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A CHARACTERIZATION OF SPECTRAL FUNCTIONS OF DEFINITIZABLE OPERATORS

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INTRODUCTION

Let $(\mathscr{H}, [\cdot,\cdot])$ be a Kreĭn space and let A be a definitizable bounded selfadjoint operator in $\mathscr{H}$, i.e. a bounded selfadjoint operator such that there exists a polynomial p with $[p(A)x, x] \geqslant 0$, $x \in \mathscr{H}$. Such a polynomial is called a *definitizing polynomial* for A. Fix a definitizing polynomial p' for A and let $\mathfrak{B}'$ denote the Boolean algebra of subsets of $\mathbf{R}$ generated by the intervals whose endpoints are different from the zeros of p'. The operator A has a spectral function (see [7], [8], [9], [4]). This is a certain homomorphism E of $\mathfrak{B}'$ into a Boolean algebra of selfadjoint projections in $\mathscr{H}$. The projections $E(\Delta)$ corresponding to intervals Δ lying between two neighbouring zeros of p' are nonnegative or nonpositive, i.e., it holds $[E(\Delta)x, x] \geqslant 0$ for all $x \in \mathscr{H}$ or $[E(\Delta)x, x] \leqslant 0$ for all $x \in \mathscr{H}$. A real point t_0 is called a *critical point* of E if $E(\Delta)$ is indefinite for every open $\Delta \in \mathfrak{B}'$ with $t_0 \in \Delta$.

In [8], H. Langer considered the problem whether a given homomorphism of the above kind is the spectral function of a definitizable operator. This problem was solved in [8] for the subclass of homomorphisms E with the property that every critical point t_0 of E satisfies one of the following conditions:

(i) The integral $\int t\,\mathrm{d}E(t)$ over some neighbourhood of t_0 converges in the strong sense.

(ii) There is an open interval Δ containing t_0 such that $E(\Delta)\mathscr{H}$ is a Pontrjagin space.

For a fixed spectral function E of this type, H. Langer also described the set of all definitizable operators whose spectral functions coincide with E.

In this paper the characterization of spectral functions is extended to a greater class of homomorphisms (Sections 2.1 and 2.2). The corresponding class of definitizable operators contains those whose root spaces belonging to the critical points are pseudo-Kreĭn spaces, but not all definitizable operators. We remark that the example given in connection with the latter fact (Section 3) shows that even if the root space belonging to a critical point is non-degenerate the spectral function can have an arbitrary finite order of growth near this critical point.

We will make use of the main tool from [8], that is to say of a certain decomposition of $\mathscr{H}$ associated with the spectral function. We characterize those definitizable operators whose spectral functions generate a fixed decomposition of $\mathscr{H}$. This can be useful for the construction of definitizable operators with special properties (Section 2.3).

Throughout this paper we shall confine ourselves to bounded definitizable selfadjoint operators such that $[A^n x, x] \geqslant 0$, $x \in \mathscr{H}$, where n is some positive integer. This is no restriction. The class of these operators is denoted by $\mathscr{D}(0, n)$. We set $\mathscr{D}(0) := \bigcup_{n \in \mathbf{N}} \mathscr{D}(0, n)$. The spectrum of an operator belonging to $\mathscr{D}(0)$ is real (see [9], [4]).

For basic facts on Kreĭn spaces and operators in these spaces we refer to [1] and [6].

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1. NOTATIONS AND PRELIMINARY RESULTS

1.1. In the following all topological notions are understood with respect to some Hilbert norm $\|\cdot\|$ on the Kreĭn space $\mathscr{H}$ such that $[\cdot,\cdot]$ is $\|\cdot\|$-continuous. For every subspace $\mathscr{L}$ of $\mathscr{H}$ we put $\mathscr{L}^{\perp} := \{x \in \mathscr{H} : [x, \mathscr{L}] = \{0\}\}$ and $\mathscr{L}^0 := \mathscr{L} \cap \mathscr{L}^{\perp}$. $\mathscr{L}^0$ is called the *isotropic part of* $\mathscr{L}$. A subspace $\mathscr{M}$ of $\mathscr{H}$ is called a *pseudo-Kreĭn* subspace if it is the direct sum of its isotropic part $\mathscr{M}^0$ and a Kreĭn subspace of $\mathscr{H}$.

Let $\mathscr{L}$ be a fixed closed neutral subspace of $\mathscr{H}$ (i.e. $\mathscr{L} = \mathscr{L}^0$). A closed neutral subspace $\mathscr{M}$ of $\mathscr{H}$ such that $\mathscr{H} = \mathscr{L} \dotplus \mathscr{M}^{\perp}$ holds (i.e. $\mathscr{H}$ is the direct sum of $\mathscr{L}$ and $\mathscr{M}^{\perp}$) is called a *closed neutral dual companion* (c.n.d.c.) of $\mathscr{L}$. If $\mathscr{M}$ is a c.n.d.c. of $\mathscr{L}$ we have also $\mathscr{H} = \mathscr{L}^{\perp} \dotplus \mathscr{M}$. If J is an arbitrary fundamental symmetry of $\mathscr{H}$ then, for example, $J\mathscr{L}$ is a c.n.d.c. of $\mathscr{L}$. In what follows, for a bounded operator T either in $\mathscr{H}$ or between subspaces of $\mathscr{H}$ the adjoint with respect to the duality $[\cdot,\cdot]$ is denoted by T^+. In this connection for a closed neutral subspace $\mathscr{L}$ a certain c.n.d.c. of $\mathscr{L}$ (which results from the context) is regarded as the dual space of $\mathscr{L}$.

Let P be the projection on $\mathscr{L}$ along $\mathscr{M}^{\perp}$. Then P^+ is the projection on $\mathscr{M}$ along $\mathscr{L}^{\perp}$. Since by the neutrality of $\mathscr{L}$ and $\mathscr{M}$ we have $P^+P = PP^+ = 0$ the operator $\hat{P} := I - P - P^+$ is a selfadjoint projection. Hence its range $\mathscr{L}^{\perp} \cap \mathscr{M}^{\perp}$ is a Kreĭn subspace of $\mathscr{H}$ and we have the following direct decomposition of $\mathscr{H}$:

$$\mathscr{H} = \mathscr{L} \dotplus (\mathscr{L}^{\perp} \cap \mathscr{M}^{\perp}) \dotplus \mathscr{M}. \tag{1.1}$$

It is called the $(\mathscr{L}, \mathscr{M})$-*decomposition* of $\mathscr{H}$. A decomposition of this form with arbitrary c.n.d.c. $\mathscr{M}$ of $\mathscr{L}$ is called an $\mathscr{L}$-*decomposition*. Obviously, we have

$$\mathscr{L}^{\perp} = \mathscr{L} \dotplus (\mathscr{L}^{\perp} \cap \mathscr{M}^{\perp}).$$

1.2. $d(0)$-HOMOMORPHISMS. Let $\mathfrak{B}_{(0)}$ denote the Boolean algebra of subsets of $\mathbf{R}$ generated by the intervals $\Delta \subset \mathbf{R}$ whose endpoints are different from 0. A homomorphism E of $\mathfrak{B}_{(0)}$ into a Boolean algebra of selfadjoint projections in $\mathscr{H}$ is called a $d(0)$-*homomorphism* if the following hold:

(i) There exist $a, b \in \mathbf{R}$, $a < 0 < b$, such that $E([a, b]) = I$.

(ii) $E(\Delta)$ is nonnegative for every real interval Δ with $\bar{\Delta} \subset (0, \infty)$. $E(\Delta)$ is either nonnegative or nonpositive for every real interval Δ with $\bar{\Delta} \subset (-\infty, 0)$.

(iii) There exists $k \in \mathbf{N}$ such that the operators $\int_{\Delta} t^k \mathrm{d}E(t)$, $\Delta \in \mathfrak{B}_{(0)}$, $0 \notin \Delta$, are uniformly bounded.

We remark that by (ii) the homomorphism E can be extended to an operator measure (generally unbounded) on $\mathbf{R} \setminus \{0\}$. This measure will also be denoted by E. In the case when $E(\Delta)$ is nonnegative for every interval $\Delta \in \mathfrak{B}_{(0)}$, $0 \notin \Delta$, we shall say that E is of *even type*. Otherwise E is said to be of *odd type*.

