosing c big enough we obtain the desired conclusion.

Let $\mathbf{B}_r = \{x = (x_1, \ldots, x_r) \in \mathbf{R}^r \mid x_1^2 + \ldots + x_r^2 \leqslant 1\}$, $\mathbf{S}^{r-1} = \partial\mathbf{B}_r$. The last lemma yields (for $J = C(\mathring{\mathbf{B}}_r) \otimes B$ and $A = C(\mathbf{B}_r) \otimes B$) the following corollary:

2.2. COROLLARY. *Let B be a C^*-algebra, $s \geqslant \mathrm{sr}(C(\mathbf{B}_r) \otimes B)$ and $f \in C(\mathbf{B}_r, B^s)$ such that $f(x) \in \mathrm{Lg}_s(B)$ for any $x \in \mathbf{S}^{r-1}$. Then for any $\varepsilon > 0$ there exists $g \in C(\mathbf{B}_r, \mathrm{Lg}_s(B))$ such that $f \mid \mathbf{S}^{r-1} = g \mid \mathbf{S}^{r-1}$ and $\|g - f\| < \varepsilon$.*

2.3. DEFINITION. Let A be a unital C^*-algebra, and $n \geqslant 1$ an integer. Then we say that n is in the absolute connected stable range of A if for any nonempty connected open set $V \subset A^n$, $V \cap \mathrm{Lg}_n(A)$ is nonempty and connected. We shall denote by $\mathrm{acsr}(A)$ the least integer n such for any $m \geqslant n$, m is in the absolute connected stable range of A; if no such integer exists we let $\mathrm{acsr}(A) = \infty$.

2.4. LEMMA. *Let n be an integer. Then n is in the absolute connected stable range of the C^*-algebra A if and only if $n \geqslant \mathrm{sr}(C(I) \otimes A)$ and, consequently, $\mathrm{acsr}(A) = \mathrm{sr}(C(I) \otimes A)$ $(I = [0, 1])$.*

Proof. A is a quotient of $C(I) \otimes A$ and thus (1.2) shows that $a = \mathrm{sr}(C(I) \otimes A) \geqslant \mathrm{sr}(A)$. Let $n \geqslant a$, $V \subset A^n$ be an open connected nonempty set. We shall show that $V \cap \mathrm{Lg}_n(A)$ is connected and we conclude that n is in the absolute connected stable range of A. Let $x_0, x_1 \in V \cap \mathrm{Lg}_n(A)$; there exists a continuous function $f : I \to V$ such that $f(0) = x_0, f(1) = x_1$. Let $\varepsilon > 0$ be less than the distance between the compact set $f([0, 1])$ and the closed set $A^n \setminus V$. By the last corollary there exists a continuous function $g : I \to \mathrm{Lg}_n(A)$ such that $g(0) = f(0) = x_0$, $g(1) = f(1) = x_1$ and $\|g - f\| < \varepsilon$. Then g takes its values in $\{y \mid \mathrm{d}(y, f([0, 1])) < \varepsilon\} \subset V$.

Conversely, let n be in the absolute connected stable range of A. It is then obvious that $n \geqslant \mathrm{sr}(A)$. Let $\varphi \in C(I, A^n)$, $\varepsilon > 0$, and choose m big enough such that $\|\varphi(t_1) - \varphi(t_2)\| < \varepsilon/3$ for any $t_1, t_2 \in I$, $|t_1 - t_2| < 2/m$. Since $n \geqslant \mathrm{sr}(A)$ we can choose $y_g \in \mathrm{Lg}_n(A)$ such that $\|y_g - f(g/m)\| < \varepsilon/3$ for $0 \leqslant g \leqslant m$. By assumption each of the sets $V_g = \{y \in \mathrm{Lg}_n(A) \mid \|y - \varphi(g/m)\| < \varepsilon \cdot 2/3\}$ is open and connected containing y_{g-1} and y_g for $1 \leqslant g \leqslant m$. Thus we may find $\psi \in C(I, \mathrm{Lg}_n(A))$, $\psi(g/m) = y_g$ and $\psi(t) \in V_j$ for $t \in [(j-1)/m, j/m]$; then $\|\psi(t) - \varphi(t)\| \leqslant \|\psi(t) - \varphi(j/m)\| + \|\varphi(j/m) - \varphi(t)\| < 2\varepsilon/3 + \varepsilon/3 = \varepsilon$ proving that $n \geqslant \mathrm{sr}(C(I) \otimes A)$.

