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EQUIVALENCE AND ISOMORPHISM FOR GROUPOID C^*-ALGEBRAS

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1. INTRODUCTION

Motivation for this work comes from two sources. The first is a desire to give in the second countable case a detailed, self-contained account of the relation that exists between the notions "equivalence of groupoids" and "strong Morita equivalence of groupoid C^*-algebras" that was developed by the second author and is discussed briefly in [15] *). The second is a desire to generalize to groupoid C^*-algebras some results of Phil Green [6] concerning the structure of the C^*-algebra of a transitive transformation group. The connection between the two sources is that to effect our generalization of Green's work, we need to know that equivalent groupoids give rise to strongly Morita equivalent C^*-algebras.

We elaborate. Set-theoretically, a groupoid is simply a small category in which each morphism is invertible. A groupoid is called *transitive* if given any two objects, there is at least one morphism connecting them. It is easy to show that if G is a transitive groupoid, then G is isomorphic to a groupoid of the form $X\times H\times X$ where X is a set and H is a group. (Two triples (x, h, y) and (x', h', y') are composable if and only if $y = x'$ and, in that case, their product is (x, hh', y'); also, $(x, h, y)^{-1} = (y, h^{-1}, x)$, by definition.) If X and H are locally compact, then the system of measures $\{\delta_x\times\lambda_H\times\mu\}_{x\in X}$, where λ_H is a fixed Haar measure on H and μ is a (Radon) measure of full support on X, constitutes a Haar system on $X\times H\times X$ in the sense of [14]. The associated groupoid C^*-algebra is, then, easily seen to be cononically isomorphic to $C^*(H)\otimes\mathcal{K}(L^2(\mu))$ where $C^*(H)$ denotes the group C^*-algebra of H and $\mathcal{K}(L^2(\mu))$ denotes the compact operators on $L^2(\mu)$. Thus the structure of transitive groupoid C^*-algebras appears to be quite straightforward. However, if G is a transitive locally compact grou-

*) We follow the notation and terminology of [14] except that we write s for the domain map; i.e., $s(\gamma) = \gamma^{-1}\gamma$ for an element γ in a groupoid.

poid with Haar system, λ, to begin with, then G need not be isomorphic to a groupoid of the form $X\times H\times X$ in such a way that λ is carried to a system $\{\delta_x\times\lambda_H\times\mu\}_{x\in X}$. Nevertheless, if G is second countable, we shall see that $C^*(G, \lambda)$ is always isomorphic to $C^*(H)\otimes\mathscr{K}(L^2(\mu))$ for an appropriate locally compact group H and a measure μ on G^0. When G is the groupoid of a transitive, transformation group, this has been proved by Phil Green in [6].

In the general case, the group H is simply the isotropy group G_u^u at any object or unit $u\in G^0$. Since G is transitive, two different units give isomorphic H's. The first step in the proof of our isomorphism theorem is to observe that G^u $(=r^{-1}(u))$ is an equivalence between H and G, in the sense of Definition 2.1 below, provided the restriction of s to G^u is open as a map from G^u onto G^0. Theorem 2.8, which implies that equivalent groupoid in the sense of Definition 2.1 have strongly Morita equivalent C^*-algebras, then enables us to exhibit a very special equivalence between $C^*(H)$ and $C^*(G)$. Using a Borel section to the map $s|G^u$ we parlay this equivalence, in Theorem 3.1, to an explicit isomorphism between $C^*(G)$ and $C^*(H)\otimes\mathscr{K}(L^2(\mu))$ for an essentially unique measure μ on G^0. The point is, the isomorphism depends only on the unit chosen and the choice of section.

To help see the scope of our results, we note that while not every transitive locally compact groupoid is (topologically) isomorphic to one of the form $X\times H\times X$ described above, the notion of equivalence (Definition 2.1) establishes, essentially, a bijection between transitive locally compact groupoids and ordinary principal group-bundles in the sense of [8]. This fact was anticipated by Seda [20] and we elaborate on the point in Example 2.2.

Pairs of equivalent groupoids appear to be ubiquitous in nature; we present a number of examples in section two. In the past, when equivalent pairs have appeared, the strong Morita equivalence of the associated C^*-algebras was proved in ad hoc ways. We hope that our Theorem 2.8 will prove useful in streamlining future investigations.

We close this section with a few words about the hypotheses that we are placing on our groupoids. Our proof of Theorem 2.8 relies heavily on a technical result of the second author [16, Proposition 4.2] concerning the disintegration of representations of groupoids. It is worth keeping in mind that this is the only place in the proof of Theorem 2.8 where the assumption of second countability will be used. For this reason we start by assuming only that each of our groupoids is locally compact and is endowed with a Haar system. We assume also that the unit space of each is paracompact — this condition is not automatically satisfied.

We would like to thank the referee for bringing to our attention a gap in our original proof of Theorem 2.8.

2. EQUIVALENCE OF GROUPOIDS

To repeat, by a groupoid we shall always mean an algebraic groupoid equipped with a locally compact, Hausdorff topology compatible with the groupoid structure. If G is a groupoid, then we shall write G^0 for the unit space. For $\gamma \in G, r(\gamma)$ and $s(\gamma)$ will denote the range and source of γ in G^0. As we will only be interested in groupoids which admit a Haar system, we shall assume from the onset that r, and hence s, are open maps from G to G^0 [14, 1.2.4]. In addition we shall *always* assume that the unit space is paracompact. For further definitions and references the reader can consult [14] and [15].

The definition of a G-space X is a straightforward generalization of that for a group action. Here we require a continuous open map from the locally compact space X onto G^0, which we call ρ or σ, according to the side on which G acts. For example, a left G-space is given by a continuous map from $G * X \to X$, where $G * X$ denotes, as in the rest of this work, the set of composable pairs (γ, x) with $s(\gamma) = \rho(x)$.

We say that the action is free if $\gamma \cdot x = x$ only when γ is a unit. We say that the action is proper if the map from $G * X$ to $X \times X$ given by $(\gamma, x) \to (\gamma \cdot x, x)$ is proper. The space X is a principal G-space if the action is both free and proper. We point out that it is a straightforward consequence of the properness that the natural map, $\pi: X \to G\backslash X$, is open, and that $G\backslash X$ is locally compact and Hausdorff. The basic example of a principal G-space is $X = G$ with either left or right multiplication.

Now suppose that X is a left principal G-space. Set $X * X = \{(x, y) \in X \times X : \rho(x) = \rho(y)\}$. Then G acts on the left via the diagonal action and we have the quotient $H = G\backslash X * X$. Notice that H has a natural groupoid structure with multiplication

$$[x, y] \cdot [y, z] = [x, z],$$

and unit space $G\backslash X$. The range and source maps are given by

$$r([x, y]) = [x] \quad \text{and} \quad s([x, y]) = [y].$$

Of course, there is an obvious right action of H on X. Specifically, the map $\sigma: X \to H^0$ (i.e. $G\backslash X$) is given by $\sigma(x) = [x]$ and so $X * H = \{(z, [x, y]) \in X \times H : [z] = [x]\}$. The action is given by the formula $z \cdot [x, y] = gy$, where g is the unique element of G such that $z = gx$. Note that this action is well defined. Indeed, if $[x', y'] = [x, y]$, there is a unique $h \in G$ such that $x' = hx$ and $y' = hy$. It follows that $[x'] = [z]$ when $[x] = [z]$ and the unique element of G which sends x' to z is gh^{-1}. Consequently, $z \cdot [x', y'] = (gh^{-1}) = gy = z \cdot [x, y]$. It is not difficult to check that, with this action, X is a (right) principal H-space. Furthermore, both actions commute and ρ induces a homeomorphism of X/H onto G^0. This situation is formalized in the following definition.

