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TRANSLATION SEMIGROUPS ON REPRODUCING KERNEL HILBERT SPACES

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INTRODUCTION

In this paper we study the one-parameter semigroup $\{S_t\}_{t\geqslant 0}$ of translation operators defined by

$$(S_t f)(\zeta) = f(\zeta + t)$$

acting on Hilbert spaces $\mathscr{B}$ equipped with a reproducing kernel. Of course $\{S_t\}_{t\geqslant 0}$ acting on L^2 spaces is a familiar and useful object; the new element here is the richness of structure that flows out of the presence of a kernel function in $\mathscr{B}$. Our initial motivation was to understand a single fascinating example in a new way, namely the semigroup of composition operators associated by Deddens [8] with the discrete Cesàro operator C_0. From this we were led to study a general framework with the hope of providing a setting in which other examples with different characteristics would appear. The last half of the paper is therefore devoted to examples, and further examples will be considered in subsequent articles. However, it is clear that much more remains to be done in this regard. Throughout this paper the phenomenon that will interest us most is the potential of $\{S_t\}_{t\geqslant 0}$ for having subnormal adjoint.

Let us consider C_0 and Deddens' semigroup. Recall that C_0 is defined on sequences $\{a_n\}$ in ℓ^2 by $C_0\{a_n\} = \{b_n\}$, where $b_n = (n+1)^{-1}\sum_{k=0}^{n} a_k$. Hardy [16] showed that C_0 is a bounded operator on ℓ^2; Brown, Halmos, and Shields [4] investigated its spectral properties and showed that C_0 is hyponormal, while Shields and Wallen [29] studied the commutant of C_0. The first author and Trutt improved "hyponormal" to "subnormal" [23] and investigated the invariant subspace lattice [24]; the main idea of the present paper flows directly out of those two papers. Further refinements were introduced by Klopfenstein [21] while Deddens [8], identifying ℓ^2 with the Hardy space H^2, studied a semigroup of composition operators whose infinitesimal generator is a function of C_0. Cowen [7] has recently turned the tables, using the semigroup to study C_0 and giving a new proof of subnormality along the way.

It will be convenient for us to consider a representation for C_0 on yet a third Hilbert space (though merely a translate of H^2). Let Ω_0 be the open disk $\{z : |z - 1| < 1\}$ in the complex plane, and denote by $P^2(\partial\Omega_0, |dz|)$ the closure of the complex polynomials in $L^2(\partial\Omega_0, |dz|)$; here $|dz|$ is arclength measure on the circle $\partial\Omega_0$. Let $e_n(z) = \frac{1}{\sqrt{2\pi}}(1 - z)^n$, $n = 0, 1, 2, \ldots$, so that $\{e_n\}_{n=0}^{\infty}$ is an orthonormal basis for $P^2(\partial\Omega_0, |dz|)$. Let A_1 denote the formal operator

$$(A_1 f)(z) = \frac{1}{z}\int_0^z f(\zeta)\mathrm{d}\zeta$$

on $P^2(\partial\Omega_0, |dz|)$. One checks that the matrix of A_1 with respect to $\{e_n\}_{n=0}^{\infty}$ is

$$\begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 & \cdots\cdots \\ 0 & 1/2 & 1/3 & 1/4 & \cdots\cdots \\ 0 & 0 & 1/3 & 1/4 & \cdots\cdots \\ 0 & 0 & 0 & 1/4 & \cdots\cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

which is precisely the matrix of C_0^*. Hence A_1 is bounded and, in fact, unitarily equivalent to C_0^*. In this setting, the semigroup studied by Deddens takes the form

$$D_t : f(z) \to f(\mathrm{e}^{-t}z), \quad t \geqslant 0.$$

For our purposes it is enough to note that $\{\mathrm{e}^{-t/2}D_t\}_{t\geqslant 0}$ is a contractive and strongly continuous semigroup on $P^2(\partial\Omega_0, |dz|)$, related to A_1 by the formulas

$$A_1 = \int_0^\infty \mathrm{e}^{-t}D_t\mathrm{d}t \quad \text{(weak integral)}$$

and

$$E_{t/2}(I - A_1) = \mathrm{e}^{-t/2}D_t, \quad t \geqslant 0,$$

where E_s is the singular inner function

(1) $$E_s(z) = \mathrm{e}^{s\frac{z+1}{z-1}}, \quad s \geqslant 0,$$

and $E_{t/2}(I - A_1)$ is defined by the H^∞ functional calculus for contractions [31]. These formulas can be made plausible by checking them on the spanning set $\{z^u : \operatorname{Re} u > -1/2\}$. See § 4.1 for details.

It is tempting to replace $P^2(\partial\Omega_0, |dz|)$ by a more general space of functions on a domain more general than Ω_0, and we will see some examples of this in § 4.1. A more fruitful point of view emerges if we set $\Lambda_0 = -\log\Omega_0 = \{-\log z : z \in \Omega_0\}$ and define

$$\mathscr{B}_0 = \{e^{-\zeta/2}f(e^{-\zeta}) : f \in P^2(\partial\Omega_0, |dz|)\},$$

a space of functions on Λ_0. Let $W : P^2(\partial\Omega_0, |dz|) \to \mathscr{B}_0$ be the map $(Wf)(\zeta) = e^{-\zeta/2}f(e^{-\zeta})$; we can norm $\mathscr{B}_0$ so as to make W a unitary operator. Then we have $W(e^{-t/2}D_t)W^{-1} = S_t$, where $\{S_t\}_{t\geqslant 0}$ is our translation semigroup above, now acting contractively on $\mathscr{B}_0$. This suggests a generalization in which we forget about A_1 and $P^2(\partial\Omega_0, |dz|)$ and replace Λ_0 and $\mathscr{B}_0$ by, respectively, a more general domain set Λ and Hilbert space $\mathscr{B}$ of functions defined on Λ. In order for S_t to make sense on $\mathscr{B}$, Λ must be translation invariant. It will turn out to be convenient, however, to work mainly with the formal cogenerator L of $\{S_t\}_{t\geqslant 0}$, rather than with $\{S_t\}_{t\geqslant 0}$ itself; the single operator L is, in most circumstances, easier to handle. We recall that the cogenerator always exists when $\{S_t\}_{t\geqslant 0}$ is a contraction semigroup [31]; it can be characterized as the unique contraction L for which 1 is not an eigenvalue, such that $E_t(L) = S_t$, $t \geqslant 0$, where E_t is given by (1).

In our example above, $P^2(\partial\Omega_0, |dz|)$ is spanned by $\{z^u : \operatorname{Re} u > -1/2\}$, which is precisely the set of eigenvectors of A_1 and D_t. It is thus reasonable to demand that $\mathscr{B}$ be spanned by the exponentials $e^{u\zeta}$ which it contains; these are exactly the eigenvectors of L and S_t. Then a well-known construction of Halmos and Shields [15] allows us to represent S_t^* and L^* as multiplication operators on a second reproducing kernel Hilbert space $\mathscr{H}$. It will turn out that S_t^* and L^* are subnormal precisely when $\mathscr{H}$ can be identified with $P^2(\mu)$, the $L^2(\mu)$ closure of the polynomials, where μ is a finite positive Borel measure with compact support in the plane. The existence of μ, and its form when it does exist, are encoded in the kernel function $k(w, z)$ for $\mathscr{B}$ in a way which allows their extraction, in principle and sometimes in practice, by inverting a certain Fourier-Laplace transform. This is the central theme of the paper. The class of operators L and $\{S_t\}_{t\geqslant 0}$ is delineated by placing axioms on $\mathscr{B}$ and $\{S_t\}_{t\geqslant 0}$; we will see that it is quite large.

The plan of the paper is as follows. In §1 we put down our axioms and establish preliminary results. Sections 2 and 3 are devoted in the main to manifestations of subnormality, while § 4 presents examples. In particular, we will return in § 4 to some variants of the operator A_1.

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1. PRELIMINARIES

We will be concerned with complex separable Hilbert spaces and operators on them which are linear and bounded. The spectrum and point spectrum of an operator T are denoted by $\sigma(T)$ and $\sigma_p(T)$, respectively. We sometimes write $S \cong T$ to indicate that the operators S and T are unitarily equivalent. Recall that T is subnormal [6, 14] if it is the restriction to an invariant subspace of a normal operator, and cosubnormal if T^* is subnormal. Our semigroup $\{S_t^*\}_{t\geq 0}$ is subnormal if it is the restriction of a normal semigroup; according to a theorem of Ito [19], this is equivalent to saying that each S_t^* is individually subnormal. The complex numbers are denoted $\mathbf{C}$ and $\mathbf{D}$ is the open unit disk $\{z \in \mathbf{C} : |z| < 1\}$. We will consider subnormal operators of the form $M_\mu : f(z) \to zf(z)$, acting on $P^2(\mu)$, where μ is a finite (positive, Borel) measure with compact support in $\mathbf{C}$. M_μ has a cyclic vector (the function 1), and every cyclic subnormal is unitarily equivalent to some M_μ [2].

Let us turn to our axioms. We hypothesize a set Λ of complex numbers and a Hilbert space $\mathscr{B}$ of complex-valued functions defined on Λ and possessing a reproducing kernel $\{k_w : w \in \Lambda\}$. Thus for each w in Λ, $k_w \in \mathscr{B}$ and $f(w) = \langle f, k_w\rangle$ for every f in $\mathscr{B}$. We use ζ for the independent variable in Λ and sometimes write $k(w, \zeta)$ for $k_w(\zeta)$. We make the following assumptions about Λ and $\mathscr{B}$.

(i) The set Λ is invariant under right translation, contains 0, and has the property that any two points in Λ can be joined by a piecewise C^1 arc in Λ on which $k(\zeta, \zeta)$ is bounded.

(ii) If $f \in \mathscr{B}$ and $f \equiv 0$ on $[0, \infty)$, then $f = 0$.

(iii) Exponentials span: If $\Gamma = \{u : e^{u\zeta} \in \mathscr{B}\}$, then $\{e^{u\zeta} : u \in \Gamma\}$ spans $\mathscr{B}$.

(iv) Λ is maximal: If v in $\mathbf{C}$ and h in $\mathscr{B}$ are such that $\langle e^{u\zeta}, h\rangle = e^{uv}$ for all u in Γ, then $v \in \Lambda$.

(v) $\{S_t\}_{t\geq 0}$ is a strongly continuous semigroup on $\mathscr{B}$ satisfying

$$\|S_t\| = O(e^{\beta t}) \quad \text{as } t \to \infty$$

for each $\beta > 0$.

DEFINITION 1. We will indicate that axioms (i)—(v) hold by saying that the space $\mathscr{B}$ or the semigroup $\{S_t\}_{t\geqslant 0}$ is *admissible*. The set Λ is called the *domain set*.

It will be a standing assumption throughout §1, §2, and §3 that $\mathscr{B}$ and $\{S_t\}_{t\geqslant 0}$ are admissible.

REMARK 1. We see by applying S_t to exponentials that $\Gamma \subset \{u : \operatorname{Re} u \leqslant 0\}$.

Now let $\operatorname{Re} v > 0$ and consider the operator B_v defined by the integral

$$B_v = \int_0^\infty \mathrm{e}^{-vt} S_t \, \mathrm{d}t, \tag{2}$$

taken in the weak sense. Axiom (v) implies that B_v is a well-defined bounded operator on $\mathscr{B}$ and moreover, for every f in $\mathscr{B}$ and w in Λ,

$$(B_v f)(w) = \mathrm{e}^{vw} \int_{[w,\infty)} \mathrm{e}^{-vs} f(s) \mathrm{d}s; \tag{3}$$

the path of integration $[w, \infty]$ is the horizontal half-line with left-hand endpoint w. Further observe that

$$B_v \mathrm{e}^{u\zeta} = \frac{1}{v-u} \mathrm{e}^{u\zeta}$$

whenever $e^{u\zeta}$ lies in $\mathscr{B}$. The formal cogenerator of $\{S_t\}_{t\geqslant 0}$ is the operator $L \equiv I - 2B_1$; we note that L acts formally as a backward shift on the sequence $\{\Phi_n\}_{n=0}^{\infty}$, where $\Phi_n(\zeta) = \mathrm{e}^{-\zeta} L_n(2\zeta)$ and L_n is the n^{th} Laguerre polynomial.

We introduce the conformal mapping $H(z) = \dfrac{z+1}{z-1}$. We see that H maps $\mathbf{D}$ onto $\{u : \operatorname{Re} u < 0\}$ and $\mathbf{C} \setminus \bar{\mathbf{D}}$ onto $\{u : \operatorname{Re} u > 0\}$. Since H is its own inverse, it also maps $\{u : \operatorname{Re} u < 0\}$ onto $\mathbf{D}$ and $\{u : \operatorname{Re} u > 0\}$ onto $\mathbf{C} \setminus \bar{\mathbf{D}}$. For any w with $H(w) \in \Gamma$ one checks that

$$L \mathrm{e}^{H(w)\zeta} = w \mathrm{e}^{H(w)\zeta}.$$

It is easy to verify the formal identity

$$(L - z)^{-1} = \frac{1}{1-z} I + \frac{2}{(1-z)^2} B_{H(z)}, \quad |z| > 1, \tag{4}$$

for each side does the same thing to $\mathrm{e}^{u\zeta}$.

PROPOSITION 1. *The spectrum $\sigma(L)$ is contained in $\bar{\mathbf{D}}$. The operator L is a contraction if and only if $\{S_t\}_{t\geqslant 0}$ is a contraction semigroup; in this case L is the cogenerator of $\{S_t\}_{t\geqslant 0}$.*

Proof. Since $\operatorname{Re} H(z) > 0$ whenever $|z| > 1$, we see from (4) that $\sigma(L) \subset \mathbf{D}$. It is easy to see from (3) that 1 is not an eigenvalue of L. Thus if $\|L\| \leqslant 1$, $\{E_t(L)\}_{t\geqslant 0}$ is a strongly continuous contraction semigroup on $\mathscr{B}$ [31] which agrees with $\{S_t\}_{t\geqslant 0}$ on exponentials. Finally, if $\{S_t\}_{t\geqslant 0}$ is contractive, it has a cogenerator which must agree with L on exponentials. Thus $\|L\| \leqslant 1$ and $S_t = E_t(L)$ for $t \geqslant 0$. ▨

The proof of the following Corollary is left for the reader.

COROLLARY 1. *The semigroup $\{S_t^*\}_{t\geqslant 0}$ is subnormal if and only if L^* is subnormal; in this case both L and $\{S_t\}_{t\geqslant 0}$ are contractive.*

The elements of $\mathscr{B}$ inherit some properties of analytic functions via axioms (ii) and (iii). The remaining results in this section present several consequences of this; the first asserts that the integral (3) expressing $B_v f$ is in a sense independent of path.

PROPOSITION 2. *Suppose that $w \in A$ and that $C(w)$ is a piecewise C^1 path from w to ∞, contained in both A and a half-strip of the form*

$$\{x + iy : x > a \quad \text{and} \quad \operatorname{Im} w_1 \leqslant y \leqslant \operatorname{Im} w_2\}$$

where a is real and w_1, w_2 in A have distinct imaginary parts. Further suppose that $k(\zeta, \zeta)$ is bounded on each compact subset of $C(w)$ and that

$$\int_{C(w)} e^{-b \operatorname{Re} z} |dz| < \infty$$

for every $b > 0$. Then

$$(B_v f)(w) = e^{vw} \int_{C(w)} e^{-vz} f(z) dz$$

whenever $\operatorname{Re} v > 0$ and $f \in \mathscr{B}$.

Proof. The conclusion of the theorem is trivially true if $f(\zeta) = e^{u\zeta}$. For an arbitrary f in $\mathscr{B}$ we are thus led to select a sequence f_n tending to f in norm, where each f_n is a finite linear combination of $\{e^{u\zeta} : u \in \Gamma\}$. The points w_1 and w_2 can be connected by an arc A as in axiom (i). Let N be a bound for $k(\zeta, \zeta)$ on A. Let us write $C(w) = C_1 \cup C_2$ where C_1 is the part of $C(w)$ from w to some intermediate point w_0 and C_2 is the remaining subarc of $C(w)$ from w_0 to ∞; we may choose w_0 so that $\operatorname{Re} s < \operatorname{Re} u$ whenever $s \in A$ and $u \in C_2$. Note that if $z \in C_1$, we have $|f_n(z) - f(z)| \leqslant \|f_n - f\| k(z, z)^{1/2}$ and so $f_n \to f$ uniformly on C_1. On C_2 something analogous happens. We choose γ with $0 < \gamma < \operatorname{Re} v$. If $z \in C_2$, then $z = s + t$ for some $s \in A$ and $t > 0$. For any g in $\mathscr{B}$ we have

$$|g(z)| = |(S_t g)(s)| \leqslant \|S_t\| \, \|g\| \, k(s, s)^{1/2} \leqslant MN^{1/2} e^{\gamma t} \|g\|,$$

where M is a bound for $e^{-\gamma t}\|S_t\|$. In particular, on taking $g = f - f_n$ we see that for an appropriate $B > 0$,

$$|e^{-vz}| \, |f(z) - f_n(z)| \leqslant B \, e^{\gamma \operatorname{Re} z - \operatorname{Re}(vz)} \|f - f_n\|$$

for all z in C_2. Since the exponential factor on the right is $|dz|$-integrable over C_2, we may conclude that

$$\int_{C(w)} e^{-vz} f(z) \, dz = \lim_{n \to \infty} \int_{C(w)} e^{-vz} f_n(z) \, dz,$$

and the conclusion follows. ▨

The above proof shows that if C is a subset of A on which $k(\zeta, \zeta)$ is bounded, then every function in $\mathscr{B}$ is continuous on C.

PROPOSITION 3. *The point spectrum $\sigma_p(L)$ is exactly $\{w : H(w) \in \Gamma\}$. Any eigenvector for L with eigenvalue w is a multiple of* $e^{H(w)\zeta}$.

