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CLASSIFICATION OF PROJECTIVE MODULES OVER THE UNITIZED GROUP C^*-ALGEBRAS OF CERTAIN SOLVABLE LIE GROUPS

ALBERT J. L. SHEU

0. INTRODUCTION

In this paper, the classification of isomorphism classes of finitely generated projective modules over the unitized group C^*-algebras of certain solvable Lie groups (regarded as some kind of "non-commutative spheres" [6]) is reduced to a long standing open problem, i.e. the classification of isomorphism classes of complex vector bundles over spheres. More precisely, it is proved that the cancellation law [4] holds for projections (identified with finitely generated projective modules in a well-known way) of dimension $\geqslant 1$ over $C^*(G)^+$ [6] of the solvable Lie groups G under consideration, and the semigroup of unitary equivalence classes of projections of dimension zero is isomorphic to the semigroup of isomorphism classes of complex vector bundles over $S^{\dim(G)-2}$.

In the first section, we prove a general cancellation property for stable C^*-algebras which is of independent interest, and in Section 2, the structure of finitely generated projective modules (identified with projections) over $(C(S^m)\otimes K)^+$ is discussed.

1. CANCELLATION PROPERTY FOR PROJECTIONS OVER $(A\otimes K)^+$

Let A be a unital C^*-algebra and $M_n(A)$ be the $n\times n$ matrix algebra over A. For x in $M_n(A)$ and y in $M_m(A)$, the element $\begin{pmatrix} x & 0\\ 0 & y\end{pmatrix}$ in $M_{n+m}(A)$ is denoted by $x\oplus y$. We define $P_\infty(A)$ (resp. $U_\infty(A)$), the set of projections over A (resp. the set of unitaries over A), to be the direct limit of $P_n(A)$ (resp. $U_n(A)$), where $P_n(A)$ (resp. $U_n(A)$) is the set of all self-adjoint idempotents (resp. all unitaries) in $M_n(A)$ and each p in $P_n(A)$ (resp. u in $U_n(A)$) is identified with $p\oplus 0$ in $P_{n+1}(A)$ (resp. $u\oplus 1$ in $U_{n+1}(A)$).

Two projections p and q over A are called unitarily equivalent (over A) if there is a unitary u over A such that $upu^{-1}=q$ (in $M_n(A)$ for some large n), and they are called stably equivalent (over A) if $p \oplus I_m$ and $q \oplus I_m$ are unitarily equivalent for some m in $\mathbf{N}$ where I_m is the identity matrix in $M_m(A)$.

It is well-known that, under direct summation $\oplus$, the set of unitary equivalence classes of projections over A form a semigroup, denoted by $P(A)$. Clearly, the cancellation law holds in a subsemigroup S containing some I_m of $P(A)$ if and only if any two stably equivalent projections p and q with $[p]$ and $[q]$ in S are indeed unitarily equivalent (over A).

For any C^*-algebra A (maybe non-unital), we denote the quotient map "mod A" from $M_n(A^+)$ to $M_n(\mathbf{C})$ by Q_A. Then the dimension $\dim(p)$ of any p in $P_n(A^+)$ is defined to be the rank of $Q_A(p)$ in $P_n(\mathbf{C})$. Clearly the set $P^n(A^+)$ of the unitary equivalence classes of projections of dimension $\geqslant n$ over A^+ is a subsemigroup of $P(A^+)$. We shall say that the cancellation law holds for projections of dimension $\geqslant n$ over A^+ if the cancellation law holds in the semigroups $P^n(A^+)$.

THEOREM 1. *The cancellation law holds for projections of dimension* $\geqslant 1$ *over* $(A\otimes K)^+$ *for any* C^*-*algebra* A.