DEFINITION 2.1. Let G and H be locally compact groupoids. We say that a locally compact space Z is a *(G, H)-equivalence* if

(i) Z is a left principal G-space,
(ii) Z is a right principal H-space,
(iii) the G and H actions commute,
(iv) the map ρ induces a bijection of Z/H onto G^0, and
(v) the map σ induces a bijection of $G\backslash Z$ onto H^0.

One should keep in mind that if Z is a (G, H)-equivalence, then H is naturally isomorphic to $G\backslash Z * Z$ and G is naturally isomorphic to $Z * Z/H$. Indeed, given $[x, y] \in G\backslash Z * Z$, $\rho(x) = \rho(y)$, then there is a unique $\gamma \in H$ such that $x\gamma = y$; the correspondence $[x, y] \to \gamma$ is the desired isomorphism between $G\backslash Z * Z$ and H.

Although we do not need this fact, we remark that the notion of equivalence induces an equivalence relation on locally compact groupoids. In fact, if Z is a (G, H)-equivalence and Y is a (H, K)-equivalence, then $Z \underset{H}{*} Y$ is a (G, H)-equivalence. Here $Z \underset{H}{*} Y$ is the quotient of $Z * Y$ where (z, y) is identified with $(z\cdot h, h\cdot y)$ for all $z \in Z$, $y \in Y$, and $h \in H$.

EXAMPLE 2.2. *Transitive Groupoids.* Our objective here is to establish the relationship between transitive groupoids and principal group bundles promised in the introduction. Recall [8, p. 41] that if H is a locally compact group and X is a locally compact left H-space, then $(X, \sigma, H\backslash X)$ is a principal H-bundle, where σ denotes the quotient map from X to $H\backslash X$, if and only if X is a principal H-space as defined above. The groupoid $G = H\backslash X * X$, then, is transitive with unit space $H\backslash X$. Indeed, to see that G is transitive, note that since H is a group, $H^0 = \{e\}$ where e is the identity. Therefore, $X * X = X\times X$ and so, given $[x]$, $[y] \in H\backslash X$, $[x, y] \in H\backslash X\times X$ is well defined and maps $[x]$ to $[y]$. (The fact that principal group bundles give rise to transitive groupoids in this way was pointed out by Seda in §4 of [20].) Suppose, on the other hand, that G is a transitive groupoid, let $u \in G^0$, and form $G^u = r^{-1}(u)$, and $H = G^u_u = \{\gamma : r(\gamma) = s(\gamma)\}$. Then H is a locally compact group acting to the left on G^u. This action is free and proper, as may be easily verified [19, Lemma 3]. Also, G acts to the right on G^u, freely and properly, and the two actions commute. If $\sigma = s|G^u$ and $\rho = r|G^u$, then a moment's reflection reveals that G^u is a (H, G)-equivalence precisely when σ is open. In general, σ need not be open; perhaps the simplest example is the transformation group groupoid determined by the obvious action of the discrete reals on the reals with the usual topology. Observe that the map Φ from $H\backslash G^u \times G^u$ to G defined by the formula $\Phi([\alpha, \beta]) = \alpha^{-1}\beta$ is a well defined, continuous, groupoid isomorphism from $H\backslash G^u\times G^u$ onto G, but it need not be a homeomorphism, as the example just given shows. However, we have the following result which shows that Φ is a homeomorphism precisely when σ is open.

THEOREM 2.2 A. *The following are equivalent:*

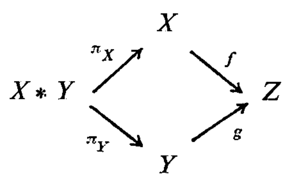
(i) *Φ is a homeomorphism.*

(ii) *σ is open.*

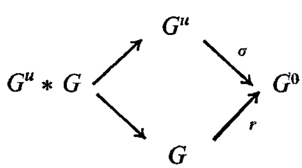
(iii) *$\pi : G \to G^0 \times G^0$ is open, where $\pi(\gamma) = (r(\gamma), s(\gamma))$.*

Proof. First we show that (iii) implies (ii). Let U be open in G. Then $\sigma(U \cap G^u)) = \pi_1(\pi(U) \cap \pi_2^{-1}(u))$ where $\pi_i, i = 1, 2$, is the projection of $G^0 \times G^0$ onto its i^{th} factor. So, if π is open, so is σ.

To see that (ii) implies (i), note that if σ is open, then as remarked above, G^u defines an equivalence between H and G. So to prove that Φ is a homeomorphism, it suffices to prove that the map $(\alpha, \beta) \to \alpha^{-1}\beta$ from $G^u \times G^u$ to G is open. Observe that this map is the composition $(\alpha, \beta) \to (\alpha, \alpha^{-1}\beta) \to \alpha^{-1}\beta$ carrying $G^u \times G^u$ to $G^u * G$ and then to G. (Recall that $G^u * G = \{(\alpha, \beta) \in G^u \times G : \sigma(\alpha) = r(\beta)\}$.) The first map is easily seen to be a homeomorphism. To prove that the second map is open, it is enlightening to appeal to the following general observation. Suppose that X, Y, and Z are topological spaces and that f and g are continuous maps from X and Y to Z, respectively. Let $X * Y = \{(x, y) \in X \times Y : f(x) = g(y)\}$ and give $X * Y$ the relative topology from $X \times Y$. Then the diagram



is commutative, where π_X and π_Y are the projections of $X * Y$ onto X and Y, respectively. The point that we wish to make is that if f is open, so is π_Y. Indeed, if U and V are open in X and Y, respectively, then $\pi_Y((U \times V) \cap (X * Y)) = g^{-1}[f(U)] \cap V$. Since sets of the form $(U \times V) \cap (X * Y)$ form a basis for the topology on $X * Y$, it is clear that the openness of f implies that for π_Y. In our special case, we have



and so the hypothesis that σ is open implies that the projection of $G^u * G$ onto G is open, as required.

Finally, we note that (i) implies (iii). This is immediate since the obvious map from $H\backslash G^u \times G^u$ to $H\backslash G^u \times H\backslash G^u$ is open.

The map σ fails to be open only in the most pathological situations; in particular, it is always open when G is second countable as we now show.

THEOREM 2.2B. *Let G be a transitive groupoid which is locally compact and second countable. Then, with the above notation, G is isomorphic and homeomorphic to $H\backslash G^u \times G^u$.*

Proof. By the preceding result, we need only show that π is open. We shall do this via a Baire category argument which is similar in spirit to the proof of the open mapping theorem for separable locally compact groups. Note that π is a groupoid homomorphism from G onto the trivial groupoid $G^0 \times G^0$. The open mapping theorem for second countable locally compact groupoids fails in general, but there is sufficient structure in our setting to push through the usual proof for groups.