Now for a fixed $d(0)$-homomorphism E we define the following linear spaces (cf. [5]):

$$\mathscr{L}'_{(0)} = \bigcup \{E(\Delta)\mathscr{H} : \Delta \in \mathfrak{B}_{(0)},\ 0 \notin \Delta\},$$

$$\mathscr{L}_{(0)} := \overline{\mathscr{L}'_{(0)}},$$

$$\mathscr{L}_0 := \bigcap \{E(\Delta)\mathscr{H} : \Delta \in \mathfrak{B}_{(0)},\ 0 \in \Delta\},$$

$$\mathscr{L}_{00} := \mathscr{L}_{(0)} \cap \mathscr{L}_0.$$

It follows that

$$\mathscr{L}_0 = \mathscr{L}_{(0)}^{\perp} \quad \text{and} \quad \mathscr{L}_{00} = \mathscr{L}_0^0 = \mathscr{L}_{(0)}^0.$$

By $\int_{(0)} t \, \mathrm{d}E(t)$ we denote the linear mapping of $\mathscr{L}'_{(0)} + \mathscr{L}_0$ in $\mathscr{H}$ which maps $x'_{(0)} + x_0$, $x'_{(0)} \in E(\Delta)\mathscr{H}$, $\Delta \in \mathfrak{B}_{(0)}, 0 \notin \Delta$, $x_0 \in \mathscr{L}_0$, into $\int_{\Delta} t \, \mathrm{d}E(t) x'_{(0)}$.

Let us fix a c.n.d.c. $\mathscr{M}$ of $\mathscr{L}_{00}$. We consider the $(\mathscr{L}_{00}, \mathscr{M})$-decomposition of $\mathscr{H}$:

$$\mathscr{H} = \mathscr{L}_{00} \dotplus \hat{\mathscr{H}} \dotplus \mathscr{M}, \quad \text{where } \hat{\mathscr{H}} := \mathscr{L}_{00}^{\perp} \cap \mathscr{M}^{\perp}.$$

Denote the corresponding projections by P, $\hat{P}$ and P^+ as in Section 1.1. We consider the mapping $\hat{E}$: $\mathfrak{B}_{(0)} \to \mathscr{L}(\hat{\mathscr{H}})$ defined by

$$\hat{E}(\Delta) := \hat{P}E(\Delta)|\hat{\mathscr{H}}, \quad \Delta \in \mathfrak{B}_{(0)}.$$

Let

$$\hat{\mathscr{L}}'_{(0)} := \bigcup\{\hat{E}(\Delta)\hat{\mathscr{H}} : \Delta \in \mathfrak{B}_{(0)}, \quad 0 \notin \Delta\},$$

$$\hat{\mathscr{L}}_{(0)} := \overline{\hat{\mathscr{L}}'_{(0)}},$$

$$\hat{\mathscr{L}}_0 := \bigcap\{\hat{E}(\Delta)\hat{\mathscr{H}} : \Delta \in \mathfrak{B}_0, \, 0 \in \Delta\}.$$

LEMMA 1.1. *$\hat{E}$ is a $d(0)$-homomorphism on the Kreĭn space $\hat{\mathscr{H}}$. The subspaces $\hat{\mathscr{L}}_{(0)}$ and $\hat{\mathscr{L}}_0$ of $\hat{\mathscr{H}}$ are non-degenerate.*

Proof. Using the relation

$$E(\Delta_1)E(\Delta_2) = E(\Delta_1)\hat{P}E(\Delta_2) \tag{1.2}$$

for $\Delta_1, \Delta_2 \in \mathfrak{B}_{(0)}$, $0 \notin \Delta_1 \cup \Delta_2$, one easily verifies that $\hat{E}$ is a $d(0)$-homomorphism. To prove that $\hat{\mathscr{L}}_{(0)}$ and $\hat{\mathscr{L}}_0$ are non-degenerate we first observe that for every $\Delta \in \mathfrak{B}_{(0)}$, $0 \notin \Delta$, we have

$$E(\Delta)\hat{\mathscr{H}} \subset E(\Delta)\mathscr{H} = E(\Delta)\hat{P}E(\Delta)\mathscr{H} \subset E(\Delta)\hat{\mathscr{H}}$$

hence

$$E(\Delta)\hat{\mathscr{H}} = E(\Delta)\mathscr{H}. \tag{1.3}$$

This implies $\hat{P}\mathscr{L}_{(0)} \subset \hat{\mathscr{L}}_{(0)}$ and $\hat{P}\mathscr{L}_0 \subset \hat{\mathscr{L}}_0$. Then we have

$$\hat{\mathscr{H}} = \hat{P}\mathscr{L}_{00}^{\perp} = \hat{P}(\overline{\mathscr{L}_{(0)} + \mathscr{L}_0}) \subset \overline{\hat{P}\mathscr{L}_{(0)} + \hat{P}\mathscr{L}_0} \subset \overline{\hat{\mathscr{L}}_{(0)} + \hat{\mathscr{L}}_0} \subset \hat{\mathscr{H}}.$$

Hence $\hat{\mathscr{L}}_{(0)}$ and $\hat{\mathscr{L}}_0$ are non-degenerate. ∎

As a consequence of Lemma 1.1 we find

$$\hat{\mathscr{H}} = \overline{\hat{\mathscr{L}}_{(0)} + \hat{\mathscr{L}}_0}.$$

LEMMA 1.2. *Let x belong to $\mathscr{H} \setminus \mathscr{L}_{00}^{\perp}$. Then there exists no $y \in \hat{\mathscr{H}}$ such that*

$$\hat{P}E(\Delta)x = \hat{P}E(\Delta)y, \tag{1.4}$$

for all $\Delta \in \mathfrak{B}_{(0)}$, $0 \notin \Delta$.

Proof. Suppose that there exists such an y. Then the relations (1.4) and (1.3) imply $[x-y, z]=0$ for all $z \in \mathscr{L}'_{(0)}$. It follows $x \in y+\mathscr{L}_0$, a contradiction to $x \in \mathscr{H} \setminus \mathscr{L}_{00}^{\perp}$. ▨

1.3. A CLASS OF DEFINITIZABLE OPERATORS. Let E be the spectral function of an operator $A \in \mathscr{D}(0)$. Obviously, the spaces $\mathscr{L}_{(0)}$, $\mathscr{L}_0$ and $\mathscr{L}_{00}$ defined above starting from the spectral function E are invariant for A and $\mathscr{L}_0$ is the root space of A corresponding to 0.

Let some $\mathscr{L}_{00}$-decomposition be given. Then it is easy to see that $\hat{E}$ is the spectral function of the operator $\hat{A}$ defined by

$$\hat{A} := \hat{P}A|\hat{\mathscr{H}},$$

which belongs to $\mathscr{D}(0)$, and A can be written in the matrix form

$$A = \begin{bmatrix} A_1 & A_3 & A_4 \\ 0 & A_2 & A_3^+ \\ 0 & 0 & A_1^+ \end{bmatrix} \quad \text{w.r.t. } \mathscr{H} = \mathscr{L}_{00} + \hat{\mathscr{H}} + \mathscr{M},$$

where $A_2 = \hat{A}$, $A_4 = A_4^+$.

In the following proposition we complete a result from [8] (see also [4, § 3.3]).