2.5. COROLLARY. 1° $\mathrm{csr}(A \otimes \mathscr{K}) \leqslant 2$ *for any C^*-algebra A.*

2° $\mathrm{csr}(C(X)) \leqslant [(m + 1)/2] + 1$ *if X is a compact space of dimension m.*

Let $A, A_1, \ldots, A_n$ be unital C^*-algebras, and $\pi_g : A_g \to A$ morfisms of unital C^*-algebras. Consider the following C^*-subalgebra of the C^*-algebra $A_1 \oplus \ldots \oplus A_n$:

$$B = \{x = (x_1, \ldots, x_n) \in A_1 \oplus \ldots \oplus A_n \mid \pi_1(x_1) = \pi_2(x_2) = \ldots = \pi_n(x_n)\}$$

and denote it by $\coprod_A A_g$ (it is the usual fiber product); denote also $\bigcap_{j=1}^{n} \pi_g(A_g)$ by A'.

2.6. PROPOSITION. *Suppose that* $s \geqslant \mathrm{sr}(A_g)$ *for* $1 \leqslant g \leqslant n$ *and* $s \geqslant \mathrm{sr}\left(\bigcap_{g=1}^{n} \pi_g(A_g)\right)$; *then* $s \geqslant \mathrm{sr}\,(\prod_A A_g)$.

Proof. Denote $\prod_A A_j$ by B. Let $x = (x_1, \ldots, x_s) \in B^s$, $\varepsilon > 0$, $x_j = (x_{j1}, \ldots \ldots, x_{jn}) \in A_1 \oplus \ldots \oplus A_n$, $1 \leqslant j \leqslant s$, such that $\pi_1(x_{g1}) = \ldots = \pi_n(x_{gn}) = t_g$. Let $t = (t_1, \ldots, t_s) \in \left(\bigcap_{j=1}^{n} \pi_g(A_g)\right)^s = A'^s$ and $t' \in \mathrm{Lg}_s(A')$ such that $\|t' - t\| < \varepsilon$. Then there exists $\|x'_{gk} - x_{gk}\| < \varepsilon$ such that $\pi_g(x'_{gk}) = t'_g$ and $(x'_{1k}, \ldots, x'_{sk}) \in \mathrm{Lg}_s(A_k)$. An obvious application of Lemma 2.1 shows that $x' = (x'_1, \ldots, x'_s) \in \mathrm{Lg}_s(B)$ if $x'_j = (x'_{j1}, \ldots, x'_{gn})$.

2.7. COROLLARY. *Let* $J_1, \ldots, J_n \subset A$ *be closed two-sided ideals in the* C^**-algebra* A. *Then* $\mathrm{sr}(A/J) \leqslant \max_{1 \leqslant k \leqslant n} \{\mathrm{sr}(A/J_k)\}$, *where* $J = J_1 \cap \ldots \cap J_n$.

Proof. Using induction on n we may suppose that $n = 2$. Let $\pi_j \colon A/J_j \to \to A/(J_1 + J_2) = B$. Then $A/(J_1 \cap J_2)$ is isomorphic with $\prod_B A/J_j$.

Let A and B be two unital C^*-algebras, and $\varphi, \psi \in \mathrm{End}(A, B)$. We say that φ and ψ are homotopic if there exists $\eta \colon I \to \mathrm{End}(A, B)$ such that $\eta(0) = \varphi$, $\eta(1) = \psi$ and the map $\eta_x \colon I = B$, $\eta_x(t) = \eta(t, x)$, is continuous for any $x \in A$. We say that B homotopically dominates A if there exists $\varphi \in \mathrm{End}(A, B)$, $\psi \in \mathrm{End}(B, A)$ such that $\psi \circ \varphi$ and id_A are homotopic.

2.8. LEMMA. *Let* A *and* B *be unital* C^**-algebras such that* B *homotopically dominates* A. *Then* $\mathrm{csr}(A) \leqslant \mathrm{csr}(B)$.

