J. OPERATOR THEORY
18(1987), 237–248

SUBNORMAL OPERATORS IN A

PATRICK SULLIVAN

1. INTRODUCTION

Let $\mathscr{H}$ be a separable, infinite-dimensional, complex Hilbert space, and let $\mathscr{L}(\mathscr{H})$ denote the algebra of all bounded linear operators on $\mathscr{H}$. Let $\mathbf{D}$ denote the open unit disc in $\mathbf{C}$ and $\mathbf{T}$ denote the unit circle. If $T \in \mathscr{L}(\mathscr{H})$, then $\sigma(T)$ denotes the spectrum of T, and if $\|T\| \leqslant 1$ we say that T is an absolutely continuous contraction if the unitary part of T is absolutely continuous (or acts on the space (0)). For such T one can define $f(T)$ for f in $H^\infty(\mathbf{D})$ using the Sz.-Nagy—Foiaş functional calculus (cf. [13, Theorem III. 2.1] and [5, Theorem 3.2]). One property of this functional calculus is that $\|f(T)\| \leqslant \|f\|_\infty$ for all $f \in H^\infty(\mathbf{D})$. The class $\mathbf{A}$ consists, by definition, of all those absolutely continuous contractions T in $\mathscr{L}(\mathscr{H})$ for which $\|f(T)\| = \|f\|_\infty$ for all f in $H^\infty(\mathbf{D})$. One reason for studying the class $\mathbf{A}$ is the following theorem of C. Apostol [1, Theorem 2.2]:

THEOREM 1.1 *If T is an absolutely continuous contraction in $\mathscr{L}(\mathscr{H})$, $\sigma(T)$ contains* $\mathbf{T}$, *and T has no nontrivial hyperinvariant subspace, then T belongs to the class* $\mathbf{A}$.

Thus to solve the invariant subspace problem for contractions T for which $\sigma(T) \supset \mathbf{T}$ one can reduce to the case where T belongs to the class $\mathbf{A}$. There is a subclass of $\mathbf{A}$ called $\mathbf{A}_1$ (cf. [3, p. 35]), and it is known that if T belongs to the class $\mathbf{A}_1$, then T has a nontrivial invariant subspace [3, p. 36]. It is natural to ask which operators of various types that belong to $\mathbf{A}$ also belong to $\mathbf{A}_1$. It is known that all subnormal operators in $\mathbf{A}$ belong to $\mathbf{A}_1$; see [10, Theorem 1]. It has been conjectured that $\mathbf{A} = \mathbf{A}_1$ and much work has been done trying to verify this conjecture (cf. [2], [6]), but many questions remain open. For example, does a hyponormal operator in $\mathbf{A}$ belong to $\mathbf{A}_1$? Before attempting to answer a question like this, one would like to know which hyponormal operators belong to $\mathbf{A}$. In this paper we will discuss which subnormal operators belong to $\mathbf{A}$.

The class $\mathbf{A}$ is defined for arbitrary absolutely continuous contractions. However in the case of absolutely continuous subnormal contractions there is an alternative characterization of the class $\mathbf{A}$ which will now be described. If μ is any Borel measure then $P^\infty(\mu)$ denotes the weak-$*$ closure of the polynomials in $L^\infty(\mu)$.

If S is any subnormal operator and μ is a scalar spectral measure for the minimal normal extension of S then S belongs to **A** if and only if $P^\infty(\mu) = H^\infty(\mathbf{D})$, (cf. [8, Corollary III.12.10 and Theorem VII.4.5]). The above result uses the characterization of $P^\infty(\mu)$ developed in [12].

The characterization of subnormal operators in **A** given below is given in the language of the functional calculus of Sz.-Nagy and Foiaş, (cf. [13, Theorem III.2.3]).

2. PRELIMINARIES

If μ is a compactly supported, Borel measure on the complex plane **C**, and if f is a μ-essentially bounded Borel measurable function, then $\|f\|_\mu$ denotes the μ-essential supremum of f. If A is a Borel set, then $\mu \mid A$ denotes the Borel measure defined by $(\mu \mid A)(B) = \mu(A \cap B)$, and $L^2(\mu, A)$ denotes the set of all Borel measurable functions such that $\int_A |f|^2 \mathrm{d}\mu < \infty$. Let M_z be the operator on $L^2(\mu, A)$ defined by $(M_z f)(z) = zf(z)$ for all $f \in L^2(\mu, A)$. If ν is another compactly supported Borel measure on **C**, then $\mu \ll \nu$ means that μ is absolutely continuous with respect to ν, and $[\mu] = [\nu]$ means that $\mu \ll \nu$ and $\nu \ll \mu$. Let m denote Lebesgue arc-length measure on **T**. Let $L^\infty(\mathbf{T})$ be the set of m-essentially bounded, complex valued, measurable functions on **T**. Let $H^\infty(\mathbf{T})$ be the set of all functions in $L^\infty(\mathbf{T})$ such that all negative Fourier coefficients are equal to 0. Let $H^\infty(\mathbf{D})$ be the space of all bounded analytic functions on **D**. If $f \in H^\infty(\mathbf{D})$ then $\|f\|_\infty = \sup\{|f(\lambda)| : \lambda \in \mathbf{D}\}$. Recall that if $f \in H^\infty(\mathbf{D})$ then there exists a unique function $\hat{f} \in H^\infty(\mathbf{T})$ such that $\|f\|_\infty = \|\hat{f}\|_m$ and $\hat{f}(\mathrm{e}^{\mathrm{i}t}) = \lim_n f(\lambda_n)$ almost everywhere on **T** when $\lambda_n \to \mathrm{e}^{\mathrm{i}t}$ non-tangentially ($\hat{f}$ is called the boundary function of f). Also, if $g \in H^\infty(\mathbf{T})$ then there exists a unique function $f \in H^\infty(\mathbf{D})$ such that $\hat{f} = g$. If f belongs to $H^\infty(\mathbf{D})$, then $\tilde{f}\colon \mathbf{D}^- \to \mathbf{C}$ is the function defined by the following rule: $\tilde{f}(\lambda) = f(\lambda)$ if λ belongs to **D** and $\tilde{f}(\lambda) = \hat{f}(\lambda)$ if λ belongs to **T**.