PROPOSITION 1.3. *Let E be the spectral function of an operator $A \in \mathscr{D}(0)$ and let $\mathscr{M}$ be a* c.n.d.c. *of $\mathscr{L}_{00}$. Then the following assertions (with respect to the $(\mathscr{L}_{00}, \mathscr{M})$-decomposition) are equivalent:*

(c$_1$) *The operator $\int_{(0)} t \, \mathrm{d}\hat{E}(t)$, which is defined on $\hat{\mathscr{L}}'_{(0)} + \hat{\mathscr{L}}_0$, is continuous in $\hat{\mathscr{H}}$.*

(c$_2$) *There is a nilpotent selfadjoint operator $\hat{N}$ in $\hat{\mathscr{H}}$ with the properties:*

(1) *$\hat{N}$ commutes with $\hat{A}$.*

(2) *$\mathscr{R}(\hat{N}) \subset \hat{\mathscr{L}}_0$.*

(3) *The root space of $\hat{A}-\hat{N}$ corresponding to 0 is equal to the kernel of $\hat{A}-\hat{N}$.*

If, moreover, $A \in \mathscr{D}(0, n)$ for some odd $n \in \mathbf{N}$, then these assertions are also equivalent to the following:

(c$_3$) *The integrals $\int_{\Delta} t \, \mathrm{d}\hat{E}(t)$, $\Delta \in \mathfrak{B}_{(0)}$, $0 \notin \Delta$, are uniformly bounded.*

If one of the assertions (c$_1$), (c$_2$), (c$_3$) *holds with respect to the $(\mathscr{L}_{00}, \mathscr{M})$-decomposition, it is also true for any other $\mathscr{L}_{00}$-decomposition.*

Proof. $(c_1) \Rightarrow (c_2)$. Assume that (c_1) holds. Let $\hat{A}_{(0)}$ be the extension by continuity of $\int_{(0)} t\, d\hat{E}(t)$ to $\hat{\mathscr{H}}$. $\hat{A}_{(0)}$ is selfadjoint and either $\hat{A}_{(0)}$ or $\hat{A}^2_{(0)}$ is positive. Obviously, the operators $\hat{A}_{(0)}$ and $\hat{A}$ commute and coincide on $\hat{\mathscr{L}}_{(0)}$. Hence $\hat{N} := \hat{A} - \hat{A}_{(0)}$ commutes with $\hat{A}$ and we have $\mathscr{R}(\hat{N}) \subset \mathscr{N}(\hat{N})^{\perp} \subset \hat{\mathscr{L}}^{\perp}_{(0)} = \hat{\mathscr{L}}_0$. There is a $k \in \mathbf{N}$ such that $\hat{A}^k|\hat{\mathscr{L}}_0 = 0$. Therefore if $x'_{(0)} \in \hat{\mathscr{L}}'_{(0)}$, $x_0 \in \hat{\mathscr{L}}_0$ we find

$$\hat{N}^k(x'_{(0)} + x_0) = (\hat{A} - \hat{A}_{(0)})^k x'_{(0)} + (\hat{A} - \hat{A}_{(0)})^k x_0 = 0,$$

hence, $\hat{N}^k = 0$.

Making use of the fact that the spectral functions of bounded definitizable operators can be approximated in the strong sense by polynomials of the operators one easily verifies that the spectral functions of $\hat{A}$ and $\hat{A}_{(0)}$ coincide. Hence the relation $\hat{A}_{(0)}\hat{\mathscr{L}}_0 = \{0\}$ implies (3) and the condition (c_2) holds.

$(c_2) \Rightarrow (c_1)$. Assume that (c_2) holds. Let m be a positive integer such that the relations $\hat{A}^m|\hat{\mathscr{L}}_0 = 0$ and $\hat{N}^m = 0$ hold. Then using (1) and (2) we find

$$(\hat{A} - \hat{N})^r = \hat{A}^r - \binom{r}{1}\hat{A}^{r-1}\hat{N} + \ldots +$$

$$+ (-1)^{m-1}\binom{r}{m-1}\hat{A}^{r-m+1}\hat{N}^{m-1} = \hat{A}^r, \quad r \geqslant 2m-1.$$

Hence $\hat{A} - \hat{N}$ belongs to $\mathscr{D}(0)$ and, by the same argument as above, the spectral functions of $\hat{A}$ and $\hat{A} - \hat{N}$ coincide. Consequently, the root space of $\hat{A} - \hat{N}$ is $\hat{\mathscr{L}}_0$. By (3) the operator $\hat{A} - \hat{N}$ restricted to $\hat{\mathscr{L}}'_{(0)} + \hat{\mathscr{L}}_0$ coincides with $\int_{(0)} t\, d\hat{E}(t)$.

$(c_1) \Rightarrow (c_3)$. Now we assume that $A \in \mathscr{D}(0, n)$ for some odd $n \in \mathbf{N}$. Then we have

$$\left[\int_{(0)} t\, d\hat{E}(t)x, x\right] \geqslant \left[\int_{\Delta} t\, d\hat{E}(t)x, x\right],$$

for every $\Delta \in \mathfrak{B}_{(0)}$, $0 \notin \Delta$, and every $x \in \hat{\mathscr{L}}'_{(0)} + \hat{\mathscr{L}}_0$. Thus (c_1) implies (c_3).

$(c_3) \Rightarrow (c_1)$. If (c_3) holds then the strong limit of the integrals $\int_{\Delta} t\, d\hat{E}(t)$, $\Delta \in \mathfrak{B}_{(0)}, 0 \notin \Delta$, where Δ tends monotonically to $\mathbf{R}\setminus\{0\}$, exists. This limit restricted to $\hat{\mathscr{L}}'_{(0)} + \hat{\mathscr{L}}_0$ coincides with $\int_{(0)} t\, d\hat{E}(t)$. Hence (c_1) holds.

To prove the last assertion assume that $\mathscr{M}_1$ is a c.n.d.c. of $\mathscr{L}_{00}$ different from $\mathscr{M}$, and P_1, $\hat{P}_1$ and P_1^+ are the projections corresponding to the $(\mathscr{L}_{00}, \mathscr{M}_1)$-decomposition and $\hat{\mathscr{H}}_1 := \hat{P}_1\mathscr{H}$. It is easy to see that $\hat{P}_1|\hat{\mathscr{H}}$ is an isometric isomorphism of the Kreĭn space $\hat{\mathscr{H}}$ onto the Kreĭn space $\hat{\mathscr{H}}_1$ and $(\hat{P}_1|\hat{\mathscr{H}})^{-1} = \hat{P}|\hat{\mathscr{H}}_1$. We have $\hat{P}_1\hat{P}|\mathscr{L}_{00}^{\perp} = \hat{P}_1|\mathscr{L}_{00}^{\perp}$ and $\hat{P}\hat{P}_1|\mathscr{L}_{00}^{\perp} = \hat{P}|\mathscr{L}_{00}^{\perp}$. This implies $\hat{P}_1E(\Delta)\hat{P}_1 = \hat{P}_1\hat{E}(\Delta)\hat{P}\hat{P}_1$. Hence the above isomorphism maps $\hat{\mathscr{L}}'_{(0)}$, $\hat{\mathscr{L}}_{(0)}$, $\hat{\mathscr{L}}_0$ onto the corresponding spaces with respect to the $(\mathscr{L}_{00}, \mathscr{M}_1)$-decomposition. These facts imply the last assertion. ▨

In what follows we shall say that a $d(0)$-homomorphism E satisfies the condition (c_1) or has the property (c_1) if E fulfils the condition (c_1) from the preceding proposition for some (or, equivalently, for every) $\mathscr{L}_{00}$-decomposition. In the same way an operator $A \in \mathscr{D}(0)$ is said to fulfil the condition (c_2) if it fulfils (c_2) for some (and then for every) $\mathscr{L}_{00}$-decomposition.

2. A CHARACTERIZATION OF SPECTRAL FUNCTIONS OF DEFINITIZABLE OPERATORS

2.1. In what follows we restrict ourselves to $d(0)$-homomorphisms with the property (c_1). Theorem 2.1 below characterizes the spectral functions of operators from $\mathscr{D}(0)$ within the class of these $d(0)$-homomorphisms. Obviously the condition (i) (see Introduction) from [8] implies (c_1). At the end of this section we shall show how the corresponding result from [8] is connected with our more general considerations.

In the following Section 2.2 we shall deal with a more restricted class of $d(0)$-homomorphisms. The results from [8] concerning the case of a Pontrjagin space are contained in the results of Section 2.2.

For a given $d(0)$-homomorphism E we use the notation $\mathscr{L}_0$, $\mathscr{L}_{00}, \ldots$ from Section 1. If some c.n.d.c. $\mathscr{M}$ of $\mathscr{L}_{00}$ is fixed we also use the notations for the spaces and the projections corresponding to the $(\mathscr{L}_{00}, \mathscr{M})$-decomposition of $\mathscr{H}$ from Section 1.