STEP 1. *If U is a neighborhood of G^0 in G and if $u \in G^0$, then $r(U \cap s^{-1}(u))$ has a nonempty interior.*

To establish this, choose a neighborhood of G^0 with $W^2 \subset U$ and assume, without loss of generality, that W is r-compact (see [14, 2.1.9]). This means that W is closed and $W \cap r^{-1}(K)$ is compact for each compact set $K \in G^0$. It follows that for each compact set $V \subseteq G$, $VW = V(W \cap r^{-1}(s(V)))$ is also compact. Next select a countable cover of G by compact neighborhoods V_n so that $V_n^{-1}V_n \subseteq W$. Then $s^{-1}(u) = \bigcup_n (V_n W) \cap s^{-1}(u)$ and so, since G is transitive, $G^0 = r(s^{-1}(u)) = \bigcup_n r(V_n W \cap s^{-1}(u))$; i.e., G^0 is covered by a sequence of compact sets. Since G^0 is locally compact, and therefore a Baire space, at least one of the sets, say $r(V_n W \cap s^{-1}(u))$, contains a nonempty open set A. But then if $A' = s(V_n \cap r^{-1}(A))$, then A' has nonempty interior and $A' \subseteq r(V_n^{-1}V_n W \cap s^{-1}(u)) \subseteq r(W^2 \cap s^{-1}(u)) \subseteq r(U \cap s^{-1}(u))$.

STEP 2. *If W is a neighborhood of G^0 and if $u \in G^0$, then there is a γ in $W \cap s^{-1}(u)$ such that $(r(\gamma), r(\gamma))$ is interior to $\pi(W)$.*

For this, choose a neighborhood V of G^0 such that $V = V^{-1}$ and $V^2 \subseteq W$. From Step 1, we can choose an open set A in G^0 and a γ in $V \cap s^{-1}(u)$ such that $r(\gamma) \in A \subseteq r(V \cap s^{-1}(u))$. We have $\pi(V) \supseteq \pi(V \cap s^{-1}(u)) \supseteq A \times \{u\}$ and we have $\pi(V) = \pi(V^{-1}) \supseteq \{u\} \times A$. Consequently, $(r(\gamma), r(\gamma)) \in A \times A = (A \times \{u\})(\{u\} \times A) \subseteq \pi(V)^2 = \pi(V^2) \subseteq \pi(W)$.

STEP 3. *If U is a neighborhood of G^0, then $\pi(U)$ is a neighborhood of the diagonal Δ in $G^0 \times G^0$.*

Fix $(u, u) \in \Delta$, and choose an open neighborhood W of G^0 such that $W = W^{-1}$ and $W^3 \subseteq U$. From Step 2, we can find a γ in $W \cap s^{-1}(u)$ and an open set $A \subseteq G^0$ such that $(r(\gamma), r(\gamma)) \in A \times A \subseteq \pi(W)$. If $A' = s(W \cap r^{-1}(A))$, then clearly A' is open and contains u, and since $W = W^{-1}$, $A' = r(W \cap s^{-1}(A))$. Moreover, we see that $(u, u) \in A' \times A' \subseteq \pi(W^3) \subseteq \pi(U)$. Indeed, choose $(v, w) \in A' \times A'$. Then there is an $\alpha \in W \cap s^{-1}(A)$ such that $r(\alpha) = v$ and there is a β in $W \cap r^{-1}(A)$ such that $s(\beta) = w$. Since $s(\alpha)$, $r(\beta) \in A$, and since $A \times A \subseteq \pi(W)$, there is a $\gamma \in W$ such that $\pi(\gamma) = (s(\alpha), r(\beta))$. It follows that $\alpha\gamma\beta \in W^3$ and $\pi(\alpha\gamma\beta) = (v, w)$.

To complete the proof of Theorem 2.2 B, let U be an open set in G, fix $\gamma \in U$ and write $x = r(\gamma)$ and $y = s(\gamma)$. Also, choose a neighborhood W of γ such that $(WW^{-1})W(W^{-1}W) \subseteq U$. Then WW^{-1} and $W^{-1}W$ are open neighborhoods in G of open sets in G^0. Hence by Step 3, we can find open sets A and B in G^0 such that $x \in A$ and $A \times A \subseteq \pi(WW^{-1})$ while $y \in B$ and $B \times B \subseteq \pi(W^{-1}W)$. It follows, then, that $(x, y) \in A \times B \subseteq \pi((WW^{-1})W(W^{-1}W)) \subseteq \pi(U)$, and the proof is complete. ▨

As a consequence of this theorem, we recover the known fact that if G is the groupoid associated to the transitive action of a locally compact group T on a locally compact space B, then for any $u \in B$, the map $t \to t \cdot u$, from T to B, is open and B can be identified with T/H, where H is the isotropy group at u, provided that T and B are second countable.

We conclude Example 2.2 by noting how locally trivial transitive groupoids [22] fit in our setting. Such a groupoid is a transitive groupoid G with the property that any of the following three conditions is satisfied:

(i) $\pi: G \to G^0 \times G^0$ has local continuous sections.

(ii) There is a unit $u \in G^0$ such that σ $(= s|G^u)$ has local continuous sections.

(iii) For every unit $u \in G^0$, σ has local continuous sections.

Evidently, for such a groupoid, σ is an open map for each unit $u \in G^0$ and so, by Theorem 2.2 A, G is isomorphic and homeomorphic to $H \diagdown G^0 \times G^0$ where H is the isotropy group at u. Moreover, as a principal H-space, G^0 is locally trivial. Conversely, if X is a principal H-space that is locally trivial, then the groupoid $H \diagdown X \times X$ is locally trivial.

We will return to this example in the next section.

EXAMPLE 2.3. *The fundamental groupoid of a space.* If we specialize the preceding example a bit, we discover that if X is a locally compact Hausdorff space that is connected, locally arcwise connected, and semilocally simply connected, then its fundamental groupoid $\Gamma(X)$ is transitive and equivalent (Definition 2.1) to the fundamental group $\pi_1(X, x_0)$ based at any point $x_0 \in X$; the equivalence is implemented by the universal covering space of X. For details on this, see Proposition 2.37 on page 133 of [13].

EXAMPLE 2.4. (cf. [17]). *Bi-transformation groups.* Suppose that H and K are groups acting freely and properly on a locally compact Hausdorff space P and assume that the actions commute. As noted above, the properness assumption implies that the orbit spaces P/H and P/K are locally compact Hausdorff spaces and the commutivity assumption implies that P/H carries a K action while P/K carries a H action. The space P, then, implements an equivalence between the transformation group groupoids $(H, P/K)$ and $(K, P/H)$.

EXAMPLE 2.5. *Equivalence relations.* Let R be an open equivalence relation of a locally compact Hausdorff space X, and suppose that as a subset of $X \times X$, R is closed. Then X implements an equivalence between R and X/R.

EXAMPLE 2.6. *Foliations.* Suppose that G is the holonomy groupoid of a foliated space (V, F) and suppose that G is Hausdorff. If N is a transverse submanifold which meets every leaf (N need not be connected), then $G_N = \{\gamma \mid s(\gamma) \in N\}$ is a (G, G_N^N)-equivalence where $G_N^N = \{\gamma \in G : r(\gamma), s(\gamma) \in N\}$.

EXAMPLE 2.7. *Abstract transversals.* Quite generally, let G be a locally compact Hausdorff groupoid and let N be a closed subset of G^0 that meets each orbit in G^0. Then, as is easy to see, G_N is a principal left G-space and a principal right G_N^N-space and the actions commute. The maps $\sigma = s|G_N$ and $\rho = r|G_N$ satisfy (iv) and (v) of Definition 2.1. So, if they are open, then G_N is a (G, G_N^N)-equivalence.

Of course Examples 2.2 and 2.6 are special cases of Example 2.7.