Proof. Let φ and ψ as above, $x = (x_1, \ldots x_s) \in \mathrm{Lg}_s(A)$, and suppose that $s \geqslant \mathrm{csr}(B)$. Then x can be joined to $\psi(\varphi(x))$ by an arc in $\mathrm{Lg}_s(A)$ and $\varphi(x)$ can be joined by an arc in $\mathrm{Lg}_s(B)$ to $(1, 0, \ldots, 0)$. This shows that x can be joined to $(1, 0, \ldots, 0)$ by an arc in $\mathrm{Lg}_s(A)$ proving that $s \geqslant \mathrm{csr}(A)$ and hence that $\mathrm{csr}(A) \leqslant \leqslant \mathrm{csr}(B)$.

2.9. COROLLARY. $\mathrm{csr}(C(\mathbf{B}_r) \otimes A) = \mathrm{csr}(A)$.

2.10. PROPOSITION. $\mathrm{csr}(M_m(A)) \leqslant \{(\mathrm{csr}(A) - 1)/m\} + 1$ *(here* $\{x\}$ *denotes the least integer greater than* x*).*

Proof. Let $x \in M_m(A)^s$. We may view x as an element of $M_{ms,m}(A) \colon x = (a_{ig})$, $1 \leqslant i \leqslant ms$, $1 \leqslant g \leqslant m$. Then $x \in \mathrm{Lg}_s(M_m(A))$ means that the equation

$$\sum_{i=1}^{ms} b_{ki} a_{ij} = \delta_{kj} \tag{1}$$

has a solution. This shows that the first column is in $\mathrm{Lg}_{ms}(A)$. If $ms \geqslant \mathrm{csr}(A)$ there exists $B_1 \in \mathrm{GL}^0(ms, A)$ such that

$$B_1x = \left(\begin{array}{c|c} 1 & x_1 \\ \hline 0 & \\ \vdots & x_2 \\ 0 & \end{array}\right)$$

where x_1 stands for $((Bx)_{12}, (Bx)_{13}, \ldots, (Bx)_{1m})$. A straightforward computation shows that the first column in the matrix x_2 is in $\mathrm{Lg}_{ms-1}(A)$. By induction we get that there exists $B \in \mathrm{GL}^0(ms, A)$ such that

$$Bx = \begin{pmatrix} 1 & & * \\ 0 & 1 & \\ \vdots & 0 & 1 \\ \vdots & & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

whenever $ms - m + 1 \geqslant \mathrm{csr}(A)$ and this matrix can be joined with the matrix $(\delta_{ij})_{1 \leqslant i \leqslant ms, 1 \leqslant j \leqslant m}$ in $\mathrm{Lg}_s(M_m(A))$.

2.11. Corollary. *Let A be an AF-C^*-algebra and B a C^*-algebra. Then* $\mathrm{sr}(A \otimes B) \leqslant \mathrm{sr}(B)$, $\mathrm{csr}(A \otimes B) \leqslant \mathrm{csr}(B)$, *with equalities if A is commutative.*

Proof. By Theorem 6.1 of [9] we know that $\mathrm{sr}(M_m(A)) = \{(\mathrm{sr}(A) - 1)/m\} + 1 \leqslant \mathrm{sr}(A)$. The last proposition shows that $\mathrm{csr}(M_m(A)) \leqslant \mathrm{csr}(A)$. Using (1.5), (1.6), (1.7) and (1.8) we get the desired inequalities. The rest is obvious.

2.12. Corollary. *Let A be a C^*-algebra. Then* $\mathrm{csr}(A) = 1$ *implies* $\mathrm{csr}(A \otimes \mathscr{K}) = 1$.

III.

In this section we obtain some technical results using mainly the inequality (1.3).

Let us fix some notations to be used from now on. We shall denote by X $(X_0, \ldots, X_n)$ a compact space of dimension m $(m_0, \ldots, m_n)$, $X = X_0 \times \ldots \times X_n$.

Let $A_1, \ldots, A_n$ be BDF-extensions of $\mathscr{K}$ by $C(X_j)$; they satisfy exact sequences:

$$0 \to K \to A_j \to C(X_j) \to 0, \quad 1 \leqslant j \leqslant n. \tag{3.1}$$

Let $A = C(X_0)\otimes A_1\otimes \ldots \otimes A_n$, $B_j = C(X_0)\otimes \ldots \otimes C(X_j)\otimes A_{j+1}\otimes \ldots \otimes A_n$, for $0 \leqslant j \leqslant n$ (thus $B_0 = A$, $B_n = C(X)$), $C_{j-1} = C(X_0)\otimes \ldots \otimes C(X_{j-1})\otimes A_{j+1}\otimes \ldots \ldots \otimes A_n$. Then from (3.1) we get the following exact sequences:

$$(3.2)\qquad 0 \to \mathscr{K}\otimes C_{j-1} \to B_{j-1} \to B_j \to 0 \quad 1 \leqslant j \leqslant n.$$