THEOREM 2.1. *Let E be a $d(0)$-homomorphism with property (c_1). Then E is the spectral function of an operator belonging to $\mathscr{D}(0)$ if and only if the following condition holds for some or, equivalently, for every* c.n.d.c. *$\mathscr{M}$ of $\mathscr{L}_{00}$:*

(*) *For every $x \in \mathscr{M}$ there exists $z \in \mathscr{H}$ such that for all $\Delta \in \mathfrak{B}_{(0)}, 0 \notin \Delta$, we have*

$$\hat{P}\int_{\Delta} t\, \mathrm{d}E(t)x = \hat{P}E(\Delta)z.$$

Proof. (1) Assume first that E is the spectral function of an operator $A \in \mathscr{D}(0)$. If $\mathscr{M}$ is a fixed c.n.d.c. of $\mathscr{L}_{00}$ and $x \in \mathscr{M}$ then we have

$$\hat{P} \int_{\Delta} t \, \mathrm{d}E(t)x = \hat{P}E(\Delta)Ax,$$

for every $\Delta \in \mathfrak{B}_{(0)}, 0 \notin \Delta$, and the condition (*) holds with $z = Ax$.

(2) Let us assume now that (*) holds for some c.n.d.c. $\mathscr{M}$ of $\mathscr{L}_{00}$. We first note that this condition can be equivalently formulated in the following way:

For every $x_0 \in \mathscr{M}$ there exist $x_1 \in \mathscr{M}$ and $y \in \hat{\mathscr{H}}$ $(=\mathscr{L}_{00}^{\perp} \cap \mathscr{M}^{\perp})$ such that for all $\Delta \in \mathfrak{B}_{(0)}, 0 \notin \Delta$, we have

$$\hat{P} \int_{\Delta} t \, \mathrm{d}E(t)x_0 - \hat{P}E(\Delta)x_1 = \hat{P}E(\Delta)y.$$

Moreover, if $\tilde{\mathscr{H}}$ denotes the quotient space $\hat{\mathscr{H}}/\hat{\mathscr{L}}_0$ then by means of Lemma 1.2 it follows that the vectors $x_1 \in \mathscr{M}$ and $y + \hat{\mathscr{L}}_0 \in \tilde{\mathscr{H}}$ are uniquely determined by x_0, hence one can define the linear mappings $C_1 : \mathscr{M} \ni x_0 \mapsto x_1 \in \mathscr{M}$ and $\tilde{C}_3 : \mathscr{M} \ni x_0 \mapsto y + \hat{\mathscr{L}}_0 \in \tilde{\mathscr{H}}$. We claim that C_1 and $\tilde{C}_3$ are bounded. Indeed, let $(x_0^{(m)})_{m \in \mathbf{N}}$ be a sequence of vectors from $\mathscr{M}$ such that $\lim_{m \to \infty} x_0^{(m)} = x_0$, $\lim_{m \to \infty} C_1 x_0^{(m)} = x_1 \in \mathscr{M}$ and $\lim_{m \to \infty} \tilde{C}_3 x_0^{(m)} = \xi \in \tilde{\mathscr{H}}$. Then for every $\Delta \in \mathfrak{B}_{(0)}, 0 \notin \Delta$, we have

$$\hat{P} \int_{\Delta} t \, \mathrm{d}E(t)x_0 - \hat{P}E(\Delta)x_1 = \hat{P}E(\Delta)y,$$

where $\xi = y + \hat{\mathscr{L}}_0$. By the closed graph theorem the claim is proved.

Let $\hat{\mathscr{L}}_1$ be a closed subspace of $\hat{\mathscr{H}}$ such that $\hat{\mathscr{H}} = \hat{\mathscr{L}}_0 \dotplus \hat{\mathscr{L}}_1$ and j the continuous linear mapping of $\tilde{\mathscr{H}}$ onto $\hat{\mathscr{L}}_1 \subset \hat{\mathscr{H}}$ such that $j(y + \hat{\mathscr{L}}_0) = y$ for all $y \in \hat{\mathscr{L}}_1$. Define $C_3 := j\tilde{C}_3 \in \mathscr{L}(\mathscr{M}, \hat{\mathscr{H}})$. Set $A_1 := C_1^+$, $A_3 := C_3^+$ and let A_2 denote the continuous extension of $\int_{(0)} t \, \mathrm{d}\hat{E}(t)$ to $\hat{\mathscr{H}}$ (which exists due to the condition (c_1)). Defining

$$A := \begin{bmatrix} A_1 & A_3 & 0 \\ 0 & A_2 & A_3^+ \\ 0 & 0 & A_1^+ \end{bmatrix} \quad \text{w.r.t. } \mathscr{H} = \mathscr{L}_{00} \dotplus \hat{\mathscr{H}} \dotplus \mathscr{M},$$

a straightforward calculation proves that A is a selfadjoint operator in $\mathscr{H}$.

We claim that

$$Ax = \int_{(0)} t\, \mathrm{d}E(t)x, \quad x \in \mathscr{L}'_{(0)}. \tag{2.1}$$

Assume that $x \in E(\Delta)\mathscr{H}$, $\Delta \in \mathfrak{B}_{(0)}$, $0 \notin \Delta$. If $u \in \mathscr{L}_{00}$ then

$$[Ax, u] = [x, Au] = 0 = \left[\int_{(0)} t\, \mathrm{d}E(t)x, u\right]. \tag{2.2}$$

If $v \in \hat{\mathscr{H}}$ we get

$$\begin{aligned}[Ax, v] &= [(\hat{P} + P)E(\Delta)x, A\hat{P}v] = \left[x, \int_{\Delta} t\, \mathrm{d}\hat{E}(t)v\right] = \\ &= \left[\int_{\Delta} t\, \mathrm{d}\hat{E}(t)\hat{P}x, v\right] = \left[\int_{\Delta} t\, \mathrm{d}E(t)x, v\right].\end{aligned} \tag{2.3}$$

Let $w \in \mathscr{M}$. Then

$$\begin{aligned}[Ax, w] &= [E(\Delta)x, C_3w + C_1w] = [x, \hat{P}E(\Delta)C_3w + \hat{P}E(\Delta)C_1w] = \\ &= \left[x, \hat{P}\int_{\Delta} t\, \mathrm{d}E(t)w\right] = \left[\int_{\Delta} t\, \mathrm{d}E(t)\hat{P}x, w\right] = \left[\int_{\Delta} t\, \mathrm{d}E(t)x, w\right].\end{aligned} \tag{2.4}$$

The relations (2.2)—(2.3) imply (2.1).

By assumption there exists an integer $k \geqslant 1$ such that the operators $\int_{\Delta} t^k\, \mathrm{d}E(t)$, $\Delta \in \mathfrak{B}_{(0)}$, $0 \notin \Delta$, are uniformly bounded. We assert that

$$A_1^k = 0. \tag{2.5}$$

Indeed, let $w_0 \in \mathscr{M}$. Setting $w_n := A_1^{+n}w_0 = C_1^n w_0 \in \mathscr{M}$, $n \in \mathbf{N}$, it follows that for every $n = 0, 1, \ldots, k-1$ there exists $x_n \in \hat{\mathscr{H}}$ such that for all $\Delta \in \mathfrak{B}_{(0)}, 0 \notin \Delta$, it holds

$$\hat{P}\int_{\Delta} t\, \mathrm{d}E(t)w_n - \hat{P}E(\Delta)w_{n+1} = \hat{P}E(\Delta)x_n.$$

Applying $\int_\Delta t^{k-n-1}\,\mathrm{d}\hat{E}(t)$ on both sides of these equality we obtain

$$\hat{P}\int_\Delta t^k\,\mathrm{d}E(t)w_0 - \hat{P}\int_\Delta t^{k-1}\,\mathrm{d}E(t)w_1 = \hat{P}E(\Delta)\int_\Delta t^{k-1}\,\mathrm{d}\hat{E}(t)x_0,$$

$$\hat{P}\int_\Delta t^{k-1}\,\mathrm{d}E(t)w_1 - \hat{P}\int_\Delta t^{k-2}\,\mathrm{d}E(t)w_2 = \hat{P}E(\Delta)\int_\Delta t^{k-2}\,\mathrm{d}\hat{E}(t)x_1,$$

$$\vdots$$

$$\hat{P}\int_\Delta t\,\mathrm{d}E(t)w_{k-1} - \hat{P}E(\Delta)w_k = \hat{P}E(\Delta)x_{k-1}.$$