Proof. We already know that $e^{H(w)\zeta}$ is an eigenvector for L with eigenvalue w, hence $\sigma_p(L) \supset \{w : H(w) \in \Gamma\}$. Suppose, on the other hand, that $Lf = wf$ for some $f \neq 0$ and complex number w. Since $L = I - 2B_1$, we may use (3) with $v = 1$ to conclude (via differentiation) that

$$\frac{\partial}{\partial x} f(x + iy) = H(w) f(x + iy)$$

for every point $x + iy$ in A. Therefore, f has the form $f(x + iy) = A(y) e^{H(w)(x+iy)}$, where $A(y)$ depends only on y. To complete the proof, it is enough to show that $A(y) = A(0)$ whenever $x + iy \in A$. Let us fix $z = x + iy$ in A, and let C_1 be a piecewise C^1 path from z to 0 as provided by axiom (i). Then the path $C(z) = C_1 \cup [0, \infty)$ satisfies the hypotheses of Proposition 2. We reinterpret the equation $Lf = wf$ using Proposition 2 (with $v = 1$) instead of (3). If we let $\xi(t)$, $0 \leqslant t \leqslant 1$ parametrize C_1 running backwards from 0 to z, we see that

$$\frac{d}{dt} f(\xi(t)) = H(w) \xi'(t) f(\xi(t))$$

wherever $\xi'(t)$ exists. It follows that $f(\xi(t)) = a\,e^{H(w)\xi(t)}$ for some constant a and $0 \leqslant t \leqslant 1$. Therefore $A(y) = a = A(0)$. ▨

Our last result of this section is a version of L. Shulman's Theorem [30] about the Cesàro operator: The eigenvectors $\{(1, w, w^2, \ldots) : |w| < 1\}$ of the backward shift in ℓ^2 are all cyclic vectors for C_0. The corresponding vectors in $\mathscr{B}$ are the kernel functions $\{k_w : w \in \Lambda\}$. We will need only the case $w = 0$.

THEOREM 1. (Shulman's Theorem). *The kernel function k_0 is a cyclic vector for L^*.*

Proof. Let $f \in \mathscr{B}$, $f \perp \{(L^*)^n k_0\}_{n=0}^{\infty}$. If $|w| > \|L\|$,

$$\langle (L-w)^{-1} f, k_0\rangle = -\sum_{n=0}^{\infty} \langle f, (L^*)^n k_0\rangle \frac{1}{w^{n+1}} = 0.$$

Since the inner product on the left is an analytic function on $|w| > 1$, it must vanish there. Now $f \perp k_0$, so $f(0) = 0$. If we invoke the formula (4), we see that the above equation reduces to $(B_{H(w)} f)(0) = 0$, $|w| > 1$. In particular, for $t > 0$

$$0 = (B_t f)(0) = \int_0^{\infty} e^{-ts} f(s)\, ds,$$

and so $f \equiv 0$ on $[0, \infty)$, which implies $f = 0$. ▨

If Λ is an open set and $\mathscr{B}$ consists of functions analytic on Λ, a trivial modification of the above argument will show that each kernel function k_w, $w \in \Lambda$, is cyclic for L^*. Details are left for the reader.

REMARK 2. Suppose now that $\{S_t^*\}_{t \geqslant 0}$ is subnormal. Proposition 1 and Corollary 1 tell us that L^* is subnormal and contractive and that $S_t^* = E_t(L^*)$. From Theorem 1 and Bram's Theorem [2], we know that there exists a finite positive Borel measure μ on $\overline{\mathbf{D}}$ and a unitary operator $W : \mathscr{B} \to P^2(\mu)$ with $W k_0 = 1$ and $W L^* = M_\mu W$. Clearly $W S_t^* = V_t^\mu W$ where $\{V_t^\mu\}_{t \geqslant 0}$ acts on $P^2(\mu)$ by $V_t^\mu : f \to E_t f$. The fact that 1 is not an eigenvalue of L implies that $\mu(\{1\}) = 0$, so the discontinuity of E_t at 1 causes no problem. The measure μ will play a vital role in what follows.

REMARK 3. One can remove the growth condition in the semigroup axiom (v) by the following trick. If $\{S_t\}_{t \geqslant 0}$ is merely strongly continuous on $\mathscr{B}$, semigroup theory [18] tells us that $\|S_t\| = O(e^{\beta t})$ for some real number β. If we put

$$\gamma = \inf\{\beta \text{ real} : \|S_t\| = O(e^{\beta t}) \text{ as } t \to \infty\},$$

then $-\infty < \gamma < \infty$. We can define a new space $\tilde{\mathscr{B}}$ of functions on Λ by $\tilde{\mathscr{B}} = \{e^{-\gamma\zeta}f(\zeta) : f \in \mathscr{B}\}$ and norm $\tilde{\mathscr{B}}$ so that the operator $V : f(\zeta) \to e^{-\gamma\zeta}f(\zeta)$ from $\mathscr{B}$ to $\tilde{\mathscr{B}}$ is unitary. If $\mathscr{B}$ satisfies (i) — (iv), so does $\tilde{\mathscr{B}}$. Moreover, $VS_t = e^{\gamma t}\tilde{S}_t V$, where $\{\tilde{S}_t\}_{t\geqslant 0}$ is our translation semigroup acting on $\tilde{\mathscr{B}}$. The definition of γ tells us that $\{\tilde{S}_t\}_{t\geqslant 0}$ satisfies (v) as well.

2. THE SPACE $\mathscr{H}$ AND SUBNORMALITY

In this section we consider the question: when are L^* and $\{S_t^*\}_{t\geqslant 0}$ subnormal? The first step is to realize L^* as a multiplication operator on a reproducing kernel Hilbert space, using a now-standard technique due to Halmos and Shields [15]. We put $\Delta = \{\bar{z} : H(z) \in \Gamma\}$, so that $\Delta \subset \bar{\mathbf{D}}\setminus\{1\}$. For each f in $\mathscr{B}$, we define a function Uf on Δ by the formula

$$(Uf)(z) = \langle f, e^{H(\bar{z})\zeta}\rangle_{\mathscr{B}}, \quad z \in \Delta.$$

Note that Uf is not identically zero if $f \neq 0$, by axiom (iii), and of course the map $f \to Uf$ is linear. We define $\mathscr{H} = \{Uf : f \in \mathscr{B}\}$ and norm $\mathscr{H}$ so as to make $U : \mathscr{B} \to \mathscr{H}$ a unitary operator. Let us calculate the operator $S \equiv UL^*U^{-1}$. Suppose $g = Uf \in \mathscr{H}$; then

$$(Sg)(z) = (UL^*f)(z) = \langle L^*f, e^{H(\bar{z})\zeta}\rangle_{\mathscr{B}} =$$

$$= \langle f, Le^{H(\bar{z})\zeta}\rangle_{\mathscr{B}} = z\langle f, e^{H(\bar{z})\zeta}\rangle_{\mathscr{B}} = zg(z).$$

Further define, for each w in Δ, $K_w \equiv Ue^{H(\bar{w})\zeta}$. Then if $g = Uf$ is in $\mathscr{H}$,

$$\langle g, K_w\rangle_{\mathscr{H}} = \langle f, e^{H(\bar{w})\zeta}\rangle = g(w),$$

so that $\{K_w : w \in \Delta\}$ is a reproducing kernel for $\mathscr{H}$.

The preceding is the general construction of Halmos and Shields applied to our setting. Let us add a new ingredient, the action of U on the kernels $\{k_a : a \in \Lambda\}$ for $\mathscr{B}$. First consider k_0. We have

$$(Uk_0)(z) = \langle k_0, e^{H(\bar{z})\zeta}\rangle_{\mathscr{B}} = 1.$$

Now k_0 is cyclic for L^* by Theorem 1. It follows that $1 = Uk_0$ is cyclic for S in $\mathscr{H}$ and therefore *the polynomials are a dense subset of* $\mathscr{H}$. More generally, if $a \in \Lambda$,

$$(Uk_a)(z) = \langle k_a, e^{H(\bar{z})\zeta}\rangle_{\mathscr{B}} = e^{\bar{a}H(z)},$$

that is, $Uk_a = e^{\bar{a}H}$. Thus, for each a and b in Λ, the unitarity of U yields the fundamental equation

$$k(a, b) = \langle e^{\bar{a}H}, e^{\bar{b}H} \rangle_{\mathscr{H}}, \tag{5}$$

which is the basis for the rest of this paper.

As a first application, consider the question of circular symmetry of L and certain functions of L. D. Trutt and the first author showed that $I - C_0$ is not similar to any weighted shift [24]. If we represent C_0 as A_1^* acting on $P^2(\partial\Omega_0, |dz|)$ as in the introduction, then the unitary map $W: P^2(\partial\Omega_0, |dz|) \to \mathscr{B}_0$ described there carries $I - A_1$ onto $(3L + I)(3 + L)^{-1}$, see § 4.1. One can show that L itself, acting on $\mathscr{B}_0$, also lacks circular symmetry in exactly the same way. On the other hand we will see that any contractive subnormal weighted shift can be represented as L^* acting on some admissible space $\mathscr{B}$. There seem to be three obstructions to L (or $g(L)$ where g is a Möbius transformation of $\mathbf{D}$) having the kind of circular symmetry exhibited by weighted shifts: the presence of non-real points in Λ, circular contact of Γ with points on the imaginary axis and a growth rate on $\|e^{u\zeta}\|_{\mathscr{B}}$ for u near those points of contact. For examples, see Remarks 4, 6, 8, 11 and Corollaries 2 and 6.

THEOREM 2. *Suppose that Γ is an open connected set and that $\|e^{u\zeta}\|_{\mathscr{B}}$ is bounded on each compact subset of Γ. Let g be a Möbius transformation of $\mathbf{D}$ onto $\mathbf{D}$ and let $\lambda \in \partial\mathbf{D}$, $\lambda \neq 1$. Suppose that Γ contains an open disk G whose boundary is tangent to the imaginary axis at the point $iy_0 \equiv H(g^{-1}(\lambda g(1)))$, such that for some positive constants c and d with $d < 1$,*

$$\|e^{u\zeta}\|_{\mathscr{B}} \leqslant c \exp\left[-\frac{1}{|\mathrm{Re}\, u|^d}\right], \quad u \in G. \tag{6}$$

Then there is no nonzero bounded operator X on $\mathscr{B}$ satisfying $Xg(L)^* = \lambda g(L)^* X$. In particular, $g(L)^*$ cannot be quasisimilar to a weighted shift.

Proof. By our hypotheses, Δ is a connected open subset of $\mathbf{D}$ and $\|K_w\|$ is bounded on each compact subset of Δ. Therefore any function in $\mathscr{H}$ is a limit of polynomials not only in norm, but uniformly on compacta in Δ, and is consequently analytic on Δ.

Since $L^* \cong S$, let us postulate a nonzero operator Y on $\mathscr{H}$ such that $Yh(S) = \lambda h(S)Y$, where $h(z) = \overline{g(\bar{z})}$. We wish to show that this is incompatible with our hypotheses. Note that the sequence $\{Q_n\}_{n=0}^{\infty}$ of partial sums of the Taylor series for h^{-1} converges uniformly in some disk $\{z: |z| < 1 + \varepsilon\}$. Since $YQ_n(h(S)) = Q_n(\lambda h(S))Y$ we have $YS = h^{-1}(\lambda h(S))Y$, and therefore $Yp(S) = p(h^{-1}(\lambda h(S)))Y$

for any polynomial p. Thus

$$(Yp)(z) = p(h^{-1}(\lambda h(z)))(Y1)(z), \quad z \in \Delta.$$

We claim that $h^{-1}(\lambda h(\Delta)) \subset \Delta$. First note that $Y1$ is not identically zero; if it were, Y would vanish on the dense set of polynomials, implying $Y = 0$. If $w \in \Delta$ and $(Y1)(w) \neq 0$, we see that

$$|p(h^{-1}(\lambda h(w)))| = \frac{|(Yp)(w)|}{|(Y1)(w)|} \leqslant \frac{\|Y\| \, \|K_w\| \, \|p\|}{|(Y1)(w)|}$$

and so the evaluation functional $p \to p(h^{-1}(\lambda h(w)))$ is bounded on polynomials p in the $\mathscr{H}$-norm. Even if $(Y1)(w) = 0$, this functional is still bounded by an application of the maximum modulus principle on a small disk about w. It follows from a standard argument [6, p. 169] that $\overline{h^{-1}(\lambda h(w))} \in \sigma_p(S^*)$, and so by Proposition 3, $h^{-1}(\lambda h(w)) \in \Delta$, as desired.

Now we can use the density of the polynomials in $\mathscr{H}$ and the claim of the preceding paragraph to assert that our formula for $(Yp)(z)$ is valid not only for polynomials, but for every function in $\mathscr{H}$. A particular function we have in mind is defined as follows. Let $a = v + is$ and $b = v + it$ be distinct points in Λ with common real part v. Consider the action of Y on

$$f = e^{\bar{b}H} - e^{\bar{a}H} = 2ie^{vH} e^{-i\left(\frac{s+t}{2}\right)H} \sin\left(\frac{s-t}{2} H\right),$$

an element of $\mathscr{H}$. Note that f is analytic on $\mathbf{D}$ and vanishes on a non-Blaschke sequence $0 < r_1 < r_2 < \ldots < 1$. Now $H^{-1}(G)$ is an open disk in $\mathbf{D}$ tangent to $\partial\mathbf{D}$ at $g^{-1}(\lambda g(1))$. The disk $G_1 \equiv \{z : \bar{z} \in H^{-1}(G)\}$ is a disk in Δ tangent to $\partial\mathbf{D}$ at $\overline{g^{-1}(\lambda g(1))} = h^{-1}(\bar{\lambda} h(1))$. It thus makes sense to state

$$(Yf)(z) = f(h^{-1}(\lambda h(z)))(Y1)(z), \quad z \in G_1. \tag{7}$$

It follows that Yf vanishes on the sequence $z_n = h^{-1}(\bar{\lambda} h(r_n))$, which is a non-Blaschke sequence in the disk G_1 (after excluding the finite number of z_n which may lie outside G_1) tending nontangentially to $h^{-1}(\bar{\lambda} h(1))$.

Our hypotheses tell us that there exist positive constants c_1 and d_1 with $d_1 < 1$ such that

$$|(Yf)(z)| \leqslant \|Y\| \, \|f\| \, \|K_z\| = \|Y\| \, \|f\| \, \|e^{H(\bar{z})\zeta}\|_{\mathscr{B}} \leqslant c_1 e^{\left(\frac{1}{1-|z|}\right)^{d_1}},$$

for z in G_1. We may now invoke a theorem of Shapiro-Shields [28] and Hayman-Korenblum [17] to conclude that $Yf \equiv 0$ on G_1. It follows from (7) that $Y1 \equiv 0$ on Δ, and we have reached a contradiction.

Now suppose that $h(S) \cong g(L)^*$ is quasisimilar to a weighted shift Q. There exist one-to-one operators Z and W with dense range such that $Zh(S) = QZ$ and $h(S)W = WQ$. Given our unimodular constant λ, we can find a unitary V with $VQ = \lambda QV$. Then $Y = WVZ$ is nonzero and $Yh(S) = \lambda h(S)Y$, again a contradiction. ∎

REMARK 4. The conclusion of Theorem 2 can fail if the hypotheses are weakened in any of three ways:

(a) If the disk G is replaced by an open triangle in $\Gamma \cap \{u : \operatorname{Re} u < 0\}$ with one vertex at iy_0, then L^* and λL^* can be unitarily equivalent (see Corollary 6 in § 4.2);

(b) if the hypotheses hold as is, except allowing $d > 1$, L^* can be a weighted shift (see Remark 6 in § 3);

(c) if Λ contains only real numbers, then L^* can be a weighted shift (see Remark 6 in § 3 and § 4.3).

We have an immediate improvement of our knowledge of the Cesàro operator:

COROLLARY 2. *$I - C_0$ is not quasisimilar to any weighted shift.*

Proof. Recall that $I - C_0 \cong g(L)^*$ where L acts on the space $\mathscr{B}_0$ in the introduction and $g(z) = (3z + 1)(3 + z)^{-1}$. The axioms for $\mathscr{B}_0$ are easy to check; the reader might consult § 4.1.1 for particulars. We have

$$\Gamma = \{u : e^{u\zeta} \in \mathscr{B}_0\} =$$

$$= \{u : z^{-\left(u + \frac{1}{2}\right)} \in P^2(\partial\Omega_0, |dz|)\} = \{u : \operatorname{Re} u < 0\}.$$

Given any open disk $G \subset \Gamma$ tangent to the imaginary axis at a point iy_0, we see that there exists $c > 0$ so that

$$\|e^{u\zeta}\|_{\mathscr{B}}^2 = \int_{\partial\Omega_0} \left|z^{-\left(u + \frac{1}{2}\right)}\right|^2 |dz| \leqslant \frac{c}{|\operatorname{Re} u|}, \quad u \in G.$$

Theorem 2 now applies. ∎

We turn to the problem of subnormality for L^*. A central role will be played by the functions

$$E_w(z) = e^{\bar{w}H(z)}, \quad z \in \mathbf{C} \setminus \{1\},$$

where w is any complex number. We have seen that $E_w \in \mathscr{H}$ whenever $w \in \Lambda$. Note that if $w \geqslant 0$, E_w is an "atomic" inner function.

REMARK 5. If μ is a measure on $\bar{\mathbf{D}}$ with $\mu(\{1\}) = 0$, then $P^2(\mu)$ contains E_w whenever $w \geqslant 0$. Indeed, if we let p_n denote the n^{th} Fejér polynomial of E_w, then $p_n \to E_w$ boundedly and pointwise on $\bar{\mathbf{D}} \setminus \{1\}$ as $n \to \infty$, hence $p_n \to E_w$ in $L^2(\mu)$.

For a finite measure μ with compact support in $\mathbf{C}$, let us write $\Delta(\mu)$ for the set of complex numbers λ for which the map $p \to p(\lambda)$ is a bounded linear functional on polynomials with respect to the $P^2(\mu)$ norm.

THEOREM 3. *The semigroup* $\{S_t^*\}_{t \geqslant 0}$ *is subnormal if and only if there is a measure* μ *on* $\bar{\mathbf{D}}$ *with* $\mu(\{1\}) = 0$ *such that*

$$k(s, t) = \int E_s \bar{E}_t \, \mathrm{d}\mu \tag{8}$$

for all $s, t \geqslant 0$. *In this case the measure* μ *is unique; moreover* $L^* \cong M_\mu$ *and* $\{S_t^*\}_{t \geqslant 0} \cong \cong \{V_t^\mu\}_{t \geqslant 0}$.

Proof. Suppose that L^*, hence S, is subnormal. Since 1 is cyclic for S, Bram's Theorem [2] tells us that there exist a measure μ on $\bar{\mathbf{D}}$ and a unitary operator $Z: P^2(\mu) \to \mathscr{H}$ with $Z1 = 1$ and $ZM_\mu = SZ$. If $\mu(\{1\}) > 0$, then S, hence L^*, would have 1 as a *normal* eigenvalue, hence $1 \in \sigma_{\mathrm{p}}(L)$, in contradiction to Proposition 3. Therefore $\mu(\{1\}) = 0$.

We may invoke [6, p. 169] to conclude that

$$\Delta(\mu) = \{w : \bar{w} \in \sigma_{\mathrm{p}}(M_\mu^*)\} = \{w : \bar{w} \in \sigma_{\mathrm{p}}(L)\} = \Delta.$$

Further, $Zp = p$ for every polynomial p, and so approximation by polynomials yields $(Zf)(w) = f(w)$, $w \in \Delta$, whenever f is in $P^2(\mu)$. We know that $E_t \in P^2(\mu)$ whenever $t \geqslant 0$, and so $(ZE_t)(w) = E_t(w)$, $w \in \Delta$. The unitarity of Z and equation (5) imply

$$\int E_s \bar{E}_t \, \mathrm{d}\mu = \langle E_s, E_t \rangle_{\mathscr{H}} = k(s, t), \quad s, t \geqslant 0,$$

as desired.

