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ON ALGEBRAIC AND PARA-REFLEXIVITY

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Let H be a Hilbert space and $B(H)$ be the set of (continuous linear) operators on H. A linear submanifold M of H is called an *operator range* if there is a Hilbert space K and an operator from K into H whose range is M. In [4] R. G. Douglas and C. Foiaş proved the following theorem.

THEOREM 1. [4]. *Suppose $S, T \in B(H)$ and T is not algebraic. If S leaves invariant every T-invariant operator range, then $S = \psi(T)$ for some entire function ψ.*

The proof given in [4] of Theorem 1 relies heavily on the Sz.-Nagy—Foiaş theory of contraction operators on Hilbert space [8]. We prove here a reflexivity theorem (Theorem 4) that generalizes Theorem 1 in several ways, i.e., the role of the set of entire functions in T is played by a more general algebra of holomorphic functions in T, the assumption of the continuity of S is dropped, and the Hilbert space H is replaced by an arbitrary Banach space. The analogue of operator ranges are still ranges of operators whose domains are Hilbert spaces. Our proof uses a reflexivity theorem from linear algebra and standard properties of the Riesz functional calculus.

We first consider the algebraic result. An algebra $\mathscr{A}$ of linear transformations on a complex vector space X is *algebraically reflexive* if $S \in \mathscr{A}$ whenever S is a linear transformation that leaves invariant every $\mathscr{A}$-invariant linear manifold (equivalently, $Sx \in \mathscr{A}x = \{Ax : A \in \mathscr{A}\}$ for every x in X). A vector x is a *separating vector* for the algebra $\mathscr{A}$ if the map $A \to Ax$ is 1-1 on $\mathscr{A}$ (equivalently, $A \in \mathscr{A}, Ax = 0$ implies $A = 0$). It should be noted that the proof of the following theorem of D. Hadwin uses only elementary techniques from linear algebra.

THEOREM 2. [6]. *Suppose X is a complex vector space and $\mathscr{A}$ is a commutative algebra of linear transformations on X with $1 \in \mathscr{A}$ such that*

(1) *$\mathscr{A}$ has no zero divisors,*

(2) $\bigcap\{\ker\rho : \rho$ *is a complex homomorphism on* $\mathscr{A}\} = \{0\}$,

(3) *$\mathscr{A}$ has a separating vector.*
Then $\mathscr{A}$ is algebraically reflexive.

We wish to apply Theorem 2 to certain algebras of operators. Suppose X is a Banach space and $T \in B(X)$. Let Ω be an open subset of the plane that contains $\sigma(T)$ (the spectrum of T). Let $H(\Omega)$ be the algebra of all complex holomorphic (analytic) functions on Ω. The Riesz functional calculus (defined in terms of the Cauchy integral formula [3]) defines a homorphism $\psi \to \psi(T)$ from $H(\Omega)$ into $B(X)$. This homomorphism is continuous with respect to the norm topology on $B(X)$ and the topology of uniform convergence on compact sets on $H(\Omega)$. In particular, if V is a component of Ω that intersects $\sigma(T)$, then $\chi_V(T)$ is an idempotent operator that commutes with T; the restriction of T to the range of $\chi_V(T)$ is called an *Ω-summand* of T. Note that since the components of Ω form an open cover of $\sigma(T)$, only finitely many components of Ω can intersect $\sigma(T)$.

LEMMA 3. *Suppose X is a Banach space, $T \in B(X)$, Ω is an open set of complex numbers containing $\sigma(T)$, and let $\mathscr{A} = \{\psi(T) : \psi \in H(\Omega)\}$. Then*

(1) *$\mathscr{A}$ has a separating vector,*

(2) *if S is a linear transformation on X such that $ST = TS$ and $Sx \in \mathscr{A}x$ for every x in X, then $S \in \mathscr{A}$, and*

(3) *if no Ω-summand of T is algebraic, then $\mathscr{A}$ is algebraically reflexive.*

Proof. Let $V_1, V_2, \ldots, V_n$ be the components of Ω that intersect $\sigma(T)$, and let M_k be the range of $\chi_{V_k}(T)$ for $1 \leqslant k \leqslant n$. Then X is the algebraic direct sum of the M_k's, and if $T_k = T|M_k$ for each k, then the algebra $\mathscr{A}$ is the direct sum of the algebras $\mathscr{A}_k = \{\psi(T_k) : \psi \in H(V_k)\}$, $1 \leqslant k \leqslant n$. Moreover, if S is a linear transformation on X and, for all x, $Sx \in \mathscr{A}x$, then S leaves each M_k invariant; thus S is a direct sum of transformations S_k, $1 \leqslant k \leqslant n$. This reduces the problem to looking at each summand; thus we may assume that Ω is connected.

