# COCYCLES ON TREES

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Let $X$ be an oriented tree, endowed with an orientation preserving action of the locally compact group $G$. In their paper [2], Julg and Valette constructed in a geometric way a Kasparov bimodule $\gamma \in \mathscr{E}_G(\mathbf{C}, \mathbf{C})$ [4] and showed that the class of $\gamma$ in $\mathrm{KK}_G(\mathbf{C}, \mathbf{C})$ coincides with $1_G$, the unit of the ring $\mathrm{KK}_G(\mathbf{C}, \mathbf{C})$. They achieved this by exhibiting a homotopy $\gamma_t$, $t \in [0, \infty]$, (i.e. an element in $\mathscr{E}_G(\mathbf{C}, C([0, \infty]))$) such that $\gamma_\infty = \gamma$ and $\gamma_0 = 1_G$. If $G = \mathbf{F}_n$ is the free group on $n$-generators acting on its natural tree, the Kasparov bimodule $\gamma$ reduces to a construction done in [7] for the computation of the K-groups of the reduced crossed products by $\mathbf{F}_n$. Moreover, the fact that $\gamma = 1_{\mathbf{F}_n}$ was needed in order to carry over the above computation. (To be more specific one needs that $j_r(\gamma)$ is the identity in the ring $\mathrm{KK}(\mathbf{F}_n \!\mid\! \times_r A, \mathbf{F}_n \!\mid\! \times_r A)$, where $j_r : \mathrm{KK}_G(\mathbf{C}, \mathbf{C}) \to \mathrm{KK}(G \!\mid\! \times_r A, G \!\mid\! \times_r A)$ is the map defined by Kasparov in [4] and [5]; (see [6]).) Lacking $G$-equivariant KK-theory, most constructions in [7] seem complicated and unnatural.

The aim of this paper is to show that the homotopy exhibited in [7] for the free group can be naturally constructed out of the tree $X$. The idea that was behind this homotopy is to perturb the action of $G$ on the vertices of $X$ by an explicit cocycle, so we construct the cocycle first. This leads to a continuous family $\gamma_w$ of Kasparov bimodules, with $w = (w_y)$ running over the direct product $\prod U(2)$ of $N = \mathrm{card}\,\Sigma^1$ copies of the unitary group $U(2)$, where $\Sigma^1$ is the orbit space of the edges of $X$. If $w = (w_y)$ is the constant $N$-tuple $w_y = 1$ we get $\gamma_w = \gamma$ and if $w_y$ is the transposition matrix $\tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, for every $y \in \Sigma^1$, we get $\gamma_w = 1_G$. The explicit knowledge of the cocycle is useful for computations. We illustrate this by computing the character of the 1-summable Fredholm module $\gamma_w$ [1]. This simplifies some of the computations of Julg and Valette in [3] and shows that their results remain true even if one drops the condition that the tree is uniform locally finite.

Let $X^0$ denote the set of vertices of the tree $X$. By an action of the locally compact group $G$ on $X$, we shall mean a continuous action $G \times X^0 \to X^0$, denoted

$(g, x) \mapsto gx$, that preserves the natural distance $d$ on $X^0$. This determines a conti-
nuous action $(g, y) \mapsto gy$ of $G$ on $X^1$, the set of edges of the tree $X$. We shall
denote by $\Sigma^1 \subset X^1$ a fixed transversal for this action of $G$ on $X^1$, that is a complete
system of representatives of the orbit space $G \backslash X^1$, and we shall denote by $\hat{y} \in \Sigma^1$
the class of the edge $y \in X^1$. The set $Y$ of oriented edges will be identified with
the pairs $(x', x'')$ of vertices such that $d(x', x'') = 1$. If $y$ is an oriented edge
$y = (x', x'')$ we shall denote by $\bar{y}$ the oriented edge $(x'', x')$ and by $|y|$ the (un-
oriented) edge determined by $y$. We shall also fix an orientation of the tree, i.e.
a cross-section of the map $Y \ni y \mapsto |y| \in X^1$, and consider $X^1$ as a subset of $Y$.
We shall moreover assume that the action of $G$ preserves this orientation, that
is the action of $G$ on $Y$ leaves $X^1$ invariant. If $x', x'' \in X^0$ are two vertices we
shall denote by $[x', x'']$ the geodesic joining $x'$ with $x''$. We shall say that the
oriented edge $y$ belongs to the geodesic $[x', x'']$ if $y$ is of the form $y = (x^i, x^{i+1})$,
$x' = x^1, x^2, \ldots, x^n = x''$ being the path that defines this geodesic.