For the converse we need a lemma.

LEMMA 1. [23, p. 220]. $\{E_t : t \geqslant 0\}$ *spans* $P^2(\mu)$.

We assume that (8) holds and apply this as follows. We define an operator Z on finite linear combinations of $\{E_t : t \geqslant 0\}$ in $P^2(\mu)$ to $\mathscr{H}$ by $ZE_t = E_t$; this Z will turn out to be the same Z as above. We see from (5) and (8) that

$$\langle ZE_s, ZE_t\rangle_{\mathscr{H}} = k(s, t) = \int E_s \bar{E}_t \mathrm{d}\mu,$$

hence Z extends to an isometry of $P^2(\mu)$ into $\mathscr{H}$. Moreover $Uk_t = E_t$ for $t \geqslant 0$, $\{k_t : t \geqslant 0\}$ spans $\mathscr{B}$ by axiom (ii), and U is unitary, so $\{E_t : t \geqslant 0\}$ spans $\mathscr{H}$ and Z is unitary. One sees as in [23, p. 222] that $Zp = p$ for every polynomial p with $p(1) = 0$; since $Z1 = 1$ (take $t = 0$ in $ZE_t = E_t$), we see that $Zp = p$ for every polynomial p. It follows that $ZM_\mu = SZ$ and S is subnormal.

Now consider uniqueness. Let μ' be another measure satisfying (8). As above, we produce a unitary $Z' : P^2(\mu') \to \mathscr{H}$ with $Z'p = p$ for every polynomial p. Then $Z^{-1}Z' : P^2(\mu') \to P^2(\mu)$ is unitary and $Z^{-1}Z'p = p$ for every polynomial p. It follows that $\int z^n \bar{z}^m \, \mathrm{d}\mu' = \int z^n \bar{z}^m \, \mathrm{d}\mu$ for $n, m \geqslant 0$, and so the Stone-Weierstrass Theorem implies that $\mu' = \mu$. The statement about V_t^μ follows from the relation $V_t^\mu = E_t(M_\mu)$. ▨

DEFINITION 2. We will say that μ is the measure *associated* with $\mathscr{B}$, $\{S_t\}_{t \geqslant 0}$ or L.

We will shortly give a positive-definiteness criterion for the existence of μ, but often the easiest way to check subnormality of L^* is to try to find μ directly. This becomes easier if we change variables. The map $-H$ carries $\bar{\mathbf{D}} \setminus \{1\}$ onto the half-plane $\{u : \operatorname{Re} u \geqslant 0\}$, and so we can replace the integral over $\bar{\mathbf{D}}$ in (8) by an integral over this half-plane with respect to $\gamma = \mu \circ (-H)^{-1}$. Thus, L^* is subnormal if and only if there exists a measure γ on $\{u : \operatorname{Re} u \geqslant 0\}$ such that

$$k(s, t) = \int_{\{\operatorname{Re} w \geqslant 0\}} \mathrm{e}^{-sw} \mathrm{e}^{-t\bar{w}} \mathrm{d}\gamma(w), \quad s, t \geqslant 0. \tag{9}$$

Let us disintegrate the measure γ with respect to the projection $x + \mathrm{i}y \to x$ [26], and write $\mathrm{d}\gamma(x + \mathrm{i}y) = \mathrm{d}\gamma_x(y)\,\mathrm{d}m(x)$ where $\mathrm{d}\gamma_x$ is a probability measure in $\mathbf{R}^1$ and $m(E) = \gamma(\{w : \operatorname{Re} w \in E\})$. On writing $u = s + t$ and $v = s - t$ we have

$$k\left(\frac{u + v}{2}, \frac{u - v}{2}\right) = \int_0^\infty \mathrm{e}^{-ux} \hat{\gamma}_x(v) \, \mathrm{d}m(x), \tag{10}$$

a formula valid for $u > 0$ and $-u \leqslant v \leqslant u$, where $\hat{\gamma}_x$ is the Fourier transform of γ_x:

$$\hat{\gamma}_x(v) = \int_{-\infty}^{\infty} e^{-ivy} d\gamma_x(y).$$

Thus, to find γ one simply inverts the Laplace transform (with v fixed) to capture the measure $\hat{\gamma}_x(v)\,dm(x)$. The value $v = 0$ gives dm, whereupon we have the Fourier transform $\hat{\gamma}_x$, inversion of which yields γ_x and thus γ! This iffy-sounding procedure can work nicely, and is illustrated in § 4.1.1 (where m is discrete) and § 4.2 (where m is continuous).

THEOREM 4. *The semigroup* $\{S_t^*\}_{t \geqslant 0}$ *is subnormal if and only if* $k(s, t)$, *considered as being defined for* $s, t \geqslant 0$, *has a continuous extension* $\tilde{k}$ *to* $\{(s, t) : s, t \in \mathbf{R}^1$ *and* $s + t \geqslant 0\}$ *satisfying*

$$\sum_{j,k} c_j \bar{c}_k \tilde{k}(u_j + v_k, u_k + v_j) \geqslant 0$$

for all finite sets of complex numbers c_j *and real numbers* u_j *and* v_j *with* $u_j + v_j \geqslant 0$.

Proof. If L^* is subnormal, equation (9) holds and the right side is defined for any real s and t with $s + t \geqslant 0$; we can use the right side then to define the extension $\tilde{k}(s, t)$. The positive definiteness condition is immediate. The other direction of the proof follows the argument of Theorem 3 in [23], using Devinatz's Theorem [9] and axiom (v). ▨

The alert reader will have noticed a resemblance to the Bram-Halmos criterion for subnormality [2]. Indeed, we have $S_t^* k_w = k_{w+t}$ and therefore the condition of the Theorem becomes

$$\sum_{j,k} c_j \bar{c}_k \langle S_{v_k}^* k_{u_j}, S_{v_j}^* k_{u_k} \rangle \geqslant 0$$

provided $u_j, v_j \geqslant 0$. However, a proof of Theorem 4 using this has eluded us.

3. SUBNORMALITY AND THE SHAPE OF Λ

In this section we will be concerned with how subnormality of $\{S_t^*\}_{t \geqslant 0}$ is reflected in the geometry of Λ, and with the related question of which cyclic subnormals can be represented as L^*. Our main results require the associated measure μ to belong to a class we call the *special* measures. We do not know whether this class includes all possible associated measures.

LEMMA 2. *Let μ be a measure on $\bar{\mathbf{D}}$ with $\mu(\{1\}) = 0$. Let w_1 and w_2 be complex numbers with distinct imaginary parts and let V be an open trapezoid bounded on the left by the segment from w_1 to w_2, and on the top and bottom by horizontal segments. If E_{w_1} and E_{w_2} lie in $L^2(\mu)$, then so does E_w for every w in $\bar{V}$. Moreover, if $f \in L^2(\mu)$, the function F defined on $\bar{V}$ by*

$$F(w) = \int f\bar{E}_w \, d\mu$$

is continuous on $\bar{V}$ and analytic on V.

Proof. Suppose that z is on the segment $[w_1, w_2]$ connecting w_1 to w_2. We can write $z = sw_1 + tw_2$ for $s, t \geqslant 0$ and $s + t = 1$. Now $E_z = E_{sw_1}E_{tw_2}$ and

$$|E_z|^2 = |E_{w_1}|^{2s}\,|E_{w_2}|^{2t} \leqslant |E_{w_1}|^2 + |E_{w_2}|^2,$$

so $E_z \in L^2(\mu)$. Now any w in $\bar{V}$ can be written as $w = z + r$ where $z \in [w_1, w_2]$ and $r \geqslant 0$. For this w we have

$$|E_w|^2 = |E_r|^2\,|E_z|^2 \leqslant |E_{w_1}|^2 + |E_{w_2}|^2,$$

hence $E_w \in L^2(\mu)$. Furthermore, the last inequality allows us to invoke the Dominated Convergence Theorem to give the continuity of F on $\bar{V}$. The analyticity follows from the theorems of Fubini and Morera. ▨

For any measure μ on $\bar{\mathbf{D}}$ with $\mu(\{1\}) = 0$, let

$$\Phi(\mu) = \{w : E_w \in P^2(\mu)\}.$$

We know from Remark 5 that $[0, \infty) \subset \Phi(\mu)$.

PROPOSITION 4. *The set $\Phi(\mu)$ is right-translation invariant and convex.*

Proof. Each E_t, $t \geqslant 0$, multiplies $P^2(\mu)$ into itself. If $w \in \Phi(\mu)$ and $t \geqslant 0$, $E_{w+t} = E_tE_w \in P^2(\mu)$, and translation invariance follows.

To verify convexity, assume that w_1 and w_2 lie in $\Phi(\mu)$. We may assume that w_1 and w_2 lie on distinct horizontal lines. By Lemma 2, $E_w \in L^2(\mu)$ for all w in $[w_1, w_2]$. Choose f in $L^2(\mu) \ominus P^2(\mu)$; to complete the proof it is enough to argue that $E_w \perp f$ for all w in $[w_1, w_2]$. Consider a trapezoid V as in Lemma 2, and put $F(w) = \int f\bar{E}_w d\mu$. Since $\Phi(\mu)$ is translation invariant and contains w_1 and w_2, $F(w) = 0$ for all w on the top and bottom edges of $\bar{V}$. Since F is analytic in V and continuous on $\bar{V}$, we conclude that $F \equiv 0$ on $\bar{V}$ and thus on $[w_1, w_2]$. ▨

Given any measure μ with compact support and any λ in $\Delta(\mu)$, there exists a kernel function J_λ in $P^2(\mu)$ with

$$p(\lambda) = \langle p, J_\lambda \rangle$$

for every polynomial p. Moreover, for any f in $P^2(\mu), f(\lambda) = \langle f, J_\lambda \rangle$ μ-a.e. on $\Delta(\mu)$; let us agree to always pick a representative for f for which this equation holds for *every* λ in $\Delta(\mu)$. We will say that $\Delta(\mu)$ is a *set of uniqueness* for $P^2(\mu)$ provided that the only f in $P^2(\mu)$ with $f(\lambda) \equiv 0$ on $\Delta(\mu)$ is $f = 0$. Equivalently, $\{J_\lambda : \lambda \in \Delta(\mu)\}$ spans $P^2(\mu)$. Recall that if μ is the measure associated with some admissible cosubnormal $\{S_t\}_{t\geqslant 0}$, then $\Delta(\mu) = \Delta$. Thus, every associated μ satisfies

(11) $$\mu(\{1\}) = 0,$$

(12) $$\Delta(\mu) \text{ is a set of uniqueness for } P^2(\mu).$$

DEFINITION 3. A measure μ satisfying (11) and (12) is *special* if in addition it has this property: if $f \in P^2(\mu)$ and w is a complex number such that $f(z) = E_w(z)$ for z in $\Delta(\mu)$, then $f = E_w$ μ-a.e.

We know of no measure μ on $\overline{\mathbf{D}}$ satisfying (11) and (12) which is not special. However, lacking knowledge that all associated measures are special, we find the following criteria useful.

PROPOSITION 5. *Let $\{S_t\}_{t\geqslant 0}$ be cosubnormal with associated measure μ. In order for μ to be special, it is sufficient that $\Gamma = \{u : \mathrm{Re}\, u < 0\}$ and either of the following hold*:

(A) $k(t, t) \to 0$ *as* $t \to +\infty$,

or

(B) *there exist positive numbers* C *and* σ *such that*

$$k(t, t) = C + O(e^{-\sigma t}) \quad \text{as } t \to +\infty.$$

Proof. Under our hypothesis on Γ, we have $\Delta(\mu) = \Delta = H^{-1}(\Gamma) = \mathbf{D}$. By (9) we have

$$k(t, t) = \int_{\{x\geqslant 0\}} e^{-2tx} d\gamma(x, y).$$

If (A) holds, then $\gamma(\{x = 0\}) = 0$ and so $\mu(\partial \mathbf{D}) = 0$. If (B) holds, then $\gamma(\{(x, y): 0 < x < \sigma/2\}) = 0$, and so $\mu(J) = 0$ where J is an open crescent of which $\partial\mathbf{D}$ is the outer boundary circle. In the first case, μ is special because $\mu(\mathbf{C} \setminus \Delta(\mu)) = 0$, and in the second, specialness follows from a theorem of Trent [32] on boundary values in $P^2(\mu)$. ▨

THEOREM 5. *Let $\{S_t\}_{t\geqslant 0}$ be cosubnormal with associated measure μ. If μ is special, then $\Lambda = \Phi(\mu)$, hence Λ is convex.*

Proof. Recall the unitary operator $U: \mathscr{B} \to \mathscr{H}$; for $\lambda \in \Lambda$, $Uk_\lambda = E_\lambda$. Let $Z: P^2(\mu) \to \mathscr{H}$ be the unitary operator from the proof of Theorem 3: $(Zf)(z) = f(z)$ for z in Δ. By the specialness of μ, $Z^{-1}E_\lambda = E_\lambda$ μ-a.e. for each λ in Λ, hence $\Lambda \subset \Phi(\mu)$.

Conversely, let $v \in \Phi(\mu)$, so $E_v \in P^2(\mu)$. We see that $E_v = ZE_v \in \mathscr{H}$, so $E_v = Uh$ for some h in $\mathscr{B}$. Let $e^{u\zeta}$ be any exponential in $\mathscr{B}$. We can write $u = H(\bar{z})$ for some z in $\Delta = \Delta(\mu)$, and so

$$\langle e^{u\zeta}, h\rangle_{\mathscr{B}} = \langle Ue^{H(\bar{z})\zeta}, Uh\rangle_{\mathscr{H}} = \langle K_z, E_v\rangle_{\mathscr{H}} =$$

$$= \overline{E_v(z)} = e^{vu}$$

Axiom (iv) now implies $v \in \Lambda$, and we have shown that $\Phi(\mu) \subset \Lambda$. ∎

THEOREM 6. *Let μ be a special measure on $\bar{\mathbf{D}}$. Then μ is the measure associated with some admissible cosubnormal $\{S_t\}_{t\geqslant 0}$, hence $M_\mu \cong L^*$.*

Proof. We define $\Lambda = \Phi(\mu)$ and construct the space $\mathscr{B}$ on which $\{S_t\}_{t\geqslant 0}$ acts with Λ as domain set. For each f in $P^2(\mu)$ we define a function Yf on Λ by

$$(Yf)(\lambda) = \int f\bar{E}_\lambda d\mu, \quad \lambda \in \Lambda.$$

Now recall that $\{E_x : x \geqslant 0\}$ spans $P^2(\mu)$, and so if $(Yf)(x) = 0$ for $x \geqslant 0$, f must be the zero function. In particular, the linear map $Y: f \to Yf$ is one-to-one. We set $\mathscr{B} = \{Yf : f \in P^2(\mu)\}$ and norm $\mathscr{B}$ so as to make $Y: P^2(\mu) \to \mathscr{B}$ a unitary operator.

We define $k_\lambda = YE_\lambda$ for $\lambda \in \Lambda$. If $g = Yf \in \mathscr{B}$,

$$\langle g, k_\lambda\rangle_{\mathscr{B}} = \langle f, E_\lambda\rangle = g(\lambda),$$

hence $\{k_\lambda : \lambda \in \Lambda\}$ is a reproducing kernel for $\mathscr{B}$. We know that Λ is convex and right-translation invariant by Proposition 4. Moreover, $k(\lambda, \lambda) = \|E_\lambda\|^2$ is uniformly bounded on the line segment connecting any two points of Λ; this by the proof of Lemma 2. Hence axiom (i) holds for $\mathscr{B}$. We have already verified axiom (ii). For any w in $\Delta(\mu)$, we see that

$$(YJ_w)(\lambda) = \int J_w\bar{E}_\lambda d\mu = e^{H(\bar{w})\lambda}, \quad \lambda \in \Lambda.$$

By assumption (12) on μ we see that $\{e^{H(\bar{w})\zeta} : w \in \Delta(\mu)\}$ spans $\mathscr{B}$ and (iii) holds.

Next we check the maximality of Λ. Suppose that for some h in $\mathscr{B}$, some complex v, and every exponential $e^{u\zeta}$ in $\mathscr{B}$, we have $\langle e^{u\zeta}, h\rangle_{\mathscr{B}} = e^{uv}$. This is, in particular, true for all $u = H(\overline{w})$ with w in $\Delta(\mu)$. For such u we have

$$e^{\bar{v}H(w)} = \langle h, e^{H(\overline{w})\zeta}\rangle_{\mathscr{B}} = \langle Y^*h, J_w\rangle_{P^2(\mu)} = (Y^*h)(w), \quad w \in \Delta(\mu),$$

hence $Y^*h = E_v$ μ-a.e. by specialness, and $v \in \Phi(\mu) = \Lambda$ as desired.

It remains to check (v). The translation operators $(S_t f)(\zeta) = f(\zeta + t)$ make sense formally since Λ is translation invariant. If $Yf \in \mathscr{B}$,

$$(S_t Yf)(\lambda) = (Yf)(\lambda + t) = \langle f, E_{\lambda+t}\rangle = \langle f, E_t E_\lambda\rangle = (Y(V_t^\mu)^* f)(\lambda).$$

We therefore have $S_t Y = Y(V_t^\mu)^*$, hence S_t is bounded with $\|S_t\| = \|V_t^\mu\| \leqslant \leqslant \|E_t\|_\infty = 1$. That μ is the measure associated with $\mathscr{B}$ now follows from the identity

$$k(s, t) = \langle YE_s, YE_t\rangle = \int E_s \overline{E}_t \mathrm{d}\mu, \quad s, t \geqslant 0.$$

▨

Note that the operator Y coincides with $U^{-1}Z$ from the proof of Theorem 3.

REMARK 6. We return to Theorem 2 and circular symmetry with two examples.

(a) Let $\mathrm{d}\mu = \mathrm{d}A$, area measure on $\mathbf{D}$. Then M_μ is a weighted shift and Theorem 6 implies that $M_\mu \cong L^*$ for some L. Now $\Delta(\mu) = \mathbf{D}$ and so $\Gamma = \{u : \operatorname{Re} u < 0\}$; from the proof of Theorem 6 we know that $(YJ_w)(\zeta) = e^{H(\overline{w})\zeta}$ and so

$$\|e^{H(\overline{w})\zeta}\|_{\mathscr{B}}^2 = J_w(w) = \frac{1}{\pi(1 - |w|^2)^2},$$

which implies that

$$\|e^{u\zeta}\|_{\mathscr{B}} \leqslant \frac{C}{|\operatorname{Re} u|}, \quad u \in G,$$

for G as in Theorem 2. The reason that Theorem 2 does not apply is that $\Lambda = = [0, \infty)$.