If $V \subset \mathbf{D}$, then NTL(V) is the set of all $\mathrm{e}^{\mathrm{i}t} \in \mathbf{T}$ such that there exists a sequence $\{\lambda_n\}_{n=1}^\infty \subset V$ with $\lambda_n \to \mathrm{e}^{\mathrm{i}t}$ non-tangentially. It is well-known that NTL(V) is a Borel subset of **T**. (A construction similar to that found in the proof of [11, Lemma 4] shows that NTL(V) is in fact a $G_{\delta\sigma}$ set.) A set $V \subset \mathbf{D}$ is called dominating for **T** if $m(\mathbf{T} \setminus \mathrm{NTL}(V)) = 0$. It is well-known that V is dominating for **T** if and only if $\|f\|_\infty = \sup\{|f(\lambda)| : \lambda \in V\}$, for all f belonging to $H^\infty(\mathbf{D})$ (cf. [4, Theorem 3]).

If α, β are two complex numbers then $[\alpha, \beta]$ denotes the closed line segment with endpoints α and β. If α is a complex number and $r > 0$, then $B(\alpha, r)$ is the open disc in **C** with center α and radius r.

3. NORMAL OPERATORS IN A

Before dealing with general subnormal operators, we must first describe the normal operators which belong to **A**.

It is known that if N is a completely non-unitary contraction that is normal, then N belongs to **A** if and only if $\sigma(N) \cap \mathbf{D}$ is dominating for $\mathbf{T}$ (cf. [3, Theorem 10.4]). We shall see shortly that if U is an absolutely continuous unitary operator, then U belongs to **A** if and only if U has a reducing subspace $\mathcal{M}$ such that $U \,|\, \mathcal{M}$ is unitarily equivalent to M_z on $L^2(m, \mathbf{T})$. In light of this the following result is not too surprising. The author would like to thank B. Chevreau for help in simplifying the proof.

THEOREM 3.1. *Let N be an absolutely continuous normal contraction in $\mathcal{L}(\mathcal{H})$. Let $N = N' \oplus U$ be the canonical decomposition where N' is completely non-unitary and U is unitary (or acts on the space (0)). Then N belongs to the class* **A** *if and only if U has a reducing subspace $\mathcal{M}$ such that $U \,|\, \mathcal{M}$ is unitarily equivalent to M_z on $L^2(m, \mathbf{T} \setminus \mathrm{NTL}(\sigma(N') \cap \mathbf{D}))$.*

Proof. We will prove the "if" part first. Let $D = \mathrm{NTL}(\sigma(N') \cap \mathbf{D})$ and $F = \mathbf{T} \setminus D$. Choose f in $H^\infty(\mathbf{D})$. To show that N belongs to **A** it suffices to show that $\|f(N)\| \geqslant \|f\|_\infty$. Since $N = N' \oplus U$, $\|f(N)\| = \max\{\|f(N')\|, \|f(U)\|\}$. Let $\hat{f}$ in $H^\infty(\mathbf{T})$ be the boundary function of f. Define $g_1, g_2 \in L^\infty(\mathbf{T})$ by $g_1 = \hat{f}\chi_F$, $g_2 = \hat{f}\chi_D$ where χ_F is the characteristic function of F and χ_D is similarly defined. Note that $\|f\|_\infty = \max\{\|g_1\|_m, \|g_2\|_m\}$, so we can show that $\|f(N)\| \geqslant \|f\|_\infty$ by showing that $\|f(U)\| \geqslant \|g_1\|_m$ and $\|f(N')\| \geqslant \|g_2\|_m$. We now show that $\|f(U)\| \geqslant \|g_1\|_m$. It is clear that $\|f(U)\| \geqslant \|f(U \,|\, \mathcal{M})\|$. Recall that $f(U \,|\, \mathcal{M})$ as defined by the Sz.-Nagy—Foiaş functional calculus agrees with $\hat{f}(U \,|\, \mathcal{M})$ as defined by a spectral integral since $U \,|\, \mathcal{M}$ is absolutely continuous. So $\|\hat{f}(U \,|\, \mathcal{M})\| = \|f(U \,|\, \mathcal{M})\|$. Moreover, since $U \,|\, \mathcal{M}$ is unitarily equivalent to M_z on $L^2(m, F)$, it is easy to see that $m|F$ is a scalar spectral measure for $U \,|\, \mathcal{M}$. Therefore, $\|\hat{f}(U \,|\, \mathcal{M})\| = \|\hat{f}\|_{m|F} = \|g_1\|_m$ (cf. [8, p. 93, II.7.6]). Thus $\|f(U)\| \geqslant \|\hat{f}(U \,|\, \mathcal{M})\| = \|g_1\|_m$. We now show that $\|f(N')\| \geqslant \|g_2\|_m$. Suppose $\lambda \in \sigma(N') \cap \mathbf{D}$ then $f(\lambda) \in \sigma(f(N'))$ (cf. [7, Lemma 3.1]). Therefore $|f(\lambda)| \leqslant \|f(N')\|$ for all λ in $\sigma(N') \cap \mathbf{D}$. However, for almost every $e^{i\theta}$ in $\mathrm{NTL}(\sigma(N') \cap \mathbf{D})$, $|\hat{f}(e^{i\theta})| \leqslant \sup\{|f(\lambda)| : \lambda \in \sigma(N') \cap \mathbf{D}\}$. Therefore $\|g_2\|_m \leqslant \sup\{|f(\lambda)| : \lambda \in \sigma(N') \cap \mathbf{D}\} \leqslant \|f(N')\|$. This concludes the proof of the "if" part of the theorem.

Now for the "only if" part of the theorem. Let μ and ν be scalar spectral measures for N' and U respectively. Using the Lebesgue decomposition of m with respect to ν, write $m = m_a + m_s$ where $m_a \ll \nu$ and m_s is singular with respect to ν. So there exist Borel sets A and $B \subset \mathbf{T}$ such that $A \cap B = \emptyset$, $A \cup B = \mathbf{T}$ and $m_s(A) = \nu(B) = 0$. Since $\nu \ll m$ we must have $\nu \,|\, A \ll m \,|\, A$; and $m_s(A) = 0$ implies that $m \,|\, A \ll \nu \,|\, A$. Therefore $[m \,|\, A] = [\nu \,|\, A]$. We claim that $m(B \cap F) = 0$. Suppose this is true. Since ν is a scalar spectral measure for U and $\nu(B) = 0$, there exists $\mathcal{M}'$, a reducing subspace for U, such that $U\,|\,\mathcal{M}'$ is unitarily equivalent to M_z on $L^2(v, A)$. Since $[v\,|\,A] = [m\,|\,A]$, M_z on $L^2(v, A)$ is unitarily equivalent to M_z on $L^2(m, A)$. Since $A \cup B = \mathbf{T}$ and $m(B \cap F) = 0$, M_z on $L^2(m, A)$ is unitarily equivalent to M_z on $L^2(m, A \cup F)$. Thus there exists $\mathcal{M}$, a reducing subspace for U, such that $\mathcal{M} \subset \mathcal{M}'$ and $U\,|\,\mathcal{M}$ is unitarily equivalent to M_z on $L^2(m, \mathbf{T}\setminus \mathrm{NTL}(\sigma(N') \cap \mathbf{D}))$.