DEFINITION. Suppose that $\pi : G \to L(H)$ is some unitary representation of
the group $G$ on the Hilbert space $H$. We shall say that a function

$$c : X^0 \times X^0 \to L(H)$$

is a *cocycle on $X$* for $\pi$, if $c$ satisfies the following conditions:
  1) $c(x, x) = 1$, $c(x^2, x^1) = c(x^1, x^2)^*$
  2) $c(x^1, x^2)c(x^2, x^3) = c(x^1, x^3)$
  3) $c(gx^1, gx^2) = \pi(g)c(x^1, x^2)\pi(g^{-1})$
for every $x, x^1, x^2, x^3 \in X^0$ and every $g \in G$.

Note that if we fix a vertex $x^0$, then the function $g \mapsto c(x^0, gx^0)$ is a cocycle
for $\pi$, that is $g \mapsto c(x^0, gx^0)\pi(g)$ is a unitary representation of $G$ on $H$.

The cocycle $c$ on $X$ is completely determined by its values on $\Sigma^1$. We shall
call the restriction of $c$ to $\Sigma^1$ the *generating function* of the cocycle. It satisfies

$$c(y)\pi(g) = \pi(g)c(y)$$

for every $g$ in the stabilizer of the edge $y$, and every $y \in \Sigma^1$. Conversely, every
unitary valued function $c : \Sigma^1 \to U(H)$ with the above property determines a
cocycle on $X$ for $\pi$, still denoted by $c$, for properties 1) and 3) uniquely determine
$c$ on the set of oriented edges, and since $X$ is a tree, property 1) shows that the
value

$$c(x^1, x^n) := c(x^1, x^2)c(x^2, x^3) \ldots c(x^{n-1}, x^n)$$

does not depend on the particular path $x^1, x^2, \ldots, x^n$ with fixed end points $x^1$
and $x^n$.

We shall construct now a particular class of cocycles on $X$ for the representation $\pi^0 : G \to U(\ell^2(X^0))$ induced by the action of $G$ on $X^0$.

If $u \in U(\mathbf{C}^2)$ is a unitary matrix and $(x', x'')$ is an oriented edge, we shall denote by

$$u(x', x'') \in U(\ell^2(X^0))$$

the unitary defined by

$$u(x', x'') = vuv^* + 1 - vv^*$$

where $v = v(x', x'') : \mathbf{C}^2 \to \ell^2(X^0)$ is the isometry defined on the basis of $\mathbf{C}^2$ by $v(e_1) = \delta_{x'}$, $v(e_2) = \delta_{x''}$.

If $w = (w_y)$, $y \in \Sigma^1$, is a card $\Sigma^1$-tuple of unitary matrices belonging to $U(\mathbf{C}^2)$, we shall denote by

$$c_w : X^0 \times X^0 \to U(\ell^2(X^0))$$

the cocycle on $X$ for the representation $\pi^0$, given by the generating function $c_w : \Sigma^1 \to U(\ell^2(X^0))$ defined for $y = (x', x'') \in \Sigma^1$ by

$$c_w(y) = w_y(x', x'').$$

It is easy to see that if we extend the function $w : \Sigma^1 \to U(\mathbf{C}^2)$ to all oriented edges by

$$w_y = w_{\hat{y}} \qquad \text{for } y \in X^1$$

$$w_y = \tau w_{\bar{y}}^* \tau \quad \text{for } y \in Y \backslash X^1$$

$\left( \tau \text{ being the transposition matrix } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)$, then the value of $c_w$ is given on every oriented edge $y = (x', x'')$ by

$$c_w(x', x'') = w_y(x', x'').$$

REMARK 1. If we fix a unitary matrix $w \in U(\mathbf{C}^2)$ such that $\tau w \tau = w^*$, then the formula

$$c_w(x', x'') = w(x', x'')$$

defines the cocycle corresponding to the constant function $w_y = w$, for every $y \in \Sigma^1$, without any reference to $\Sigma^1$.

Let us record for further use the following two straightforward properties of the cocycle $c_w$.

i) If $x'$, $x''$ are arbitrary vertices of the tree $X$ and $p(x', x'')$ denotes the orthogonal projection on the space spanned by the vectors $\{\delta_x \mid x \in [x', x'']\}$, then $p(x', x'')$ commutes with $c_w(x', x'')$ and

$$p(x', x'')c_w(x', x'') + 1 - p(x', x'') = c_w(x', x'').$$

ii) If $x' \neq x''$ belong to $X^0$, then:

$$\langle c_w(x', x'')\delta_{x''}, \delta_{x'} \rangle = \prod_{y \in [x', x'']} (w_y)_{1, 2}$$

where $(w_y)_{1, 2}$ is the 1, 2 entry of the matrix $w_y$.