(b) In another direction, let

$$\mathrm{d}\mu(z) = e^{-\frac{1}{1-|z|}} \mathrm{d}A(z) \quad \text{on } \mathbf{D}.$$

In this case Λ contains many nonreal points and $\Gamma = \{u : \operatorname{Re} u < 0\}$. However, the relation $\|e^{H(\bar{w})\zeta}\|^2_{\mathscr{B}} = J_w(w)$ and elementary estimates on the kernel function $J_w(z)$ for $P^2(\mu)$ imply that (6) fails for $d < 1$ though it holds for all $d > 1$. Moreover, $L^* \cong M_\mu$, which is a weighted shift.

THEOREM 7. *Let* $-\infty \leqslant a < b \leqslant \infty$ *and suppose* g *is a piecewise* C^1 *function defined on* (a, b) *such that the domain*

$$\Lambda = \{x + iy : a < y < b \text{ and } x > g(y)\}$$

is convex and contains $[0, \infty)$. *Then there exists an admissible cosubnormal semigroup* $\{S_t\}_{t \geqslant 0}$ *whose associated measure* μ *is special and whose domain set is* Λ.

Proof. Let us construct the space $\mathscr{B}$. We claim that it is enough to find a special measure μ such that

$$\Lambda = \{w : E_w \in L^2(\mu)\}, \tag{13}$$

for suppose that we have done so. If f is any element of $L^2(\mu)$, Lemma 2 tells us that the function $F(w) = \int f\bar{E}_w d\mu$ is analytic on Λ. If $f \perp P^2(\mu)$, we see that $F(x) = 0$ for $x \geqslant 0$, so $F \equiv 0$ on Λ. It follows that $E_w \in P^2(\mu)$ for each w in Λ, and therefore $\Phi(\mu) = \Lambda$. This fact, when combined with Theorems 5 and 6, yields Theorem 7.

It remains to construct μ satisfying (13). The hypothesis on g means that (a, b) is a finite or countable union of closed bounded intervals J_k, $k \geqslant 1$, any two of which share at most an endpoint, such that g is of class C^1 on each J_k. Of course $g'(y)$ is a one-sided derivative if y is an endpoint of some J_k, and $g'(y)$ will have two meanings if y belongs to two J_k. Let J be one of the intervals J_k, and let $T = \{g(y) + iy : y \in J\}$ be the corresponding arc of $\partial\Lambda$. For t in J, let $x = d + sy$ be the equation of the line tangent to T at $g(t) + it$. We have $s = s(t) = g'(t)$ and $d = d(t) = g(t) - tg'(t)$, both continuous functions of t on J. The convexity of Λ tells us that

$$g(y) \geqslant d(t) + s(t)y \tag{14}$$

for all t in J and $a < y < b$. For $\delta > 0$, let $I(y, \delta) = J \cap [y - \delta, y + \delta]$, and let $\omega(\delta)$ be a positive function of δ which strictly decreases to 0 as $\delta \downarrow 0$, such that

$$\omega(\delta) \geqslant \sup_{\substack{t \in I(y,\delta) \\ y \in J}} |d(t) - d(y)| + |y|\,|s(t) - s(y)|.$$

Choose a sequence $\delta_k \downarrow 0$ and a function analytic on $\mathbf{D}$ with a simple zero at 0 and having value k/δ_k at $e^{-\omega(\delta_k)}$. On replacing the Taylor coefficients of this function by their absolute values, we get a new function $h(z) = \sum_{n=1}^{\infty} c_n z^n$ analytic on $\mathbf{D}$ with $c_n \geqslant 0$ and

$$h(e^{-\omega(\delta_k)}) \geqslant k/\delta_k, \quad k = 1, 2, 3, \ldots .$$

Now for each $n = 1, 2, \ldots$ define a measure γ_n on the line $\{u + iv : u = n/2\}$ by the formula

$$\int f \, d\gamma_n = c_n \int_J e^{nd(t)} f\left(\frac{n}{2}(1 - is(t))\right) dt$$

and put $\chi = \sum_{n=1}^{\infty} \gamma_n$. We recall that $-H$ maps $\mathbf{D}$ onto $\{u + iv : u > 0\}$; we define $\nu = \chi \circ (-H)$, a measure on $\mathbf{D}$. The total mass of ν is

$$\sum_{n=1}^{\infty} c_n \int_J e^{nd(t)} \, dt.$$

However, $d(t) \leqslant g(0) < 0$ by (14) and so the total mass of ν does not exceed $|J| h(e^{g(0)}) < \infty$, where $|J|$ is the length of J. We are using here the fact that h is increasing on $[0,1)$.

CLAIM 1. If $a < y < b$ and $x > g(y)$, then $E_{x+iy} \in L^2(\nu)$ and

$$\int |E_{x+iy}|^2 d\nu \leqslant |J| \, h(e^{g(y)-x}) < \infty. \tag{15}$$

To see this, let us write $-H = u + iv$ so that $|E_{x+iy}| = e^{-(ux+vy)}$. We have

$$\int |E_{x+iy}|^2 d\nu = \sum_{n=1}^{\infty} \int e^{-(2ux+2vy)} d\gamma_n(u, v) =$$

$$= \int_J \sum_{n=1}^{\infty} c_n (e^{d(t)+s(t)y-x})^n \, dt.$$

If $x > g(y)$, we see from (14) that $d(t) + s(t)y - x \leqslant g(y) - x < 0$, which yields (15).

CLAIM 2. If $y \in J$ and $x \leqslant g(y)$, then $E_{x+iy} \notin L^2(\nu)$.

To see this, fix y in J, $x \leqslant g(y)$, and suppose $t \in I(y, \delta_k)$, for some $k \geqslant 1$. By the definition of $\omega(\delta)$ and the identity $g(y) = d(y) + ys(y)$ we have

$$d(t) + s(t)y - x \geqslant d(t) + s(t)y - g(y) \geqslant -\omega(\delta_k).$$

Thus for any positive integer N,

$$\int |E_{x+iy}|^2 \mathrm{d}\nu \geqslant \int_J \sum_{n=1}^{N} c_n (\mathrm{e}^{d(t)+s(t)y-x})^n \, \mathrm{d}t \geqslant$$

$$\geqslant \int_{I(y,\delta_k)} \sum_{n=0}^{N} c_n \mathrm{e}^{-n\omega(\delta_k)} \mathrm{d}t.$$

On letting $N \to \infty$, we find

$$\int |E_{x+iy}|^2 \mathrm{d}\nu \geqslant \delta_k h(\mathrm{e}^{-\omega(\delta_k)}) \geqslant k,$$

and because k is arbitrary, the claim is proved.

Return now to our collection $\{J_k\}$ of intervals. For each k, associate h_k and ν_k to J_k as h and ν were associated to J above. We may choose positive numbers ε_k so that $\sum \varepsilon_k h_k$ converges uniformly on compact subsets of $\mathbf{D}$. Further, choose a finite positive measure m on $\mathbf{R}^1$ such that $\int \mathrm{e}^{-2yv} \mathrm{d}m(v) < \infty$ exactly when $a < y < b$, and define a measure χ_0 on the line $\{1 + \mathrm{i}y : y \in \mathbf{R}\}$ by the formula

$$\int f \, \mathrm{d}\chi_0 = \int f(1 + \mathrm{i}y) \, \mathrm{d}m(y).$$

We put $\nu_0 = \chi_0 \circ (-H)$ and note that $E_{x+iy} \in L^2(\nu_0)$ exactly when $a < y < b$. We easily see from Claims 1 and 2 that

$$\nu_0 + \sum_{k \geqslant 1} \frac{\varepsilon_k}{|J_k|} \nu_k$$

is a (finite positive) measure on $\mathbf{D}$ such that

$$E_{x+iy} \in L^2\left(\nu_0 + \sum_{k \geqslant 1} \frac{\varepsilon_k}{|J_k|} \nu_k \right)$$

if and only if $x + iy \in \Lambda$. Finally, let $\tilde{\mu}$ be this measure on $\mathbf{D}$:

$$d\tilde{\mu} = \exp\left(-\exp\frac{1}{1-|z|}\right) dA.$$

The measure μ we want is

$$\mu = \tilde{\mu} + \nu_0 + \sum_{k\geqslant 1} \frac{\varepsilon_k}{|J_k|} \nu_k .$$

Note that $\Delta(\tilde{\mu}) = \mathbf{D}$, hence $\tilde{\mu}$ is special. Moreover, μ is carried by $\mathbf{D}$ and $\mu \geqslant \tilde{\mu}$, so $\Delta(\mu) = \mathbf{D}$ and μ is special also. Since every E_w belongs to $L^2(\tilde{\mu})$, we see that (13) holds, as desired. ▨

When Λ is an open set, analyticity of $\partial\Lambda$ is sometimes another consequence of subnormality of $\{S_t^*\}_{t\geqslant 0}$. Consider the situation where the kernel function for $\mathscr{B}$ has the form

$$k(w, \zeta) = \overline{u(w)}u(\zeta)\overline{\varphi'(w)}^{\delta}\varphi'(\zeta)^{\delta}h(\overline{\varphi(w)}\varphi(\zeta)), \tag{16}$$

where u is analytic on Λ, h is analytic on $\mathbf{D}$, $\varphi : \Lambda \to \mathbf{D}$ is a conformal mapping and $\delta > 0$. This will be true of all of our examples, except § 4.3.

THEOREM 8. *Suppose that $\mathscr{B}$ is admissible with open domain set Λ such that $\partial\Lambda$ is a simple arc. Suppose that the kernel function k for $\mathscr{B}$ has the form* (16), *where u is analytic across and nonvanishing on $\partial\Lambda$, h is analytic across $\partial\mathbf{D}$ except for a pole at* 1, *and h is never zero on $\partial\mathbf{D}\setminus\{1\}$. Suppose also that φ is real on $\Lambda \cap \mathbf{R}$ and that $\rho_0 \equiv \inf \Lambda \cap \mathbf{R} > -\infty$. If $\{S_t^*\}_{t\geqslant 0}$ is subnormal, then each component of $\{\lambda : \operatorname{Re}\lambda > \rho_0\} \cap \partial\Lambda$ is an analytic arc.*

REMARK 7. This theorem applies to the Hardy and Bergman spaces in Sections 4.1.1, 4.1.2, and 4.2. Theorem 14 shows that $\partial\Lambda$ can fail to be analytic at points λ with $\operatorname{Re}\lambda = \rho_0$.

LEMMA 3. *Let $\{S_t\}_{t\geqslant 0}$ be cosubnormal with associated measure μ and domain set Λ. Then*

$$k(s, t) = \int E_s\overline{E_t}\, d\mu$$

for all real s and t in Λ.

Proof. We know this already for $s, t \geqslant 0$. By equation (5),

$$k(s, t) = \langle E_s, E_t\rangle_{\mathscr{H}}$$

for $s, t \in A \cap \mathbf{R}$. Let $t < 0$ be in A and suppose that $Z : P^2(\mu) \to \mathscr{H}$ is as in the proof of Theorem 3. We have the operator V^{μ}_{-t} on $P^2(\mu)$, and we know that $ZE_x = E_x$ for $x \geqslant 0$, and that $\{E_x : x \geqslant 0\}$ spans $P^2(\mu)$. It follows that $ZV^{\mu}_{-t}f = E_{-t}Zf$ for every f in $P^2(\mu)$; in particular, $ZV^{\mu}_{-t}Z^{-1}E_t = E_{-t}ZZ^{-1}E_t = 1$. Since $Z1 = 1$, we see that $V^{\mu}_{-t}Z^{-1}E_t = 1$ μ-a.e. Therefore, $Z^{-1}E_t = E_t$ μ-a.e. for $0 > t \in A$, a fact we already know for $t \geqslant 0$. Thus for any real s, t in A, E_s and E_t lie in $P^2(\mu)$ and

$$\langle E_s, E_t \rangle_{\mathscr{H}} = \int E_s \overline{E}_t \mathrm{d}\mu. \qquad ▨$$

Proof of Theorem 8. Let $\zeta_0 \in \partial A$ with $\operatorname{Re}\zeta_0 > \rho_0$. Our object is to show that φ has an analytic continuation to a neighborhood of ζ_0. Clearly, φ maps $A \cap \mathbf{R}$ onto $(-1, 1)$; we may assume that $\varphi(\infty) = 1$ and $\varphi(\rho_0) = -1$. We see that $\varphi(\overline{\zeta}) = \overline{\varphi(\zeta)}$ and so A is symmetric about the real axis. With the notation $\tilde{u}(\lambda) = \overline{u(\overline{\lambda})}$ we have

$$k(\overline{\zeta}, \zeta) = \tilde{u}(\zeta)u(\zeta)\varphi'(\zeta)^{2\delta}h(\varphi(\zeta)^2), \qquad \zeta \in A;$$

this function gives an analytic continuation of $k(t, t)$ from $A \cap \mathbf{R}$ to all of A. Lemma 3 gives another analytic continuation of $k(t, t)$, namely

$$\int e^{2\zeta \operatorname{Re} H} \mathrm{d}\mu,$$

to the region $\{\zeta : \operatorname{Re}\zeta > \rho_0\}$, which contains ζ_0, and so the function

$$\varphi'(\zeta)h(\varphi(\zeta)^2)^{1/2\delta} = \left[\frac{k(\overline{\zeta}, \zeta)}{\tilde{u}(\zeta)u(\zeta)}\right]^{1/2\delta}$$

continues to a neighborhood of ζ_0. We select an antiderivative $F(\zeta)$ of this continuation.

The map φ extends to a homeomorphism of $\overline{A}$ onto $\overline{\mathbf{D}}\setminus\{1\}$, so $z_0 \equiv \varphi(\zeta_0) \notin \{-1, 1\}$. It follows that $h(z^2)^{1/2\delta}$ continues to a neighborhood of z_0, with corresponding antiderivative $G(z)$. We see that $G \circ \varphi$ and F have the same derivative, and so we may adjust a constant of integration to give $G \circ \varphi = F$ at all points near ζ_0. Since $h(z_0^2)^{1/2\delta} \neq 0$, G has a local inverse at $G(z_0)$ and $G^{-1} \circ F$ gives the desired continuation of φ at ζ_0. ▨

4. EXAMPLES AND APPLICATIONS

4.1. DILATION SEMIGROUPS. We pursue an idea from the introduction and generalize the semigroup $\{D_t\}_{t \geq 0}$ and the operator A_1 acting on $P^2(\partial\Omega_0, |dz|)$. Let us suppose that Ω is a simply connected domain in $\mathbf{C}$ with $0 \in \partial\Omega$ and with $\Omega \cup \{0\}$ starlike about 0. Suppose that $\mathscr{A}$ is a reproducing kernel Hilbert space of functions on Ω. We will study the situation in which, for appropriate $\alpha > 0$, the collection

$$\mathscr{B} = \{e^{-\frac{\alpha}{2}\zeta} f(e^{-\zeta}) : f \in \mathscr{A}\} \tag{17}$$

is a space of functions on $\Lambda = -\log \Omega$ satisfying our axioms. Here it is understood that $\mathscr{B}$ is normed so that the map $W: f(z) \to e^{-\frac{\alpha}{2}\zeta} f(e^{-\zeta})$ is a unitary operator from $\mathscr{A}$ to $\mathscr{B}$. The role of the operators B_v on $\mathscr{B}$ will be played by the operators A_v on $\mathscr{A}$ defined formally for $\operatorname{Re} v > \alpha/2$ by

$$(A_v f)(z) = \frac{1}{z^v} \int_0^z f(w) w^{v-1} dw,$$

where the path of integration is the segment from 0 to z. Note that

$$A_v z^u = \frac{1}{v+u} z^u \quad \text{if} \quad \operatorname{Re}(v+u) > 0.$$

Let us agree to define z^u, $z \in \Omega$, by requiring $\arg z = 0$ if $z > 0$.

We impose some axioms on $\mathscr{A}$, analogous to those for $\mathscr{B}$; they could be made more general, but these will suffice for our examples.

(i)′ Ω contains $(0,1]$.

(ii)′ Every f in $\mathscr{A}$ is analytic on Ω.

(iii)′ If $\hat{\Gamma} = \{u : z^u \in \mathscr{A}\}$, then $\{z^u : u \in \hat{\Gamma}\}$ spans $\mathscr{A}$.

(iv)′ If v is a complex number and h is an element of $\mathscr{A}$ such that $\langle z^u, h \rangle = e^{uv}$ for all u in $\hat{\Gamma}$, then $e^v \in \Omega$.

(v)′ The dilation semigroup $\{D_t\}_{t \geq 0}$ is strongly continuous on $\mathscr{A}$ and satisfies

$$\|D_t\|^2 = O(e^{\beta t}) \quad \text{as } t \to \infty$$

for all β exceeding a fixed positive number α.

Of course, the strong continuity of $\{D_t\}_{t\geqslant 0}$ implies the existence of many α's; let us understand that we will choose a particular α, which we call the *parameter* of $\mathscr{A}$ or $\{D_t\}_{t\geqslant 0}$. In practice we always take α to be as small as possible. We will indicate that (i)′—(v)′ hold by saying that $\mathscr{A}$ or $\{D_t\}_{t\geqslant 0}$ is *admissible*.

Let us see how this meshes with our theory of $\mathscr{B}$, L and $\{S_t\}_{t\geqslant 0}$. Given $\mathscr{A}$ satisfying (i)′ — (v)′, define $\mathscr{B}$ by (17) using, of course, the α from (v), with corresponding unitary operator $W : \mathscr{A} \to \mathscr{B}$. The reader will readily verify that $\mathscr{B}$ satisfies (i)—(v). In particular, we note a few useful facts. First, $Wz^u = e^{-(u+\alpha/2)\zeta}$ so that $\Gamma + \alpha/2 = -\hat{\Gamma}$. Moreover, $e^{-(\alpha/2)t}WD_t = S_tW$ and $WA_v = B_{v-\alpha/2}W$ for $\mathrm{Re}\, v > \alpha/2$. Thus we have, via W, $I - 2A_{1+\alpha/2} \cong L$.