To complete the proof we must show that $m(B \cap F) = 0$. We will assume that $m(B \cap F) > 0$ in order to obtain a contradiction. Using the regularity of m, we can find $B' \subset B \cap F$ such that B' is closed and $m(B') > 0$. We will now state two lemmas whose proofs will be postponed.

LEMMA 3.2. *There exists an open set* $W \subset \mathbf{D}$ *such that* $W \cap \sigma(N') = \emptyset$ *and* $B' \cap \mathrm{NTL}(\mathbf{D}\setminus W) = \emptyset$.

LEMMA 3.3. *There exists an open set* $V \subset \mathbf{D}$ *such that* $\mathrm{NTL}(V) = \mathbf{T}\setminus B'$.

We now assume the lemmas are true and complete the proof of the theorem. Suppose $f \in H^\infty(\mathbf{D})$, then

$$\|f\|_\infty = \|f(N)\| = \max\{\|f(U)\|, \|f(N')\|\}.$$

Also $\|f(U)\| = \|\hat{f}\|_v = \|\hat{f}\|_{v|A}$ (since $v(B) = 0$) $= \|\hat{f}\|_{m|A}$ (since $[m\,|\,A] = [v\,|\,A]$) $\leqslant \sup\{|f(\lambda)| : \lambda \in V\}$ (since $\mathrm{NTL}(V) \supset A$). Let μ be a scalar spectral measure for N'. Since N' is completely non-unitary we know that $\mu(\mathbf{T}) = 0$. Moreover

$$\|f(N')\| = \|f\|_\mu \leqslant \sup\{|f(\lambda)| : \lambda \in \sigma(N') \cap \mathbf{D}\} \leqslant$$

$$\leqslant \sup\{|f(\lambda)| : \lambda \in \mathbf{D}\setminus W\}.$$

The first inequality is true because $\mu(\mathbf{D}^-\setminus(\sigma(N') \cap \mathbf{D})) = 0$, the second is true because $(\sigma(N') \cap \mathbf{D}) \subset (\mathbf{D}\setminus W)$. Therefore, $\|f\|_\infty \leqslant \sup\{|f(\lambda)| : \lambda \in V \cup (\mathbf{D}\setminus W)\} \leqslant \|f\|_\infty$. This implies that $V \cup (\mathbf{D}\setminus W)$ is dominating for $\mathbf{T}$ (cf. [4, Theorem 3]). But $B' \cap \mathrm{NTL}(V \cup (\mathbf{D}\setminus W)) = \emptyset$ and $m(B') > 0$. This is a contradiction and completes the proof of the theorem. We now proceed with the proofs of the lemmas.

Proof of Lemma 3.2. If $e^{it} \in B'$, and $0 < \varphi < \pi$, then $\Delta_{t,\varphi} \subset \mathbf{D}$ is an open isosceles triangle with the following properties: one vertex at e^{it}, the angle at e^{it} equal to φ, $[0, e^{it}]$ bisecting the angle φ and the altitude measured from e^{it} sufficiently small that $\Delta_{t,\varphi} \cap \sigma(N') = \emptyset$. Since $B' \cap \mathrm{NTL}(\sigma(N') \cap \mathbf{D}) = \emptyset$ such a triangle exists. Let $W = \bigcup\{\Delta_{t,\varphi} : e^{it} \in B',\ 0 < \varphi < \pi\}$, and we are finished.

Proof of Lemma 3.3. Since B' is closed, $\mathbf{T}\setminus B' = \bigcup_{n \in \mathfrak{N}} \{I_n\}$, where $\mathfrak{N}$ is a subset of the positive integers and $\{I_n\}_{n \in \mathfrak{N}}$ is a pairwise disjoint collection of open arcs. Suppose e^{ia_n} and e^{ib_n} are the end-points of I_n and $m(I_n) = b_n - a_n$. Let $c_n =$

$= (1/2)(b_n + a_n)$. Choose $\alpha_n \in \mathbf{D}$ such that:

$$[\alpha_n, e^{ic_n}] \subset [0, e^{ic_n}], \quad |\alpha_n - e^{ic_n}| < 1, \quad \text{and} \quad \frac{|\alpha_n - e^{ic_n}|}{c_n - a_n} < \frac{1}{n}.$$

Let $\Gamma_{n,1}$ be a curve from α_n to e^{ia_n} such that: $\Gamma_{n,1}$ is tangent to $\mathbf{T}$ at e^{ia_n}; and if $a_n < t < c_n$, $\beta_t = \Gamma_{n,1} \cap [0, e^{it}]$ then $\dfrac{|\beta_t - e^{it}|}{t - a_n} \leqslant \dfrac{|\alpha_n - e^{ic_n}|}{c_n - a_n}$. See Figure 1.

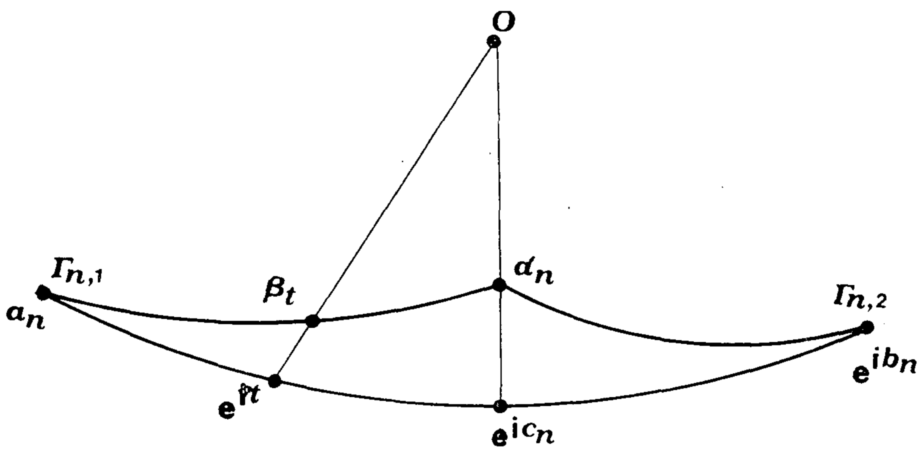

Figure 1. The region V_n.