We turn now to the main theorem of this section, Theorem 2.8, which asserts that a groupoid equivalence between two groupoids induces a strong Morita equivalence between their C^*-algebras. We fix two groupoids, G and H, and Haar systems $\{\lambda^u\}_{u\in G^0}$ and $\{\beta^v\}_{v\in H^0}$. We shall write $C^*(G, \lambda)$ and $C^*(H, \beta)$ for the groupoid C^*-algebras formed with respect to the universal C^*-norm (cf. [14] or [15]).

THEOREM 2.8. *Suppose that (G, λ) and (H, β) are second countable locally compact groupoids with Haar systems λ and β. Then for any (G, H)-equivalence Z, $C_c(Z)$ can naturally be completed into a $C^*(G, \lambda)$-$C^*(H, \beta)$ imprimitivity bimodule. In particular, $C^*(G, \lambda)$ and $C^*(H, \beta)$ are strongly Morita equivalent.*

Before beginning the proof, we require several preliminary results. First a technical lemma.

LEMMA 2.9. *Let Ω be a principal left G-space.*

a) *If $F \in C_c(\Omega \times G)$, then*

$$\varphi(\omega, u) = \int_G F(\omega, \gamma)\,d\lambda^u(\gamma)$$

defines an element of $C_c(\Omega \times G^0)$.

b) *If $f \in C_c(\Omega)$, then*

$$\lambda(f)([\omega]) = \int_G f(\lambda^{-1}\cdot\omega)\mathrm{d}\lambda^{\rho(\omega)}(\gamma)$$

defines a surjection of $C_c(\Omega)$ onto $C_c(G\diagdown\Omega)$.

Proof. For part a), it suffices to consider functions F of the form $F(\omega, \gamma) = f(\omega)g(\gamma)$ with $f \in C_c(\Omega)$ and $g \in C_c(G)$. The assertion is now immediate since $\{\lambda^u\}_{u\in G^0}$ is a Haar system.

For part b), view $\lambda(f)$ as a function on Ω. It is constant on orbits, by left invariance. Since the statement about supports is obvious, it will suffice to show that $\lambda(f)$ is continuous at $\omega \in \Omega$. However by using the properness of the G-action, there is an $F \in C_c(G\times\Omega)$ such that $F(z, \gamma) = f(\gamma^{-1}\cdot z)$ near ω. The result follows from part a). ▨

As usual, we work with the pre-C^*-algebras $A = C_c(G, \lambda)$ and $B = C_c(H, \beta)$. We define the left A-action and the right B-action as follows:

$$f\cdot\varphi(z) = \int_G f(\gamma)\varphi(\gamma^{-1}\cdot z)\mathrm{d}\lambda^{\rho(z)}(\gamma),$$

and

$$\varphi\cdot g(z) = \int_H \varphi(z\cdot\eta)g(\eta^{-1})\mathrm{d}\beta^{\sigma(z)}(\eta),$$

where $\varphi \in C_c(Z), f \in A$, and $y \in B$. Note that $f\cdot\varphi$ and $\varphi\cdot g$ are in $C_c(Z)$ by virtue of Lemma 2.9 a). Now $Z \underset{\sigma}{*} Z = \{(z, w) : \rho(z) = \rho(\omega)\}$ is a principal G-space when G is given the diagonal action. Thus if φ and ψ are in $C_c(Z)$,

$$\Phi(z, w) = \int_G \overline{\varphi(\gamma^{-1}\cdot z)}\psi(\gamma^{-1}\cdot w)\mathrm{d}\lambda^{\rho(z)}(\gamma)$$

defines an element of $C_c(G\diagdown Z \underset{\sigma}{*} Z)$.

On the other hand,

$$\beta(\eta) = \int_G \overline{\varphi(\gamma^{-1}\cdot z)}\psi(\gamma^{-1}\cdot z\cdot\eta)\mathrm{d}\lambda^{\rho(z)}(\gamma)$$

is independent of choice of z with $\sigma(z) = r(\eta)$, and clearly has compact support in H. Now if $\eta_\alpha \to \eta$ in H, then since σ and r are open, we can find $z_\alpha \to z$ (passing to a subnet if necessary) so that $\sigma(z_\alpha) = r(\eta_\alpha)$. Thus, $\beta(\eta_\alpha) = \Phi(z_\alpha, z_\alpha\cdot\eta_\alpha)$ converges to $\beta(\eta)$. In short

$$\langle\varphi,\psi\rangle_B(\eta)=\int_G \overline{\varphi(\gamma^{-1}\cdot z)}\psi(\gamma^{-1}\cdot z\cdot\eta)\,\mathrm{d}\lambda^{\rho(z)}(\gamma),$$

where $\sigma(z)=r(\eta)$, defines a B-valued inner-product on $C_c(Z)$.

Similarly

$$\langle\varphi,\psi\rangle_A(\gamma)=\int_H \varphi(\gamma^{-1}\cdot z\cdot\eta)\overline{\psi(z\cdot\eta)}\,\mathrm{d}\beta^{\sigma(z)}(\eta),$$

where $\rho(z)=r(\gamma)$, defines an A-valued inner-product.

The verification of the following identities is straightforward.

$$f\cdot\langle\varphi,\psi\rangle_A=\langle f\cdot\varphi,\psi\rangle_A,\quad f\in A,\ \varphi,\psi\in C_c(Z).$$

$$\langle\varphi,\psi\rangle_B\cdot g=\langle\varphi,\psi\cdot g\rangle_B,\quad g\in B,\ \varphi,\psi\in C_c(Z).$$

$$(f*f_1)\cdot\varphi=f\cdot(f_1\cdot\varphi),\quad f,f_1\in A,\ \varphi\in C_c(Z).$$

$$\varphi\cdot(g*g_1)=(\varphi\cdot g)\cdot g_1,\quad g,g_1\in B,\ \varphi\in C_c(Z).$$

$$\langle\varphi,\psi\rangle_A^*=\langle\psi,\varphi\rangle_A,$$

$$\langle\varphi,\psi\rangle_B^*=\langle\psi,\varphi\rangle_B,$$

$$\langle\varphi,\psi\rangle_A\cdot\xi=\varphi\cdot\langle\psi,\xi\rangle_B,\quad \varphi,\psi,\text{ and }\xi\in C_c(Z).$$

It is important to realize that arguments similar to those of Lemma 2.9 show that these operations are continuous with respect to the inductive limit topology.

The crucial point is the following result (cf. [17]).

PROPOSITION 2.10. *There is a net e_α in $C_c(G)$ of elements of the form*

$$e_\alpha=\sum_1^{n_\alpha}\langle\varphi_i^\alpha,\varphi_i^\alpha\rangle_A,$$

with each $\varphi_i^\alpha\in C_c(Z)$, which is an approximate identity with respect to the inductive limit topology for both the left action of $C_c(G)$ on itself and on $C_c(Z)$. A similar statement holds for H.

This proposition answers a question raised in [14, 2.1.9].

COROLLARY 2.11. *For any locally compact groupoid with paracompact unit space and Haar system λ, the $*$-algebra $C_c(G,\lambda)$ has a two-sided approximate identity with respect to the inductive limit topology.*

Before proving the proposition we require three lemmas. First recall from [14] and the proof of Theorem 2.2 B that a subset, L, of G is called *r-relatively compact* if $L \cap r^{-1}(K)$ is relatively compact for every compact $K \subseteq G$. It follows from the proof of [14, 2.1.9], that if G^0 is paracompact, then G^0 has a fundamental system of r-relatively compact neighborhoods in G. The next lemma is a trivial modification of [14, 2.1.9], so we omit the proof.