Proof. Since the algebraic direct limit of $M_n(A)$'s is dense in $A\otimes K$, it is easy to see (by using functional calculus) that every projection over $(A\otimes K)^+$ is unitarily equivalent (over $(A\otimes K)^+$) to a "standard" projection p over $M_n(A)^+$ for some large n, that is, $Q_{M_n(A)}(p) = I_k$ in $P_\infty(M_n(A)^+)$, and any two projections over $M_n(A)^+$ $(\subseteq (A\otimes K)^+)$ are stably equivalent over $M_N(A)^+$ $(\supseteq M_n(A)^+)$ for some large N if and only if they are stably equivalent over $(A\otimes K)^+$.

So, in order to prove the theorem, we only need to prove that if $p, q \in P_m(M_n(A)^+)$ are standard and stably equivalent over $M_n(A)^+$ then p and q are unitarily equivalent over $M_N(A)^+$ $(\supseteq M_n(A)^+)$ for some large N. Without loss of generality, we may assume $n = 1$.

Let I_j be the identity of $M_j(A^+)$ and u be an element of $U_{m+j}(A^+)$ such that $u\cdot(p\oplus I_j)\cdot u^{-1} = q\oplus I_j$. Then $\pi(p\oplus I_j) = I_k\oplus 0_{m-k}\oplus I_j = \pi(q\oplus I_j)$ where $\pi = Q_A$ and 0_{m-k} is the zero element of $M_{m-k}(A^+)$. Hence $\pi(u)$ is of the form $\begin{pmatrix} a & 0 & b\\ 0 & x & 0\\ c & 0 & d\end{pmatrix}$ where $\begin{pmatrix} a & b\\ c & d\end{pmatrix} \in U_{k+j}(\mathbf{C})$ $(\subseteq U_{k+j}(A^+))$ and $x \in U_{m-k}(\mathbf{C})$.

Let $\begin{pmatrix} a(t) & b(t)\\ c(t) & d(t)\end{pmatrix}$ and $x(t)$ be paths of unitaries (of the right sizes) over $\mathbf{C}$ such that $\begin{pmatrix} a(1) & b(1)\\ c(1) & d(1)\end{pmatrix} = \begin{pmatrix} a & b\\ c & d\end{pmatrix}$, $\begin{pmatrix} a(0) & b(0)\\ c(0) & d(0)\end{pmatrix} = I_{k+j}$, $x(1) = x$ and $x(0) = I_{m-k}$. We define $u_t = \begin{pmatrix} a(t) & 0 & b(t)\\ 0 & x(t) & 0\\ c(t) & 0 & d(t)\end{pmatrix}^{-1}\cdot u$. Then $\pi(u_t(p\oplus I_j)u_t^{-1}) = I_k\oplus 0_{m-k}\oplus I_j$.

For $T \in M_{m(j+1)}(A^+)$, we define $\varphi(T)$ to be $v \cdot T \cdot v^* \in M_{m(j+1)}(A^+) = M_m(M_{j+1}(A^+))$, where v is a unitary in $U_{m(j+1)}(\mathbf{C})$ such that $v(e_i)$ is equal to $e_{(i-1)(j+1)+1}$ if $1 \leqslant i \leqslant m$ and is equal to $e_{h(j+1)+i-m-hj+1} = e_{h+i-m+1}$ if $hj < i - m \leqslant (h+1)j$, where $\{e_i\}$ is the standard orthonormal basis for $\mathbf{C}^{m(j+1)}$.

Let $q_t = (u_t(p \oplus I_j)u_t^{-1}) \oplus I_{(k-1)j} \oplus 0_{(m-k)j}$ which is meaningful since $k \geqslant 1$. Then $\varphi(q_t)$ is a path of projections in $M_m(M_{j+1}(A^+))$. Since $\pi(q_t) = I_k \oplus 0_{m-k} \oplus I_j \oplus I_{(k-1)j} \oplus 0_{(m-k)j} = I_k \oplus 0_{m-k} \oplus I_{kj} \oplus 0_{(m-k)j}$, we get $\pi(\varphi(q_t)) = I_{k(j+1)} \oplus 0_{(m-k)(j+1)}$, so $\varphi(q_t) \in M_m(M_{j+1}(A)^+) \subseteq M_m((A \otimes K)^+)$. Furthermore, $q_0 = q \oplus I_j \oplus I_{(k-1)j} \oplus 0_{(m-k)j}$, so $\varphi(q_0) = q$ if we embed A^+ into $M_{j+1}(A)^+ = M_{j+1}(A)^+$ canonically by sending a in A to $a \oplus 0_j$ in $M_{j+1}(A)$ (since $\pi(q) = I_k \oplus 0_{m-k}$). Similarly, we have $\varphi(p \oplus I_j \oplus I_{(k-1)j} \oplus 0_{(m-k)j}) = p$ if we embed A^+ into $M_{j+1}(A)^+ = M_{j+1}(A)^+$ as above. Now