Adding up these equalities we get

$$\hat{P}\int_\Delta t^k\,\mathrm{d}E(t)w_0 - \hat{P}E(\Delta)w_k = \hat{P}E(\Delta)\left(\sum_{j=0}^{k-1}\int_\Delta t^j\,\mathrm{d}\hat{E}(t)x_{k-1-j}\right),$$

whence making use of (1.2) and of the fact that $\operatorname*{s-lim}_{\Delta'\to\mathbf{R}\setminus\{0\}}\int_{\Delta'} t^k\,\mathrm{d}E(\mathrm{t}) =: S$ exists,

we find

$$\hat{P}E(\Delta)\hat{P}Sw_0 - \hat{P}E(\Delta)w_k = \hat{P}E(\Delta)\left(\sum_{j=0}^{k-1} A_2^j x_{k-1-j}\right).$$

Thus we have

$$\hat{P}E(\Delta)w_k = \hat{P}E(\Delta)x',$$

for some $x'\in\hat{\mathscr{H}}$ and all $\Delta\in\mathfrak{B}_{(0)}$, $0\notin\Delta$. Then Lemma 1.2 implies $w_k=0$. This proves $A_1^{+k}=0$ hence $A_1^k=0$.

Now let $x\in\mathscr{L}_0$. Since $A_1^kPx=0$ and $A_2\hat{P}x=0$ we have $A^{k+1}x=0$. This fact and (2.1) imply

$$A^r x = \lim_{n\to\infty}\int_{\Delta^{(n)}} t^r\,\mathrm{d}E(t)x,\quad x\in\overline{\mathscr{L}_{(0)}+\mathscr{L}_0},$$

for $r\geqslant k+1$, where $\Delta^{(n)} := \mathbf{R}\setminus(-1/n,1/n)$, $n\in\mathbf{N}$, and moreover

$$A^{r+k}x = A^rA^kx = \lim_{n\to\infty}\int_{\Delta^{(n)}} t^r\,\mathrm{d}E(t)A^kx = \lim_{n\to\infty}\int_{\Delta^{(n)}} t^{r+k}\,\mathrm{d}E(t)x,\quad x\in\mathscr{H}, \tag{2.6}$$

for $r\geqslant k+1$.

If E is of odd type this relation implies

$$[A^{2k+1}x,x]\geqslant 0,\quad x\in\mathscr{H}.$$

If E is of even type then

$$[A^{2k+2}x, x] \geqslant 0, \quad x \in \mathscr{H}.$$

Thus A is definitizable. Also by (2.6) it follows that for every polynomial p we have

$$p(A)A^{2k+1}x = \lim_{n\to\infty} \int_{\Delta^{(n)}} p(t)t^{2k+1}\,\mathrm{d}E(t)x, \quad x \in \mathscr{H},$$

whence according to the fact that the spectral projections of A can be approximated in the strong operator topology by operators of the form $p(A)A^{2k+1}$ (see e.g. [4, Theorem 4]) it follows that the spectral function of A coincides with E. ▨

REMARK. By means of (1.2) it can easily be proved, independently of the preceding theorem, that the condition (*) is equivalent to the following:

For every $x \in \mathscr{M}$ there exists $z \in \mathscr{H}$ such that for all $\Delta \in \mathfrak{B}_{(0)}$, $0 \notin \Delta$, we have

$$\int_{\Delta} t\,\mathrm{d}E(t)x = E(\Delta)z.$$

COROLLARY 2.2. *Let E be a $d(0)$-homomorphism and let $\mathscr{M}$ be a* c.n.d.c. *of $\mathscr{L}_{00}$. If $\hat{P}\int_{\Delta} t\,\mathrm{d}E(t)$, $\Delta \in \mathfrak{B}_{(0)}$, $0 \notin \Delta$, converges strongly in $\mathscr{H}$ for $\Delta \to \mathbf{R}\setminus\{0\}$, then E has property* (c_1), (*) *holds and, hence, E is the spectral function of an operator $A \in \mathscr{D}(0)$.*

REMARK. Under the assumptions of Corollary 2.2 the operators A_1 and A_3 defined in the proof of Theorem 2.1 fulfil the relations $A_1 = 0$, $\mathscr{N}(A_3) \supset \hat{\mathscr{L}}_0$ and $\mathscr{R}(A_3^+) \subset \hat{\mathscr{L}}_{(0)}$.

Along the lines of the proof of Theorem 2.1 one easily verifies the following description of all operators having a given $d(0)$-homomorphism as its spectral function.

THEOREM 2.3. *Let E be the spectral function with property* (c_1) *of an operator belonging to $\mathscr{D}(0)$. Let $\mathscr{M}$ be a* c.n.d.c. *of $\mathscr{L}_{00}$.*

Then E is the spectral function of an operator $B \in \mathscr{D}(0)$ if and only if B has a matrix form

$$B = \begin{bmatrix} A_1 & B_3 & B_4 \\ 0 & A_2 + N & B_3^+ \\ 0 & 0 & A_1^+ \end{bmatrix} \quad \text{w.r.t. } \mathscr{H} = \mathscr{L}_{00} \dotplus \hat{\mathscr{H}} \dotplus \mathscr{M},$$

where

(i) A_1 *and* A_2 *are the operators defined by* E *as in part* (2) *of the proof of Theorem* 2.1.

(ii) N *is an arbitrary nilpotent selfadjoint operator in* $\hat{\mathscr{H}}$ *commuting with* A_2 *such that* $\mathscr{R}(N) \subset \hat{\mathscr{L}}_0$.

(iii) $B_3 \in \mathscr{L}(\hat{\mathscr{H}}, \mathscr{L}_{00})$ *is an arbitrary operator satisfying the relations*

$$\hat{E}(\Delta)B_3^+ = \hat{E}(\Delta)A_3^+ \quad \textit{for all } \Delta \in \mathfrak{B}_{(0)},\ 0 \notin \Delta,$$

where A_3 *is defined by* E *as in part* (2) *of the proof of Theorem* 2.1.

(iv) $B_4 \in \mathscr{L}(\mathscr{M}, \mathscr{L}_{00})$ *is an arbitrary operator with* $B_4 = B_4^+$.

COROLLARY 2.4. *Let* E *be as in Corollary* 2.2. *Fix some* c.n.d.c. $\mathscr{M}$ *of* $\mathscr{L}_{00}$. *Then among the operators belonging to* $\mathscr{D}(0)$ *with spectral function* E *there is a "simplest" one with respect to the* $(\mathscr{L}_{00}, \mathscr{M})$*-decomposition:*

$$A' := \begin{bmatrix} 0 & A_3 & 0 \\ 0 & A_2 & A_3^+ \\ 0 & 0 & 0 \end{bmatrix},$$

where A_2 *and* A_3 *are defined in the proof of Theorem* 2.1 *(see the remark after Corollary* 2.2). *Moreover, the following assertions are equivalent:*

(i) $B \in \mathscr{D}(0)$ *has* E *as its spectral function.*

(ii) $B = A' + N'$, *where* N' *is a nilpotent selfadjoint operator commuting with* A' *such that* $\mathscr{R}(N') \subset \mathscr{L}_0$.

(iii) *If* $A_4 \in \mathscr{L}(\mathscr{M}, \mathscr{L}_{00})$ *is an arbitrary operator with* $A_4 = A_4^+$ *and*

$$A'' = \begin{bmatrix} 0 & A_3 & A_4 \\ 0 & A_2 & A_3^+ \\ 0 & 0 & 0 \end{bmatrix}, \tag{2.7}$$

then $B = A'' + N''$, *where* N'' *is a nilpotent selfadjoint operator commuting with* A'' *such that* $\mathscr{R}(N'') \subset \mathscr{L}_0$.