Let us denote by $c_j = \mathrm{csr}(B_j)$, $s_j = \mathrm{sr}(B_j)$, $0 \leqslant j \leqslant n$. From (1.3) and (1.9) we get

$$(3.3)\qquad \begin{gathered} c_{j-1} \leqslant \max\{c_j, \mathrm{csr}(\mathscr{K}\otimes C_{j-1})\} \leqslant \max\{c_j, \mathrm{csr}(C_{j-1})\} \\ s_j \leqslant s_{j-1} \leqslant \max\{c_j, s_j, \mathrm{sr}(\mathscr{K}\otimes C_{j-1})\}. \end{gathered}$$

So we obtain $c_0 \leqslant \max_{1\leqslant j\leqslant n} \{c_n, \mathrm{csr}(\mathscr{K}\otimes C_{j-1})\}$, $s_n \leqslant s_0 \leqslant \max_{1\leqslant j\leqslant n} \{s_n, c_n, \mathrm{sr}(\mathscr{K}\otimes C_{j-1}), \mathrm{csr}(\mathscr{K}\otimes C_{j-1})\}$. But $s_n = [m/2] + 1$, $c_n \leqslant [(m + 1)/2] + 1$. If $m = 0$ we can prove by induction on n (using (3.3), (2.5) and (2.12)) that $\mathrm{csr}(\mathscr{K}\otimes C_{j-1}) = 1$, $\mathrm{sr}(\mathscr{K}\otimes C_{j-1}) = 1$ and thus $s_0 = c_0 = 1$. (This also follows from a theorem of L.G. Brown which implies that A is an AF-C^*-algebra.)

3.4. PROPOSITION. *Let $m \neq 1$; then*

$$[m/2] + 1 \leqslant \mathrm{sr}(A) \leqslant \max\{[m/2] + 1, \mathrm{csr}(C(X))\},$$

$$\mathrm{csr}(A) \leqslant \mathrm{csr}(C(X)).$$

For $m = 1$, we have that $\mathrm{sr}(A)$ *and* $\mathrm{csr}(A) \in \{1,2\}$.

Proof. We have proved everything except $\mathrm{csr}(A) \leqslant \mathrm{csr}(C(X))$. But this follows from (3.3) by induction on n since C_{j-1} has at most $n - 1$ extension terms in the tensor product.

3.5. COROLLARY. $[m/2] + 1 \leqslant \mathrm{sr}(A) \leqslant [(m + 1)/2] + 1$.

3.6. REMARK. This corollary is the analogue of Theorem 6 of [8]. Its proof is inspired by that in [8].

To improve the last corollary we shall make from now on the following assumptions: X_j is the inverse limit of m_j-dimensional finite CW-complexes, $\dim(X_j) = m_j$ for $0 \leqslant j \leqslant n$ and $\dim(X) = \dim(X_0) + \dim(X_1) + \ldots + \dim(X_n) = m$.