$$\varphi(q_1) = \varphi(u_1(p \oplus I_j)u_1^{-1} \oplus I_{(k-1)j} \oplus 0_{(m-k)j}) = = w \cdot \varphi(p \oplus I_j \oplus I_{(k-1)j} \oplus 0_{(m-k)j}) \cdot w^{-1} = w \cdot p \cdot w^{-1},$$

where $w = \varphi(u_1 \oplus I_{(k-1)j} \oplus I_{(m-k)j})$ is in $U_m(M_{j+1}(A)^+)$ since $\pi(u_1) = \pi(\pi(u)^{-1}u) = I_{m+j} \in U_{m+j}(\mathbf{C})$.

So $q = \varphi(q_0)$ is connected to $w \cdot p \cdot w^{-1} = \varphi(q_1)$ by a path of projections in $P_m(M_{j+1}(A)^+) \subseteq P_m((A \otimes K)^+)$ where $w \in U_m(M_{j+1}(A)^+) \subseteq U_m((A \otimes K)^+)$, so q is unitarily equivalent to p over $M_{j+1}(A)^+$ (hence over $(A \otimes K)^+$). Thus the proof is completed. Q.E.D.

REMARK. (1) The above proof can be modified to show that if p and q are projections of dimension k such that $p \oplus I_j$ is unitarily equivalent to $q \oplus I_j$ over A^+, then p is unitarily equivalent to q over $M_{N+1}(A)^+$ where N is the least integer greater than or equal to j/k and A^+ is embedded into $M_{N+1}(A)^+$ canonically as a unital subalgebra. Indeed, we can use each diagonal entry of I_k to "absorb" an I_N instead of an I_j.

(2) Since $\mathrm{K}_0((A \otimes K)^+) \simeq \mathrm{K}_0(A) \oplus \mathbf{Z}$, it is not hard to prove that $\mathrm{K}_0((A \otimes K)^+)_+$ is equal to $(\mathrm{K}_0(A)_+ \oplus 0) \cup (\mathrm{K}_0(A) \oplus \mathbf{N})$ (preserving the natural semigroup structure on each component), where $\mathrm{K}_0(A)_+$ denotes the positive cone of $\mathrm{K}_0(A)$, namely the set of classes $[p]$ in $\mathrm{K}_0(A)$ such that p is a projection in some $M_n(A)$, for any C^*-algebra A unital or not. In fact, let p_n be the identity in $M_n(A)$ if A is unital and be the identity in $M_n(A^+)$ if A is non-unital. Then, for unital A, any element of the form $([p] - [p_n], k)$ in $\mathrm{K}_0(A) \oplus \mathbf{N}$ with p in $P_\infty(A)$ is identified with $[p \oplus (I_1 - p_n) \oplus I_{k-1}]$ in $\mathrm{K}_0((A \otimes K)^+)_+$. For non-unital A, any element of $\mathrm{K}_0(A)$ is of the form $[p] - [p_n]$ where $p = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} p_n & 0 \\ 0 & 0 \end{pmatrix}$ with a in $M_n(A)$ and b, c, d, matrices over A, and we identify $([p] - [p_n], k)$ in $\mathrm{K}_0(A) \oplus \mathbf{N}$ with $\begin{pmatrix} a + I_1 & b \\ c & d \end{pmatrix} \oplus I_{k-1}$ where $\begin{pmatrix} a + I_1 & b \\ c & d \end{pmatrix}$ is regarded as a two by two matrix over $(A \otimes K)^+$.