In our example $\mathscr{A} = P^2(\partial\Omega_0, |dz|)$, the appropriate choice of α is 1, and $A_{1+\alpha/2} = A_{3/2}$, whereas we were considering A_1 as the "natural" operator on that space. To see how to switch the emphasis from $A_{1+\alpha/2}$ to A_α in general, recall the assertion from §2 that in this example,

$$I - A_1 \cong \left(L + \frac{1}{3}\right)\left(I + \frac{1}{3}L\right)^{-1}.$$

This is a special instance of the general fact, valid for any admissible space $\mathscr{B}$, that if $|w| < 1$ and g_w is the Möbius transformation

$$g_w(z) = \left(\frac{1-\bar{w}}{1-w}\right)\frac{z-w}{1-\bar{w}z}$$

then

$$g_w(L) = I - 2(\mathrm{Re}\, v)B_v,$$

where $v = -H(\bar{w})$; this is an immediate consequence of (3). Thus, in general,

$$I - \alpha A_\alpha \cong I - \alpha B_{\alpha/2} = (L - w)(I - wL)^{-1}$$

where $w = (\alpha - 2)(\alpha + 2)^{-1}$.

Since L is the cogenerator of $\{S_t\}_{t\geqslant 0}$, we see that $I - 2A_{1+\alpha/2}$ is the cogenerator of $\{e^{-(\alpha/2)t}D_t\}_{t\geqslant 0}$, at least when this semigroup is contractive. However, the perhaps more natural operator $I - \alpha A_\alpha$ plays a similar role, namely

$$e^{-(\alpha/2)t}D_t = E_{(\alpha/2)t}(I - \alpha A_\alpha),$$

a formula valid whenever $\{e^{-(\alpha/2)t}D_t\}_{t\geqslant 0}$ is contractive, or equivalently, whenever $\|I - \alpha A_\alpha\| \leqslant 1$. To verify this, one checks that both sides agree on $\{z^u : u \in \hat{\Gamma}\}$.

We may summarize our main conclusions as follows.

PROPOSITION 6. *If $\mathscr{A}$ is admissible with parameter α, then the space $\mathscr{B}$ defined by* (17) *is admissible in the sense of Definition* 1. *In this case,* $e^{-(\alpha/2)t}D_t$ *and A_v are unitarily equivalent, via W, to S_t and $B_{v-\alpha/2}$, respectively.*

We turn to subnormality of $\{D_t^*\}_{t\geq 0}$, or equivalently, of A_α^*. First observe that the kernel functions $\hat{k}$ for $\mathscr{A}$ and k for $\mathscr{B}$ are related by

$$\hat{k}(e^{-w}, e^{-\zeta}) = e^{(\alpha/2)(\bar{w}+\zeta)}\, k(w, \zeta), \quad w, \zeta \in \Lambda.$$

On putting this together with Theorem 3 as interpreted in equation (9), we immediately deduce the following:

PROPOSITION 7. *Let $\mathscr{A}$ be admissible with parameter α. In order that A_α^* and $\{D_t^*\}_{t\geq 0}$ be subnormal, it is necessary and sufficient that there exist a measure β on $\{w : \operatorname{Re} w \geq -\alpha/2\}$ such that*

$$\hat{k}(a, b) = \int a^w b^{\bar{w}} d\beta(w), \quad 0 < a, b \leq 1.$$

In this case $\{e^{-(\alpha/2)t}D_t^*\}_{t\geq 0}$ and $I-\alpha A_\alpha^*$ are unitarily equivalent to $\{V_t^\mu\}_{t\geq 0}$ and $\left(M_\mu - \frac{\alpha-2}{\alpha+2}\right)\left(I - \left(\frac{\alpha-2}{\alpha+2}\right)M_\mu\right)^{-1}$ respectively, where $\mu = \beta \circ \Phi$ and $\Phi(z) = \frac{1+z}{1-z} - \frac{\alpha}{2}$.

In some cases that will interest us $\hat{k}(\bar{z}, z)$, which is defined and analytic on $\Omega \cap \{\bar{z} : z \in \Omega\}$, extends to be meromorphic at $z = 0$. Proposition 7 can then be cast in a very useful form.

PROPOSITION 8. *Suppose that $\mathscr{A}$ is admissible and that the parameter α is a positive integer. Suppose that $\hat{k}(\bar{z}, z)$ has a meromorphic extension with pole of order α at $z = 0$. Then A_α^* and $\{D_t^*\}$ are subnormal if and only if there exists a sequence $\{\beta_n\}_{n=-\alpha}^{\infty}$ of finite positive Borel measures on $\mathbf{R}$ such that*

$$\hat{k}(re^{-v/2}, re^{v/2}) = \sum_{n=-\alpha}^{\infty} \hat{\beta}_n(v) r^n$$

whenever $0 < r \leq 1$ and $2\log r \leq v \leq -2\log r$, where

$$\hat{\beta}_n(v) = \int_{-\infty}^{\infty} e^{-ivy} d\beta_n(y).$$

In this case

$$d\beta(x+iy) = \sum_{n=-\alpha}^{\infty} d\beta_n(y)\,d\delta_{n/2}(x)$$

where $\delta_{n/2}$ is a unit point mass at $n/2$; consequently μ is supported on a sequence of circles in $\bar{\mathbf{D}}$ which are tangent to $\partial\mathbf{D}$ at the point 1.

Proof. Suppose that A_α^* is subnormal, and that $\hat{k}(\bar{z}, z)$ has a Laurent series expansion

$$\hat{k}(\bar{z}, z) = \sum_{n=-\alpha}^{\infty} b_n z^n$$

for z near 0. From Proposition 7 we see that

$$\hat{k}(re^{-v/2}, re^{v/2}) = \int e^{-ivy} r^{2x}\,d\beta \tag{18}$$

for $0 < r \leqslant 1$, $2\log r \leqslant v \leqslant -2\log r$. If we set $v = 0$ and compare with the Laurent expansion, we see that β is supported on the union of the lines $\{x + iy : 2x = n\}$, $n = -\alpha, -\alpha+1, \ldots$, and the β-measure of the n^{th} line is b_n. The Proposition now follows from (18). ▨

4.1.1. WEIGHTED HARDY SPACES ON Ω. Let Ω be a Jordan domain containing $(0, 1]$ whose boundary is rectifiable and contains 0. Further assume that $\Omega \cup \{0\}$ is starlike about 0 and that $\partial\Omega$ is analytic at 0. Let $\tau : \mathbf{D} \to \Omega$ be a conformal map with $\tau(1) = 0$, and denote by $\psi : \Omega \to \mathbf{D}$ the inverse map. Since $\partial\Omega$ is rectifiable we have $\tau' \in H^1$ [10]. Lemmas 5 and 6 in § 4.2 applied to $\sigma \equiv -\log\tau$ show that τ' is an outer function, i.e., that Ω is a Smirnov domain. Let q be an outer function in $P^2(\partial\Omega, |dz|)$, which means that $q \circ \tau$ is outer on $\mathbf{D}$; see [10] for a treatment of these spaces. We will consider the spaces $\mathscr{A} = P^2(\partial\Omega, |q|^2|dz|)$. The Smirnov condition guarantees that a function f analytic on Ω lies in $\mathscr{A}$ if and only if

$$(q\circ\tau)(f\circ\tau)(\tau')^{1/2} \tag{19}$$

lies in $H^2 = P^2\left(\partial\mathbf{D}, \frac{1}{2\pi}|dz|\right)$, the classical Hardy space on $\mathbf{D}$. Here we are simultaneously thinking of $\mathscr{A}$ as a space of functions analytic on Ω and as the subspace of $L^2(\partial\Omega, |q|^2\,dz|)$ spanned by the polynomials; the connection is of course via boundary values. The kernel function $\hat{k}$ for $\mathscr{A}$ is

$$\hat{k}(w, z) = \frac{1}{2\pi}\,\frac{\overline{\psi'(w)^{1/2}}\psi'(z)^{1/2}}{\overline{q(w)}q(z)}\,\frac{1}{1-\overline{\psi(w)}\psi(z)}\,. \tag{20}$$

Throughout this section we will assume that the above requirements on Ω are in force.

THEOREM 9. *Suppose that, in addition to the above requirements on* Ω, *there exist* $c_1, c_2 > 0$ *with*

$$c_1 |\psi'| \leqslant |q|^2 \leqslant c_2 |\psi'| \quad \text{on } \Omega. \tag{21}$$

Then $P^2(\partial\Omega, |q|^2|dz|)$ *is admissible with* $\alpha = 1$.

Proof. We have a natural unitary operator $P: \mathscr{A} \to H^2$ such that $(2\pi)^{-1/2}P$ maps f in $\mathscr{A}$ to the H^2 function given by (19). That this operator is onto follows from the facts that $(g \circ \tau)(\tau')^{1/2}$ is outer and that $\{\tau^n\}_{n=0}^{\infty}$ spans H^2, the latter being a consequence of a theorem of Carathéodory and Walsh [33]. It follows that axiom (iii)′ holds. Axiom (iv)′ we leave for the reader; an argument as in Proposition 11 below (but easier) will do the trick. Axioms (i)′ and (ii)′ are, of course, automatic.

It remains to verify (v)′. We readily compute that $PD_tP^{-1} = QR_tC_{\varphi_t}$ where C_{φ_t} is the composition operator on H^2 given by $C_{\varphi_t}f = f \circ \varphi_t$ with $\varphi_t(z) = \psi(e^{-t}\tau(z))$, and Q and R_t are the multiplication operators defined on H^2 by

$$(Qf)(z) = \frac{q(\tau(z))}{\psi'(\tau(z))^{1/2}} f(z) \quad \text{and} \quad (R_tf)(z) = \frac{\psi'(e^{-t}\tau(z))^{1/2}}{q(e^{-t}\tau(z))} f(z) .$$

The hypothesis (21) implies that $\|Q\|^2 \leqslant c_2$ and $\|R_t\|^2 \leqslant 1/c_1$. Moreover, a theorem of Ryff [27] implies that $\|C_{\varphi_t}\|^2 \leqslant 2(1 - |\varphi_t(0)|)^{-1}$. Since $\psi'(0)$ exists and is nonzero (by the analyticity of $\partial\Omega$ at 0), we see that $1 - |\varphi_t(0)|$ behaves like a positive multiple of e^{-t} as $t \to \infty$. It follows that $\|C_{\varphi_t}\|^2 \leqslant ae^t$ for appropriate $a > 0$ and all t; thus $\|D_t\|^2 \leqslant c_2c_1^{-1}ae^t$ and the proof is complete. ▨

REMARK 8. (a) One checks that $\hat{\Gamma} = \{u : \operatorname{Re} u > -1/2\}$, which implies that $\Gamma = \{u : \operatorname{Re} u < 0\}$, whence $\Delta = \mathbf{D}$ and $\sigma_p(L) = \mathbf{D}$. It follows that $\sigma(L) = \bar{\mathbf{D}}$ and, since $I - A_1$ is a Möbius transformation of L, that $\sigma(I - A_1) = \bar{\mathbf{D}}$.

(b) Theorem 2 and the argument in Corollary 2 show that $I - A_1$, acting on $P^2(\partial\Omega, |q|^2|dz|)$, cannot be quasisimilar to a weighted shift.

(c) If q is analytic at 0 and $q(0) \neq 0$, then $\hat{k}(\bar{z}, z)$ has a simple pole at $z = 0$ and Proposition 8 applies. In addition Proposition 5(B) is in effect: If $\{D_t^*\}_{t \geqslant 0}$ is subnormal, then the measure μ associated with the corresponding space $\mathscr{B}$ is special, and therefore $\log \Omega$ is convex, by Theorem 5.

(d) If $\{D_t^*\}_{t \geqslant 0}$ is subnormal, if Ω is symmetric about the real axis, and if q is analytic on an open set containing $\bar{\Omega}$, then Theorem 8 implies that $\partial\Omega \cap \cap \{z : |z| < R\}$ is a single analytic arc, where $R = \max\{|z| : z \in \bar{\Omega}\}$.

We consider a subclass of examples. Suppose that $|q|^2|dz| = d\omega_\lambda$, harmonic measure at some point λ in Ω, so that

$$|q(z)|^2 = \frac{1}{2\pi}|\psi'(z)|\frac{1-|\psi(\lambda)|^2}{|1-\overline{\psi(\lambda)}\psi(z)|^2} \tag{22}$$

is the Poisson kernel at λ. Note that q satisfies (21).

PROPOSITION 9. *Suppose that for $i = 1, 2$, the domains Ω_i satisfy the hypotheses of Theorem 9. Suppose that $\lambda_i \in \Omega_i$ and $d\omega_{\lambda_i}$ is harmonic measure on $\partial\Omega_i$ for the point λ_i. Then A_1 acting on $P^2(\partial\Omega_1, d\omega_{\lambda_1})$ is unitarily equivalent to A_1 acting on $P^2(\partial\Omega_2, d\omega_{\lambda_2})$ if and only if $\lambda_2\Omega_1 = \lambda_1\Omega_2$.*

Proof. First suppose that the map $\eta(z) = (\lambda_1/\lambda_2)z$ carries Ω_2 onto Ω_1. Let us associate $|q_i|^2$ with Ω_i and λ_i as $|q|^2$ is associated with Ω and λ in (22). A change of variable will show that for any polynomial f,

$$\int_{\partial\Omega_2}\left|f\left(\frac{\lambda_1}{\lambda_2}z\right)\right|^2|q_2(z)|^2|dz| = \int_{\partial\Omega_1}|f(z)|^2\left|\frac{\lambda_2}{\lambda_1}\right|\left|q_2\left(\frac{\lambda_2}{\lambda_1}z\right)\right|^2|dz|$$

and it is easy to check that

$$|q_1(z)|^2 = \left|\frac{\lambda_2}{\lambda_1}\right|\left|q_2\left(\frac{\lambda_2}{\lambda_1}z\right)\right|^2.$$

Thus the map $X: f \to f \circ \eta$ is a unitary operator from $\mathscr{A}_1 = P^2(\partial\Omega_1, d\omega_{\lambda_1})$ to $\mathscr{A}_2 = P^2(\partial\Omega_2, d\omega_{\lambda_2})$. One checks that the operators XA_1 and A_1X agree on powers of z, and so the desired unitary equivalence is established.

Conversely, suppose that A_1 defines unitarily equivalent operators on the two spaces, with the equivalence implemented by a unitary $X: \mathscr{A}_1 \to \mathscr{A}_2$. If $\operatorname{Re} u > -1/2$, we know that z^u is an eigenvector for A_1 on $\mathscr{A}_1$ with eigenvalue $(1+u)^{-1}$. Thus Xz^u is an eigenvector for A_1 on $\mathscr{A}_2$ with the same eigenvalue; by Proposition 3, we have $Xz^u = a_u z^u$ for some complex a_u. We therefore have

$$a_u\overline{a}_v\langle z^u, z^v\rangle_{\mathscr{A}_2} = \langle Xz^u, Xz^v\rangle_{\mathscr{A}_2} = \langle z^u, z^v\rangle_{\mathscr{A}_1}.$$

Now observe that for any f in $\mathscr{A}_i$,

$$\langle f, 1\rangle_{\mathscr{A}_i} = \int_{\partial\Omega_i} f\, d\omega_{\lambda_i} = f(\lambda_i),$$

so if we set $v = 0$ in our above equation, we find $a_u \bar{a}_0 \lambda_2^u = \lambda_1^u$. Putting $u = 0$ yields $|a_0|^2 = 1$, and so

$$a_u = a_0 \lambda_1^u \lambda_2^{-u} = a_0 e^{(v_1 - v_2)u},$$

where $v_i = \log \lambda_i$. Let w be in Ω_2 and write $t = \log w$. We have

$$a_0 e^{(v_1 - v_2 + t)u} = \langle X z^u, \hat{k}_w^2 \rangle_{\mathscr{A}_2} = \langle z^u, X^* \hat{k}_w^2 \rangle_{\mathscr{A}_1}$$

where $\hat{k}_w^2$ is the kernel function for $\mathscr{A}_2$ at w. On applying axiom (iv)′ with $h = a_0 X^* \hat{k}_w^2$, we see that $\lambda_1 \lambda_2^{-1} w = e^{v_1 - v_2 + t}$ lies in Ω_1. This yields the inclusion $\lambda_1 \Omega_2 \subset \lambda_2 \Omega_1$; the reverse inclusion follows by symmetry. ▨

Let us return to general q and consider subnormality. For the remainder of this section we suppose that the hypotheses of Theorem 9 are in force, and that q is analytic at 0 with Taylor expansion $q(z) = q_0 + q_1 z + \ldots$, where $q_0 \neq 0$. In this case $\hat{k}(\bar{z}, z)$ has a simple pole at $z = 0$ and we may invoke Proposition 8, with $\alpha = 1$.

PROPOSITION 10. *Suppose that $\partial\Omega$ is normal to the real axis at 0. If $\{D_t\}_{t \geq 0}$ acting on $P^2(\partial\Omega, |q|^2 |dz|)$ is cosubnormal, then $q_1/q_0 \leq 0$ and the measures β_{-1} and β_0 are given by*

$$d\beta_{-1}(y) = \frac{1}{4\pi^2 |q_0|^2} \left| \Gamma\left(\frac{1}{2} + iy\right) \right|^2 dy$$

and

$$d\beta_0(y) = -\frac{1}{2\pi |q_0|^2} \left(\frac{q_1}{q_0}\right) d\delta_0(y) + \frac{\varkappa}{4\pi^2 |q_0|^2} |\Gamma(1 + iy)|^2 dy,$$

where δ_0 is a unit point mass at 0 and $\varkappa$ is the curvature of $\partial\Omega$ at 0.

Proof. We assume that $\{D_t^*\}_{t \geq 0}$ is subnormal. From our hypotheses we see that $\psi(z) = 1 + c_1 z + c_2 z^2 + \ldots$ with $c_1 < 0$. We want to recover $\hat{\beta}_{-1}(v)$ and $\hat{\beta}_0(v)$ from the expansion for $\hat{k}(re^{-v/2}, re^{v/2})$ in Proposition 8. Let us abbreviate $e^{v/2} = t$, $t + t^{-1} = \lambda$ and $\hat{\beta}_n(v) = \hat{\beta}_n$. On plugging in our formula for $\hat{k}$, we see that this expansion becomes

$$\overline{\psi'(r/t)}^{1/2} \psi'(rt)^{1/2} = 2\pi \overline{q(r/t)} q(rt)(1 - \overline{\psi(r/t)} \psi(rt)) \sum_{n=-1}^{\infty} \hat{\beta}_n r^n. \tag{23}$$

We calculate

$$\psi'(z)^{1/2} = i|c_1|^{1/2} - i|c_1|^{-1/2} c_2 z + \ldots$$

and so

$$\overline{\psi'(r/t)^{1/2}}\psi'(rt)^{1/2} = |c_1| + (c_2 t + \bar{c}_2 t^{-1})r + O(r^2).$$

Furthermore, we have

$$1 - \overline{\psi(r/t)}\psi(rt) = -[c_1(t + t^{-1}) + (c_2 t^2 + c_1^2 + \bar{c}_2 t^{-2})r^2 + O(r^3)].$$

We may temporarily assume (without affecting $|q|^2$) that $q_0 > 0$. We now multiply out low order terms in the right side of (23) and equate constant terms and r-terms on the left and right to find $\hat{\beta}_{-1}$ and $\hat{\beta}_0$. If we put $c_2 = a + ib$ and $q_1 = d + ie$, the result is $\hat{\beta}_{-1} = (2\pi q_0^2 \lambda)^{-1}$ and

$$\hat{\beta}_0 = \frac{1}{2\pi q_0^2}\left\{\left[\frac{2a - c_1^2}{c_1}\right]\frac{1}{\lambda^2} - \frac{d}{q_0} - \frac{ie}{q_0}\left[\frac{t - t^{-1}}{t + t^{-1}}\right]\right\}.$$

By [13, p. 39] we have

$$\frac{1}{\lambda^k} = \frac{1}{2\pi(k-1)!}\int_{-\infty}^{\infty} e^{-ivy}\left|\Gamma\left(\frac{k}{2} + iy\right)\right|^2 dy \tag{24}$$

for $k = 1, 2, 3, \ldots$, which tells the form of $\hat{\beta}_{-1}$ and part of $\hat{\beta}_0$.