Let $\Gamma_{n,2}$ be the curve obtained by reflecting $\Gamma_{n,1}$ over $[0, e^{ic_n}]$. Let V_n be the open set bounded by I_n, $\Gamma_{n,1}$, and $\Gamma_{n,2}$. See Figure 1. Let $V = \bigcup_{n \in \mathfrak{N}} \{V_n\}$. Notice that $\{V_n\}_{n\in\mathfrak{N}}$ is a pairwise disjoint collection of open sets. Clearly $\mathrm{NTL}(V) \supset \mathbf{T} \setminus B'$. So to finish the proof it suffices to show that $B' \cap \mathrm{NTL}(V) = \emptyset$. Suppose $e^{it} \in B'$. If the distance from e^{it} to I_n^- along $\mathbf{T}$ is $\geqslant \delta > 0$ for every $n \in \mathfrak{N}$, then the sector of $\mathbf{D}$ bounded by $[0, e^{i(t-\delta)}]$, $[0, e^{i(t+\delta)}]$ and the shorter arc from $e^{i(t+\delta)}$ to $e^{i(t-\delta)}$ is disjoint from V. Moreover, any sequence of points converging non-tangentially to e^{it} must pass through this sector. So $e^{it} \notin \mathrm{NTL}(V)$. So we may assume that there exist intervals I_n with the disance from e^{it} to I_n^- along $\mathbf{T}$ arbitrarily small. There are four cases:

(i) $e^{it} = e^{ia_n}$ for some $n \in \mathfrak{N}$ and $e^{it} = e^{ib_k}$ for some $k \in \mathfrak{N}$.

(ii) $e^{it} \neq e^{ia_n}$ for every $n \in \mathfrak{N}$ and $e^{it} \neq e^{ib_k}$ for every $k \in \mathfrak{N}$.

(iii) $e^{it} = e^{ia_n}$ for some $n \in \mathfrak{N}$ and $e^{it} \neq e^{ib_k}$ for every $k \in \mathfrak{N}$.

(iv) $e^{it} \neq e^{ia_n}$ for every $n \in \mathfrak{N}$ and $e^{it} = e^{ib_k}$ for some $k \in \mathfrak{N}$.

Case (i). Since $\Gamma_{n,1}$ is tangent to $\mathbf{T}$ at $e^{ia_n} = e^{it}$ and $\Gamma_{k,2}$ is tangent to $\mathbf{T}$ at $e^{ib_k} = e^{it}$ it is clear that $e^{it} \notin \mathrm{NTL}(V)$.

Case (ii). Assume $e^{it} \in \mathrm{NTL}(V)$ in order to obtain a contradiction. Let $\{\lambda_k\}_{k=1}^{\infty} \subset V$ be a sequence such that $\lambda_k \to e^{it}$ non-tangentially. Let θ be an angle with the following properties: $0 < \theta < \pi$, vertex at e^{it}, $[0, e^{it}]$ bisecting θ, and $\{\lambda_k\}_{k=1}^{\infty}$ in the interior of θ. Let L_1, L_2 be the sides of θ. See Figure 2.

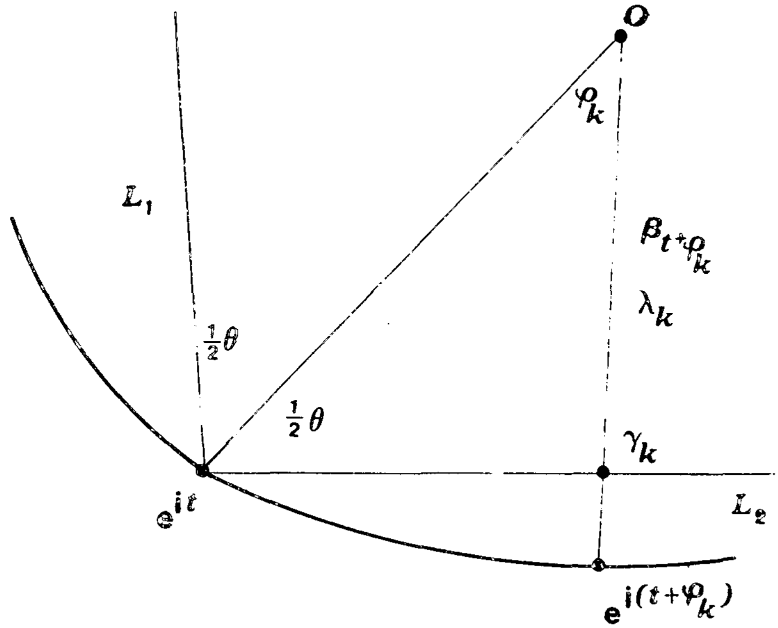

Figure 2. The points λ_k, γ_k and $\beta_{t+\varphi_k}$.

If $\lambda_k = r_k e^{i(t+\varphi_k)}$ with $-\pi < \varphi_k < \pi$, then $\varphi_k \to 0$ as $k \to \infty$. Choose $\delta_1 > 0$ such that the distance from e^{it} to I_n^- along $\mathbf{T}$ is $> \delta_1$ for $1 \leqslant n \leqslant 2(\tan(1/2)\theta)^{-1}$. Recall that $e^{it} \notin I_n$ for all $n \in \mathfrak{N}$. Choose δ_2 with $0 < \delta_2 < \delta_1$ such that $|\varphi| < \delta_2$ implies that:

$$\frac{\sin\left(\frac{1}{2}\theta + \varphi\right) - \sin\left(\frac{1}{2}\theta\right)}{\varphi \sin\left(\frac{1}{2}\theta + \varphi\right)} > \frac{3}{4}\tan\left(\frac{1}{2}\theta\right).$$

This is possible since:

$$\lim_{\varphi\to 0} \frac{\sin\left(\frac{1}{2}\theta + \varphi\right) - \sin\left(\frac{1}{2}\theta\right)}{\varphi \sin\left(\frac{1}{2}\theta + \varphi\right)} = \tan\left(\frac{1}{2}\theta\right).$$

Choose k sufficiently large that $|\varphi_k| < \delta_2$. Since $\lambda_k \in V$ there exists a unique positive integer $n_k \in \mathfrak{N}$ such that $\lambda_k \in V_{n_k}$. This implies that $e^{i(t+\varphi_k)} \in I_{n_k}$. Since $|\varphi_k| < \delta_1$ we must have $n_k > 2(\tan(1/2)\theta)^{-1}$. Without loss of generality assume $\varphi_k > 0$. Let $\beta_{t+\varphi_k} = \Gamma_{n_k,1} \cap [0, e^{i(t+\varphi_k)}]$ and $\gamma_k = [\lambda_k, e^{i(t+\varphi_k)}] \cap L_1$. (See Figure 2.) Note that $\gamma_k \in V_{n_k}$ and $|\gamma_k - e^{i(t+\varphi_k)}| \geqslant |\beta_{t+\varphi_k} - e^{i(t+\varphi_k)}|$. Also $\varphi_k = t + \varphi_k - t > > t + \varphi_k - a_{n_k}$ which implies:

$$\frac{|\gamma_k - e^{i(t+\varphi_k)}|}{\varphi_k} < \frac{|\beta_{t+\varphi_k} - e^{i(t+\varphi_k)}|}{t+\varphi_k - a_{n_k}} < \frac{|\alpha_{n_k} - e^{i(t+\varphi_k)}|}{c_{n_k} - a_{n_k}} < \frac{1}{n_k} < \frac{1}{2}\tan\left(\frac{1}{2}\theta\right).$$