From Theorem 1, it is easy to prove that $P((A \otimes K)^+) = (P(A) \oplus 0) \cup \cup (K_0(A) \oplus \mathbf{N})$ where $P(A) = \{[p] \text{ in } P(A^+) \mid p \text{ is a projection in some } M_n(A)\}$ if A is not unital. In case $A = \mathbf{C}$, we have $P(K^+) = \{p_n \oplus I_k \mid n \geqslant 0, k \geqslant 0\} \cup \cup \{p_n \oplus I_{k-1} \mid n < 0, k \geqslant 1\}$ where $p_n = I_1 - p_{-n}$ in K^+ if $n < 0$.

2. THE STRUCTURE OF PROJECTIONS OVER $(C(S^m) \otimes K)^+$

It is well-known that $C(S^m) \otimes K \simeq C(S^m, K)$ and $K_0((C(S^m) \otimes K)^+)$ is either $\mathbf{Z} \oplus \mathbf{Z}$ (if m is odd) or $\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$ (if m is even). In either case, let us assume that the last copy of $\mathbf{Z}$ corresponds to the dimensions of projections over $(C(S^m) \otimes K)^+$.

By Remark 2 of Section 1, we have $P(C(S^m, K)^+)$ equal to $\mathrm{VB}(S^m) \cup (\mathbf{Z} \oplus \mathbf{N})$ if m is odd and equal to $\mathrm{VB}(S^m) \cup (\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{N})$ if m is even, where $\mathrm{VB}(S^m)$ is the semigroup of isomorphism classes of complex vector bundles over S^m which is well known to be isomorphic to $P(C(S^m))$. In order to prove Theorem 2 of Section 3, we need more detailed description of projections over $C(S^m, K)^+$.

Let $S^m = \mathbf{R}^m \cup \{\infty\}$. Then clearly the exact sequence $0 \to C_0(\mathbf{R}^m, K) \to \to C(S^m, K)^+ \xrightarrow{\pi} K^+ \to 0$ splits $(K^+ \subseteq C(S^m, K)^+)$ where $\pi(f) = f(\infty)$ for $f \in C(S^m, K)^+ \subseteq C(S^m, K^+)$. So we may assume that the induced map $\pi_*: K_0(C(S^m, K)^+) \to K_0(K^+)$ sends (n, k) to (n, k) if m is odd and (n, r, k) to (n, k) if m is even.

We shall analyze the structure of projections of dimension $\geqslant 1$ over $C(S^m, K)^+$, for which, by Theorem 1, the cancellation law holds. For simplicity, we shall assume that m is even, hence $K_0(C(S^m, K)^+) = \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$. The case in which m is odd can be treated by exactly the same argument.

Let p be a projection of dimension $k \geqslant 1$ over $C(S^m, K)^+$, say $[p] = (n, r, k) \in \in K_0(C(S^m, K)^+)$, then $[p(\infty)] = \pi_*([p]) = (n, k) \in K_0(K^+)$. So by the classification of projections over K^+ in Section 1, we get $p(\infty)$ unitarily equivalent over K^+ to either $I_k \oplus p_n$ (if $n \geqslant 0$) or $I_{k-1} \oplus p_n$ (if $n < 0$). Since $K^+ \subseteq C(S^m, K)^+$, we get p unitarily equivalent to some projection p' such that $p'(\infty) = I_k \oplus p_n \oplus 0_{j-1}$ or $I_{k-1} \oplus p_n \oplus 0_j$ for some j. By a homotopy, p' is again unitarily equivalent to some "standard" projection, i.e. a projection p'' over $C(S^m, K)^+$ such that $p''(x) = = p''(\infty)$ equal to either $I_k \oplus p_n \oplus 0_{j-1}$ (if $n \geqslant 0$) or $I_{k-1} \oplus p_n \oplus 0_j$ (if $n < 0$) for all x in $\mathbf{R}^m$ with $|x| \geqslant 1$, where $\dim(p'') = k \geqslant 1$. Thus without loss of generality, we may only consider "standard" projections (of dimension $\geqslant 1$) over $C(S^m, K)^+$.