If T is algebraic, then (1) is obvious, and (2) follows from a result of L. Brickman and P. A. Fillmore [2]. Therefore, we can assume that T is not algebraic.

Suppose $\psi \in H(\Omega)$, $\psi \neq 0$, and $\psi(T) = 0$. Since $\psi(\sigma(T)) = \sigma(\psi(T))$, we conclude that $\sigma(T)$ is finite. Hence there is a polynomial p and a φ in $H(\Omega)$ such that $\psi = p\varphi$ and φ is nonzero in a neighborhood of $\sigma(T)$. Thus $\varphi(T)$ is invertible and $0 = \psi(T) = \varphi(T)p(T)$. Hence $p(T) = 0$, contradicting the fact that T is not algebraic. Hence the algebra $\mathscr{A}$ is isomorphic to $H(\Omega)$. Since Ω is connected, it follows that $\mathscr{A}$ satisfies (1) and (2) in Theorem 2. The proof will be complete once we show $\mathscr{A}$ has a separating vector.

Suppose $x \in X$, $\psi \in H(\Omega)$, $\psi \neq 0$, and $\psi(T)x=0$. Thus $\psi(T)\mathscr{A}x = \mathscr{A}\psi(T)x = \{0\}$. Let M be the closure of $\mathscr{A}x$. Then M is $\mathscr{A}$-invariant and $\psi(T)|M=0$. Since $(z-\lambda)^{-1} \in H(\Omega)$ for each $\lambda \notin \Omega$, it follows that $(T-\lambda)^{-1}$ leaves M invariant for each $\lambda \notin \Omega$. Hence $\sigma(T|M) \subset \Omega$, and $\psi(T|M) = \psi(T)|M = 0$. It follows from the argument above that there is a polynomial p such that $p(T|M)=0$ and $p \neq 0$. It follows that if $\mathscr{A}$ has no separating vector, then T must be locally algebraic, which, by a theorem of I. Kaplansky [7], implies that T is algebraic. Thus $\mathscr{A}$ has a separating vector. It now follows from Theorem 2 that $\mathscr{A}$ is algebraically reflexive. ▨

If Ω is a bounded open subset of the complex plane, let $A^2(\Omega)$ be the Bergman space of all holomorphic functions ψ on Ω that are square integrable with respect to planar Lebesgue measure ($dx\,dy$). It is well-known that $A^2(\Omega)$ is a closed subspace of $L^2(\Omega)$ and that the inclusion map from $A^2(\Omega)$ into $H(\Omega)$ is continuous (see, e.g., [3]).

We are now ready for the main reflexivity result. Throughout the remainder of the paper, X will be a Banach space. A linear submanifold M of X is a *Hilbert-range* (resp. *Fréchet-range*) if M is the range of a continuous linear transformation whose domain is a separable Hilbert space (resp. Fréchet space).

THEOREM 4. *Suppose $T \in B(X)$ and Ω is an open subset of the complex plane containing $\sigma(T)$ such that*

(1) *every component of Ω intersects $\sigma(T)$, and*

(2) *every algebraic Ω-summand of T is reflexive.*

Suppose $\mathscr{A}$ is a closed subalgebra of $H(\Omega)$ with $1 \in \mathscr{A}$, and let $\mathscr{A}(T) = \{\psi(T) : \psi \in \mathscr{A}\}$. If S is a (not necessarily continuous) linear transformation on X that leaves invariant every $\mathscr{A}(T)$-invariant Hilbert-range, then $S \in \mathscr{A}(T)$.

Proof. Choose a sequence $\{\Omega_n\}$ of bounded open subsets of Ω such that

(3) $\sigma(T) \subset \Omega_1 \subset \Omega_1^- \subset \Omega_2 \subset \Omega_2^- \subset \ldots$,

(4) the intersection of Ω_n with each component of Ω is connected, for $n = 1, 2, \ldots$,

(5) Ω is the union of the Ω_n's.

Since Ω has at most finitely many components (they form a minimal open cover of $\sigma(T)$), the Ω_n's can be chosen to be finite unions of open discs.