Now consider the function S, defined by $S(v) = -1$ for $v < 0$, $S(0) = 0$ and $S(v) = 1$ for $v > 0$. Then $S(v) - \tanh\left(\frac{v}{2}\right) \in L^2(-\infty, \infty)$ and so $\hat{\beta}_0$ can be written as

$$\hat{\beta}_0(v) = C + DS(v) + f(v)$$

where C and D are constants and $f \in L^2(-\infty, \infty)$. It follows that $\hat{\beta}_0$ is the Fourier transform of the distribution

$$C d\delta_0(y) + \frac{iD}{\pi y}dy + h(y)dy,$$

where $h \in L^2(-\infty, \infty)$ and the middle term is taken in the principal value sense. This distribution is a finite measure only if $D = 0$, and thus $e = 0$. It follows from (24) that $\hat{\beta}_0$ is the Fourier transform of the real measure

$$d\beta_0(y) = \frac{1}{2\pi q_0^2}\left\{-\frac{d}{q_0}d\delta_0(y) + \frac{1}{2\pi}\left[\frac{2a - c_1^2}{c_1}\right]|\Gamma(1 + iy)|^2 dy\right\}.$$

The positivity of β_0 implies that $d/q_0 \leqslant 0$. Furthermore, $(2a - c_1^2)c_1^{-1}$ is exactly $\varkappa$, the curvature of $\partial\Omega$ at 0, which is automatically nonnegative. Indeed the standard formula for the curvature of a parametric curve yields

$$\varkappa = \left| \frac{\operatorname{Re}\psi''(0) - \psi'(0)^2}{\psi'(0)} \right| = \left| \frac{2a - c_1^2}{c_1} \right| ;$$

we may remove the absolute value signs as follows: for z near 1, $\tau(z)$ has a Taylor series expansion from which we find, for t small and real,

$$\operatorname{Re}\tau(e^{it}) = \tau'(1)(\cos t - 1) + \frac{\operatorname{Re}\tau''(1)}{2}(\cos(2t) - 2\cos t + 1) +$$

$$+ \frac{\operatorname{Im}\tau''(1)}{2}(2\sin t - \sin(2t)) + \dots =$$

$$= -\frac{1}{2}(\tau'(1) + \operatorname{Re}\tau''(1))t^2 + O(t^3).$$

Since $\Omega \cup \{0\}$ is starlike about $0, \Omega$ lies in the right half-plane. It follows that $\tau'(1) + \operatorname{Re}\tau''(1) \leqslant 0$, that is,

$$\frac{\operatorname{Re}\psi''(0) - \psi'(0)^2}{\psi'(0)} \geqslant 0$$

as desired.

Finally, we may relax our assumption that $q_0 > 0$, which replaces q_0^2 by $|q_0|^2$, d/q_0 by $\operatorname{Re}(q_1/q_0)$, and the condition $e = 0$ by $\operatorname{Im}(q_1/q_0) = 0$. ▨

As in the introduction we let $\Omega_0 = \{z : |z - 1| < 1\}$. We conclude this section with a characterization of those measures $|q|^2|dz|$ for which $\{D_t\}_{t\geqslant 0}$, acting on $P^2(\partial\Omega_0, |q|^2|dz|)$, is cosubnormal. Note that our hypotheses on q take the form: $|q(z)|^2$ is bounded above and below in Ω_0, and q is analytic at 0.

THEOREM 10. *If q is as above, then $\{D_t^*\}_{t\geqslant 0}$ and A_1^* are subnormal on $P^2(\partial\Omega_0$, $|q|^2|dz|)$ if and only if $|q(z)|^2|dz|$ is a multiple of harmonic measure $d\omega_\rho$ on $\partial\Omega_0$ for some point ρ with $0 < \rho \leqslant 1$.*

Proof. First suppose that $|q|^2|dz| = d\omega_\rho$ for some $\rho, 0 < \rho \leqslant 1$; if $\rho = 1$, then $|q|^2 = \dfrac{1}{2\pi}$ and we know that then $A_1^* \cong C_0$, which is subnormal. If we require $q_0 > 0$, we have

$$q(z) = \left[\frac{\rho(2-\rho)}{2\pi}\right]^{1/2} \frac{1}{\rho + (1-\rho)z} .$$

We may take $\psi(z) = 1 - z$, and therefore by (20)

$$\hat{k}(w, z) = C\frac{[\rho + (1-\rho)\bar{w}][\rho + (1-\rho)z]}{\bar{w} + z - \bar{w}z}$$

for appropriate $C > 0$. For the purpose of this calculation, let us take $C = 1$, which multiplies $|q|^2|dz|$ by a constant. With $t = e^{i\theta/2}$ and $\lambda = t + t^{-1}$ we have

$$\hat{k}(r/t, rt) = \frac{\rho^2 + \rho(1-\rho)\lambda r + (1-\rho)^2 r^2}{\lambda r - r^2} =$$

$$= \left[\frac{\rho^2}{r} + \rho(1-\rho)\lambda + (1-\rho)^2 r\right]\sum_{n=0}^{\infty}\frac{1}{\lambda^{n+1}}r^n =$$

$$= \frac{\rho^2}{\lambda}r^{-1} + \left[\rho(1-\rho) + \frac{\rho^2}{\lambda^2}\right] + \sum_{n=1}^{\infty}\left[\frac{\rho^2}{\lambda^{n+2}} + \frac{1-\rho}{\lambda^n}\right]r^n.$$

Therefore we read off

$$\hat{\beta}_{-1} = \frac{\rho^2}{\lambda}, \quad \hat{\beta}_0 = \rho(1-\rho) + \frac{\rho^2}{\lambda^2}, \quad \hat{\beta}_n = \frac{\rho^2}{\lambda^{n+2}} + \frac{1-\rho}{\lambda^n}, \qquad n \geqslant 1.$$

That these are truly Fourier transforms of positive measures β_n is immediate from (24). It follows that A_1^* is subnormal on $P^2(\partial\Omega_0, d\omega_\rho)$.

Conversely, let us suppose that A_1^* is subnormal on $P^2(\partial\Omega_0, |q|^2|dz|)$. We may assume that $q_0 > 0$, and then Proposition 10 tells us that $q_1 \leqslant 0$. Our equation (23) becomes

$$1 = 2\pi\overline{q(r/t)}q(rt)(r\lambda - r^2)\sum_{n=-1}^{\infty}\hat{\beta}_n r^n. \tag{25}$$

We will see that merely the boundedness of each $\hat{\beta}_n$, for $n \geqslant 1$, is enough to determine the form of q.

We may write

$$2\pi\overline{q(r/t)}q(rt) = \sum_{m=0}^{\infty} Q_m r^m$$

where

$$Q_m = 2\pi\sum_{j=0}^{m} q_j\overline{q_{m-j}}t^{2j-m}.$$

Furthermore,

$$(\lambda r - r^2) \sum_{n=-1}^{\infty} \hat{\beta}_n r^n = \sum_{n=0}^{\infty} D_n r^n$$

where $D_0 = \lambda\hat{\beta}_{-1}$ and $D_n = \lambda\hat{\beta}_{n-1} - \hat{\beta}_{n-2}$ for $n \geqslant 1$. Let us now agree to normalize q so that $2\pi q_0^2 = 1$, that is, $Q_0 = 1$. Equation (25) becomes

$$1 = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} D_k Q_{n-k} \right) r^n;$$

for $n \geqslant 1$ the coefficient of r^n is of course zero, and by our formulas for D_0, D_1, $\hat{\beta}_{-1}$ and $\hat{\beta}_0$ (from Proposition 10) we have

$$\hat{\beta}_{n-1} = \frac{1}{\lambda}\hat{\beta}_{n-2} - \frac{1}{\lambda}\sum_{k=0}^{n-1} D_k Q_{n-k} =$$

$$= \frac{1}{\lambda}\left\{\hat{\beta}_{n-2} - Q_n - (\lambda\hat{\beta}_0 - \hat{\beta}_{-1})Q_{n-1} - \sum_{k=2}^{n-1} (\lambda\hat{\beta}_{k-1} - \hat{\beta}_{k-2})Q_{n-k}\right\}.$$

Now

$$Q_m = 2\pi q_m q_0 t^m + O(t^{m-2})$$

as $v \to +\infty$, and so for $n \geqslant 2$,

$$\hat{\beta}_{n-1} = 2\pi(q_{n-1}q_1 - q_n q_0)t^{n-1} + O(t^{n-2})$$

as $v \to +\infty$. Since $\hat{\beta}_{n-1}$ must be bounded we have $q_{n-1}q_1 = q_n q_0$, or

$$q_n = \left(\frac{q_1}{q_0}\right)^{n-1} q_1, \quad n = 2, 3, 4, \dots .$$

Let us put $\rho = q_0(q_0 - q_1)^{-1}$; we have $0 < \rho \leqslant 1$ since $q_1 \leqslant 0$. Moreover, $q_1/q_0 = (\rho - 1)/\rho$, so

$$q(z) = q_0 + q_1 z + \sum_{n=2}^{\infty} \left(\frac{q_1}{q_0}\right)^{n-1} q_1 z^n = \frac{q_0\rho}{\rho + (1-\rho)z},$$

and $|q|^2|dz|$ is a constant multiple of $d\omega_\rho$ as desired. ▨

Suppose that in Theorem 10, $|q(z)|^2|dz| = d\omega_\rho$ with $0 < \rho \leqslant 1$. Then $\{e^{-t/2}D_t^*\}_{t\geqslant 0}$ and $I - A_1^*$ are unitarily equivalent to $\{V_t^\mu\}_{t\geqslant 0}$ and

$(M_\mu + 1/3)(I + (1/3)M_\mu)^{-1}$ respectively, where μ is as described in Propositions 7 and 8 with

$$d\beta_{-1}(y) = \rho^2 \frac{C}{2\pi} \left| \Gamma\left(\frac{1}{2} + iy\right) \right|^2 dy,$$

$$d\beta_0(y) = C\rho(1-\rho)d\delta_0(y) + \rho^2 \frac{C}{2\pi} |\Gamma(1+iy)|^2 dy,$$

$$d\beta_n(y) = \frac{C}{2\pi(n-1)!} \left[\frac{\rho^2}{n(n+1)} \left(\frac{n^2}{4} + y^2 \right) + (1-\rho) \right] \left| \Gamma\left(\frac{n}{2} + iy\right) \right|^2 dy,$$

for $n \geqslant 1$, where C is a positive constant. This follows from the expression for $\hat{k}(r/t, rt)$ in the proof together with (24).

4.1.2. THE BERGMAN SPACE ON Ω. Let Ω be a Jordan domain containing $(0, 1]$ and with $0 \in \partial\Omega$. We let $A^2(\Omega)$ denote the Bergman space of all functions f analytic on Ω with

$$\|f\|^2 = \int_\Omega |f|^2 \, dA < \infty,$$

where dA denotes area measure.

THEOREM 11. *If $\Omega \cup \{0\}$ is starlike about* 0, *then $A^2(\Omega)$ is admissible with $\alpha = 2$. In this case $\{e^{-t}D_t\}_{t\geqslant 0}$ defines a contraction semigroup on $A^2(\Omega)$ whose cogenerator is $I - 2A_2$.*

Proof. Since Ω is a Jordan domain, the polynomials span $A^2(\Omega)$ and no point outside of Ω defines a bounded point evaluation on the polynomials in $A^2(\Omega)$, see [3]. Thus axioms (i)′—(iv)′ hold. A simple change of variables yields the inequality $\|D_t f\| \leqslant e^t \|f\|$, valid for every f in $A^2(\Omega)$, so axiom (v)′ holds as well. The assertion about the cogenerator follows from Proposition 1 and general remarks beginning § 4.1. ▨

We will see in the next section that A_2^* and $\{D_t^*\}_{t\geqslant 0}$ are subnormal on $A^2(\Omega_0)$, where $\Omega_0 = \{z : |z-1| < 1\}$. A characterization of all Ω which yield subnormal $\{D_t^*\}_{t\geqslant 0}$ will be the subject of a separate article [22]. The associated space

$$\mathscr{B} = \{e^{-\zeta} f(e^{-\zeta}) : f \in A^2(\Omega)\}$$

is also of interest; it is precisely $A^2(\Lambda)$, where $\Lambda = -\log\Omega$. Of course $e^{-t}D_t$ on $A^2(\Omega)$ corresponds to S_t on $A^2(\Lambda)$. In § 4.2 we will study $A^2(\Lambda)$ for domain sets Λ which do not necessarily arise as $-\log\Omega$.

4.1.3. Weighted Bergman spaces on Ω_0 and operators of Kay, Soul, and Trutt. For $\alpha > 0$ we consider the Hilbert space $\mathscr{A}_\alpha$ of functions analytic on $\Omega_0 = \{z : |z-1| < 1\}$ whose reproducing kernel is

$$\hat{k}^\alpha(w, z) = \frac{1}{(\bar{w} + z - \bar{w}z)^\alpha}.$$

The change of variables $z \to 1 - z$ will take us to the more usual version of this space, based on the disk $\mathbf{D}$, where the kernel function has the form $(1 - \bar{w}z)^{-\alpha}$. We can write any f in $\mathscr{A}_\alpha$ as $f(z) = \sum_0^\infty a_n e_n(z)$, where $e_n(z) = (1-z)^n$, $n = 0, 1, 2, \ldots$. One checks that $\sum a_n e_n$ lies in $\mathscr{A}_\alpha$ if and only if

$$\|f\|_\alpha^2 \equiv \sum_0^\infty p_n^\alpha |a_n|^2 < \infty,$$

where $p_0^\alpha = 1$ and

$$p_n^\alpha = (-1)^n \binom{-\alpha}{n}^{-1} = \frac{n!}{\alpha(\alpha+1)\ldots(\alpha+n-1)}, \quad n \geqslant 1.$$

The inner product in $\mathscr{A}_\alpha$ is of course given by

$$\langle f, g\rangle_\alpha = \sum_{n=0}^\infty p_n^\alpha a_n \overline{b_n}$$

where $f = \sum a_n e_n$, $g = \sum b_n e_n$. We have the expansion

$$\hat{k}^\alpha(w, z) = \sum_{n=0}^\infty \frac{1}{p_n^\alpha} \overline{e_n(w)} e_n(z).$$

We see that $\mathscr{A}_1 = P^2\left(\partial\Omega_0, \frac{1}{2\pi}|\mathrm{d}z|\right)$, the translate of H^2 from $\mathbf{D}$ to Ω_0, while for $\alpha > 1$,

$$\|f\|_\alpha^2 = \frac{\alpha - 1}{\pi} \int_{\Omega_0} |f(z)|^2 (1 - |1-z|^2)^{\alpha-2} \mathrm{d}A(z).$$

Of course $\mathscr{A}_2 = A^2(\Omega_0)$ with norm divided by π. When $0 < \alpha < 1$, $\|f\|_\alpha$ and $(|f(0)|^2 + \|f'\|_{\alpha+2}^2)^{1/2}$ define equivalent (but not equal) norms on $\mathscr{A}_\alpha$. Clearly the space $\mathscr{A}_\alpha$ satisfies (i)′—(iv)′. It also satisfies the semigroup axiom (v)′ (with parameter α); for $\alpha > 1$ this can be seen by applying the proof of Proposition 3.4 of

MacCluer and Shapiro [25]. However, we will establish this, and more, via the following identity:

$$1 + \sum_{k=1}^{\infty} \binom{s}{k}\binom{t}{k} \frac{k!}{\alpha(\alpha+1)\dots(\alpha+k-1)} = F(-s, -t, \alpha, 1)$$

$$= \frac{\Gamma(\alpha)\Gamma(\alpha+s+t)}{\Gamma(\alpha+s)\Gamma(\alpha+t)}$$

where F denotes the hypergeometric function, $\alpha > 0$ and $s, t \geqslant 0$, see p. 56 and 104 of [12]. For $s \geqslant 0$ we have

$$z^s = (1-(1-z))^s = \sum_{k=0}^{\infty} (-1)^k \binom{s}{k} e_k(z).$$

Thus, if m and n are nonnegative integers and

$$a_n = \frac{\Gamma(\alpha)^{1/2}\Gamma\left(\frac{\alpha+1}{2}+n\right)}{\Gamma(\alpha+n)}$$

we have

$$\langle z^m, z^n\rangle_\alpha = \sum_{k=0}^{\infty}\binom{m}{k}\binom{n}{k} p_k^\alpha = \frac{\Gamma(\alpha)\Gamma(\alpha+m+n)}{\Gamma(\alpha+m)\Gamma(\alpha+n)} =$$

$$= a_m a_n \frac{\Gamma\left(1+\left[m+\frac{\alpha-1}{2}\right]+\left[n+\frac{\alpha-1}{2}\right]\right)}{\Gamma\left(1+m+\frac{\alpha-1}{2}\right)\Gamma\left(1+n+\frac{\alpha-1}{2}\right)} =$$

$$= a_m a_n \sum_{k=0}^{\infty}\binom{m+\frac{\alpha-1}{2}}{k}\binom{n+\frac{\alpha-1}{2}}{k} p_k^1 = a_m a_n \langle z^{m+\frac{\alpha-1}{2}}, z^{n+\frac{\alpha-1}{2}}\rangle_1.$$

Now $z^{(\alpha-1)/2}$ is an outer function in $\mathscr{A}_1 = P^2\left(\partial\Omega_0, \frac{1}{2\pi}|dz|\right)$ and therefore $\{z^{n+(\alpha-1)/2}\}_{n=0}^{\infty}$ spans $\mathscr{A}_1$. We thus have a unitary operator $Y: \mathscr{A}_\alpha \to \mathscr{A}_1$ determined by the action

$$Y: z^n \to a_n z^{n+(\alpha-1)/2}, \quad n = 0, 1, 2, \dots .$$

What about the operators D_t and A_θ? We calculate that $e^{-(\alpha/2)t} Y D_t = e^{-t/2} D_t Y$

and $YA_v = A_{v+(1-\alpha)/2}Y$, $\operatorname{Re} v > \alpha/2$, by checking these equations on $\{z^n\}_{n=0}^{\infty}$. On putting this together with our knowledge of A_v and D_t acting on $\mathscr{A}_1$, we may state:

THEOREM 12. *For $\alpha > 0$, the space $\mathscr{A}_\alpha$ is admissible with parameter α; moreover, $\{e^{-(\alpha/2)t}D_t\}_{t\geqslant 0}$ is a cosubnormal contraction semigroup on $\mathscr{A}_\alpha$, unitarily equivalent to $\{e^{-t/2}D_t\}_{t\geqslant 0}$ acting on $P^2(\partial\Omega_0, |dz|)$. A_v acting on $\mathscr{A}_\alpha$ is unitarily equivalent to $A_{v+(1-\alpha)/2}$ on $P^2(\partial\Omega_0, |dz|)$; in particular $A_{(1+\alpha)/2}$ acting on $\mathscr{A}_\alpha$ is unitarily equivalent to C_0^*.*

In [20], Kay, Soul, and Trutt introduced the weighted averaging operators $\mathscr{C}_\alpha$ on ℓ^2 defined by $\mathscr{C}_\alpha\{a_n\} = \{b_n\}$, where

$$b_n = \frac{1}{(n+1)d_n}\sum_{k=0}^{n} a_k d_k \quad \text{with } d_k^2 = \frac{\Gamma(\alpha)\Gamma(k+1)}{\Gamma(\alpha+k)};$$

here $0 < \alpha \leqslant 1$. They showed that $\|\mathscr{C}_\alpha\| \leqslant 2$, computed the point spectrum and found that $\mathscr{C}_\alpha$ is subnormal. The choice $\alpha = 1$ yields C_0. A comparison of our work and their setup shows that $\mathscr{C}_\alpha$ is unitarily equivalent to a linear fractional map of A_α^* acting on $\mathscr{A}_\alpha$. Though Kay, Soul, and Trutt never introduce the space $\mathscr{A}_\alpha$, they do use the kernel $\hat{k}^\alpha$, and their proof of subnormality involves producing the measure that goes with $\hat{k}^\alpha$ as in Proposition 7. Using our unitary map Y and the discussion at the beginning of § 4.1, we can go further.