By a similar argument as used in [6] for the case of $C_0(\mathbf{R}^m, K)^+$, for any "standard" projection p of dimension $k \geqslant 1$ in $P_{k+j}(C(S^m, K)^+)$, there is a unitary u in $U_{k+j}(C_b(\mathbf{R}^m, K^+))$ such that $u \equiv I_{k+j} \bmod K$ and $u(x)p(x)u(x)^{-1} = p(1, 0, 0, \ldots \ldots, 0) = p(\infty)$ for all x, hence, at $|x| \geqslant 1$, $u = v + w$ where $v = p(\infty) \cdot u \cdot p(\infty)$ and

$w = (I_{k+j} - p(\infty))u(I_{k+j} - p(\infty))$ are partial isometries such that $v \cdot w = w \cdot v = 0$. As in [6] it can be shown that the class $[v|S^{m-1}]$ in $\pi_0(V_1(p(\infty))) \simeq K^1(S^{m-1})$ (note that $k \geqslant 1$) is independent of the choice of u, where $V_1(p(\infty)) = \{T \mid T \text{ is a unitary in } p(\infty)M_{k+j}(C(S^{m-1}, K)^+)p(\infty) \text{ such that } T \equiv I_k \oplus 0_j \mod K\}$. In fact, if $[p]=(n, r, k)$, then $[v|S^{m-1}] = r \in \mathbf{Z} \simeq K^1(S^{m-1})$ in the case of even m. By abuse of language, we shall denote such v by v_p.

Conversely, for any v in $V_1(I_k \oplus p_n \oplus 0_{j-1})$ with $n \geqslant 0$ or $V_1(I_{k-1} \oplus p_n \oplus 0_j)$ with $n < 0$, we may construct a standard p with $v_p \mid S^{m-1} = v$, when j is sufficiently large, say $j \geqslant k+1$, by reversing the above process and using v^* as in [6].

3. THE CLASSIFICATION OF PROJECTIONS OVER $C^*(G)^+$ FOR CERTAIN SOLVABLE LIE GROUPS G

By the works of J. Rosenberg [5] and P. Green [2], we have a short exact sequence of C^*-algebras

$$0 \to C(S^{m-1}) \otimes K \xrightarrow{i} C^*(G) \to C_0(\mathbf{R}) \to 0$$

for the solvable Lie groups G of the form $\mathbf{R}^m \times \mathbf{R}$ with all roots of the $\mathbf{R}$-action contained in the right open half plane of $\mathbf{C}$. In the following discussion, G shall be such a Lie group.

By Lemma 5.1 of [6], every projection over $C^*(G)^+$ is unitarily equivalent (over $C^*(G)^+$) to a projection over $C(S^{m-1}, K)^+ \subseteq C^*(G)^+$ since $K_0(C_0(\mathbf{R})) = 0$ and the cancellation law holds for projections over $C_0(\mathbf{R})^+ \cong C(S^1)$.

Since $K_0(C^*(G)) = K_0(C_0(\mathbf{R}^{m-1}))$ by Connes' Thom isomorphism theorem [1], we get $K_0(C^*(G)) \oplus \mathbf{Z} \cong K_0(C(S^{m-1}) \otimes K)$ and $K_0(C^*(G))$ is either $\mathbf{Z}$ (if m is even) or $\mathbf{Z} \oplus \mathbf{Z}$ (if m is odd). Moreover the map $i_*\colon K_0(C(S^{m-1}, K)^+) \to$ $\to K_0(C^*(G)^+)$ is surjective.