One of the results of [8] (which has a simple direct proof) is contained in the Corollaries 2.2 and 2.4: If $\int_\Delta t\mathrm{d}E(t)$, $\Delta \in \mathfrak{B}_{(0)}$, $0 \notin \Delta$, converges strongly in $\mathscr{H}$ for $\Delta \to \mathbf{R}\setminus\{0\}$, then the assumptions of these corollaries are fulfilled and $\operatorname{s-lim}_{\Delta \to \mathbf{R}\setminus\{0\}} \int_\Delta t\,\mathrm{d}E(t)$ is of the form (2.7).

Hence in this case an operator $B \in \mathscr{D}(0)$ has E as its spectral function if and only if

$$B = \underset{\Delta \to \mathbf{R}\setminus\{0\}}{\text{s-lim}} \int_{\Delta} t \, dE(t) + N$$

where N is a nilpotent selfadjoint operator with $NE(\Delta) = E(\Delta)N = 0$ for every $\Delta \in \mathfrak{B}_{(0)}, 0 \notin \Delta$.

2.2. Now we assume additionally that for the given $d(0)$-homomorphism E the space $\mathscr{L}_0$ is a pseudo-Kreĭn space. Then for an arbitrary $\mathscr{L}_{00}$-decomposition of $\mathscr{H}$ as above we have $\hat{\mathscr{H}} = \hat{\mathscr{L}}_0 \dotplus \hat{\mathscr{L}}_{(0)}$, where $\hat{\mathscr{L}}_0$ and $\hat{\mathscr{L}}_{(0)}$ are Kreĭn subspaces (see [2]). The corresponding selfadjoint projections (in $\mathscr{H}$) are denoted by $\hat{P}_0$ and $\hat{P}_{(0)}$, respectively. Thus the following decomposition of $\mathscr{H}$ holds:

$$\mathscr{H} = \mathscr{L}_{00} \dotplus \hat{\mathscr{L}}_0 \dotplus \hat{\mathscr{L}}_{(0)} \dotplus \mathscr{M}.$$

If, in addition, E is of even type the condition (c_1) is automatically fulfilled. Indeed, in this case $\hat{\mathscr{L}}_{(0)}$ is a Hilbert space and we have $\|\hat{E}(\Delta)\|_{\hat{\mathscr{L}}_{(0)}} \leq 1$ for all $\Delta \in \mathfrak{B}_{(0)}, 0 \notin \Delta$. This implies that E has the property (c_1). Consequently, if for some $\Delta' \in \mathfrak{B}_{(0)}, 0 \in \Delta'$, the restriction of E to $E(\Delta')\mathscr{H}$ is of even type then (c_1) is fulfilled again.

If E is an arbitrary $d(0)$-homomorphism with values in $\mathscr{L}(\mathscr{H})$ and $\mathscr{H}$ is a Pontrjagin space or, more generally, $E(\Delta')\mathscr{H}$ is a Pontrjagin space for some interval $\Delta' \in \mathfrak{B}_{(0)}, 0 \in \Delta'$, then $\mathscr{L}_0$ is a pseudo-Kreĭn space and, according to the above considerations, E has the property (c_1). The same holds for the pseudo-regular $d(0)$-homomorphisms considered in [3].

PROPOSITION 2.5. *Let E be a $d(0)$-homomorphism such that $\mathscr{L}_0$ is a pseudo-Kreĭn space. Then the condition* (c_1) *is necessary for the fact that E is the spectral function of an operator belonging to $\mathscr{D}(0)$.*

Proof. Let E be the spectral function of $A \in \mathscr{D}(0)$. If E is of even type then by the preceding observations (c_1) holds. If E is of odd type the operator $\hat{P}_{(0)}\hat{A}\hat{P}_{(0)}$ is nonnegative and this fact implies (c_1) (e.g. by [5, Theorem 2]). ▨

If $\mathscr{L}_0$ is a pseudo-Kreĭn space then, on account of Proposition 2.5, we can restrict ourselves to $d(0)$-homomorphisms with property (c_1). From Theorems 2.1 and 2.3 we obtain the following results:

THEOREM 2.1′. *Let, in addition to the assumptions of Theorem 2.1, $\mathscr{L}_0$ be a pseudo-Kreĭn space. Then E is the spectral function of an operator belonging to $\mathscr{D}(0)$ if and only if the following condition holds for some or, equivalently, for every* c.n.d.c. *$\mathscr{M}$ of $\mathscr{L}_{00}$:*

(**) *For every* $x \in \mathcal{M}$ *there exists* $z \in \mathcal{H}$ *such that for all* $\Delta \in \mathfrak{B}_{(0)}, 0 \notin \Delta$, *we have*

$$\hat{P}_{(0)} \int_{\Delta} t \, \mathrm{d}E(t) x = \hat{P}_{(0)} E(\Delta) z. \tag{2.8}$$

REMARK 1. $P^{+}z$ and $\hat{P}_{(0)}z$ are uniquely determined by x (see the proof of Theorem 2.1).

REMARK 2. If the $d(0)$-homomorphism E is pseudo-regular in the sense of [3] (e.g. if E is of even type or $\mathcal{H}$ is a Pontrjagin space) then $\hat{E}$ is bounded (cf. [3, Lemma 3.4]) and the condition (**) can be replaced by the following one (which appears in [8]):

For every $x_0 \in \mathcal{M}$ *there exists* $x_1 \in \mathcal{M}$ *such that the limit*

$$\lim_{n \to \infty} \left(\hat{P}_{(0)} \int_{\Delta^{(n)}} t \, \mathrm{d}E(t) x_0 - \hat{P}_{(0)} E(\Delta^{(n)}) x_1 \right)$$

exists, where $\Delta^{(n)} := \mathbf{R} \setminus (-1/n, 1/n)$, $n \in \mathbf{N}$.

THEOREM 2.3′. *Let, in addition to the assumptions of Theorem* 2.3, $\mathcal{L}_0$ *be a pseudo-Kreĭn space. Then* $B \in \mathcal{D}(0)$ *has the spectral function* E *if and only if it has a matrix form*

$$B = \begin{bmatrix} A_{11} & B_{12} & A_{13} & B_{14} \\ 0 & B_{22} & 0 & B_{12}^{+} \\ 0 & 0 & A_{33} & A_{13}^{+} \\ 0 & 0 & 0 & A_{11}^{+} \end{bmatrix} \quad \text{w.r.t. } \mathcal{H} = \mathcal{L}_{00} \dotplus \hat{\mathcal{L}}_0 \dotplus \hat{\mathcal{L}}_{(0)} \dotplus \mathcal{M},$$

where

(i) A_{11}^{+} *coincides with the operator* $x \mapsto P^{+}z$ *defined with the help of* (2.8).

(ii) A_{13}^{+} *coincides with the operator* $x \mapsto \hat{P}_{(0)}z$ *defined with the help of* (2.8).

(iii) A_{33} *is the extension by continuity of* $\int_{(0)} t \, \mathrm{d}(\hat{P}_{(0)} E(t) \hat{P}_{(0)})$.

(iv) B_{22} *is an arbitrary nilpotent bounded selfadjoint operator in* $\hat{\mathcal{L}}_0$.

(v) $B_{12} \in \mathcal{L}(\hat{\mathcal{L}}_0, \mathcal{L}_{00})$ *is arbitrary.*

(vi) $B_{14} \in \mathcal{L}(\mathcal{M}, \mathcal{L}_{00})$ *is arbitrary with* $B_{14}^{+} = B_{14}$.

REMARK. Theorem 2.3′ shows that if $\mathcal{L}_0$ is a pseudo-Kreĭn space then among the operators which have the same spectral function there is a "simplest" one with respect to the given $\mathscr{L}_{00}$-decomposition, namely

$$A' = \begin{bmatrix} A_{11} & 0 & A_{13} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & A_{33} & A_{13}^+ \\ 0 & 0 & 0 & A_{11}^+ \end{bmatrix}$$

(compare with Corollary 2.4).

As an application of the preceding results we consider a $d(0)$-homomorphism E with the following properties:

(i) There is an interval $\Delta' \in \mathfrak{B}_{(0)}, 0 \in \Delta'$, such that E restricted to $E(\Delta')\mathscr{H}$ is of even type.