It follows from (4) above that the Ω_n-summands of T are equal to the Ω-summands of T for $n = 1, 2, \ldots$. It follows from (2) above and Lemma 3 that the algebra $\mathscr{A}_n = \{\psi(T) : \psi \in H(\Omega_n)\}$ is algebraically reflexive for $n = 1, 2, \ldots$, and has a separating vector x_n.

Fix a positive integer n. Since $\Omega_n^- \subset \Omega$, it follows that $\psi|\Omega_n$ is bounded for each ψ in $H(\Omega)$. Hence $\{\psi|\Omega_n : \psi \in H(\Omega)\} \subset A^2(\Omega_n)$. Let $\mathscr{S}_n$ be the closure of $\{\psi|\Omega_n : \psi \in H(\Omega)\}$ in $A^2(\Omega_n)$. Then $\mathscr{S}_n$ is a separable Hilbert space, and it follows that, for each x in $X, \mathscr{S}_n(T)x$ is a Hilbert-range, since the map $\psi \to \psi(T)x$ is continuous and linear. Furthermore, if $\psi \in \mathscr{A}$, it follows from the boundedness of $\psi|\Omega_n$ that $\psi\mathscr{S}_n \subset \mathscr{S}_n$; whence, $\psi(T)\mathscr{S}_n(T)x \subset \mathscr{S}_n(T)x$ for each x in X. Hence $\mathscr{S}_n(T)x$ is an $\mathscr{A}(T)$-invariant Hilbert-range for each x in X.

It follows that $Sx \in \mathscr{S}_n(T)x \subset \mathscr{A}_n x$ for each x in X. Since $\mathscr{A}_n$ is algebraically reflexive, we conclude that $S \in \mathscr{A}_n$. Since x_n is a separating vector for $\mathscr{A}_n$ and $Sx_n \in \mathscr{S}_n(T)x_n$, it follows that $S \in \mathscr{S}_n(T)$.

Hence, for each positive integer n, there is a ψ_n in $\mathscr{S}_n$ such that $S = \psi_n(T)$. Since $(\psi_n - \psi_{n+1}|\Omega_n)(T) = 0$ and no Ω_n-summand of T is algebraic, we conclude that $\psi_{n+1}|\Omega_n = \psi_n$ for $n = 1, 2, \ldots$. Thus there is a ψ in $H(\Omega)$ such that $\psi|\Omega_n = \psi_n$ for $n = 1, 2, \ldots$. Clearly, $S = \psi(T)$.

To show that $\psi \in \mathscr{A}$, first note that Ω_n^- is a compact subset of Ω_{n+1} for each n, and since $\psi|\Omega_{n+1}$ is in the $A^2(\Omega_{n+1})$-closure of $\{\varphi|\Omega_{n+1} : \varphi \in \mathscr{A}\}$, it follows that there is a sequence of functions in $\mathscr{A}$ that converge to ψ uniformly on Ω_n^-. Thus, for each n, there is a φ_n in $\mathscr{A}$ such that $|\varphi_n(z) - \psi(z)| < 1/n$ for every z in Ω_n. Clearly, the sequence $\{\varphi_k\}$ converges to ψ uniformly on each Ω_n. However, it follows from (3) above that each compact subset of Ω is contained in some Ω_n; whence, $\varphi_k \to \psi$ uniformly on compact subsets of Ω. Since $\mathscr{A}$ is closed in $H(\Omega)$, we conclude that $\psi \in \mathscr{A}$, and, therefore, $S \in \mathscr{A}(T)$. ▨

Note that in the following corollary the assumption in Theorem 4 that every component of Ω intersects $\sigma(T)$ is dropped.

COROLLARY 5. *Suppose $T \in B(X)$ and Ω is an open set in the plane containing $\sigma(T)$ such that no Ω-summand of T is algebraic. Let $\mathscr{R} = \{r(T) : r$ is a rational function with poles off $\Omega\}$. Then a linear transformation S leaves invariant every $\mathscr{R}$-invariant Hilbert-range if and only if $S = \psi(T)$ for some ψ in $H(\Omega)$.*

Proof. First suppose that S leaves invariant every $\mathscr{R}$-invariant Hilbert-range. If we let Ω_0 be the union of the components of Ω that intersect $\sigma(T)$, then $\{\psi(T) : \psi \in H(\Omega)\} = \{\psi(T) : \psi \in H(\Omega_0)\}$, and if we denote this set by $\mathscr{A}$, it follows from $\mathscr{R} \subset \mathscr{A}$ that S leaves invariant every $\mathscr{A}$-invariant Hilbert-range. Thus, by Theorem 4, $S \in \mathscr{A}$.