Using P. Green's method [2] of constructing an isometry (essentially a unilateral shift) of index one in $(C_0((-\infty, \infty]) \times_\tau \mathbf{R})^+$, we can construct an isometry of index one in $C^*(G)^+$. More precisely, we can decompose $\mathscr{H}$, on which the K is acting, into $\mathscr{H}_1 \oplus \mathscr{H}_2$ ($\dim(\mathscr{H}_1) = \dim(\mathscr{H}_2) = \infty$) and find an $\mathscr{S} \in C^*(G)^+ \subseteq$ $\subseteq C(S^{m-1}, \mathscr{L}(\mathscr{H}))$ such that $\mathscr{S}(x)|\mathscr{H}_1$ is the unilateral shift and $\mathscr{S}(x)|\mathscr{H}_2$ is the identity map, for all $x \in S^{m-1}$. In fact, without loss of generality, we may assume that $C^*(G) \simeq C_0(\mathbf{R}^m) \times \mathbf{R}$ with $\mathbf{R}$ acting on $\mathbf{R}^m$ by $r \cdot (x_1, \ldots, x_m) = (e^r x_1, \ldots, e^r x_m)$ [5]. For each x in S^{m-1}, if we identify tx with $(-\ln t)x$ for $t > 0$, then the action of $\mathbf{R}$ on $R_x = \{tx \mid t \geqslant 0\}$ can be identified with the translation action τ of $\mathbf{R}$ on $(-\infty, \infty]$. Under this identification, the restriction of elements of $C_0(\mathbf{R}^m)$ to R_x for x in S^{m-1} gives rise to a homomorphism π_x from $C^*(G)$ to $C_0((-\infty, \infty]) \times_\tau \mathbf{R}$ which acts on $L^2(\mathbf{R})$ faithfully. These π_x's define a faithful homomorphism π from

$C^*(G)$ to $C(S^{m-1})\otimes(C_0((-\infty,\infty])\times_\tau \mathbf{R})$. Let $F(r,x) = e^{-r/2}\chi(r)\chi(-\ln(|x|)-r)$ for r in $\mathbf{R}$ and x in $\mathbf{R}^m$. (It is understood that $\chi(-\ln(0)-r) = 1$ for all r.) Then as in [2], $F(r,x)$ determines an element T of $C_0(\mathbf{R}^m)\times\mathbf{R}\simeq C^*(G)$ such that $I-\pi_x(T)$ acts on $L^2((-\infty,0))$ as the identity operator and acts on $L^2((0,\infty))$ as the unilateral shift [2, 3] for any x in S^{m-1}. Thus we may take $\mathscr{S} = 1 - T$ in $C^*(G)^+$, $\mathscr{H}_1 = L^2((0,\infty))$ and $\mathscr{H}_2 = L^2((-\infty,0))$.

Let p_n be the projection of rank $n\geqslant 0$ as before (in Section 1) and the range of p_n is assumed to be contained in $\mathscr{H}_1$. Let p be a standard projection of dimension $\geqslant 1$ over $C(S^{m-1},K)^+$ as discussed in Section 2, say $[p] = (n,r,k)\in$ $\in \mathrm{K}_0(C(S^{m-1},K)^+) = \mathbf{Z}\oplus\mathbf{Z}\oplus\mathbf{Z}$ in case m is odd (resp. $(n,k)\in \mathrm{K}_0(C(S^{m-1},K)^+) =$ $= \mathbf{Z}\oplus\mathbf{Z}$ in case m is even), where $k = \dim(p)$.

Case 1. $n\geqslant 0$. $p(\infty) = I_k\oplus p_n\oplus 0_{j-1}$ for some $j\geqslant 1$. We define $q = T\cdot p\cdot T^{-1}$ where

$$T = \begin{pmatrix} \mathscr{S}^n & 0 & p_n & 0 \\ 0 & I_{k-1} & 0 & 0 \\ 0 & 0 & (\mathscr{S}^*)^n & 0 \\ 0 & 0 & 0 & I_{j-1} \end{pmatrix} \quad (k\geqslant 1)$$

is a unitary in $M_{k+j}(C^*(G)^+)$. Then q is also a standard projection which is unitarily equivalent over $C^*(G)^+$ to p. Since $q(\infty) = T(\infty)\cdot p(\infty)\cdot T(\infty)^{-1} = I_k\oplus 0_j$ and $[v_q|S^{m-2}] = [(T\cdot v_p\cdot T^{-1})\,|\,S^{m-2}] = [v_p\,|\,S^{m-2}]$ in $\mathrm{K}^1(S^{m-2})$, we have $[q] = (0,r,k)$ in $\mathrm{K}_0(C(S^{m-1},K)^+)$ if m is odd (resp. $[q] = (0,k)$ if m is even), where v_p and v_q are as defined in Section 2.