However using the Law of Sines on the triangle determined by 0, e^{it}, and γ_k we see that:

$$|\gamma_k - e^{i(t+\varphi_k)}| = \frac{\sin\left(\frac{1}{2}\theta + \varphi_k\right) - \sin\left(\frac{1}{2}\theta\right)}{\sin\left(\frac{1}{2}\theta + \varphi_k\right)}.$$

(See Figure 2.) So

$$\frac{|\gamma_k - e^{i(t+\varphi_k)}|}{\varphi_k} = \frac{\sin\left(\frac{1}{2}\theta + \varphi_k\right) - \sin\left(\frac{1}{2}\theta\right)}{\varphi_k \sin\left(\frac{1}{2}\theta + \varphi_k\right)} > \frac{3}{4}\tan\left(\frac{1}{2}\theta\right).$$

This is a contradiction. This concludes the proof of case (ii). Cases (iii) and (iv) are similar and left to the reader.

4. SUBNORMAL OPERATORS IN A

The following notation will be in use during this section: S in $\mathscr{L}(\mathscr{H})$ will be a subnormal operator, $\mathscr{K}$ containing $\mathscr{H}$ will be another separable Hilbert space, and N in $\mathscr{L}(\mathscr{K})$ will be the minimal normal extension of S. The projection-valued spectral measure of N will be denoted by $E(\cdot)$. The goal of this section is to prove the following theorem which completely characterizes the subnormal operators which belong to **A**.

THEOREM 4.1. *Let S be an absolutely continuous subnormal contraction. Let $S = U \oplus S'$ be the canonical decomposition where U is unitary (or acts on the space*

(0)) and S' is completely non-unitary. Then S belongs to **A** *if and only if there exists $\mathscr{M}$, a reducing subspace for U, such that $U|\mathscr{M}$ is unitarily equivalent to M_z on* $L^2(m, \mathbf{T}\setminus \mathrm{NTL}(\sigma(S)\cap \mathbf{D}))$.

Before proving this theorem we need the following lemma which is a generalization of [8, Theorem VI.4.12].

LEMMA 4.2. *Suppose S in $\mathscr{L}(\mathscr{H})$ is subnormal, and there exists an open set X in* **C** *such that $\sigma(S)\cap X$ is a non-empty subset of* **T**. *Then S has a non-zero unitary part.*

Proof. Without loss of generality assume $X = B(e^{it}, r)$ for some $r > 0$ and some e^{it} in $\mathbf{T}$, and $\sigma(S)\cap X^- \subset \mathbf{T}$. Clearly $\sigma(S)\cap X \subset \partial\sigma(S) \subset \sigma(N)$. This implies that $\sigma(S)\cap X = \sigma(N)\cap X \neq \emptyset$. Let $I = X\cap \mathbf{T}$. Let $\mathscr{K}_1 = E(I)\mathscr{K} \neq 0$ and $\mathscr{H}_1 = (E(I)\mathscr{H})^-$. Since N is the minimal normal extension of S, $\mathscr{H}_1 \neq 0$.

Let $N_1 = N|\mathscr{K}_1$, and $S_1 = N|\mathscr{H}_1$, then S_1 is subnormal, N_1 is the minimal normal extension of S_1, and $\sigma(S_1)\subset \sigma(S)\cap I^-$ (cf. [8, Lemma VI.4.11]). Since $\sigma(S_1)\subset \mathbf{T}$, S_1 must be normal (cf. [8, Theorem V.3.1]). This implies that $N_1 = S_1$, which implies that $\mathscr{H}_1 = \mathscr{K}_1$ is reducing for N. Moreover S_1 being normal and $\sigma(S_1)\subset \mathbf{T}$ implies that S_1 is unitary. If we can show that $\mathscr{H}_1 \subset \mathscr{H}$, then $S_1 = N|\mathscr{H}_1 = S|\mathscr{H}_1$ will be a non-zero unitary part of S. Thus, to complete the proof of the lemma, we must show that $E(I)\mathscr{H} \subset \mathscr{H}$. Let $R > 0$ be sufficiently large that $B(e^{it}, R) \supset \sigma(S)$. Recall that $X = B(e^{it}, r)$. Let $K = \{\lambda : r \leqslant |\lambda - e^{it}| \leqslant R\} \cup (\sigma(S)\cap I^-)$. Then K is compact and $\sigma(S)\subset K$. Let $\mathscr{R}(K)$ be the set of all complex-valued continuous functions on K uniformly approximable by rational functions with no poles in K. If $f\in \mathscr{R}(K)$, then $f(N)$ can be defined using a spectral integral, since $K\supset \sigma(N)$. Moreover, $f(N)\mathscr{H} \subset \mathscr{H}$ (cf. [8, p. 207]). Suppose one can construct a sequence $\{f_n\}_{n=1}^{\infty} \subset \mathscr{R}(K)$ such that f_n converges to χ_I boundedly and pointwise on K. Take $x, y \in \mathscr{K}$, then $(f_n(N)x, y) = = \int_K f_n(\lambda)\, d(E(\lambda)x, y)$ which converges to $\int_K \chi_I(\lambda)\, d(E(\lambda)x, y)$ using the Lebesgue dominated convergence theorem. However, $\int_K \chi_I(\lambda)\, d(E(\lambda)x, y) = (\chi_I(N)x, y) = = (E(I)x, y)$. Thus $(f_n(N)x, y)$ converges to $(E(I)x, y)$ for all x, y in $\mathscr{K}$. Choosing x in $\mathscr{H}$ and y in the orthogonal complement of $\mathscr{H}$, we see that $E(I)\mathscr{H}\subset\mathscr{H}$. Thus, to complete the proof we must construct the sequence $\{f_n\}$. Suppose $m(I) = 2a$. Take $n > a^{-1}$. If $\lambda \in K$, then either

(i) $r \leqslant |\lambda - e^{it}| \leqslant R$ or

(ii) $\lambda = e^{is}$ with $t - a < s < t + a$, and $\lambda \in \sigma(S)$.