3.7. LEMMA. *Suppose that $m_0 > 0$, then* $\mathrm{sr}(A) = [m/2] + 1$.

Proof. Using (1.5) we may suppose that X_0 is a finite CW-complex of dimension $m_0 > 0$. Let $X_0 = X_0' \cup U_1 \cup \ldots \cup U_k$, where X_0' is a CW-complex of dimension less than m_0 and U_j are disjoint open sets such that there exist functions $\varphi_j : \mathbf{B}_{m_0} \to U_j$ verifying $\varphi_j(\partial \mathbf{B}_{m_0}) \subset X_0'$ and $\varphi_j|\mathring{\mathbf{B}}_{m_0}$ is a homeomorphism of $\mathring{\mathbf{B}}_{m_0}$ onto U_j ($1 \leqslant j \leqslant k$) and $X_0' \cap U_j = \emptyset$. Then $\mathrm{sr}(C(X_0')\otimes A_1\otimes \ldots \otimes A_n) \leqslant [m/2] + 1 = s$ by Corollary 3.5. Let $f \in C(X_0,(A_1\otimes A_2\otimes \ldots \otimes A_n)^s)$, and $\varepsilon > 0$. There exists $g \in C(X_0', (A_1\otimes \ldots \otimes A_n)^s)$, $g(x) \in \mathrm{Lg}_s(A_1\otimes \ldots \otimes A_n)$, $\|g - f|X_0^1\| < \varepsilon/2$. Extend g to X_0 such that the last inequality be fulfilled. Using the functions φ_j we pullback g to a function $g_j = g\circ\varphi_j \in C(\mathbf{B}_{m_0}, (A_1\otimes \ldots \otimes A_n)^s)$. But $\mathrm{csr}(C(\mathbf{B}_{m_0})\otimes A_1\otimes \ldots \otimes A_n) = \mathrm{csr}(A_1 \otimes \ldots \otimes A_n)$ by Corollary 2.9, thus $\mathrm{csr}(C(\mathbf{B}_{m_0})\otimes A_1\otimes \ldots \otimes A_n) \leqslant [m/2] + 1 = s$ by Corollary 2.5. By Corollary 2.2 there exists $h_j \in C(\mathbf{B}_{m_0}, \mathrm{Lg}_s(A_1 \otimes \ldots \otimes A_n))$, $\|h_j - g_j\| < \varepsilon/2$, $h_j|\mathbf{S}^{m_0-1} = g_j|\mathbf{S}^{m_0-1}$. Then the function $h : X_0 \to \mathrm{Lg}_s(A_1\otimes \ldots \otimes A_n)$ given by $h(x) = g(x)$ for $x \in X_0'$ and $h(x) = h_j(\varphi_j^{-1}(x))$, $x \in U_j$ satisfies $\|h - f\| < \varepsilon$ and $h|X_0'$, $h\circ\varphi_j$ are continuous. This shows that h is continuous.

IV.

This section contains results concerning the irreducible representations of A and the proof of the main theorem.

We know that there exists a unique class of irreducible representations of $\mathscr{K}$ and hence a unique class of irreducible representations of A_j not vanishing on $\mathscr{K}$ (see [2]); denote by ρ_j an element of this class. Let $\pi_j : A_j \to C(X_j)$ the quotient map and $\sigma_j = \pi_j\otimes 1$; here 1 stands for the identity map on $A_1\otimes \ldots \otimes A_{j-1}\otimes A_{j+1}\otimes \ldots \otimes A_n$.

4.1. LEMMA. *Let π be an irreducible representation of $A = A_1 \otimes \ldots \otimes A_n$. Then one of the following statements is true:*

1° *π is unitarily equivalent to $\rho_1 \otimes \ldots \otimes \rho_n = \rho$;*

2° *π factors through σ_j (i.e. there exists $j \in \{1, \ldots, n\}$ and π_0 such that $\pi = \pi_0\circ\sigma_j$).*

Proof. Let $J_j = \ker \sigma_j$. There are exactly two possibilities (see [2]): 1° $\pi|J_1J_2 \ldots J_n$ is irreducible, and hence unitarily equivalent to $\rho_1\otimes \ldots \otimes\rho_n/J_1 \ldots J_n$ since $J_1 \ldots J_n \cong \mathscr{K}$; this shows ([2]) that π and ρ are unitarily equivalent; 2° there exists j, $1 \leqslant j \leqslant n$ such that $\pi|J_j = 0$; but this means that π factors through σ_j.

Let us observe that $\rho(J) = \mathscr{K}$ ($J = J_1J_2 \ldots J_n$).

The following lemma shows why the case $m = 1$ is an exceptional one. Let $\rho = \rho_1\otimes \ldots \otimes\rho_n$, and $\mathscr{H} = \mathscr{H}_\rho$ (see [2] for notation).

4.2. LEMMA. *Let $\varepsilon > 0$, and $x, y \in A = A_1 \otimes \ldots \otimes A_n$. Then there exists $x_1, y_1 \in A$ such that $\|x - x_1\| < \varepsilon$, $\|y - y_1\| < \varepsilon$ and* $\ker \rho(x_1) \cap \ker \rho(x_2) = \{0\}$.

Proof. We shall denote by $[\mathcal{M}]$ the ortogonal projection onto the closed subspace $\mathcal{M}$ of a Hilbert space $\mathcal{H}$.