Case 2. $n<0$. $p(\infty) = I_{k-1}\oplus p_n\oplus 0_j$ for some j. Similarly we define $q = T\cdot p\cdot T^{-1}$ where

$$T = \begin{pmatrix} I_{k-1} & 0 & 0 & 0 \\ 0 & (\mathscr{S}^*)^n & 0 & 0 \\ 0 & p_{-n} & \mathscr{S}^n & 0 \\ 0 & 0 & 0 & I_{j-1} \end{pmatrix} \quad (k\geqslant 1)$$

is a unitary in $M_{k+j}(C^*(G)^+)$. Then q is a standard projection which is unitarily equivalent over $C^*(G)^+$ to p. Since $q(\infty) = I_k\oplus 0_j$ and $[v_q\,|\,S^{m-2}] =$ $[(T\cdot v_p\cdot T^{-1})\,|\,S^{m-2}] = [v_p\,|\,S^{m-2}]$ in $\mathrm{K}^1(S^{m-2})$, we have $[q] = (0,r,k)$ if m is odd (resp. $[q] = (0,k)$ if m is even).

Thus, in either case, we have $i_*([p]) = i_*([q])$. So $i_*((n,r,k)) = i_*((0,r,k))$ if m is odd and $i_*((n,k)) = i_*((0,k))$ if m is even, for all $k\geqslant 1$ and $n,r\in\mathbf{Z}$. Thus the map $i_*:\mathrm{K}_0(C(S^{m-1},K)^+)\to \mathrm{K}_0(C^*(G)^+)$ sends (n,r,k) to (r,k) if m is odd and sends (n,k) to k if m is even, for all $n,r,k\in\mathbf{Z}$.

Let p and p' be two standard projections of dimension $k \geqslant 1$ over $C(S^{m-1}, K)^+$ which are stably equivalent over $C^*(G)^+$. Then $i_*([p]) = i_*([p'])$ and hence by the above description of i_*, we get $[p] = (n, r, k)$ and $[p'] = (n', r, k)$ for some n, n' and r in $\mathbf{Z}$ if m is odd (resp. $[p] = (n, k)$ and $[p'] = (n', k)$ if m is even) in $K_0(C(S^{m-1}, K)^+)$. But p and p' are unitarily equivalent over $C^*(G)^+$ to some standard q and q' with $[q] = (0, r, k) = [q']$ if m is odd (resp. $[q] = (0, k) = [q']$ if m is even) in $K_0(C(S^{m-1}, K)^+)$, by the above discussion. Since the cancellation law holds for projections of dimension $\geqslant 1$ over $C_0(S^{m-1}, K)^+$, we get q unitarily equivalent over $C_0(S^{m-1}, K)^+$ to q'. Thus p and p' are unitarily equivalent over $C^*(G)^+$, and the cancellation law holds also for projections of dimension $\geqslant 1$ over $C^*(G)^+$.

We have seen that every projection over $C^*(G)^+$ "comes" from one over $C(S^{m-1}, K)^+$, in particular, every projection of dimension zero over $C^*(G)^+$ is unitarily equivalent (over $C^*(G)^+$) to a projection in $C(S^{m-1}, M_N(\mathbf{C}))$ for some large N. Any two such projections p and q in $C(S^{m-1}, M_N(\mathbf{C}))$ are unitarily equivalent over $C(S^{m-1}, \mathscr{L}(\mathscr{H})) \supseteq C(S^{m-1}, M_N(\mathbf{C}))$ only if the corresponding complex vector bundles E_p and E_q are isomorphic. Conversely, $E_p \cong E_q$ implies that p and q are unitarily equivalent over $C(S^{m-1}, K)^+$. So we get that two projections p and q in $C(S^{m-1}, M_N(\mathbf{C}))^+$ are unitarily equivalent over $C^*(G)^+$ if and only if $E_p \cong E_q$.