(ii) $\mathscr{L}_0$ is a pseudo-Kreĭn space.

(iii) $\operatorname{s-lim}_{\Delta \to \mathbf{R}\setminus\{0\}} \int_\Delta t^2 \, \mathrm{d}E(t)$ exists.

Such a $d(0)$-homomorphism was also considered in [8] (in the case of a Pontrjagin space $\mathscr{H}$, where (i) and (ii) are fulfilled automatically).

We fix some $\mathscr{L}_{00}$-decomposition. E has property (c_1) and by the relation

$$\left[\hat{P}_{(0)} \int_\Delta t \, \mathrm{d}E(t)x, \; \hat{P}_{(0)} \int_\Delta t \, \mathrm{d}E(t)x\right] = \left[\int_\Delta t^2 \, \mathrm{d}E(t)x, x\right],$$

$\Delta \in \mathfrak{B}_{(0)}$, $0 \notin \Delta \subset \Delta'$, $x \in \mathscr{H}$, the strong limit of $\hat{P}_{(0)} \int_\Delta t \, \mathrm{d}E(t)$ for $\Delta \to \mathbf{R}\setminus\{0\}$ exists. Then, according to Corollary 2.2, E is the spectral function of an operator belonging to $\mathscr{D}(0)$. By Corollary 2.4 and Theorem 2.3′ E is the spectral function of the operator A' defined by

$$A' = \operatorname{s-lim}_{\Delta \to \mathbf{R}\setminus\{0\}} (P + \hat{P}_{(0)}) \int_\Delta t \, \mathrm{d}E(t) \hat{P}_{(0)} + \operatorname{s-lim}_{\Delta \to \mathbf{R}\setminus\{0\}} \hat{P}_{(0)} \int_\Delta t \, \mathrm{d}E(t) P^+ .$$

An operator $B \in \mathscr{D}(0)$ has E as its spectral function if and only if $B = A' + N'$, where N' is a nilpotent selfadjoint operator with $E(\Delta)N' = N'E(\Delta)$ for every $\Delta \in \mathfrak{B}_{(0)}$, $\Delta \not\ni 0$. This result contains that from [8].

2.3. In this section let $\mathscr{L}$ be a closed neutral subspace of $\mathscr{H}$ and $\mathscr{M}$ a c.n.d.c. of $\mathscr{L}$. Assume that in the $(\mathscr{L}, \mathscr{M})$-decomposition of $\mathscr{H}$ the intersection $\mathscr{L}^\perp \cap \mathscr{M}^\perp$ is the direct sum of two Kreĭn subspaces $\mathscr{H}_2$ and $\mathscr{H}_3$:

(2.9) $$\mathscr{H} = \mathscr{L} \dotplus \mathscr{H}_2 \dotplus \mathscr{H}_3 \dotplus \mathscr{M}.$$

We shall say that an operator $A \in \mathscr{D}(0)$ is *associated with the decomposition* (2.9) if for the spectral function E of A we have $\mathscr{L}_{00} = \mathscr{L}$, $\hat{\mathscr{L}}_0 = \mathscr{H}_2$, $\hat{\mathscr{L}}_{(0)} = \mathscr{H}_3$, where $\hat{\mathscr{L}}_0$ and $\hat{\mathscr{L}}_{(0)}$ are defined with respect to the $(\mathscr{L}, \mathscr{M})$-decomposition.

Let the operator $A \in \mathscr{D}(0)$ be associated with the decomposition (2.9). Assume that $\mathscr{M}_1$ is a c.n.d.c. of $\mathscr{L}$ different from $\mathscr{M}$. Denote by $\hat{P}_1$ the projection on $\mathscr{L}^\perp \cap \mathscr{M}_1^\perp$ corresponding to the $(\mathscr{L}, \mathscr{M}_1)$-decomposition. Then as in the proof of Proposition 1.3 one can show that A is also associated with the decomposition

$$\mathscr{H} = \mathscr{L} \dotplus \hat{P}_1\mathscr{H}_2 \dotplus \hat{P}_1\mathscr{H}_3 \dotplus \mathscr{M}_1.$$

We shall find necessary and sufficient conditions in terms of the elements of the matrix representation of A w.r.t. (2.9) for the fact that A is associated with the decomposition (2.9).

PROPOSITION 2.6. *Let $A \in \mathscr{D}(0)$ have the matrix form*

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ 0 & A_{22} & 0 & A_{12}^+ \\ 0 & 0 & A_{33} & A_{13}^+ \\ 0 & 0 & 0 & A_{11}^+ \end{bmatrix} \quad \text{w.r.t. (2.9)}$$

with $A_{22} = A_{22}^+$, $A_{33} = A_{33}^+$, $A_{14} = A_{14}^+$, nilpotent A_{11} and A_{22}, and $\mathscr{N}(A_{33}) = \{0\}$.

Then A is associated with the decomposition (2.9) if and only if there is a $k \in \mathbf{N}$ such that the relations $A_{11}^k = 0$,

$$\mathscr{R}(A_{33}^k) \cap \mathscr{R}\left(\sum_{\nu=0}^{k-1} A_{33}^\nu A_{13}^+ (A_{11}^+)^{k-1-\nu}\right) = \{0\} \tag{2.10}$$

and

$$\mathscr{N}\left(\sum_{\nu=0}^{k-1} A_{33}^\nu A_{13}^+ (A_{11}^+)^{k-1-\nu}\right) = \{0\} \tag{2.11}$$

hold.

Proof. Let A be associated with the decomposition (2.9). Then

$$A' = \begin{bmatrix} A_{11} & 0 & A_{13} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & A_{33} & A_{13}^+ \\ 0 & 0 & 0 & A_{11}^+ \end{bmatrix}$$

is also associated with the decomposition (2.9). We choose an arbitrary $k \in \mathbf{N}$ such that $A_{11}^k = 0$. Then

$$A'^k = \begin{bmatrix} 0 & 0 & \sum_{\nu=0}^{k-1} A_{11}^{k-1-\nu} A_{13} A_{33}^{\nu} & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & A_{33}^k & \sum_{\nu=0}^{k-1} A_{33}^{\nu} A_{13}^{+} (A_{11}^{+})^{k-1-\nu} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Suppose that at least one of the relations (2.10) and (2.11) does not hold. Then we find an $x \in \mathscr{H}_3 + \mathscr{M}$, $x \neq 0$, such that $A'^k x \in \mathscr{L}$. This contradicts the fact that A' is associated with (2.9).

Conversely, assume that the conditions $A_{11}^k = 0$, (2.10) and (2.11) hold for some $k \in \mathbf{N}$. It is easy to see that the root space of A' corresponding to 0 is $\mathscr{L} \dotplus \mathscr{H}_2$. Hence A' is associated with (2.9). By Theorem 2.2 A is also associated with the decomposition (2.9). ▨

COROLLARY 2.7. *If under the assumptions of Proposition* 2.6 *we have*

$$\mathscr{R}(A_{33}) \cap \mathscr{R}(A_{13}^{+}) = \{0\}$$

then A is associated with the decomposition (2.9).

3. A DEFINITIZABLE OPERATOR WHICH DOES NOT SATISFY CONDITION (c_2)

In this section we give an example of a compact operator $B \in \mathscr{D}(0)$ in a Kreĭn space $\mathscr{K}$ which does not satisfy the condition (c_2). More precisely, we show that for every integer $\varkappa > 2$ there exists a compact operator B with $\sigma(B) \subset [0, \infty)$ and the following properties

(i) $[B^{2\varkappa}\mathrm{x}, \mathrm{x}] \geqslant 0, \quad \mathrm{x} \in \mathscr{K}$.

(ii) The integrals $\int_{[1/n,\infty)} t^{2\varkappa-2}\, \mathrm{d}E_B(t)$, $n = 1, 2, \ldots$, are not uniformly bounded. Here E_B is the spectral function of B.

(iii) The root space of B corresponding to 0 is non-degenerate and not orthocomplemented.