If (i) occurs then $f_n(\lambda) = 0$.

If (ii) occurs then:

$$f_n(e^{is}) = \begin{cases} n(s-(t-a)) & \text{if } t-a < s < t-a+n^{-1} \\ 1 & \text{if } t-a+n^{-1} \leqslant s \leqslant t+a-n^{-1} \\ n(t+a-s) & \text{if } t+a-n^{-1} < s < t+a. \end{cases}$$

Notice that the interior of K is exactly those λ for which $r \leqslant |\lambda - e^{it}| \leqslant R$, and $f_n = 0$ on the interior of K. Thus f_n is analytic on the interior of K and continuous on K. This implies that $f_n \in \mathscr{R}(\mathscr{K})$ (see [9, Corollary VIII.8.4]). This completes the proof of the lemma.

Proof of Theorem 4.1. Let $D = \mathrm{NTL}(\sigma(S) \cap \mathbf{D})$, and $F = \mathbf{T} \setminus D$. We will prove the "if" part first. Choose $f \in H^\infty(\mathbf{D})$. To show that S belongs to $\mathbf{A}$ it suffices to show that $\|f(S)\| \geqslant \|f\|_\infty$. Since $S = S' \oplus U$, $\|f(S)\| = \max\{\|f(S')\|, \|f(U)\|\}$. Let $\hat{f} \in H^\infty(\mathbf{T})$ be the boundary function of f. Define $g_1, g_2 \in L^\infty(\mathbf{T})$ by $g_1 = \hat{f}\chi_F$, $g_2 = \hat{f}\chi_D$ where χ_F is the characteristic function of F and χ_D is similarly defined. Note that $\|f\|_\infty = \max\{\|g_1\|_m, \|g_2\|_m\}$, so we can show that $\|f(S)\| \geqslant \|f\|_\infty$ by showing that $\|f(U)\| \geqslant \|g_1\|_m$ and $\|f(S') \geqslant \|g_2\|_m$. We now show that $\|f(U)\| \geqslant \|g_1\|_m$. It is clear that $\|f(U)\| \geqslant \|(U|\mathscr{M})\|$. Recall that $f(U|\mathscr{M})$ as defined by the Sz.-Nagy—Foiaş functional calculus agrees with $\hat{f}(U|\mathscr{M})$ as defined by a spectral integral since $U|\mathscr{M}$ is absolutely continuous. So $\|\hat{f}(U|\mathscr{M})\| = \|f(U|\mathscr{M})\|$. Moreover, since $U|\mathscr{M}$ is unitarily equivalent to M_z on $L^2(m, F)$ it is easy to see that $m|F$ is a scalar spectral measure for $U|\mathscr{M}$. Therefore $\|\hat{f}(U|\mathscr{M})\| = \|\hat{f}\|_{m|F}$ (cf. [8, II.7.6]). Clearly $\|\hat{f}\|_{m|F} = \|g_1\|_m$. We can now conclude that $\|f(U)\| \geqslant \|g_1\|_m$. We now show that $\|f(S')\| \geqslant \|g_2\|_m$. Suppose $\lambda \in \sigma(S') \cap \mathbf{D}$, then $f(\lambda) \in \sigma(f(S'))$ (cf. [7, Lemma 3.1]). Therefore $|f(\lambda)| \leqslant \|f(S')\|$ for all $\lambda \in \sigma(S') \cap \mathbf{D}$. However for every e^{it} in $\mathrm{NTL}(\sigma(S') \cap \mathbf{D})$, $|\hat{f}(e^{it})| \leqslant \sup\{|f(\lambda)| : \lambda \in \sigma(S') \cap \mathbf{D}\}$. Therefore $\|g_2\|_m \leqslant \sup\{|f(\lambda)| : \lambda \in \sigma(S') \cap \mathbf{D}\} \leqslant \|f(S')\|$. This concludes the proof of the "if" part of the theorem.

We now prove the "only if" part of the theorem. Let ν be a scalar spectral measure for U. Using the Lebesgue decomposition of m with respect to ν, write $m = m_a + m_s$ where $m_a \ll \nu$ and m_s is singular with respect to ν. So there exists Borel sets A and B contained in $\mathbf{T}$ such that $A \cup B = \mathbf{T}$, $A \cap B = \emptyset$, and $m_s(A) = \nu(B) = 0$. Since $\nu \ll m$ we must have $\nu|A \ll m|A$; and $m_s(A) = 0$ implies that $m|A \ll \nu|A$. Therefore, $[m|A] = [\nu|A]$. We claim that $m(B \cap F) = 0$. Suppose this is true. Since ν is a scalar spectral measure for U and $\nu(B) = 0$, there exists $\mathscr{M}'$, a reducing subspace for U, such that $U|\mathscr{M}'$ is unitarily equivalent to M_z on $L^2(\nu, A)$. Since $[\nu|A] = [m|A]$, M_z on $L^2(\nu, A)$ is unitarily equivalent to M_z on $L^2(m, A)$. Since $A \cup B = \mathbf{T}$ and $m(B \cap F) = 0$, M_z on $L^2(m, A)$ is unitarily equivalent to M_z on $L^2(m, A \cup F)$. Therefore, there exists $\mathscr{M}$, a reducing subspace for U, such that $\mathscr{M} \subset \mathscr{M}'$ and $U|\mathscr{M}$ is unitarily equivalent to M_z on $L^2(m, F)$, which is what we are trying to prove. To complete the proof we must sho that $m(B \cap F) = 0$. We will assume that $m(B \cap F) > 0$ in order to obtain a contradiction. Recall now that $F = \mathbf{T} \setminus \mathrm{NTL}(\sigma(S) \cap \mathbf{D})$. If $e^{i\theta} \in B \cap F$, then for $0 < \alpha < \pi$ there exists an isosceles triangle $T_{\theta,\alpha}$ such that: $e^{i\theta}$ is one vertex of $T_{\theta,\alpha}$, $\mathrm{int}(T_{\theta,\alpha}) \subset \mathbf{D}$, the angle of $T_{\theta,\alpha}$ at $e^{i\theta}$ is equal to α, $[0, e^{i\theta}]$ bisects this angle and $\mathrm{int}(T_{\theta,\alpha}) \cap \sigma(S) = \emptyset$. Note that this implies that $\mathrm{int}(T_{\theta,\alpha}) \cap \sigma(N) = \emptyset$. For each $e^{i\theta}$ in $B \cap F$ we can find exactly one bounded component of $\mathbf{C} \setminus \sigma(N)$ denoted by V_θ such that V_θ contains $\mathrm{int}(T_{\theta,\alpha})$ for $0 < \alpha < \pi$. Since $\mathbf{C} \setminus \sigma(N)$ has only countably many components, one of these components, denoted by V, must have the property that $m(\{e^{i\theta} : V = V_\theta\}) > 0$. Let $B' \subset B \cap F$ be the set of $e^{i\theta}$ such that $V_\theta = V$. Clearly $m(B') > 0$. Let B'' be a subset of B' such that $m(B'') > 0$ and $e^{i\theta}$ in B'' implies that the height of $T_{\theta,9\pi/10}$ measured from $e^{i\theta}$ is $\geqslant \beta > 0$. Let $T_\theta = T_{\theta,9\pi/10}$. Without loss of generality assume that the height of $T_\theta = \beta$ for all $e^{i\theta}$ in B''. The following construction is based on a construction found in [11, pp. 122—123, 130]. We now choose a closed subset $C \subset B''$ and an open arc $\Gamma \subset \mathbf{T}$ such that: $m(C) > 0$, $C \subset \Gamma^-$, the endpoints of Γ belong to C, C has exactly two isolated points — the endpoints of Γ, $m(\Gamma) < \min(\pi/100, \beta/10)$, and the corresponding sides of the T_θ's for $e^{i\theta}$ in C do not intersect. This can all be accomplished using the regularity of m and choosing Γ sufficiently small. For more details see [11, pp. 122, 130]. Since $\Gamma \setminus C$ is an open set, $\Gamma \setminus C = \bigcup_n \{M_n\}$ where $\{M_n\}$ is a collection of pairwise disjoint collection of open arcs. If M_n has endpoints e^{ia} and e^{ib}, let R_a and S_b denote the sides of T_a and T_b respectively which intersect. Let P_{ab} denote the point of intersection of R_a and S_b. See Figure 3.