Suppose first that at least one of the operators $\rho(x)$ and $\rho(y)$, say $\rho(x)$, has the property that $\dim \mathcal{H}/\rho(x)\mathcal{H} = \infty$. Then there exists a compact operator T, $\|T\| < \varepsilon/2$, such that $\dim \mathcal{H}|\overline{(\rho(x) + T)\mathcal{H}} = \infty$, (see [7]). There exists also a compact operator T_1, $\|T_1\| < \varepsilon/2$, with right support $r(T_1) = \ker(\rho(x) + T) \cap \ker \rho(y)$ and left support $l(T_1) \leqslant 1 - [\mathrm{Ran}(\rho(x) + T)]$. If we let $y_1 = y$ and x_1 such that $\|x_1 - x\| < \varepsilon$ and $\rho(x_1 - x) = T + T_1$ we obtain the desired conclusion.

Suppose now that $\mathcal{H}/\rho(x)\mathcal{H}$ and $\mathcal{H}/\rho(y)\mathcal{H}$ are finite dimensional. Then there exists an $\delta > 0$ such that $(0, \delta) \cap \sigma(\rho(x)\rho(x)^*) = \emptyset$. Let $f(0) = 1$ and $f(t) = 0$ for $t \geqslant \delta$; then $f(\rho(x)\rho(x)^*) = 1 - [\rho(x)\mathcal{H}]$ and $f(\rho(x)^*\rho(x)) = [\ker \rho(x)]$. By assumption there exists $a \in J$ such that $\rho(a) = \rho(f(xx^*))$. Since $A/(J_1 + \ldots + J_n)$ is commutative we get that $f(xx^*) - f(x^*x) \in J_1 + \ldots + J_n$ thus obtaining that there exists $b \in J_1 + \ldots + J_n$ such that $\rho(b) = [\ker \rho(x)]$. We want to show that $\mathrm{Ran}\, \rho(y)|\ker \rho(x) = \mathrm{Ran}\, \rho(yb)$ has infinite codimension to conclude the proof as above. Suppose the contrary. Then the same will be true for any T close enough to $\rho(xb)$ and we may find such a T of the following particular type $T = \sum_{j=1}^{n}\left(\sum_{k=1}^{m_j} A_{jk} \otimes F_{jk} \otimes B_{jk}\right)$ with F_{jk} a finite rank operator on $\mathcal{H}_{\rho_j}$ and A_{j_k} and B_{j_k} bounded operators. Let $\xi_j \perp \mathrm{Ran}\, F_{jk}$ $(1 \leqslant j \leqslant n,\ 1 \leqslant k \leqslant m_j)$; then $\xi = \xi_1 \otimes \otimes \ldots \otimes \xi_n$ is ortogonal to $\mathrm{Ran}\, T$ and these vectors span an infinite dimensional vector space, a contradiction.

4.3. Remark. Suppose that $m_n = \dim(X_n) = 1$; then $\mathrm{sr}(A_n) \in \{1, 2\}$ by Proposition 3.4. Let $x \in A_n$, $\varepsilon > 0$; there exists $x_1 \in A_n$ such that $\|x_1 - x\| < \varepsilon/2$ and $\pi_n(x_1) \in \mathrm{Lg}_1(C(X_n)) = \mathrm{GL}(1, C(X_n))$. This shows that $\rho_n(x_1)$ is a Fredholm operator which will be left-invertible if and only if $\ker \rho_n(x_1) = 0$. If $\mathrm{ind}(\rho_n(x_1)) = \dim \ker(\rho_n(x_1)) - \dim \ker(\rho_n(x_1)^*) \leqslant 0$ we can find a finite rank operator F such that $\|F\| < \varepsilon/2$ and $\ker(\rho_n(x_1) + F) = \{0\}$ thus finding $x_2 = x_1 + a$ $(\rho_n(a) = F)$ such that $\rho_n(x_2)$ is left invertible. Lemma 4.1 shows that $x_2 \in \mathrm{Lg}_1(A_n)$. We obtain that $\mathrm{sr}(A_n) = 1$ if and only if any Fredholm operator in A_n has index 0, if and only if the composed map $\mathrm{H}^1(X_n) \to \mathrm{K}_1(C(X_n)) \to \mathrm{K}_0(\mathcal{K}) = \mathbf{Z}$ is trivial (see [10]).

Suppose that $X, X_0, \ldots, X_n$ satisfy the assumptions preceding Lemma 3.7, $A = C(X_0) \otimes A_1 \otimes \ldots \otimes A_n$, $m_j = \dim(X_j)$, $0 \leqslant j \leqslant n$.