LEMMA 2.12. *Suppose that for each triple (K, U, ε) consisting of a compact subset $K \subseteq G^0$, an open r-relatively compact neighborhood, U, of G^0 in G, and a positive number $\varepsilon > 0$, we have $e = e_{K,U,\varepsilon} \in C_c^+(G)$ such that*

(i) $\mathrm{supp}(e) \subseteq U$, *and*

(ii) $\left|\int e(\gamma)\, d\lambda^u(\gamma) - 1\right| \leqslant \varepsilon$ *for* $u \in K$.

Then the net $e_{K,U,\varepsilon}$ directed by increasing K and decreasing U and ε is an approximate identity for the left action of $C_c(G)$ on $C_c(Z)$.

The next two lemmas are also routine, so we omit their proofs as well.

LEMMA 2.13. *Let X and Y be locally compact spaces and $\pi : X \to Y$ a continuous, open surjection. Let λ^y be a family of measures on X such that the support of λ^y is exactly $\pi^{-1}(y)$, and which defines a map λ from $C_c(X)$ onto $C_c(Y)$. Then for any open set $U \subseteq X$ and any $b \in C_c^+(Y)$ with* $\mathrm{supp}(b) \subseteq \pi(U)$, *there is a $g \in C_c^+(X)$ with* $\mathrm{supp}(g) \subseteq U$ *and* $\lambda(g) = b$.

LEMMA 2.14. *Let X, Y, π, and λ be as above. For any U open in X, any $g \in C_c^+(X)$ with* $\mathrm{supp}(g) \subseteq U$, *and any $\varepsilon > 0$, there is an $f \in C_c^+(X)$ with* $\mathrm{supp}(f) \subseteq U$ *and*

$$|g(x) - f(x)\lambda(f) \circ \pi(x)| \leqslant \varepsilon$$

for every x.

Proof of Proposition 2.10. We will consider only the G-action, the proof for $C_c(H)$ being analogous. Furthermore, it will suffice to consider only the action of $C_c(G)$ on $C_c(Z)$, as G always implements an equivalence of G with itself.

Let (K, U, ε) be given. By the properness of the G-action on Z, we can find open, relatively compact sets V_i, $i = 1, \ldots, n$ in Z such that $\{\rho(V_i)\}_{i=1}^n$ cover K, and such that $(\gamma \cdot z, z) \in V_i \times V_i$ implies $\gamma \in U$.

Let $\{b_i\}_{i=1}^n$ be a partition of unity subordinate to $\{\rho(V_i)\}$. Hence, we may assume that

$$b_i \in C_c^+(G^0),$$

$$\mathrm{supp}(b_i) \subseteq \rho(V_i),$$

and

$$\sum_{i=1}^{n} b_i(u) = 1 \qquad \text{for each } u \in K.$$

Lemma 2.9 implies that the hypotheses of Lemma 2.13 are satisfied and, therefore, that there exist $\psi_i \in C_c^+(z)$ with $\mathrm{supp}(\psi_i) \subseteq V_i$ and

$$\int_H \psi_i(z \cdot \eta)\, \mathrm{d}\beta^{\sigma(z)}(\eta) = b_i(\rho(z))$$

for all $z \in Z$. By Lemma 2.14, there exist $\varphi_i \in C_c^+(Z)$ with $\mathrm{supp}(\varphi_i) \subseteq V_i$ and

$$\left| \psi_i(z) - \varphi_i(z) \int_G \varphi_i(\gamma^{-1} \cdot z)\, \mathrm{d}\lambda^{\rho(z)}(\gamma) \right| \leqslant \frac{\varepsilon}{M},$$

where

$$M = \sup_z \sum_{i=1}^n \int 1_{V_i}(z \cdot \eta)\, \mathrm{d}\beta^{\sigma(z)}(\eta).$$

We claim that $e = \sum_{i=1}^n \langle \varphi_i, \varphi_i \rangle_A$ satisfies the hypothesis of Lemma 2.12. In fact, $e(\gamma) = 0$ off U by construction, and if $\rho(z) \in K$, then

$$\left| \int_G e(\gamma)\, \mathrm{d}\lambda^{\rho(z)}(\gamma) - 1 \right| = \left| \sum_{i=1}^n \int_G \int_H \varphi_i(z\eta)\varphi_i(\gamma^{-1}z\eta)\, \mathrm{d}\beta^{\sigma(z)}(\eta)\, \mathrm{d}\lambda^{\rho(z)} - \sum_{i=1}^n b_i(\rho(z)) \right| =$$

$$= \left| \sum_{i=1}^n \int_H \left\{ \varphi_i(z\eta) \int_G \varphi_i(\gamma^{-1}z\eta)\, \mathrm{d}\lambda^{\rho(z)}(\gamma) - \psi_i(z\eta) \right\} \mathrm{d}\beta^{\sigma(z)}(\eta) \right| \leqslant \varepsilon.$$

▨

End of the proof of Theorem 2.8. In order to show that $C_c(Z)$ is a $C_c(G, \lambda)$-$C_c(H, \beta)$-imprimitivity bimodule with respect to the usual C^*-norms we have left to check only the following:

(1) the positivity of the inner products,

(2) the density of the range of the inner products,

and

(3) the boundedness of the actions.

However, since the inductive limit topology is finer than the C^*-norm topology, (1) and (2) follow from Proposition 2.10 by what are now standard means (cf. [4] remarks following Lemma 2 or [17]).

Thus, it remains to show

$$(*) \qquad \langle f \cdot \varphi, f \cdot \varphi \rangle_B \leqslant \|f\|^2_{C^*(G)} \langle \varphi, \varphi \rangle_B,$$

and

$$(**) \qquad \langle \varphi \cdot g, \varphi \cdot g \rangle_A \leqslant \|g\|^2_{C^*(H)} \langle \varphi, \varphi \rangle_A,$$

where $f \in C_c(G)$, $g \in C_c(H)$, and $\varphi \in C_c(Z)$. To establish $(*)$, we take a state ζ on $C^*(H)$. Then, $C_c(Z)$ becomes a pre-Hilbert space with respect to the inner product

$$\langle \cdot, \cdot \rangle_\zeta = \zeta(\langle \cdot, \cdot \rangle_B).$$

Let D be the (dense) image of $C_c(Z)$ in the (Hausdorff) completion, V_ζ, of $C_c(Z)$ with respect to $\langle \cdot, \cdot \rangle_\zeta$. Of course, the left action of $C_c(G)$ on $C_c(Z)$ defines a $*$-representation, L, of $C_c(G)$ by operators on D. Moreover, by Proposition 2.10 and the fact that the module actions as well as the inner products are continuous with respect to the inductive limit topologies, we have the following facts.

a) L is non-degenerate in the sense that elements of the form $L(f)\xi$ are dense in V_ζ, where $f \in C_c(G)$ and $\xi \in D$.

b) L is continuous in the sense that for all $\xi, \eta \in D$ the linear functional

$$L_{\xi,\eta}(f) = \langle \xi, f \cdot \eta \rangle_\zeta$$

is continuous with respect to the inductive limit topology on $C_c(G)$.