Conversely, suppose $S = \psi(T)$ for some ψ in $H(\Omega)$. Suppose M is an $\mathscr{R}$-invariant Hilbert-range. We use standard arguments due to Foiaş [5]. Then there is a Hilbert space H and an operator $A : H \to X$ whose range is M. Since $H/\ker A$ is a Hilbert space, we can assume that A is 1-1. If $W \in B(X)$ and $W(M) \subset M$, then $A^{-1}WA$ is a linear transformation on H, and it follows from the closed graph theorem that $A^{-1}WA \in B(H)$. If $\mathscr{S}$ is the algebra of all operators in $B(X)$ that leave M invariant, then the map $\pi : \mathscr{S} \to B(H)$ by $\pi(W) = A^{-1}WA$ is an algebra homomorphism, and $\pi(1) = 1$. Since T and $(T - \lambda)^{-1} \in \mathscr{S}$ for each λ not in Ω, it follows that $\sigma(\pi(T)) \subset \Omega$. Using Runge's theorem, we can choose a sequence

$\{r_n\}$ of rational functions with poles off Ω that converges uniformly on compact subsets of Ω to the function ψ. Thus, by taking limits of both sides of the equation $Ar_n(\pi(T)) = r_n(T)A$, we obtain $A\psi(\pi(T)) = \psi(T)A$; whence, $\psi(T) = S$ leaves $M = \operatorname{ran} A$ invariant. ▨

COROLLARY 6. *Suppose $T \in B(X)$ and T is not algebraic. A linear transformation S on X leaves invariant every T-invariant Hilbert-range if and only if $S = \psi(T)$ for some entire function ψ.*

COROLLARY 7. *Suppose $T \in B(X)$, T is not algebraic, T is invertible. Then a linear transformation S on X leaves invariant every $\{T, T^{-1}\}$-invariant Hilbert range if and only if $S = \psi(T)$ for some ψ holomorphic in the complement of $\{0\}$*, i.e., $\psi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, *where the series converges for $z \neq 0$.*

What happens in the preceding theorems if we replace Hilbert-ranges by Fréchet-ranges? Since every Hilbert-range is a Fréchet-range, it follows that Theorem 4 remains true (but less interesting) if we replace Hilbert-ranges by Fréchet-ranges. However, the second half of the proof of Corollary 5 breaks down because the Riesz functional calculus does not work for continuous linear transformations on a Fréchet space.

EXAMPLE. Let Y be the Fréchet space of all complex sequences with the topology of termwise convergence, and define $T: Y \to Y$ by $T(z_1, z_2, \ldots) = (z_2, z_3, \ldots)$. Then any continuous linear transformation on Y that commutes with T must be a polynomial in T (a standard computation). In addition, every complex number is a eigenvalue for T. Thus there is no reasonable way to define an expression like e^T.

The following theorem, which was motivated by the preceding example, shows that the real reason that the proof of Corollary 5 breaks down when Hilbert-ranges are replaced by Fréchet-ranges is that the result does not remain true. In fact, even Corollary 6 fails to remain true in this case.

THEOREM 8. *Suppose $T \in B(X)$ and T is not algebraic. If S is a linear transformation on X, then S leaves invariant every T-invariant Fréchet-range if and only if $S = p(T)$ for some polynomial p.*

Proof. The "if" part is obvious. Suppose S leaves invariant every T-invariant Fréchet-range. Let $\mathscr{P}$ be the vector space of all complex polynomials, and, for each positive integer k, we define a norm $\| \ \|_k$ on $\mathscr{P}$ by $\|p\|_k = \sup\{|p(z)| : |z| \leqslant k\}$.

Suppose $d = \{d(n)\}$ is a sequence of positive integers. We define norms $\| \ \|_{d,m}$ for each non-negative integer m, by

$$(1) \qquad \|p\|_{d,0} = \sum_n d(n)\,|p^{(n)}(0)|/n!$$

and

(2) $\|p\|_{d,m} = \|z^m p(z)\|_{d,0} = \sum_n d(n+m)\|p^{(n)}(0)\|/n!$ for $m = 1, 2, \ldots$.