4.4. Theorem. *Let* $m \neq 1$; *then* $\mathrm{sr}(A) = [m/2] + 1$. *If* $m = 1$, *let* $m_1 \leqslant \ldots \leqslant m_n$; *then* $\mathrm{sr}(A) = \mathrm{sr}(A_n)$.

Proof. Let $s = [m/2] + 1$. For $m = 0$ the theorem has already been proved. For $m = 1$ there are two possibilities: 1° $\dim(X_0) = 1$, $\dim(X_1) = \ldots = \dim(X_n) = 0$; then $\mathrm{sr}(A) = 1 = \mathrm{sr}(A_n)$ by Lemma 3.7. 2° $\dim(X_0) = \ldots = \dim(X_{n-1}) = 0$, $\dim(X_n) = 1$; then $C(X_0) \otimes A_1 \otimes \ldots \otimes A_{n-1}$ is an AF-C^*-algebra (see [5]) and thus, by Corollary 2.11, $\mathrm{sr}(A) \leqslant \mathrm{sr}(A_n)$. The reverse inequality follows from the fact that A_n is a quotient of A.

Suppose now that $m \geqslant 2$. We shall use induction on n; for $n = 0$ there is nothing to prove. Let $x = (x_1, \ldots, x_s) \in A^s$, $\varepsilon > 0$. If $m_0 > 0$ the theorem follows from Lemma 3.7. If $m_0 = 0$, $C(X_0)$ is an AF-C^*-algebra and thus $\mathrm{sr}(A) = \mathrm{sr}(A_1 \otimes \otimes \ldots \otimes A_n)$ by Corollary 2.11; we may suppose then that $A = A_1 \otimes \ldots \otimes A_n$. By the induction hypothesis there exists $x' = (x'_1, \ldots, x'_s) \in A^s$ such that $\|x - x'\| < \varepsilon/2$ and $\sigma_j(x') \in \mathrm{Lg}_s(A_1 \otimes \ldots \otimes A_{j-1} \otimes C(X_j) \otimes \ldots \otimes A_n)$. We may also find x'' such that the last condition be fulfilled, $\|x'' - x'\| < \varepsilon/2$ and $\ker \rho(x''_1) \cap \ker \rho(x''_2) = \{0\}$; then by Lemma 4.1, $x'' \in \mathrm{Lg}_s(A)$.

4.5. REMARK. It is obvious that if A and B are C^*-algebras as in Theorem 4.4 then $\mathrm{sr}(A \otimes B) \leqslant \mathrm{sr}(A) + \mathrm{sr}(B)$ (this answers a question from [9] in a particular case).

REFERENCES

1. BROWN, L. G.; DOUGLAS, R. G.; FILLMORE, P. A., Unitary equivalence modulo the compact operators and extensions of C^*-algebras, in *Springer Lecture Notes in Math.*, **345**(1973), 58–128.
2. DIXMIER, J., *Les C^*-algèbres et leurs représentations*, Gauthier-Villars et Cie, 1984.
3. DOUGLAS, R. G., *Banach algebra techniques in the theory of Toeplitz operators*, CBMS Regional Conference Series in Math., No, **15**.
4. DOUGLAS, R. G.; HOWE, R., On the C^*-algebra of Toeplitz operators on the quarter-plane, *Trans. Amer. Math. Soc.*, **158**(1971), 203–217.
5. EFFROS, E., Dimension and C^*-algebras, preprint.
6. HERMAN, R. H.; VASERSTEIN, L. N., The stable rank of C^*-algebras, *Invent. Math.*, **77**(1984), 553–555.
7. LEBOW, A.; SCHECHTER, M., Semigroups of operators and measures of noncompactness, *J. Funct. Anal.*, **7**(1971), 1–26.
8. NAGY, G., Stable rank of C^*-algebras of Toeplitz operators on polydisks, in *Operators in indefinite metric spaces, scattering theory and other topics*, Birkhäuser-Verlag, 1986, pp. 227–235.
9. RIEFFEL, M. A., Dimension and stable rank in the K-theory of C^*-algebras, *Proc. London Math. Soc. (3)*, **46**(1983), 301–333.
10. TAYLOR, J. L., Banach algebras and topology, in *Algebras in Analysis* (Ed. J. H. Williamson), Academic Press, 1975, pp. 118–186.

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