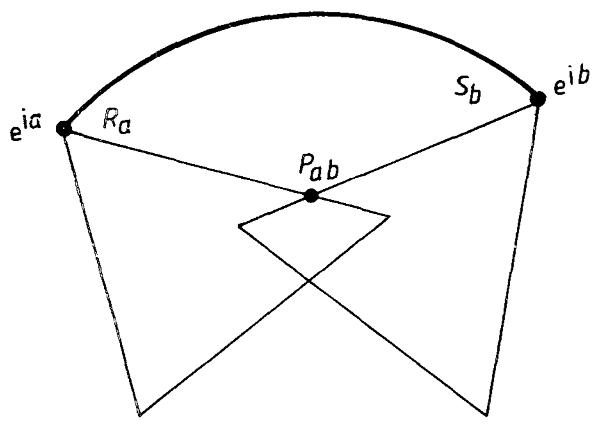

Figure 3. The point P_{ab}.

Let C_n denote the intersection of $\mathbf{D}$ with the circle passing through the points e^{ia}, e^{ib}, and P_{ab}. Elementary geometry shows that the length of C_n is less than or equal to a constant times the length of M_n. Let $G = C \cup (\mathbf{T} \setminus \Gamma) \cup (\bigcup_n C_n)$. It is easy to see that G is a rectifiable Jordan curve. Suppose the endpoints of Γ are denoted by e^{ic} and e^{id}. Let Y be the closed region bounded by $[(1-\beta/5)e^{ic}, e^{ic}]$, $[(1-\beta/5)e^{id}, e^{id}]$, $\{(1-\beta/5)e^{i\theta} : e^{i\theta} \in \Gamma\}$, and Γ. We claim that $(\sigma(S) \cap \text{int}(Y)) \subset \mathbf{D} \setminus (\text{int}(G) \cup G)$. (Note that $\mathbf{D} \setminus (\text{int}(G) \cup G)$ is just the union of the open regions bounded by M_n and C_n for each n.) This follows from the fact that $\{re^{i\varphi} : e^{i\varphi} \in \Gamma, 1-\beta/2 < r \leqslant 1-\beta/20\}$ is contained in $\bigcup \{\text{int}(T_\theta) : e^{i\theta} \in C\}$ and $\bigcup \{\text{int}(T_\theta) : e^{i\theta} \in C\}$ is disjoint from $\sigma(S)$. See [11, p. 122]. Let u be a function defined on G by:

$$u(z) = \begin{cases} 0 & z \in (\mathbf{T} \setminus \Gamma) \cup (\cup C_n) \\ 1 & z \in C. \end{cases}$$