Let now $x^0 \in X^0$ be a fixed origin of the tree $X$.

DEFINITION. We shall denote by $\pi_w^0$ the unitary representation of $G$ on $\ell^2(X^0)$ defined by

$$\pi_w^0(g) = c_w(x^0, gx^0)\pi^0(g).$$

In the sequel, $V : \ell^2(X^1) \rightarrow \ell^2(X^0)$ will denote the isometry defined by Julg and Valette in [2], induced by the map that sends each vertex $x \in X^0 \setminus \{x^0\}$ to the (unique) edge $y$ that has one extremity $x$ and lies on the geodesic $[x, x^0]$.

LEMMA 1. a) *If* $w : \Sigma^1 \rightarrow U(\mathbf{C}^2)$ *is the constant function* $w_y = 1$, *for every* $y \in \Sigma^1$, *then* $\pi_w^0 = \pi^0$. *If* $w$ *is the constant function* $w_y = \tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, *then* $\pi_w^0(g) =$ $= V\pi^1(g)V^* + p(x^0)$, *where* $\pi^1 : G \rightarrow U(\ell^2(X^1))$ *is the unitary representation induced by the action of* $G$ *on* $X^1$, *and* $p(x^0)$ *is the orthogonal projection on the space* $\mathbf{C}\delta_{x^0}$.

b) $$\pi_w^0(g) - \pi_{w'}^0(g) = p(x^0, gx^0)(\pi_w^0(g) - \pi_{w'}^0(g))$$

*for every* $w, w' : \Sigma^1 \rightarrow U(\mathbf{C}^2)$ *and every* $g \in G$.

c) *The function*

$$U(\mathbf{C}^2)^{\Sigma^1} \ni w \mapsto \pi_w^0(g) \in U(\ell^2(X^0))$$

*is norm continuous for every* $g \in G$.

*Proof.* a) If $w_y = \tau$ for every $y \in \Sigma^1$, then it is easy to see that

$$c_w(x', x'')\delta_x = \delta_x \quad \text{if} \quad x \notin [x', x'']$$

while if $x = x^i$ belongs to the geodesic $x' = x^1, x^2, \ldots, x^n = x''$ then

$$c_w(x', x'')\delta_{x^i} = \begin{cases} \delta_{x^{i+1}} & \text{for } i < n \\ \delta_{x^1} & \text{for } i = n. \end{cases}$$

An easy computation shows now that $c_w(x^0, gx^0)\pi^0(g) = V\pi^1(g)V^* + p(x^0)$.

Point b) follows from property i) of the cocycle $c_w$, while c) is obvious.

$$Q.E.D.$$

We shall denote by $H_w$, $w \in U(\mathbf{C}^2)^{\Sigma^1}$, the graded Hilbert space $\ell^2(X^0) \oplus \ell^2(X^1)$ endowed with the continuous $G$ action $\pi_w(g) = \pi_w^0(g) \oplus \pi^1(g)$. The Kasparov bimodule determined by the isometry $V$ will be denoted by $\gamma_w \in \mathscr{E}_G(\mathbf{C}, \mathbf{C})$. The preceding lemma obviously implies the following proposition.

PROPOSITION 2. *The above constructed family* $\gamma_w$, $w \in U(\mathbf{C}^2)^{\Sigma^1}$, *belongs to* $\mathscr{E}_G(\mathbf{C}, C(U(\mathbf{C}^2)^{\Sigma^1}))$. *Its restriction to the point* $w_y = 1$, *for every* $y \in \Sigma^1$, *is the Kasparov bimodule* $\gamma$ *constructed by Julg and Valette, while its restriction to the point* $w_y = \tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, *for every* $y \in \Sigma^1$ *is* $1_G$.