These properties imply that B does not satisfy condition (c_2). Indeed, suppose that B has property (c_2). Then, on account of (iii) $\int_{(0)} t\, \mathrm{d}E_B(t)$ is continuous in $\mathscr{K}$

and condition (i) implies that the integrals $\int_\Delta t^2\,\mathrm{d}E_B(t)$, $\Delta \in \mathfrak{B}_{(0)}$, $0 \notin \Delta$, are uniformly bounded (see proof of Proposition 1.3, $(c_1) \Rightarrow (c_3)$), which contradicts (ii).

EXAMPLE. Let $\mathscr{H}$ be a Pontrjagin space of index $\varkappa$, let $\mathscr{L}$ be a $\varkappa$-dimensional neutral subspace of $\mathscr{H}$ and let $\mathscr{M}$ be a c.n.d.c. of $\mathscr{L}$. We set $\mathscr{H}_2 := \{0\}$, $\mathscr{H}_3 := \mathscr{L}^\perp \cap \mathscr{M}^\perp$ and consider the decomposition

$$\mathscr{H} = \mathscr{L} \dotplus \mathscr{H}_2 \dotplus \mathscr{H}_3 \dotplus \mathscr{M}. \tag{3.1}$$

$\mathscr{H}_3$ is a Hilbert space. Further, we consider an operator

$$A = \begin{bmatrix} A_{11} & A_{13} & 0 \\ 0 & A_{33} & A_{13}^+ \\ 0 & 0 & A_{11}^+ \end{bmatrix} \quad \text{w.r.t. } \mathscr{H} = \mathscr{L} \dotplus \mathscr{H}_3 \dotplus \mathscr{M}$$

with the following properties: A_{33} is a compact positive operator in the Hilbert space $\mathscr{H}_3$ such that $0 \in \sigma_c(A_{33})$ and all eigenvalues of A_{33} are simple. A_{11} is a bounded operator in $\mathscr{L}$ with

$$A_{11}^{\varkappa} = 0 \quad \text{and} \quad A_{11}^{\varkappa-1} \neq 0. \tag{3.2}$$

$A_{13} \in \mathscr{L}(\mathscr{H}_3, \mathscr{L})$ satisfies the condition

$$\mathscr{R}(A_{33}) \cap \mathscr{R}(A_{13}^+) = \{0\}.$$

Then, by Corollary 2.7, A is associated with the decomposition (3.1). It is easy to see that $\sigma(A)$ is contained in $[0, \infty)$ and all eigenvalues $\lambda \neq 0$ of A are algebraically simple. Since $\mathscr{L}$ is an A-invariant maximal nonpositive subspace of $\mathscr{H}$ we find (see [9, I.3.(b)])

$$[A^{2\varkappa}x, x] \geqslant 0, \quad x \in \mathscr{H}.$$

Let $\lambda_1 > \lambda_2 > \ldots$ be the eigenvalues of A. By E_i, $i = 1, 2, \ldots$, we denote the selfadjoint projection on the eigenspace corresponding to λ_i. $\mathscr{L} = \mathscr{L}^0$ coincides with the root space of A corresponding to 0. Then the relation $A_{11}^{\varkappa-1} \neq 0$ implies that the operators

$$\sum_{i=1}^{j} \lambda_i^{2\varkappa-2} E_i, \quad j = 1, 2, \ldots,$$

are not uniformly bounded ([7], [5, Theorem 5]). Let (ε_i) and (δ_i) be two monotonically decreasing sequences converging to 0 with the following properties:

(i) The intervals $(\lambda_i - \varepsilon_i, \lambda_i + \varepsilon_i)$, $i = 1, 2, \ldots$, are pairwise disjoint.

(ii) The relations $|\lambda_i' - \lambda_i| < \varepsilon_i$, $i = 1, 2, \ldots$, and $\|E_i' - E_i\| < \delta_i$, $i = 1, 2, \ldots$, where E_i' are positive projections of rank 1, imply that the operators $\sum_{i=1}^{j} \lambda_i'^{2\varkappa-2} E_i'$, $j = 1, 2, \ldots$, are not uniformly bounded. For all $\eta > 0$ we set

$$f_\eta(t) = \begin{cases} \eta & \text{if } t \leqslant \eta \\ t & \text{if } \eta < t. \end{cases}$$

Now we define a sequence (A_j) of selfadjoint operators in $\mathscr{H}$. We set

$$A_j := \begin{bmatrix} A_{11} & A_{13} & 0 \\ 0 & f_\eta(A_{33}) & A_{13}^+ \\ 0 & 0 & A_{11}^+ \end{bmatrix}$$

where η is chosen so small that

$$|\lambda_i(A_j) - \lambda_i| < \varepsilon_i, \quad \|E_i(A_j) - E_i\| < \delta_i, \quad i = 1, \ldots, j,$$

where $\lambda_i(A_j)$ is the i-th eigenvalue of A_j and $E_i(A_j)$ is the corresponding spectral projection. As above, we find

$$[A_j^{2\varkappa} x, x] \geqslant 0, \quad x \in \mathscr{H}. \tag{3.3}$$

Since 0 is an isolated eigenvalue of A_j, $j = 1, 2, \ldots$, the root space of A_j corresponding to 0 is non-degenerate. We have $\sigma(A_j) \subset [0, \infty)$.

Now we choose a compatible Hilbert norm $\|\cdot\|$ in $\mathscr{H}$. Obviously, the linear space

$$\mathscr{K} := \left\{ (x_j)_{j=1}^{\infty} : x_j \in \mathscr{H}, \sum_{j=1}^{\infty} \|x_j\|^2 < \infty \right\}$$

provided with the indefinite form $[\cdot,\cdot]$ defined by

$$[\mathbf{x}, \mathbf{y}] = \sum_{j=1}^{\infty} [x_j, y_j], \quad \mathbf{x} := (x_j),\ \mathbf{y} := (y_j),$$

is a Kreĭn space. By P_j, $j = 1, 2, \ldots$, we denote the projections defined by $P_j((x_l)_{l=1}^{\infty}) = x_j$. Let B denote the bounded selfadjoint operator in $\mathscr{K}$ defined by

$$P_j B P_j = A_j, \quad P_l B P_j = 0, \quad l, j = 1, 2, \ldots, \quad l \neq j.$$

By (3.3) we have

$$[B^{2\varkappa}\mathbf{x}, \mathbf{x}] \geqslant 0, \quad \mathbf{x} \in \mathscr{K}. \tag{3.4}$$

The relations $\sigma(A_j) \subset [0,\infty)$, $j = 1, 2, \ldots$, imply $\sigma(B) \subset [0,\infty)$. Since the root spaces of A_j corresponding to 0 are non-degenerate, the root space of B corresponding to 0 is also non-degenerate. Let E_B be the spectral function of B. From (3.4) we find that $\int t^{2\varkappa}\,\mathrm{d}E_B(t)$ converges. We claim that $\int t^{2\varkappa-2}\,\mathrm{d}E_B(t)$ does not converge. Indeed, choose $z \in \mathscr{H}$ so that the sequence

$$\left(\left[\sum_{i=1}^{j} \lambda_i(A_j)^{2\varkappa-2}E_i(A_j)z, z\right]\right), \quad j = 1, 2, \ldots, \tag{3.5}$$

is not bounded. Define elements $\mathbf{z}^{(j)} \in \mathscr{K}$, $j = 1, 2, \ldots$, by $P_j\mathbf{z}^{(j)} = z$ and $P_l\mathbf{z}^{(j)} = 0$, $l \neq j$. The sequence (3.5) coincides with

$$\left(\left[\int_{[\lambda_j-\varepsilon_j,\infty)} t^{2\varkappa-2}\,\mathrm{d}E_B(t)\mathbf{z}^{(j)}, \mathbf{z}^{(j)}\right]\right), \quad j = 1, 2, \ldots .$$

Since the sequence $(\|\mathbf{z}^{(j)}\|)$, $j = 1, 2, \ldots$, is bounded in $\mathscr{K}$ the operators $\int_{[\lambda_j-\varepsilon_j,\infty)} t^{2\varkappa-2}\,\mathrm{d}E_B(t)$, $j = 1, 2, \ldots$, are not uniformly bounded. This fact, the relations (3.4), and $\sigma(B) \subset [0,\infty)$ imply that the root space of B corresponding to 0 is not orthocomplemented.

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