Let p be a projection in $C(S^{m-1}, M_N(\mathbf{C}))$, say $[p] = (n, r, 0)$ in $K_0(C(S^{m-1}, K)^+)$ if m is odd (resp. $[p] = (n, 0)$ if m is even). Then $[E_p] = (n, r)$ in $K^0(S^{m-1})$ (resp. $[E_p] = n$), and $i_*([p]) = (r, 0)$ in $K_0(C^*(G)^+)$ (resp. $i_*([p]) = 0$). So the cancellation law always fails for projections of dimension zero over $C^*(G)^+$ since we can always find projections p and q in $C(S^{m-1}, M_N(\mathbf{C}))$ such that $[E_p] = (n, r)$ and $[E_q] = (n + 1, r)$ (resp. $[E_p] = n$ and $[E_q] = n + 1$) for some large N and n and some r, and hence p and q are stably equivalent (since $i_*([p]) = i_*([q])$) but not unitarily equivalent ($E_p \ncong E_q$) over $C^*(G)^+$.

Now we may summarize what we got in the following theorem.

THEOREM 2. *For the solvable Lie groups G specified in the beginning of this section, we have*

(a) *The positive cone of* $K_0(C^*(G)^+)$ *is either* $\{(r, k) \in \mathbf{Z} \oplus \mathbf{Z} \mid k \geqslant 0\}$ *if* $\dim(G)$ *is even, or* $\{k \in \mathbf{Z} \mid k \geqslant 0\}$ *if* $\dim(G)$ *is odd.*

(b) *If* $\dim(G) = m + 1$ *is even,* $P(C^*(G)^+)$ *is equal to* $\mathrm{VB}(S^{m-1}) \cup (\mathbf{Z} \oplus \mathbf{N})$ *and, for* E_p *in* $\mathrm{VB}(S^{m-1})$ *with* $[E_p] = (n, r)$ *in* $K^0(S^{m-1})$ *and* (s, k) *in* $\mathbf{Z} \oplus \mathbf{N}$, $E_p + (s, k) = (r + s, k)$.

(c) *If* $\dim(G) = m + 1$ *is odd,* $P(C^*(G)^+)$ *is equal to* $\mathrm{VB}(S^{m-1}) \cup \mathbf{N}$ *and, for* E_p *in* $\mathrm{VB}(S^{m-1})$ *and* k *in* $\mathbf{N}$, $E_p + k = k$.

REMARK. (1) Theorem 2 shows that the classification of projections over $C^*(G)^+$ up to unitary equivalence is as hard as that of isomorphism classes of complex vector bundles over S^{m-1}.

(2) The group C^*-algebras $C^*(G)^+$ of the solvable Lie groups considered here and the nilpotent Lie groups considered in [6] are regarded as some kind of "non-commutative spheres" since they have the same K-groups as $C(S^{\dim(G)})$. However, if $\dim(G)$ is even, the positive cone of $K_0(C^*(G)^+)$ is equal to $\mathbf{Z}\oplus(\mathbf{N}\cup\{0\})$ if G is a solvable Lie group considered above, and equal to $(\mathbf{Z}\oplus\mathbf{N})\cup\cup\{(0,0)\}$ if G is a non-commutative nilpotent Lie group considered in [6], while the positive cone of $K_0(C(S^{\dim(G)}))$ is a proper subset of $(\mathbf{Z}\oplus\mathbf{N})\cup\{(0,0)\}$ in general since certain homotopy classes of clutching maps defined on $S^{\dim(G)-1}$ can only be used to construct vector bundles of sufficiently high dimension over $S^{\dim(G)}$. So the positive cones of the K_0-groups do give more information as expected.

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ALBERT J. L. SHEU
Department of Mathematics,
The University of Kansas,
Lawrence, Kansas 66045–2142,
U.S.A.

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