REMARK 2. Julg and Valette defined in [2] besides $V$ the isometry $U$ by $U = VS$, where $S : \ell^2(X^1) \to \ell^2(X^1)$ is given by $S \delta_{(x', x'')} = \varepsilon(x', x'')\delta_{(x', x'')}$, $\varepsilon$ being $\varepsilon(x', x'') = 1$ (resp. $-1$) if $d(x', x^0) > d(x'', x^0)$ (resp. $d(x', x^0) < d(x'', x^0)$). If we denote by $\gamma_w'$ the Kasparov bimodules determined on $H_w$ by $U$ we still get an element $\gamma' \in \mathscr{E}_G(\mathbf{C}, C(U(\mathbf{C}^2)^{\Sigma^1}))$. In this case $1_G$ corresponds to the constant function $w_y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, for every $y \in \Sigma^1$, while the restriction to the constant function $w_y = 1$ yields an element that obviously equals $\gamma$ in $KK_G(\mathbf{C}, \mathbf{C})$. Moreover it is easy to see that their homotopy corresponds to the path $[0, \infty] \ni t \mapsto w^t \in U(\mathbf{C}^2)^{\Sigma^1}$, where $w^t$ is the constant function

$$w_y^t = \begin{pmatrix} (1 - e^{-2t})^{1/2} & e^{-t} \\ -e^{-t} & (1 - e^{-2t})^{1/2} \end{pmatrix}$$

for every $y \in \Sigma^1$. Note that the above matrices satisfy $\tau w_y^t \tau = w_y^{t*}$ (see Remark 1).

It is easy to see from Lemma 1 b) that the Kasparov bimodules $\gamma_w$ are 1-summable Fredholm modules in the sense of Connes [1], so we pass to the computation of their characters.

LEMMA 3. a) *The trace of the finite rank operator*

$$p(x, gx)\pi_w^0(g)$$

*does not depend on the vertex* $x \in X^0$.

b) *The trace of the above operator is equal to*

$$\prod_{y \in [x, gx]} (w_y)_{1, 2}$$

*for every vertex* $x \in X^0$ *satisfying* $x \neq gx$ *and* $[g^{-1}x, x] \cap [x, gx] = \{x\}$. *If* $x$ *is fixed by* $g$, *then the above trace is* 1.

*Proof.* Let $x \in X^0$ be a vertex such that $[g^{-1}x, x] \cap [x, gx] = \{x\}$. Using property i) of the cocycle $c_w$ we get:

$$\text{tr}(p(x, gx)\pi^0_w(g)) = \text{tr}(p(x, gx)c_w(x, gx)p(x, gx)\pi^0(g)) =$$

$$= \text{tr}(p(x, gx)c_w(x, gx)\pi^0(g)p(g^{-1}x, x)) =$$

$$= \langle c_w(x, gx)\pi^0(g)\delta_x, \delta_x \rangle = \langle c_w(x, gx)\delta_{gx}, \delta_x \rangle.$$

Thus point b) of the lemma follows from property ii) of the cocycle $c_w$.

To prove a) note first that if $x \in X^0$ satisfies $gx \neq x$ and $[g^{-1}x, x] \cap \cap [x, gx] = \{x\}$, then every $x' \in X^0$ satisfying $[g^{-1}x', x'] \cap [x', gx'] = \{x'\}$ has to lie on the geodesic $[g^i x, g^{i+1}x]$ for some $i \in \mathbf{Z}$. Point b) then implies that

$$\text{tr}(p(x', gx')\pi^0_w(g)) = \text{tr}(p(x, gx)\pi^0_w(g)).$$

This shows that the trace of the operator $p(x, gx)\pi^0_w(g)$ is constant on the set of vertices satisfying $[g^{-1}x, x] \cap [x, gx] = \{x\}$.

Let now $x \in X^0$ be any vertex and let $x = x^1, \ldots, x^n = gx$ be the path that defines the geodesic $[x, gx]$. If $[g^{-1}x, x] \cap [x, gx] \neq x$, it follows that $g(x^2) = x^{n-1}$. The properties of the cocycle $c_w$ imply that

$$\text{tr}(p(x, gx)c_w(x, gx)\pi^0(g)) = \text{tr}(p(x, gx)c_w(x, x^2)c_w(x^2, gx)\pi^0(g)) =$$

$$= \text{tr}(c_w(x, x^2)p(x, gx)c_w(x^2, gx)\pi^0(g)) = \text{tr}(p(x, gx)c_w(x^2, gx)\pi^0(g)c_w(x, x^2)) =$$

$$= \text{tr}(p(x, gx)c_w(x^2, gx)c_w(gx, gx^2)\pi^0(g)) = \text{tr}(p(x, gx)c_w(x^2, gx^2)\pi^0(g)).$$

It is now easy to see that the last trace is equal

$$\text{tr}(p(x^2, gx^2)c_w(x^2, gx^2)\pi^0(g)).$$

Since $G$ acts without inversion we may proceed inductively until we find $x^i \in [x, gx]$ such that $[g^{-1}x^i, x^i] \cap [x^i, gx^i] = \{x^i\}$.                        Q.E.D.