COROLLARY 3. *For each α, $0 < \alpha < 1$, there is a linear fractional map g_α such that $\mathscr{C}_\alpha \cong g_\alpha(C_0)$.*

4.2. TRANSLATION INVARIANT BERGMAN SPACES. Let Λ be a right-translation invariant domain in $\mathbf{C}$ containing $[0, \infty)$. In this section we investigate the Bergman space $\mathscr{B} = A^2(\Lambda)$, on which $\{S_t\}_{t\geqslant 0}$ acts as a contraction semigroup. The Bergman spaces $A^2(\Omega)$ of § 4.1.2 can, of course, be put in this form: if $\Lambda = -\log\Omega$, then $W: f(z) \to e^{-\zeta}f(e^{-\zeta})$ is a unitary operator from $A^2(\Omega)$ onto $A^2(\Lambda)$ with $e^{-t}WD_t = S_tW$. However, we will be interested in examples for which Λ does not arise as $-\log\Omega$.

The study of $A^2(\Lambda)$ and $\{S_t\}_{t\geqslant 0}$ for general Λ seems to us of great interest. For the moment we will see what can be said when we impose a few additiona- conditions on Λ to bring $A^2(\Lambda)$ within the purview of our theory. We assume:

(A) $\partial\Lambda$ is a simple curve.

(B) Λ is contained in some sector $\mathscr{S}$ of the form

$$\mathscr{S} = \{re^{i\theta} - c : -\gamma < \theta < \gamma \text{ and } r > 0\}$$

where $c > 0$ and $0 < \gamma < \pi/2$.

(C) There is a conformal map $\varphi: \Lambda \to \mathbf{D}$ such that $\varphi(x) \to 1$ nontangentially as x tends to $+\infty$ along $[0, \infty)$.

We note for future reference that the kernel function for $A^2(\Lambda)$ has the form

$$k(w, z) = \frac{1}{\pi} \overline{\varphi'(w)}\varphi'(z) \frac{1}{(1 - \overline{\varphi(w)}\varphi(z))^2}.$$

Our axioms are trivially satisfied except for (iii) and (vi). We therefore need only show that $A^2(\Lambda)$ is spanned by the exponentials it contains and that the maximal set of bounded point evaluations for those exponentials is exactly Λ. We proceed via several lemmas. Our condition (B) implies that $A^2(\Lambda)$ contains $e^{-t\zeta}$ whenever $t > 0$. Let us write $\sigma = \varphi^{-1} : \mathbf{D} \to \Lambda$; σ extends to a homeomorphism of $\bar{\mathbf{D}} \setminus \{1\}$ onto $\bar{\Lambda}$. Note that there is a natural unitary operator $Q: A^2(\Lambda) \to A^2(\mathbf{D})$ given by $Qf = (f \circ \sigma)\sigma'$, with $Q^{-1}g = (g \circ \varphi)\varphi'$.

LEMMA 4. *Suppose that* $\operatorname{Re} w < -c$, $n \geqslant 0$ *and* $g \in A^2(\mathbf{D})$. *Then* $t^n e^{-t(\sigma - w)}\sigma' g$ *is integrable over* $[1, \infty) \times \mathbf{D}$ *with respect to* $dt \times dA$.

Proof. We may write $g = (f \circ \sigma)\sigma'$ for some f in $A^2(\Lambda)$ and restate the assertion as: $t^n e^{-t(\zeta - w)} f$ is $dt \times dA$-integrable over $[1, \infty) \times \Lambda$. We have

$$\int_{\Lambda} |e^{-t(\zeta - w)} f| dA \leqslant \left[\int_{\Lambda} |e^{-t(\zeta - w)}|^2 dA\right]^{1/2} \left[\int_{\Lambda} |f|^2 dA\right]^{1/2} \leqslant$$

$$\leqslant \|f\| \left[\int_{\mathscr{S}} e^{-2t(x - \operatorname{Re} w)} dy dx\right]^{1/2} = \|f\| \left(\frac{\tan \gamma}{2}\right)^{1/2} \frac{e^{t(\operatorname{Re} w + c)}}{t}.$$

Now multiply by t^n, integrate over $[1, \infty)$ and apply Tonelli's Theorem. ▨

PROPOSITION 11. *If* v *in* $\mathbf{C}$ *and* h *in* $A^2(\Lambda)$ *are such that* $\langle e^{u\zeta}, h \rangle = e^{uv}$ *for all exponentials* $e^{u\zeta}$ *in* $A^2(\Lambda)$, *then* $v \in \Lambda$.

Proof. Suppose that $h \in A^2(\Lambda)$ and v is a complex number such that $e^{-tv} = \langle e^{-t\zeta}, h \rangle$ for all $t > 0$. We want to show that $v \in \Lambda$. Let us write the inner product as an integral over $\mathbf{D}$, via Q:

$$e^{-tv} = \int_{\mathbf{D}} e^{-t\sigma} \overline{h \circ \sigma}\, |\sigma'|^2 dA.$$

Now choose ε with $\varepsilon > c$ and $\varepsilon > -\operatorname{Re} v$. If $n \geqslant 1$, we may multiply by $(t - 1)^{n-1} e^{-\varepsilon t}$ and employ Lemma 4 and Fubini's theorem to integrate over $[1, \infty)$, yielding

$$e^{-v}\left(\frac{1}{v + \varepsilon}\right)^n = \int_{\mathbf{D}} \left(\frac{1}{\sigma + \varepsilon}\right)^n e^{-\sigma} \overline{h \circ \sigma}\, |\sigma'|^2 dA \tag{26}$$

for $n = 0, 1, 2, \ldots$. Now the function $\Psi = \dfrac{1}{\sigma + \varepsilon}$ maps $\mathbf{D}$ onto a Jordan domain G and extends to a homeomorphism of $\bar{\mathbf{D}}$ onto $\bar{G}$. Given any $m = 0, 1, \ldots$ we may choose, by the theorem of Carathéodory and Walsh [33], a sequence of polynomials p_k with $p_k \to (\Psi^{-1})^m$ uniformly on $\bar{G}$; if $v \notin \bar{\Lambda}$ $\left(\text{so that } \dfrac{1}{v + \varepsilon} \notin \bar{G}\right)$, we may simultaneously arrange that $p_k\left(\dfrac{1}{v + \varepsilon}\right) = 1$. From (26) we see that

$$e^{-v} p_k\left(\frac{1}{v + \varepsilon}\right) = \int_{\mathbf{D}} (p_k \circ \Psi) e^{-\sigma} \overline{h \circ \sigma} |\sigma'|^2 \mathrm{d}A;$$

note that $e^{-\sigma} \overline{h \circ \sigma} |\sigma'|^2 \in L^1(\mathbf{D}, \mathrm{d}A)$. Now we define λ in $\bar{\mathbf{D}}$ by this: $\Psi(\lambda) = \dfrac{1}{v + \varepsilon}$ if $v \in \bar{\Lambda}$ and otherwise $\lambda = 1$. In either case letting $k \to \infty$ in our equation yields

$$e^{-v} \lambda^m = \int_{\mathbf{D}} z^m e^{-\sigma} \overline{h \circ \sigma} |\sigma'|^2 \mathrm{d}A$$

for $m = 0, 1, 2, \ldots$. On letting $m \to \infty$, we see that $|\lambda| < 1$, that is $v \in \Lambda$. ▨

LEMMA 5. $e^{-\sigma}$ *is a bounded outer function on* $\mathbf{D}$.

Proof. The function $e^{-\sigma}$ is bounded since $\Lambda \subset \mathscr{S}$. If $e^{-\sigma}$ has an inner factor, it must have the form E_s for some $s > 0$ since 1 is the only point in $\partial\mathbf{D}$ which σ maps to ∞. Let $\mathscr{C}$ denote the image of $[0, \infty)$ under φ. Then for z in $\mathscr{C}$ we have

$$e^{-\sigma(z)} \leqslant \|e^{-\sigma}\|_\infty e^{-s\frac{1-|z|^2}{|1-z|^2}};$$

since $\mathscr{C}$ tends to 1 nontangentially,

$$\sigma(z) \geqslant \delta(1 - |z|)^{-1}, \quad z \in \mathscr{C}$$

for some $\delta > 0$. Now $(\sigma + c)^{\frac{\pi}{2\gamma}}$ has positive real part in $\mathbf{D}$, and since σ is real on $\mathscr{C}$, there exists $D > 0$ with

$$(\sigma(z) + c)^{\frac{\pi}{2\gamma}} \leqslant D(1 - |z|)^{-1}, \quad z \in \mathscr{C}.$$

Since $\dfrac{\pi}{2\gamma} > 1$, this is incompatible with our lower bound on $\sigma(z)$, a contradiction. ▨

LEMMA 6. *σ' is an outer function lying in H^p whenever $0 < p < 1/3$.*

Proof. Since Λ is translation invariant, $\mathbf{C}\setminus\Lambda$ is a union of closed half-lines whose relative interiors are disjoint. It follows from a theorem of Lewandowski [11] that σ is a close-to-convex univalent function, that is, there exists a conformal map g of $\mathbf{D}$ onto a convex region such that $\mathrm{Re}\,\frac{\sigma'(z)}{g'(z)} > 0$ on $\mathbf{D}$. Now a theorem of Clunie, Duren, and Leung [11] implies that $\sigma' \in H^p$ for any $p < 1/3$; we need to argue that σ' has no inner factor. Since g is convex, a theorem of Alexander [11] implies that zg' is univalent. However, a theorem of Lohwater and Ryan [10] then implies that g' (which lies in H^p for the same reason as σ') has no inner factor. It follows that $h \equiv \sigma'/g'$ lies in the Smirnov class N^+ [10]; since $\mathrm{Re}\,h > 0$, h itself is outer, hence $\sigma' = hg'$ is outer as well.

THEOREM 13. *$A^2(\Lambda)$ is spanned by those exponentials which it contains.*

Proof. It will be enough to show that $\{e^{-t\zeta} : t \geqslant 1\}$ spans $A^2(\Lambda)$, or equivalently, that $Qe^{-t\zeta} = e^{-t\sigma}\sigma'$, $t \geqslant 1$, spans $A^2(\mathbf{D})$. Suppose then that $f \in A^2(\mathbf{D})$ and

$$0 = \int_{\mathbf{D}} e^{-t\sigma}\sigma'\bar{f}\,\mathrm{d}A, \quad t \geqslant 1.$$

As in the proof of Proposition 11, we choose $\varepsilon > c$, multiply by $(t-1)^{n-1}e^{-\varepsilon t}$, integrate over $[1,\infty)$, and use the Carathéodory-Walsh theorem in the same way to conclude that

$$0 = \int_{\mathbf{D}} z^m e^{-\sigma}\sigma'\bar{f}\,\mathrm{d}A, \quad m = 0, 1, 2, \ldots .$$

Now Lemmas 5 and 6 imply that $e^{-\sigma}\sigma'$ is an outer function in $N^+ \cap A^2(\mathbf{D})$, and therefore, by recent results of Berman, Brown, and Cohn [1], $\{z^m e^{-\sigma}\sigma'\}_{m=0}^{\infty}$ spans $A^2(\mathbf{D})$. It follows that $f = 0$, as desired. ▨

COROLLARY 4. *The space $A^2(\Lambda)$ is admissible.*

We turn to an example. Fix an integer $n \geqslant 2$, a real number θ with $|\theta| < \frac{\pi}{2n}$, and let Λ be the sector

$$\Lambda = \left\{ re^{it} - 1 : \theta - \frac{\pi}{2n} < t < \theta + \frac{\pi}{2n} \text{ and } r > 0 \right\}. \tag{27}$$

We will study $\mathscr{B} = A^2(\Delta)$. One easily checks that

$$\Gamma = \left\{ s + it : s < 0, \frac{1}{\beta} s < t < \frac{1}{\alpha} s \right\}$$

where $\alpha = \tan\left(\theta - \frac{\pi}{2n}\right)$ and $\beta = \tan\left(\theta + \frac{\pi}{2n}\right)$, and therefore $\Delta = \{\bar{z} : H(z) \in \Gamma\}$ is the intersection of two disks with centers on the imaginary axis (one below $-\mathrm{i}$ and one above i) whose boundary circles pass through -1 and 1.

THEOREM 14. *If Δ is given by (27), then $\{S_t^*\}_{t \geqslant 0}$ and L^* are subnormal and the associated measure μ is*

$$\mathrm{d}\mu(z) = \frac{1 - |z|^2}{|1 - z|^4} \mathrm{e}^{-2\frac{1 - |z|^2}{|1 - z|^2}} \sum_{k=1}^{n} C_k \mathrm{d}s_k ,$$

where each C_k is a positive number and $\mathrm{d}s_k$ is arclength measure on a circular arc (or line segment) Σ_k with endpoints -1 and 1, and passing through the imaginary axis at

$$\mathrm{i} \frac{\cos\left(\frac{2k - 1}{2n} \pi - \theta\right)}{1 + \sin\left(\frac{2k - 1}{2n} \pi - \theta\right)} .$$

The top and bottom arcs, Σ_1 and Σ_n respectively, together comprise $\partial\Delta$.

Proof. We will use Theorem 3 as expressed in equation (10) to find μ. If we put $\lambda = \mathrm{e}^{\mathrm{i}\theta}$, our conformal map $\sigma : \mathbf{D} \to \Delta$ takes the form

$$\sigma(z) = \lambda \left(\frac{1 + z}{1 - z} \right)^{1/n} - 1, \quad z \in \mathbf{D},$$

and so

$$\varphi(\zeta) = \frac{(\zeta + 1)^n - \lambda^n}{(\zeta + 1)^n + \lambda^n}, \quad \zeta \in \Delta.$$

From the form of the kernel function for $A^2(\Delta)$ we readily compute

$$k(w, z) = \frac{n^2}{\pi} \frac{(\bar{w} + 1)^{n-1}(z + 1)^{n-1}}{[\lambda^n(\bar{w} + 1)^n + \bar{\lambda}^n(z + 1)^n]^2} .$$

We want to use equation (10), so we calculate that

$$k\left(\frac{u+v}{2}, \frac{u-v}{2}\right) = \frac{4n^2}{\pi} \frac{(u+v+2)^{n-1}(u-v+2)^{n-1}}{[\lambda^n(u+v+2)^n + \bar{\lambda}^n(u-v+2)^n]^2} .$$

We will find $\hat{\gamma}_x(v)\mathrm{d}m(x)$ in the right side of (10). If we make the change of variable $p = u + 2$, equation (10) becomes

$$\text{(28)} \qquad \frac{4n^2}{\pi} \frac{(p^2 - v^2)^{n-1}}{[\lambda^n(p+v)^n + \bar{\lambda}^n(p-v)^n]^2} = \int_0^\infty e^{-px}\hat{\gamma}_x(v)e^{2x}\mathrm{d}m(x).$$

The denominator, as a function of p, has double roots at those p for which $(p+v)(p-v)^{-1} = \bar{\lambda}^2\omega_k$, $k = 1, 2, \ldots, n$, where $\omega_1, \ldots, \omega_n$ are the n^{th} roots of -1. In other words, the roots are the imaginary numbers $\mathrm{i}vr_k$, $k = 1, 2, \ldots, n$ where

$$\mathrm{i}r_k = H(\bar{\lambda}^2\omega_k) = \frac{\omega_k + \lambda^2}{\omega_k - \lambda^2}.$$

Note that $\omega_k - \lambda^2 \neq 0$, for if $\lambda^2 = \omega_k$, then $\lambda^{2n} = -1$, contradicting $|\theta| < \dfrac{\pi}{2n}$.