In short, we see that the conditions a), b), and c) of [16, Proposition 4.2] are satisfied and we conclude (with the help of [14, 2.1.7]) that L is bounded with respect to the C^*-norm on $C_c(G)$. That is

$$\zeta(\langle f \cdot \varphi, f \cdot \varphi \rangle_B) \leqslant \|f\|^2_{C^*(G)} \zeta(\langle \varphi, \varphi \rangle_B)$$

for all states ζ on $C^*(H)$. The inequality $(*)$ follows. The proof of $(**)$ is analogous. ▨

We remark in passing that when the equivalent pairs of groupoids in Examples 2.2—2.7 have Haar systems and are second countable, then by Theorem 2.8 they yield strongly Morita equivalent C^*-algebras.

It should be pointed out that Rieffel [18] showed that the C^*-algebras, $C^*(H, P/K)$ and $C^*(K, P/H)$, associated with the transformation groups in Example 2.4 are strongly Morita equivalent. His proof is similar in certain respect to ours. A different proof may be found in [2]. If, in Example 2.5, R is the orbit equivalence relation determined by the free action on X of a locally compact group, and if X and G are second countable, then Green [4] showed that $C^*(G, X)$ is strongly Morita equivalent to $C^0(X/G)$ if and only if the action of G is proper; i.e., if and only if R satisfies the hypotheses of Example 2.5. Of course, as we just noted, Theorem 2.8 implies that if R is an arbitrary equivalence relation on X that satisfies the hypotheses of Example 2.5, and if λ is a Haar system on R, then $C^*(R, \lambda)$ is strongly Morita equivalent to $C_0(X/R)$. We expect the converse assertion to be true, but we do not have a proof at this time. Finally, we note that in [11], the authors associate a certain groupoid to the C^*-algebra generated by a family of Wiener-Hopf operators. The groupoid is obtained from a certain transformation group groupoid by cutting down to a closed set that meets every orbit, as in Example 2.7. In this situation, the hypotheses of Example 2.7 are not difficult to verify and we conclude from Theorem 2.8 that the C^*-algebras of Wiener-Hopf operators described in [11] are strongly Morita equivalent to the corresponding transformation group C^*-algebras.

3. TRANSITIVE GROUPOIDS

In this section, we apply the equivalence theorem of the preceding section, Theorem 2.8, to show that the C^*-algebra of a transitive groupoid has a special form. This is our generalization of Green's result [6] mentioned in the introduction. As before (G, λ) will be a fixed second countable groupoid with Haar system λ. In the main theorem of this section, we assume G is transitive.

THEOREM 3.1. *Let G be a second countable, transitive groupoid, let $u \in G^0$, and set $H = G_u^u$. Then there is a positive measure μ on G^0 such that $C^*(G)$ is isomorphic to $C^*(H) \otimes \mathscr{K}(L^2(G^0, \mu))$ where $\mathscr{K}(L^2(G^0, \mu))$ denotes the algebra of compact operators on $L^2(G^0, \mu)$.*

The proof is broken into a series of lemmas. The first is a general statement about strongly Morita equivalent C^*-algebras. The proof is straightforward and, therefore, we will omit it.

LEMMA 3.2. *Let A be a C^*-algebra and for $i = 1,2$ let X_i be a left A-rigged space with A-valued inner product denoted $\langle\ ,\ \rangle_i$. Denote the imprimitivity algebra of X_i by E_i. If $W : X_1 \to X_2$ is an A-linear map such that $\langle Wx, Wy\rangle_2 = \langle x, y\rangle_1$, for all $x, y \in X_1$, then W implements a C^*-isomorphism from E_1 onto E_2 via the formula*

$$WT_{x\otimes y}W^{-1} = T_{(Wx)\otimes(Wy)}$$

where $T_{x\otimes y}$ is the element of E_i such that

$$T_{x\otimes y}z = \langle z, x\rangle_i y.$$

By Theorem 2.8, we know that $C^*(H)$ and $C^*(G)$ are strongly Morita equivalent via the $C^*(H)$ module X, which is the completion of $C_c(G^u)$ in the $C^*(H)$-valued inner product

$$\langle x, y\rangle_{C^*(H)}(t) = \int_{G^u} x(t\gamma)\overline{y(\gamma)}\, d\lambda^u(\gamma)$$

for $x, y \in C_c(G^u)$; i.e., $C^*(G)$ is the imprimitivity algebra of this $C^*(H)$-rigged space, X_1. Thus, by Lemma 3.2, all we need to do is to produce another $C^*(H)$-rigged space X_2 such that the imprimitivity algebra of X_2 is $C^*(H) \otimes \mathscr{K}(L^2(G^0, \mu))$, for a suitable measure μ, and a $C^*(H)$-module map $W: X_1 \to X_2$ such that $\langle Wx, Wy \rangle_2 = \langle x, y \rangle_1$ for all $x, y \in X_1$. (From now on, we call such a W a $C^*(H)$-isometry.)

To form X_2, consider the spaces $B_c(H) \otimes B_c(G^0)$ where, for any locally compact space M, $B_c(M)$ stands for the space of all bounded, compactly supported Borel functions on M and $B_c(H) \otimes B_c(G^0)$ denotes the linear span of the functions $\varphi \otimes \xi$ on $H \times G^0$ defined on $H \times G^0$ by the formula $\varphi \otimes \xi(t, \omega) = \varphi(t)\xi(\omega)$. Of course $C_c(H)$ acts on $B_c(H) \otimes B_c(G^0)$ by left convolution in the first variable: $\psi \cdot (\varphi \otimes \xi) = (\psi * \varphi) \otimes \xi$. For each (Radon) measure μ on G^0, we obtain a $C^*(H)$-valued inner product on $B_c(H) \otimes B_c(G^0)$ according to the formula

$$\langle \varphi_1 \otimes \xi_1, \varphi_2 \otimes \xi_2 \rangle_{C^*(H)}(t) = \left(\int_H \varphi_1(ts)\overline{\varphi_2(s)}\, \mathrm{d}\lambda_H(s) \right) \left(\int_{G^0} \xi_1(\omega)\overline{\xi_2(\omega)}\, \mathrm{d}\mu(\omega) \right)$$

where λ_H is Haar measure on H — which we fix, once and for all. Observe that the function of t, $\int_H \varphi_1(ts)\varphi_2(s)\, \mathrm{d}\lambda_H(s)$, belongs to $C_c(H)$ for each pair of functions $\varphi_1, \varphi_2 \in B_c(H)$ and that this is a bonafide $C^*(H)$-inner product yielding, upon completion, a $C^*(H)$-rigged space $X_2(\mu)$. It is evident that $X_2(\mu)$ is the so-called Hilbert space over $C^*(H)$ in the sense of [9, p. 136]; i.e., $X_2(\mu)$ is the completion of the algebraic tensor product $C^*(H) \otimes L^2(G^0, \mu)$ in the norm determined by the $C^*(H)$-inner product

$$\langle \varphi_1 \otimes \xi_1, \varphi_2 \otimes \xi_2 \rangle_{C^*(H)} = (\varphi_1 * \varphi_2^*)(\xi_1, \xi_2)_{L^2(G^0, \mu)}.$$

According to [9, Lemma 4], then, the imprimitivity algebra of $X_2(\mu)$ is

$$C^*(H) \otimes \mathscr{K}(L^2(G^0, \mu)).$$

Thus, we are left with the problem of pinning down μ and finding a $C^*(H)$-isometry from X_1 onto $X_2(\mu)$.

Let X and Y be locally compact spaces and let $\pi : X \to Y$ be a continuous surjection. We will call a Borel map $\rho : Y \to X$ a *regular cross section* for π if $\pi \circ \rho(y) = y$ for all $y \in Y$, and $\rho(K)$ has compact closure in X for each compact set K in Y. The following lemma was proved by Mackey, [10, Lemma 1.1].