PROPOSITION 4. *The character of the* 1-*summable Fredholm module* $\gamma_w$ *is given by*

$$\tau_w(g) = \begin{cases} 1 & \text{if } g \text{ has a fixed point in } X^0 \\ \prod\limits_{y \in [x, gx]} (w_y)_{1,2} & \text{otherwise, where } x \in X^0 \text{ is any vertex satisfying } gx \neq x \text{ and } [g^{-1}x, x] \cap [x, gx] = \{x\}. \end{cases}$$

*Proof.* We have to compute the trace of $\pi^0_w(g) - V\pi^1(g)V^*$. If we denote by $w_0$ the constant function $(w_0)_y = \tau$ for every $y \in \Sigma^1$, then

$$\text{tr}(\pi^0_w(g) - V\pi^1(g)V^*) = \text{tr}(\pi^0_w(g) - \pi^0_{w_0}(g) + p(x^0)) = \text{tr}(\pi^0_w(g) - \pi^0_{w_0}(g)) + 1.$$

By Lemma 1 $\mathrm{tr}(\pi_w^0(g) - \pi_{w_0}^0(g)) = \mathrm{tr}(p(x^0, gx^0)(\pi_w^0(g) - \pi_{w_0}^0(g)))$ and by the preceding lemma $\mathrm{tr}(p(x^0, gx^0)\pi_{w_0}^0(g)) = 1$. This shows that

$$\tau_w(g) = \mathrm{tr}(p(x^0, gx^0)\pi_w^0(g))$$

which together with the preceding lemma concludes the proof of the proposition.

Q.E.D.

REMARK 3. The same formula (with the same proof) holds for the characters $\tau_w'$ of the 1-summable Fredholm modules $\gamma_w'$ of Remark 2.

COROLLARY 5. *If* $w : \Sigma^1 \to U(\mathbf{C}^2)$ *satisfies* $(w_y)_{1, 2} = \lambda \in \mathbf{R}$, *for every* $y \in \Sigma^1$, *then the character of* $\gamma_w$ *is given by*

$$\tau_w(g) = \lambda^{p(g)}$$

*where* $p(g) = \inf\limits_{x \in X^0} d(x, gx)$.

*Proof.* The proof is a straightforward consequence of the preceding proposition once we notice that $(\tau w^* \tau)_{1, 2} = \overline{w}_{1, 2}$, for every $w \in L(\mathbf{C}^2)$, and that $p(g)$ is equal to the length of the geodesic $[x, gx]$, for every vertex satisfying $[g^{-1}x, x] \cap$ $\cap [x, gx] = \{x\}$. Q.E.D.

REMARK 4. In view of Remark 3, the preceding corollary holds for the 1-summable Fredholm modules $\gamma_w'$ too. Remark 2 then shows that the formula obtained in Chapter 5 of [3] holds for arbitrary trees.

Moreover, Corollary 5 shows that the results of Julg and Valette in [3] concerning the Selberg principle are true without any assumption on the tree $X$.

REFERENCES

1. CONNES, A., Non-commutative diferential geometry, *Publ. Math. I.H.E.S.*, **62**(1985), 41 - 144.

2. JULG, P.; VALETTE, A., K-amenability for $SL_2(\mathbf{Q}_p)$ and the action on the associated tree, *J. Funct. Anal.*, **58**(1984), 194—215.

3. JULG, P.; VALETTE, A., Twisted coboundary operator on a tree and the Selberg principle, *J. Operator Theory*, **16**(1986), 285—304.

4. KASPAROV, G. G., K-theory, group $C^*$-algebras and higher signatures (conspectus), preprint Chernogolovka, 1981.

5. KASPAROV, G. G., Operator K-theory and its applications (Russian), in *Itogi Nauki i Tehniki, Sovremenie problem matematiki*, **27**(1985), 3—31.

6. PIMSNER, M., KK-groups of crossed products by groups acting on trees, *Invent. Math.*, **86**(1986), 603—634.

7. PIMSNER, M.; VOICULESCU, D., K-groups of reduced crossed products by free groups, *J. Operator Theory*, **8**(1982), 131 156.

8. SERRE, J.-P., *Arbres, amalgames*, $SL_2$, Astérisque, **46**(1977).

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