Factoring the denominator in (28) yields

$$\frac{n^2}{\pi(\cos n\theta)^2} \frac{(p^2 - v^2)^{n-1}}{\prod_{j=1}^{n} (p - \mathrm{i}vr_j)^2} = \int_0^\infty e^{-px}\hat{\gamma}_x(v)e^{2x}\mathrm{d}m(x).$$

Since $\hat{\gamma}_x(0) = 1$, we have

$$\left(\frac{n^2}{\pi(\cos n\theta)^2}\right)\frac{1}{p^2} = \int_0^\infty e^{-px}e^{2x}\mathrm{d}m(x)$$

from which we find

$$e^{2x}\mathrm{d}m(x) = \frac{n^2}{\pi(\cos n\theta)^2}\, x\,\mathrm{d}x.$$

Now let $v \neq 0$. For each $k = 1, 2, \ldots, n$ we put

$$F_k(p) = \frac{(p^2 - v^2)^{n-1}}{\prod_{j \neq k} (p - \mathrm{i}vr_j)^2}.$$

From Laplace transform tables we find that

$$\hat{\gamma}_x(v)x = \sum_{k=1}^{n} (F_k(\mathrm{i}vr_k)x + F_k'(\mathrm{i}vr_k))\mathrm{e}^{\mathrm{i}vr_k x}. \tag{29}$$

One checks that

$$A_k \equiv F_k(\mathrm{i}vr_k) = \frac{(1 + r_k^2)^{n-1}}{\prod_{j \neq k} (r_j - r_k)^2},$$

a quantity both positive and independent of v. Logarithmic differentiation yields

$$F_k'(\mathrm{i}vr_k) = A_k \frac{2\mathrm{i}}{v} \left[(1 - n) \frac{r_k}{r_k^2 + 1} + \sum_{j \neq k} \frac{1}{r_k - r_j} \right].$$

We wish to show that this quantity vanishes, which is trivially equivalent to the following statement about $\{\omega_1, \ldots, \omega_n\}$; we are indebted to Doug Costa for supplying the proof.

COSTA'S LEMMA. *For each* $k = 1, 2, \ldots, n$,

$$\sum_{j \neq k} \frac{1 - \bar{\lambda}^2 \omega_j}{\omega_k - \omega_j} = \left(\frac{n-1}{2} \right) \frac{1 + \bar{\lambda}^2 \omega_k}{\omega_k}.$$

Proof. We put $\omega = \mathrm{e}^{\frac{2\pi \mathrm{i}}{n}}$ and observe that for k fixed, $\{\omega_1, \ldots, \omega_n\} = \{\omega_k \omega^j : j = 0, \ldots, n-1\}$. Thus we must prove

$$\sum_{j=1}^{n-1} \frac{1 - z\omega^j}{1 - \omega^j} = \frac{1}{2}(n-1)(z+1),$$

where $z = \bar{\lambda}^2 \omega_k$. Since $\omega^n = 1$, the sum on the left can be rewritten as

$$\sum_{j=1}^{n-1} \frac{\omega^j - z}{\omega^j - 1} = (n-1) + (z-1) \sum_{j=1}^{n-1} \frac{1}{1 - \omega^j},$$

so our task reduces to showing that

$$\sum_{j=1}^{n-1} \frac{1}{1 - \omega^j} = \frac{n-1}{2}.$$

This, however, follows from the identity $(1 - \zeta)^{-1} + (1 - \bar{\zeta})^{-1} = 1$, valid for $|\zeta| = 1$, $\zeta \neq 1$. ▨

Returning to the theorem, we have $F_k'(\mathrm{i}vr_k) = 0$ by the Lemma, and putting $t_k = -r_k$ we see from (29) that

$$\hat{\gamma}_x(v) = \sum_{k=1}^{n} A_k e^{-\mathrm{i}vt_k x}, \quad v \neq 0. \tag{30}$$

We need to know that this formula agrees with $\hat{\gamma}_x(0) = 1$ when $v = 0$. But we now know that

$$\frac{(p^2 - v^2)^{n-1}}{\prod_{j=1}^{n} (p + \mathrm{i}vt_j)^2} = \int_0^\infty e^{-px} \left(\sum_{k=1}^{n} A_k e^{-\mathrm{i}vt_k x} \right) x \, dx$$

for $v \neq 0$. Both sides are clearly continuous functions of v, so we may take $v \to 0$ to get

$$\frac{1}{p^2} = \left(\sum_{k=1}^{n} A_k \right) \int_0^\infty e^{-px} x \, dx.$$

Hence $\sum_{k=1}^{n} A_k = 1$ and the formula (30) holds for all v. It follows that $\hat{\gamma}_x$ is the Fourier transform of the measure

$$d\gamma_x(y) = \sum_{k=1}^{n} A_k d\delta_{xt_k}(y)$$

where δ_{xt_k} is a unit point mass at xt_k. Thus the measure γ exists and

$$d\gamma(x + \mathrm{i}y) = d\gamma_x(y) \, dm(x) =$$

$$= \frac{n^2}{\pi(\cos n\theta)^2} \left[\sum_{k=1}^{n} A_k d\delta_{xt_k}(y) \right] e^{-2x} x \, dx =$$

$$= (\mathrm{Re}\, z) e^{-2 \mathrm{Re}\, z} \sum_{k=1}^{n} B_k \, d\tilde{s}_k(z)$$

where B_k is a positive number and $d\tilde{s}_k$ is arclength measure on the ray $L_k = \{(1 + \mathrm{i}t_k)x : x \geqslant 0\}$. Since $\gamma = \mu \circ (-H)^{-1}$, we see that μ has the stated form and Σ_k is the image of L_k under the map $(-H)^{-1}(z) = \dfrac{z-1}{z+1}$. One easily checks that Σ_k is a circular arc (or line segment) with endpoints -1 and 1, lying in the closed disk $\bar{\mathbf{D}}$, and passing through the imaginary axis at $it_k((1+t_k^2)^{1/2}+1)^{-1}$. If $t_k \neq 0$, the center of the corresponding circle is $-i/t_k$; if $t_k = 0$, then $\Sigma_k = [-1, 1]$. We label our roots ω_k so that $\omega_k = e^{i\frac{2k-1}{n}\pi}$, $k = 1, 2, \ldots, n$ and compute $t_k = \cot\left(\frac{2k-1}{2n}\pi - \theta\right)$ where $0 < \frac{2k-1}{2n} \cdot \pi - \theta < \pi$. Therefore Σ_k intersects the imaginary axis at the claimed point. Since t_k decreases as k increases from 1 to n, we see that Σ_{k+1} lies below Σ_k for each k. We therefore have $\partial\Delta = \Sigma_1 \cup \Sigma_n$. ▨

COROLLARY 5. *If Λ is given by (27), then the invariant subspace lattice of $\{S_t\}_{t\geqslant 0}$ acting on $A^2(\Lambda)$ is isomorphic to the lattice of the simple unilateral backward shift on H^2.*

REMARK 9. This corollary is in marked contrast to the case $A^2(\Lambda_0)$ where $\Lambda_0 = -\log \Omega_0$ and $\Omega_0 = \{z : |z - 1| < 1\}$. For that space L^* is unitarily equivalent to a linear fractional map of the Cesàro operator C_0 (by Theorem 12) and therefore the lattice of $\{S_t\}$ on $A^2(\Lambda_0)$ is the lattice of C_0^*, which is known to be much more complicated that the shift lattice [24].

Proof of Corollary 5. The map $\eta = (-H)^{-1}$, i.e., $\eta(z) = (z-1)(z+1)^{-1}$, carries the open sector bounded by the rays L_1 and L_n onto $\Delta \subset \mathbf{D}$. We can thus construct a conformal map $\Psi(z) = \eta(\omega(-H(z))^\delta)$ from $\mathbf{D}$ to Δ for appropriate ω and δ, $|\omega| = 1$ and $0 < \delta < 1$. Let $\nu = \mu \circ \Psi$ be the transplant of μ from $\bar{\Delta}$ to $\bar{\mathbf{D}}$. Then the map defined on polynomials by $p \to p \circ \Psi$ extends to a unitary operator $Y : P^2(\mu) \to P^2(\nu)$ satisfying $YM_\mu = \Psi(M_\nu)Y$. One checks that $d\nu = w\,d\theta + d\nu_0$ where $\log w \in L^1(\partial\mathbf{D}, d\theta)$ and ν_0 is carried by $\mathbf{D}$. A theorem of Clary [5] implies that the invariant subspace lattice of M_ν, which is the same as the lattice of $\Psi(M_\nu)$, is isomorphic to the shift lattice. ▨

Let us consider some examples which shed light on Theorem 2. If we let Λ be given by (27) with $\theta = 0$ and let L act on $A^2(\Lambda)$, then $\sigma(L^*) = \bar{\Delta}$ admits precisely one circular symmetry, the map $z \to -z$. However, Theorem 2 does not apply because although Γ touches the imaginary axis at $iy_0 = H(-1) = 0$, it touches there too sharply. In fact, the conclusion of Theorem 2 fails spectacularly.

COROLLARY 6. *Let Λ be given by (27) with $\theta = 0$ and $n = 2$. Then there exists a unitary operator X on $A^2(\Lambda)$ such that $XL^* = -L^*X$.*

Proof. We know that $L^* \cong M_\mu$ where μ is as in Theorem 14. Since $n = 2$, μ is supported on $\Sigma_1 \cup \Sigma_2 = \partial\Delta$, hence $d\mu(z) = h(z)|dz|$ where $|dz|$ is arclength measure on $\partial\Delta$ and $0 \leqslant h \in L^1(\partial\Delta, |dz|)$. As in the proof of Corollary 5, $M_\mu \cong$

$\cong \Psi(M_\nu)$, but now $d\nu = w d\theta$ has no mass in $\mathbf{D}$. We may write $w = |g|^2$ for some outer function g in H^2 and define a unitary operator $R: P^2(\nu) \to H^2$ by $Rf = \sqrt{2\pi}\, gf$. Clearly, $R\Psi(M_\nu) = TR$ where T acts on H^2 by $T: f \to \Psi f$.

So far we have $L^* \cong T$. To conclude the proof we will construct a unitary Q on H^2 with $QT = -TQ$. Our conformal map Ψ is

$$\Psi(z) = \frac{(1+z)^{1/2} - (1-z)^{1/2}}{(1+z)^{1/2} + (1-z)^{1/2}};$$

since $\Psi(-z) = -\Psi(z)$, we take $(Qf)(z) = f(-z)$. ▨

REMARK 10. An examination of the above proof shows that if $\theta = 0$ and $n = 2$, $\{S_t^*\}_{t \geqslant 0}$ acting on $A^2(\Lambda)$ is unitarily equivalent to the semigroup $\{X_t\}_{t \geqslant 0}$ of multiplication operators on H^2 given by

$$X_t : f(z) \to e^{-t\left(\frac{1+z}{1-z}\right)^{1/2}} f(z), \quad f \in H^2.$$

REMARK 11. What happens to circular symmetry when we restrict the rate at which Λ "opens up"? If

$$\Lambda = \{x + iy : -\infty < y < \infty,\ x > g(y)\}, \tag{31}$$

then the requirement that $g(y)/|y| \to +\infty$ as $|y| \to \infty$ will imply that $\Gamma = \{u : \operatorname{Re} u < 0\}$. Indeed, if $s > 0$ and $-\infty < t < \infty$, then

$$\|e^{(-s+it)\xi}\|_{\mathscr{B}}^2 = \frac{1}{2s}\int_{-\infty}^{\infty} e^{-2(sg(y)+ty)}\, dy < \infty.$$

Suppose $g(y) \geqslant |y|^\alpha - c$ for some $\alpha > 1$ and $c > 0$. Then the above integral is dominated by $s^{-1}e^{2sc}(AM+1)$, where $[A, \infty)$ is the subinterval of $[0,\infty)$ where $2(|t|y - sy^\alpha) \leqslant -y$ and M is the maximum value of $e^{2(|t|y - sy^\alpha)}$ on $[0,\infty)$. The result is

$$\|e^{(-s+it)}\|_{\mathscr{B}}^2 \leqslant \frac{e^{2sc}}{s}\left\{\left(\frac{2|t|+1}{2s}\right)^{1/(\alpha-1)} \exp\left[D\,\frac{|t|^{\alpha/(\alpha-1)}}{s^{1/(\alpha-1)}}\right] + 1\right\}$$

where D depends only on α. Now t^2/s remains bounded if $-s+it$ is in a disk $G \subset \Gamma$ which touches the imaginary axis at $iy_0 = 0$, and thus the hypothesis (6) of Theorem 2 (with $g(z) = z$) holds for $\lambda = -1$. Theorem 2 thus implies that when $\alpha > 1$, there is no nonzero bounded operator X on $A^2(\Lambda)$ satisfying $XL^* = -L^*X$. If $\alpha > 2$, Theorem 2 applies to all λ in $\partial\mathbf{D}$ with $\lambda \neq 1$.

REMARK 12. Let us reconsider special measures. Suppose that $\{S_t^*\}_{t\geq 0}$ acting on $A^2(\Lambda)$ is subnormal with associated measure μ. If Λ arises as $-\log\Omega$, where Ω is as in § 4.1.2, and if $\partial\Omega$ is analytic at 0, then $k(t, t)$ satisfies the hypotheses of Proposition 9(B) because $\hat{k}(\bar{z}, z)$ has a pole at $z = 0$, hence μ is special. On the other hand, the measure μ associated with a sector Λ as in Theorem 14 is special by a theorem of Trent [32]. Between these two extremes is the situation where Λ is given by (31) with $g(y)/|y| \to +\infty$ as $|y| \to \infty$. In this case, too, μ (if it exists) is special—this time by Proposition 9(A). Indeed, $\Gamma = \{u : \mathrm{Re}\, u < 0\}$ and if $x \geqslant 0$, the area mean value theorem implies that $k(x, x) \leqslant \dfrac{1}{\pi R(x)^2}$ where $R(x)$ is the radius of the largest open disk in Λ with center x. Proposition 9(A) applies since $R(x) \to \infty$ as $x \to \infty$.

4.3. REPRESENTING THE SHIFT AS L^*. In all of our examples so far, Λ is an open set. Here we start with $d\mu = (1/2\pi)d\theta$ on $\partial\mathbf{D}$, so that $P^2(\mu) = H^2$ and M_μ is the simple unilateral shift. We construct $Y: P^2(\mu) \to \mathscr{B}$ as in Theorem 6. Since μ is special, $\Lambda = \Phi(\mu) = [0, \infty)$. We have

$$(Yf)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta})\overline{E_x(e^{i\theta})}d\theta,$$

and the kernel function for $\mathscr{B}$ is

$$k(t, x) = \frac{1}{2\pi} \int E_t \bar{E}_x d\theta = e^{-|x-t|}.$$

The space $\mathscr{B}$ is then the unique Hilbert space of functions on $[0, \infty)$ with this reproducing kernel; the reader can easily verify that $\mathscr{B}$ consists of all bounded absolutely continuous functions f such that $f' - f \in L^2(0, \infty)$. The norm is given by

$$\|f\|_{\mathscr{B}}^2 = \frac{1}{2} \int_0^\infty |f'(x) - f(x)|^2 dx.$$

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REFERENCES

1. BERMAN, R.; BROWN, L.; COHN, W., Cyclic vectors of bounded characteristic in Bergman spaces, forthcoming.
2. BRAM, J., Subnormal operators, *Duke J. Math.*, 22(1955), 75–94.

3. BRENNAN, J., Invariant subspaces and weighted polynomial approximation, *Ark. Mat.*, **11**(1973), 167--189.

4. BROWN, A,; HALMOS, P.; SHIELDS, A., Cesàro operators, *Acta. Sci. Math. (Szeged)*, **26**(1965), 125--137.

5. CLARY, W. S., Quasi-similarity and subnormal operators, Dissertation, Univ. of Michigan, 1973.

6. CONWAY, J. B., *Subnormal operators*, Pitman, Plainfield, Mass., 1981.

7. COWEN, C., Subnormality of the Cesàro operator and a semigroup of composition operators, *Indiana Univ. Math. J.*, **33**(1984), 305--318.

8. DEDDENS, J., Analytic Toeplitz and composition operators, *Canad. J. Math.*, **24**(1972), 859--865.

9. DEVINATZ, A., Integral representations of positive definite functions, *Trans. Amer. Math. Soc.*, **74**(1953), 56--77.

10. DUREN, P., *Theory of H^p spaces*, Academic Press, New York, 1970.

11. DUREN, P., *Univalent functions*, Springer-Verlag, New York, 1983.

12. ERDELYI, A., et al., *Higher transcendental functions, Vol. 1*, McGraw-Hill, New York, 1953.

13. ERDELYI, A., et al., *Tables of integral transforms, Vol. 1*, McGraw-Hill, New York, 1954.

14. HALMOS, P., Normal dilations and extensions of operators, *Summa Bras. Math.*, **2**(1950), 125--134.

15. HALMOS, P., *A Hilbert space problem book*, second edition, Springer-Verlag, New York, 1982.

16. HARDY, G. H., Note on a theorem of Hilbert, *Math. Z.*, **6**(1920), 314--317.

17. HAYMAN, W.; KORENBLUM, B., A critical growth rate for functions regular in a disk, *Michigan Math. J.*, **27**(1980), 21--30.

18. HILLE, E.; PHILLIPS, R., *Functional analysis and semigroups*, rev. edition, Amer. Math. Soc. Colloq. Publ., Vol. **31**, Providence, RI, 1957.

19. ITO, T., On the commutative family of subnormal operators, *J. Fac. Sci. Hokkaido Univ.*, **14**(1958), 1--15.

20. KAY, E.; SOUL, H.; TRUTT, D., Some subnormal operators and hypergeometric kernel functions, *J. Math. Anal. Appl.*, **53**(1976), 237--242.

21. KLOPFENSTEIN, K. F., A note on Hilbert spaces of factorial series, *Indiana Univ. Math. J.*, **25**(1976), 1073--1081.

22. KRIETE, T. L., Cosubnormal dilation semigroups on Bergman spaces, forthcoming.

23. KRIETE, T. L.; TRUTT, D., The Cesàro operator in ℓ^2 is subnormal, *Amer. J. Math.*, **93**(1971), 215--225.

24. KRIETE, T. L.; TRUTT, D., On the Cesàro operator, *Indiana Univ. Math. J.*, **24**(1974--5), 197--214.

25. MACCLUER, B.; SHAPIRO, J. H., Angular derivatives and compact composition operators on the Hardy and Bergman spaces, *Canad. J. Math.*, to appear.

26. PARTHASARATHY, K., *Probability measures on metric spaces*, Academic Press, New York, 1967.

27. RYFF, J., Subordinate H^p functions, *Duke J. Math.*, **33**(1966), 347--354.

28. SHAPIRO, H. S.; SHIELDS, A., On the zeros of functions with finite Dirichlet integral and some related function spaces, *Math. Z.*, **80**(1962), 217--229.

29. SHIELDS, A.; WALLEN, L., The commutants of certain Hilbert space operators, *Indiana Univ. Math. J.*, **20**(1971), 777--788.

30. SHULMAN, L., *Notices Amer. Math. Soc.*, 70T--B189, 1970.

31. SZ.-NAGY, B.; FOIAŞ, C., *Harmonic analysis of operators in Hilbert space*, American Elsevier, New York, 1970.

32. TRENT, T., $H^2(\mu)$ spaces and bounded point evaluations, *Pacific J. Math.*, **80**(1979), 279–292.

33. WALSH, J., *Interpolation and approximation by rational functions in the complex domain*, 2nd edition, Amer. Math. Soc. Colloq. Publ., Vol. **20**, Amer. Math. Soc., Providence, RI, 1956.

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