This is well-defined with respect to arc-length measure on G. Thus we can extend u to be a harmonic function on $\text{int}(G)$. Note that $u \geqslant 0$ on $\text{int}(G)$. Suppose z belongs to $\mathbf{D} \setminus (\text{int}(G) \cup G)$, then z is contained in the open region bounded by M_n and C_n for some n. Since u is harmonic on $\text{int}(G)$ and continuous on $\text{int}(G) \cup C_n$ and $u = 0$ on C_n, we can define $u(z) = -u(z^*)$ where z^* is the point in $\text{int}(G)$ symmetric to z with respect to C_n. Now u is a harmonic function on $\mathbf{D}$. Moreover $u \leqslant 0$ on $\mathbf{D} \setminus \text{int}(G)$, which implies that $u \leqslant 0$ on $\sigma(S) \cap \text{int}(Y)$. Using the maximum modulus principle we can find $r < 1$ such that $u(z) \leqslant r$ for $z \in \sigma(S) \cap (\mathbf{D} \setminus \text{int}(Y))$. Therefore $u(z) \leqslant r$ for all $z \in \sigma(S) \cap \mathbf{D}$. Let v be a harmonic conjugate for u and $f = e^{u+iv}$. We conclude that f belongs to $H^\infty(\mathbf{D})$ and $\|f\|_\infty = e$, because $|f(z)| = e^{u(z)}$ and $\sup\{u(z) : z \in \mathbf{D}\} = 1$. Let $g(z) = f(z)/e$, then $g \in H^\infty(\mathbf{D})$ and $\|g\|_\infty = 1$. Let us consider the operator $g(S)$, which is subnormal. We shall now show that $g(S)$ is completely non-unitary. Since $S = U \oplus S'$ we can write $g(S) = g(U) \oplus g(S')$. Since v is a scalar spectral measure for U, $\|g(U)\| = \|\hat{g}(U)\| = \|\hat{g}\|_v$ (cf. [8, p. 93, II.7.6]). Recall also that $v(B) = 0$ and $[v \mid A] = [m \mid A]$, so we can conclude that $\|\hat{g}\|_v = \|\hat{g}\|_{m|A}$. Since $C \subset B$ and $A \cap B = \emptyset$, we must have $A \subset (\mathbf{T} \setminus \Gamma) \cup (\cup M_n)$. If $z \in \mathbf{T} \setminus \Gamma$ then $u(z) = 0$, so $|\hat{g}(z)| = 1/e$. If $z \in M_n$ for some n, and if $\lambda_k \to z$ non-tangentially then for sufficiently large k, λ_k must be contained in the region bounded by C_n and M_n, which implies that $u(\lambda_k) \leqslant 0$; therefore $|g(\lambda_k)| \leqslant 1/e$ and $|\hat{g}(z)| \leqslant 1/e$. Therefore, $\|g(U)\| = \|\hat{g}\|_{m|A} \leqslant 1/e$, which implies that $g(U)$ is completely non-unitary. We now show that $g(S')$ is completely non-unitary. Write $S' = S_1 \oplus N_1$, where S_1 is completely non-normal and N_1 is normal; then $g(S') = g(S_1) \oplus g(N_1)$. Since S' is completely non-unitary, so is N_1. Since $g \in H^\infty(\mathbf{D})$ and $\|g\|_\infty = 1$, $g(N_1)$ is completely non-unitary (cf. [13, Theorem III.2.1(e)]). Since g is analytic on $\mathbf{D}$ and S_1 is completely non-normal $g(S_1)$ is completely non-normal (cf. [8, Corollary VIII.2.14]). So we have shown that $g(S)$ is completely non-unitary. We now apply [8, Theorem VIII.4.2] to obtain $\sigma(g(S)) = \tilde{g}(\sigma(N)) \cup [g(\sigma(S) \cap \mathbf{D})]^-$. Since N belongs to $\mathbf{A}$ we know that $\mathbf{T} \subset \sigma(N)$. It is easy to see that $\tilde{g}(\sigma(N)) = [g(\sigma(N) \cap \mathbf{D})]^- \cup \hat{g}(\mathbf{T})$. If $z \in C$, then $|\hat{g}(z)| = 1$. Moreover, since N belongs to $\mathbf{A}$, Theorem 3.1 tells us that if U_1 is the unitary part of N, then U_1 has a reducing subspace $\mathcal{M}$ such that $U_1 \mid \mathcal{M}$ is unitarily equivalent to M_z on $L^2(m, \mathbf{T} \setminus \text{NTL}(\sigma(N) \cap \mathbf{D}))$. Since C has positive measure and is contained in $\mathbf{T}\setminus \mathrm{NTL}(\sigma(S)\cap \mathbf{D})$ (which is contained in $\mathbf{T}\setminus \mathrm{NTL}(\sigma(N)\cap \mathbf{D})$), the spectral mapping theorem for normal operators tells us that $\hat{g}(C)$ is a non-empty subset of $\sigma(U_1|\mathcal{N})$, hence a non-empty subset of $\sigma(N)$. We now conclude that $\hat{g}(C)$ is a non-empty subset of $\mathbf{T}$. If $z \in \mathbf{T}\setminus C$, then as we saw before $|\hat{g}(z)| \leqslant 1/\mathrm{e}$. We also noted before that if $z \in \sigma(S)\cap \mathbf{D}$ that $u(z) \leqslant r < 1$, so $|g(z)| \leqslant \mathrm{e}^{r-1}$ for $z \in \sigma(S)\cap \mathbf{D}$. Therefore $g(\sigma(N)\cap \mathbf{D}) \subset g(\sigma(S)\cap \mathbf{D}) \subset \subset B(0, \mathrm{e}^{r-1})$. Putting it all together we see that $\sigma(g(S)) \subset B(0, \mathrm{e}^{r-1})^- \cup \mathbf{T}$ and $\sigma(g(S))\cap \mathbf{T} \neq \emptyset$. Choose $z \in \sigma(g(S))\cap \mathbf{T}$ and let $V = B(z, (1+\mathrm{e}^{r-1})/2)$, then $\emptyset \neq [V\cap \sigma(g(S))] \subset \mathbf{T}$. Therefore $g(S)$ must have a non-zero unitary part by Lemma 4.2. This is a contradiction and concludes the proof of the theorem.

REFERENCES

1. Apostol, C., Ultraweakly closed operator algebras, *J. Operator Theory*, **2**(1979), 49–61.
2. Apostol, C.; Bercovici, H.; Foiaş, C.; Pearcy, C., Invariant subspaces, dilation theory, and the structure of the predual of a dual algebra. I, *J. Funct. Anal.*, **63**(1985), 369–404.
3. Bercovici, H.; Foiaş, C.; Pearcy, C., *Dual algebras with applications to invariant subspaces and dilation theory*, CBMS Regional Conf. Ser. in Math., no. **56**, Amer. Math. Soc., Providence, R.I., 1985.
4. Brown, L.; Shields, A.; Zeller, K., On absolutely convergent sums, *Trans. Amer. Math. Soc.*, **96**(1960), 162–183.
5. Brown, S.; Chevreau, B.; Pearcy, C., Contractions with rich spectrum have invariant subspaces, *J. Operator Theory*, **1**(1979), 123–136.
6. Chevreau, B.; Pearcy, C., On the structure of contraction operators with applications to invariant subspaces, *J. Funct. Anal.*, **67**(1986), 360–379.
7. Chevreau, B.; Pearcy, C.; Shields, A., Finitely connected domains G, representations of $H^\infty(G)$ and invariant subspaces, *J. Operator Theory*, **6**(1981), 375–405.
8. Conway, J. B., *Subnormal operators*, Pitman, Boston, Mass., 1981.
9. Gamelin, T., *Uniform algebras*, Prentice-Hall, Englewood Cliffs, N. J., 1969.
10. Olin, R.; Thomson, J., Algebras of subnormal operators, *J. Funct. Anal.*, **37**(1980), 271–301.
11. Olin, R.; Thomson, J., Some index theorems for subnormal operators, *J. Operator Theory*, **3**(1980), 115–142.
12. Sarason, D., Weak-star density of polynomials, *J. Reine Angew. Math.*, **252**(1972), 1–15.
13. Sz.-Nagy, B.; Foiaş, C., *Harmonic analysis of operators on Hilbert space*, North Holland, Amsterdam, 1970.

PATRICK J. SULLIVAN
Department of Mathematics and Computing Science, Valparaiso University, Valparaiso, IN 46383, U.S.A.

Received September 4, 1986; revised March 12, 1987.

Added in proofs. Since this paper was accepted, H. Bercovici (Factorization theorems and the structure of operators on Hilbert space, preprint) and B. Chevreau (Sur les contractions à calcul fonctionnel isométrique. II, preprint) have independently shown that $\mathrm{A} = \mathrm{A}_1$. Thus, any absolutely continuous contraction whose spectrum contains the unit circle has a non-trivial invariant subspace.