LEMMA 3.3. *If X and Y are second countable, locally compact spaces, then each continuous surjection $\pi : X \to Y$ has a regular cross section.*

Recall that $\sigma = s|G^u$ is a continuous map on G^u that is onto G^0 since G is transitive. Consequently, by our separability assumption and Lemma 3.3, there is a regular cross section $\rho : G^0 \to G^u$ for σ. The following proposition will complete the proof of Theorem 3.1.

PROPOSITION 3.4. *For each regular cross section ρ for σ, there is measure μ on G^0 and a $C^*(H)$-isometry from X_1 onto $X_2(\mu)$.*

Proof. Extend ρ to a map, keeping the same name, from G^u onto G^u by the formula $\rho(\gamma) = \rho(\sigma(\gamma))$, and define the map $\psi : G^u \to G$ by the formula $\psi(\gamma) = \gamma\rho(\gamma)^{-1}$. Observe that ψ is well defined and maps G^u into H. Indeed, since $\sigma(\rho(\gamma)) = \sigma(\gamma)$ and $r(\rho(\gamma)^{-1}) = \sigma(\gamma) = s(\gamma)$, $(\gamma, \rho(\gamma)) \in G^{(2)}$ for all $\gamma \in G^u$; but $r(\psi(\gamma)) = r(\gamma) = u$, by assumption, and $s(\psi(\gamma)) = s(\rho(\gamma)^{-1}) = r(\rho(\gamma)) = u$; therefore, $\psi(\gamma) \in G^u_u = H$ for all $\gamma \in G^u$. Now define φ mapping G^u to $H\times G^0$ by the formula $\varphi(\gamma) = (\psi(\gamma), \sigma(\gamma))$, $\gamma \in G^u$, and let H act on $H\times G^0$ by translation in the first coordinate.

ASSERTION. *φ is an H-equivariant, Borel isomorphism from G^u onto $H\times G^0$.*

For $t \in H$, $\gamma \in G^u$, we have $\sigma(t\gamma) = s(\gamma)$ and $\rho(t\gamma) = \sigma(\gamma)$. So $\varphi(t\gamma) = (\psi(t\gamma), \sigma(t\gamma)) = ((t\gamma)\rho(t\gamma)^{-1}, \sigma(t\gamma)) = t(\gamma\rho(\gamma)^{-1}, \sigma(\gamma))$, and φ is equivariant. An easy calculation shows that $\varphi^{-1}(t, \omega) = t\rho(\omega)$, so φ is a Borel isomorphism.

Next observe that since ρ is a regular cross section for $\sigma, f\circ\varphi \in B_c(G^u)$ for each $f \in B_c(H\times G^0)$. Also, $\varphi_*\lambda^u$ is a Radon measure on $H\times G^0$ where, for a Borel set $E \subseteq H\times G^0$, $(\varphi_*\lambda^u)(E) = \lambda^u(\varphi^{-1}(E))$.

ASSERTION. *There is a positive Radon measure μ on G^0 such that $\varphi_*\lambda^u = \lambda_H\times\mu$.*

Since φ is equivariant, and λ^u is an H-invariant measure on G^u, $\varphi_*\lambda^u$ is an H-invariant measure on $H\times G^0$. By Theorem 5.4 in [20], there is a measurable family of measures $\{\nu_\omega\}_{\omega\in G^0}$ on $H\times G^0$ (i.e., $\omega\to\nu_\omega(E)$ is a Borel function on G^0 for each Borel set $E \subseteq H\times G^0$), and a measure $\tilde{\mu}$ on G^0 such that ν_ω is supported on $H\times\{\omega\}$, each ν_ω is H-invariant, and $\varphi_*\lambda^u(E) = \int_{G^0} \nu_\omega(E)\,d\tilde{\mu}$. By the uniqueness of Haar measure, there is, for each $\omega \in G^0$, a positive constant $C(\omega)$ so that $\nu_\omega = C(\omega)(\lambda_H\times\delta_\omega)$. Since $\{\lambda_H\times\delta_\omega\}_{\omega\in G^0}$ and $\{\nu_\omega\}_{\omega\in G^0}$ are both measurable families, it follows that C is measurable. Thus we simply set $\mu = C\tilde{\mu}$ to obtain a positive Borel measure μ on G^0 such that $\varphi_*\lambda^u = \lambda\times\mu$. Finally, to see that μ is a Radon measure, let $K_2 \subseteq G^0$ be compact and choose a compact set $K_1 \subseteq H$ so that $\lambda_H(K_1) = 1$. Then $\mu(K_2) = \lambda_H \times \mu(K_1 \times K_2) = \lambda^u(\varphi^{-1}(K_1 \times K_2)) < \infty$ because $\varphi^{-1}(K_1\times K_2)$ has compact closure and λ^u is a Radon measure.

Next we observe that $B_c(G^u)$ and $B_c(H\times G^0)$ may be viewed as (dense) subspaces of X_1 and $X_2(\mu)$, respectively. We show this for $B_c(H\times G^0)$ and $X_2(\mu)$. The argument for $B_c(G^u)$ and X_1 is similar and will be omitted. Let $\xi \in B_c(H\times G^0)$ and choose a sequence $\{\xi_n\}_{n=1}^{\infty}$ in $B_c(H)\otimes B_c(G^0)$ having common compact support that converges pointwise boundedly to ξ. It suffices to show that $\{\xi_n\}_{n=1}^{\infty}$ is Cauchy in $X_2(\mu)$-norm. But

$$\|\xi_n - \xi_m\|^2_{C^*(H)}(t) = \int_{G^0}\left(\int_H \xi_n(ts,\omega)\overline{\xi_n(s,\omega)}\,\mathrm{d}\lambda_H(s)\right)\mathrm{d}\mu(\omega) -$$

$$- 2\,\mathrm{Re}\left\{\int_{G^0}\left(\int_H \xi_n(ts,\omega)\overline{\xi_m(s,\omega)}\,\mathrm{d}\lambda_H(s)\right)\mathrm{d}\mu(\omega)\right\} +$$

$$+ \int_{G^0}\left(\int_H \xi_m(ts,\omega)\overline{\xi_m(s,\omega)}\,\mathrm{d}\lambda_H(s)\right)\mathrm{d}\mu(\omega).$$

The hypotheses on the convergence of $\{\xi_n\}_{n=1}^{\infty}$ and Fubini's theorem imply that $\|\xi_n - \xi_m\|^2_{C^*(H)}(\cdot)$ is arbitrarily small in $L^1(H)$-norm when n and m are sufficiently large; and since the L^1-norm dominates the C^*-norm, $\{\xi_n\}_{n=1}^{\infty}$ is Cauchy in $X_2(\mu)$.