Let Y_d be the completion of $\mathscr{P}$ with respect to the family $\{\|\ \|_k : k \geqslant 1\} \cup \cup \{\|\ \|_{d,m} : m \geqslant 0\}$ of seminorms. Since the completion of $\mathscr{P}$ with respect to the family $\{\|\ \|_k : k \geqslant 1\}$ is precisely the entire functions with the topology of uniform convergence on compact sets, standard arguments show that the space Y_d is the set of all entire functions f for which the formulas (1) and (2) are finite when p is replaced by f. Furthermore, convergence in Y_d implies uniform convergence on compact subsets. Let $\mathscr{S}_d = \{f(T) : f \in Y_d\}$. Then, for every x in X, $\mathscr{S}_d x$ is a Fréchet-range. Since $1 \in \mathscr{S}_d$, we know $x \in \mathscr{S}_d x$. Furthermore, it follows that if $f \in Y_d$, then $zf(z) \in Y_d$. Thus $\mathscr{S}_d x$ is a T-invariant Fréchet-range for every x in X. Let $\mathscr{A} = \{f(T) : f$ is entire$\}$. Since, by Lemma 3, $\mathscr{A}$ is algebraically reflexive, and $Sx \in \mathscr{S}_d x \subset \mathscr{A}x$ for every x in X, it follows that $S \in \mathscr{A}$. But $\mathscr{A}$ has a separating vector x, and $Sx \in \mathscr{S}_d x$; whence, $S \in \mathscr{S}_d$.

Since T is not algebraic, it follows (see the proof of Lemma 3) that if f and g are entire functions and $f(T) = g(T)$, then $f = g$. Since the sequence d was arbitrary, it follows that there is an entire function ψ such that $S = \psi(T)$ and ψ is in every Y_d. This clearly implies that ψ is a polynomial. ▨

The role played by Hilbert spaces in our results is not an essential one. Suppose Y is a Banach space. A linear submanifold of a Banach space X is a *Y-range* if it is the range of a continuous linear transformation from Y into X. Note that the conditions below are met by most of the classical Banach spaces.

THEOREM 9. *Suppose Y is a Banach space and U is the open unit disk in the plane. The following are equivalent:*

(1) *for every Banach space X, every e in X, and every non-algebraic operator T on X,*

(a) *there is a T-invariant Y-range containing e,*

and

(b) *the only linear transformations on X leaving invariant every T-invariant Y-range are entire functions in T.*

(2) *there is a continuous linear transformation $A : Y \to H(U)$ whose range contains 1 and is closed under multiplication by z.*

(3) *there is a biorthogonal system $\{(y_n, y_n^*)\}_{n=0}^{\infty}$ in $Y \times Y^*$ such that*

(a) $\limsup_n |y_n^*(y)|^{1/n} \leqslant 1$ *for every y in Y,*

and

(b) *for every y in Y there is a w in Y such that $y_{n+1}^*(w) = y_n^*(y)$ for $n = 0, 1, \ldots$.*

Proof. The proof of (2) ⇒ (1) is obtained from the proof of Theorem 4 by letting $\Omega_n = nU$, replacing $A^2(\Omega_n)$ with Y, and replacing the inclusion map from $A^2(\Omega_n)$ into $H(\Omega_n)$ by the map $y \to Ay(z/n)$. The proof of (1) $\Rightarrow$ (2) follows from letting $X = A^2(U)$, $e = 1$, and letting T be multiplication by z. The proof of (2) $\Leftrightarrow$ (3) is obtained by letting $A(y)$ be the power series $\sum_n y_n^*(y)z^n$. ▨

The techniques of this paper give a more direct proof of a theorem of E. Nordgren, M. Radjabalipour, H. Radjavi, and P. Rosenthal that states that every operator on an infinite-dimensional Hilbert space has a family $\{M_t : t \in [0,1]\}$ of invariant operator ranges such that $M_s \cap M_t = \{0\}$ whenever $s \neq t$.

PROPOSITION 10. *Each operator on an infinite-dimensional Banach space has a family* $\{M_t : t \in [1,2]\}$ *of linearly independent invariant Hilbert-ranges.*

Proof. The proof is easy when the operator is nilpotent; hence we can assume that the operator T is not algebraic and that $\|T\| < 1$. Let U be the open unit disk, and let V be the disk centered at 0 with radius 3. Choose f in $H(U)$ so that f is bounded and f cannot be meromorphically extended to a larger disk. For each t in $[1,2]$, let $f_t(z) = f(z/t)$. By Lemma 3, $H(U)(T)$ has a separating vector g. For each t in $[1,2]$, let $M_t = A^2(V)(T)f_t(T)g$. Clearly, M_t is a T-invariant Hilbert-range. Moreover if $t_1 < t_2 < \ldots < t_n$ and $h_1, h_2, \ldots, h_n \in A^2(V)$, $h_1 \neq 0$, and $h_1(T)f_{t_1}(T)g + \ldots + h_n(T)f_{t_n}(T)g = 0$, then $f_{t_1} = (-1/h_1)\sum_{i>1} h_i f_{t_i}$. This contradicts the fact that f_{t_1} cannot be meromorphically extended to a disk centered at 0 with radius larger than t_1. ▨

Our final theorem shows how much a non-algebraic operator is determined by its set of invariant Hilbert-ranges.

THEOREM 11. *Suppose that T is a non-algebraic operator on a Banach space X, and that S is a linear transformation on X. Then S and T leave invariant exactly the same set of Hilbert-ranges in X if and only if there are scalars a, b, with $b \neq 0$, such that $S = a + bT$.*

Proof. The "if" part is obvious. If S and T have the same set of invariant Hilbert-ranges, it follows that S is bounded and that $S = f(T)$ for some entire function f. It follows that S is not algebraic, and thus $T = g(S)$ for some entire function g. Since $g(f(T)) = T$ and no non-zero entire function annihilates T, we conclude that $g(f(z)) = z$ for every complex number z. Thus f is 1-1, which implies that $f(z) = a + bz$ for scalars a, b, with $b \neq 0$. ▨

QUESTIONS AND COMMENTS

1. Note that Theorem 8 is a sharp improvement over the theorem in [4] (see also [6]) that asserts that if T is not algebraic, then $\{p(T) : p \text{ is a polynomial}\}$ is algebraically reflexive. It follows from Souslin's theorem [1] that every Fréchet-

-range in a Banach space is a Borel set. Thus if X is separable and infinite-dimensional, then X has $2^{\aleph_0} = c$ Fréchet-ranges and 2^c linear submanifolds.

2. It follows from part (2) of Lemma 3 that if the assumption that no Ω-summand is algebraic (or that T is not algebraic) is replaced by the assumption that $ST = TS$, then Corollaries 5, 6, 7 and Theorem 8 remain valid.

3. There are many questions that arise from Corollary 5. For example, suppose $T \in B(X)$ and Λ is a subset of the complex plane that is disjoint from $\sigma(T)$. Let $R_\Lambda(T) = \{r(T) : r$ is a rational function with poles in $\Lambda\}$. What are the linear transformations on X that leave invariant every $R_\Lambda(T)$-invariant Hilbert-range? If Λ is a closed set, then the answer is given by Corollary 5. Things are much more complicated when the set is not closed; for example, if T is the unilateral shift operator on ℓ^2, and A is a diagonal operator with eigenvalues $1, 1/2, 1/2^2, \ldots$, then T leaves the range of A invariant, $A^{-1}TA = 2T$, and $(T - \lambda)^{-1}$ leaves the range of A invariant if and only if $|\lambda| > 2$. Hence, if $\Lambda = \{-1 + 3e^{i\theta} : 0 < \theta < 2\pi\}$, then the range of A is $R_\Lambda(T)$-invariant, but it is not invariant under $(T - 2)^{-1}$, even though 2 is in the closure of Λ. What happens if Λ is the complement of $\sigma(T)$? Is the answer in this case the set of functions of T that are holomorphic in a neighborhood of $\sigma(T)$? Of course we need some assumption akin to "no Ω-summand is algebraic"; perhaps the assumption that T have no eigenvalues would be good for a start.

4. What can be done with pairs of operators? Suppose $A, B \in B(X)$ and $AB = BA$. Is there some analogue of Theorem 4 for the pair A, B? Are there reasonable assumptions on A, B so that the linear transformations leaving invariant every $\{A, B\}$-invariant Hilbert-ranges are precisely the entire functions (of two complex variables) in A, B?

5. Which Banach spaces satisfy the conditions on Theorem 9? It seems conceivable that every infinite-dimensional Banach space does.

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