Finally, we define $W_\rho : B_c(G^u) \to B_c(H\times G^0)$ by the formula $(W_\rho\xi) = \xi\circ\varphi^{-1}$. As we noted earlier, W_ρ maps $B_c(G^u)$ isomorphically onto $B_c(H\times G^0)$. It therefore suffices to check that $W_\rho(\psi\cdot\xi) = \psi(W_\rho\xi)$ for all $\psi \in C_c(H)$ and $\xi \in B_c(G^u)$ and that $\|W_\rho\xi\|_{X_2(\mu)} = \|\xi\|_{X_1}$ for all $\xi \in B_c(G^u)$. The first is a straightforward calculation; and so is the second, once it is realized that the functions $\langle W_\rho\xi, W_\rho\xi\rangle_{X_2(\mu)}(t)$ and $\langle\xi,\xi\rangle_{X_1}(t)$ are the same. This completes the proof of Proposition 3.4 and, with it, the proof of Theorem 3.1. ▨

It should be emphasized, that in Theorem 3.1, once the unit $u \in G^0$ is fixed, the isomorphism constructed depends only on the section to σ. Takesaki has remarked to us that it is known to the cognoscenti that the C^*-algebra of the fundamental groupoid $\Gamma(X)$ of a space X (cf. Example 2.3) is isomorphic to $C^*(\pi_1(X, x_0)) \otimes \mathcal{K}$ and that the isomorphism depends upon the choice of the base point x_0 and section to the covering map.

We conclude with a rather long remark which is intended to put Theorem 3.1 into a broader context. We omit most of the details because the hypotheses are somewhat technical and there does not seem to be a global setting in which they are all satisfied. Rather they must be checked, in an ad hoc fashion, in each situation. As before, G is a locally compact groupoid with left Haar system $\lambda = \{\lambda^u\}_{u\in G^0}$. We suppose, too, that X is a locally compact, proper (left) G-space and that $\alpha = \{\alpha^u\}_{u\in G^0}$ is a continuous equivariant system of measures on X. This means that $\mathrm{supp}(\alpha^u) = \rho^{-1}(u)$, that

$$\alpha(f) = \int_X f(x)\,\mathrm{d}\alpha^u(x)$$

defines a map from $C_c(X)$ onto $C_c(G^0)$, and that

$$\int f(\gamma x)\, d\alpha^{s(\gamma)}(x) = \int f(x)\, d\alpha^{r(\gamma)}(x),$$

$f \in C_c(X)$ and $\gamma \in G$ (cf. Lemma 2.11). Finally, we suppose that H is a G-Hilbert bundle over X. As with group actions, this means that $\pi : H \to X$ is an ordinary Hilbert bundle over X in the sense of Fell [3] and that G acts continuously on H in such a way that the following conditions are satisfied:

a) $(\gamma, h) \in G*H$ if and only if $(\gamma, \pi(h)) \in G*X$;

b) for $(\gamma, x) \in G*X, \gamma H(x) = H(\gamma x)$, where $H(x) = \pi^{-1}(x)$;

and

c) for $(\gamma, x) \in G*X$, the map from $H(x)$ to $H(\gamma x)$ determined by γ is a Hilbert space isomorphism.

The data, (X, α, H), give rise to a $C^*(G, \lambda)$ module in the following way. Let $\Gamma_c(H)$ denote the compactly supported, continuous sections of H. For $f \in C_c(G)$ and $\xi \in \Gamma_c(H)$, we define

$$f*\xi(x) = \int f(\gamma)\gamma\xi(\gamma^{-1}x)\, d\lambda^{\rho(x)}(\gamma),$$

and for $\xi, \eta \in \Gamma_c(H)$, we define

$$\langle \xi, \eta \rangle_{C^*(G)}(\gamma) = \int \langle \xi(\gamma x), \eta(x) \rangle_{H(x)}\, d\alpha^{s(\gamma)}(x).$$

Parts of the proof of Theorem 2.8 apply mutatis mutandis to show that these definitions do give rise to a $C^*(G, \lambda)$-module, which we denote by $C^*(X, \alpha, H)$. (The reader is urged to consult [21] where some useful and related results are presented.) Observe, in particular, that when $X = G$, $\alpha = \lambda$, and H is the one dimensional trivial bundle, $C^*(X, \alpha, H) = C^*(G, \lambda)$ as a module over $C^*(G, \lambda)$. We will denote this module by $C^*(G, \lambda)$ also.

For another example, which is germane for our purposes, suppose that X is a principal G-space, that σ is a regular cross section for the quotient map $X \to X/G$, and that there is a continuous family of measures, $\tau = \{\tau^u\}_{u \in G^0}$ defining a map from $C_c(X/G)$ onto $C_c(G^0)$, as in Lemma 2.11, such that for $f \in C_c(X)$,

$$\int f\, d\alpha^u(x) = \iint f(\gamma\sigma(\dot{x}))\, d\tau^{s(\gamma)}(\dot{x})\, d\lambda^u(\gamma), \tag{3.1}$$

where $\dot{x}$ denotes the image of x in X/G. Thus the map $(\gamma, \dot{x}) \to \gamma\sigma(\dot{x})$ sets up a Borel isomorphism between $G \times X/G$ and X that transforms α to $\lambda \times \tau = \{\lambda^u \times \tau^u\}_{u \in G^0}$. In general, it seems to be difficult to decide if a continuous family

τ exists given σ and α. In the special case when G is a group, this is covered by the argument in Proposition 3.4. The family τ defines, in a natural way, a Hilbert bundle over G^0, denoted $L^2(X/G, \tau)$. The fibres are the spaces $L^2(X/G, \tau^u)$, $u \in G^0$, and the basic fields which go to define the bundle structure on $L^2(X/G, \tau)$ are simply the continuous, compactly supported functions on X/G. If $C^*(X, \alpha)$ denotes the $C^*(G, \lambda)$-module determined by X, α, and the trivial, one-dimensional bundle over X, then the equation (3.1) shows that $C^*(X, \alpha)$ is isomorphic to $C^*(G, \lambda, s^*(L^2(X/G, \tau)))$ where $s^*(L^2(X/G, \tau))$ denotes the pull-back to G of $L^2(X/G, \tau)$ determined by the source map s. This latter module is easily seen to be isomorphic to $C^*(G, \lambda)\otimes_{C_0(G^0)}\Gamma_0(L^2(X/G, \tau))$, where $\Gamma_0(L^2(X/G, \tau))$ is the space of continuous section of $L^2(X/G, \tau)$ that vanish at infinity. (Recall that $C_0(G^0)$ acts as right multipliers on $C^*(G, \lambda)$, and $\Gamma_0(L^2(X/G, \tau))$ is a left $C_0(G^0)$-module; so tensoring over $C_0(G^0)$ makes perfectly good sense.) If it happens that $L^2(X/G, \tau)$ is trivial, so that we may view $\Gamma_0(L^2(X/G, \tau))$ as $C_0(G^0, H)$ for a suitable Hilbert space H, then we conclude that our original module $C^*(X, \alpha)$ is isomorphic to $C^*(G, \lambda)\otimes \otimes H$, a Hilbert space over $C^*(G)$ in the sense of [9]. Of course the imprimitivity algebra of this module is $C^*(G)\otimes\mathcal{K}(H)$. It is now quite easy to see how Theorem 3.1 fits into this set up. In that case, the groupoid is H, and its unit space consists of one point, X is G^u, α is λ^u, λ is Haar measure on H, and τ is μ. On the other hand the stability of the C^*-algebra of a foliation established by Hilsum and Skandalis [7] can be recovered along these lines. Let Γ be the holonomy groupoid of the foliation and T the transversal they construct in Lemma 2. If $G = \Gamma_T^T$ and $X = \Gamma^T$, then the principal G-space X carries a continuous equivariant system of measures $\{\alpha^u\}_{u\in G^0}$ ($G^0=T$) that decomposes as $\lambda\times\tau$ for a certain cross section to the quotient map $X \to X/G$ and continuous system τ on X/G. This yields a direct proof of the triviality of the C^*-module associated with X.

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