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K-SPECTRAL VALUES FOR SOME FINITE MATRICES

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1. INTRODUCTION

Let X be a compact set in the complex plane and let $R(X)$ denote the algebra of quotients of polynomials whose poles lie off X. We let $M_n(R(X))$ denote the algebra of n by n matrices with entries from $R(X)$. If we let the n by n matrices have the operator norm that they inherit as linear transformations on the n-dimensional Hilbert space $\mathbf{C}^n$, then we can endow $M_n(R(X))$ with the norm,

$$\|(f_{i,j})\| = \sup\{\|(f_{i,j}(x))\| : x \in X\} = \sup\{\|(f_{i,j}(x))\| : x \in \partial X\}$$

where ∂X denotes the boundary of the set X. In a similar fashion, if $L(H)$ denotes the algebra of bounded linear operators on a Hilbert space H, then we endow $M_n(L(H))$ with the norm it inherits by regarding an element $(T_{i,j})$ in $M_n(L(H))$ as an operator on $H \oplus \ldots \oplus H$ (n copies).

If one is given a bounded linear map $\rho: R(X) \to L(H)$ then one can define bounded linear maps $\rho_n: M_n(R(X)) \to M_n(L(H))$ by $\rho_n((f_{i,j})) = (\rho(f_{i,j}))$. One has, $\|\rho\| \leqslant \|\rho_2\| \leqslant \ldots$, and in general the supremum of $\|\rho_n\|$ need not be finite. Maps for which it is are called *completely bounded* and $\|\rho\|_{\mathrm{cb}}$ is used to denote this supremum.

If T is an element of $L(H)$ whose spectrum is contained in X then there is a homomorphism $\rho: R(X) \to L(H)$ defined by $\rho(f) = f(T)$. If ρ is a bounded map, then X is called a *K-spectral set* for T and we set $K_X(T) = \|\rho\|$. When ρ is completely bounded, X is called a *complete K-spectral set for T* and we set $M_X(T) = \|\rho\|_{\mathrm{cb}}$.

There are two outstanding questions concerned with developing a structure theory for the class of operators which has a set as a K-spectral set. The first question asks if $K_X(T) = 1$, then is T necessarily the compression of a normal operator whose spectrum is contained in the boundary of X to a rationally semi-invariant subspace, i.e., does T have a ∂X-normal dilation. No counterexamples are known, and since being discussed in Sz.-Nagy and Foiaş [14] this question has only been resolved affirmatively for sets which are "close" to being simply connected. Except for the additional case of the annulus [1].

The most general result is due to Arveson [2], who proved that this question is equivalent to asking if $K_X(T) = 1$, then must $M_X(T) = 1$?

The second question asks if X is a K-spectral set for T, then is T similar to an operator which has a ∂X-normal dilation? When X is the closed unit disk, this second question is equivalent to asking if $K_X(T)$ is finite, then is T similar to a contraction? This latter question is one of Halmos' "Ten Problems" [5].

Generalizing Arveson's result we have shown that the second question is equivalent to asking if $K_X(T)$ is finite, then is it necessarily the case that $M_X(T)$ is finite? In fact, if, for any invertible S, we let $c(S)$ denote the condition number of S, that is, the product of the norm of S and the norm of S^{-1}, then [10],

$$M_X(T) = \min\{c(S) : M_X(S^{-1}TS) = 1\}.$$

Both of the above problems thus become questions about the relationship between $K_X(T)$ and $M_X(T)$. Very little is known about this relationship even for particular finite matrices, except that there are examples where $M_X(T) \neq K_X(T)$ [7]. Yet we believe that if counter-examples exist to the first question, then there should be counter-examples which are matrices. Analogously, the second question would be false unless one can find a bound for $M_X(T)$ in terms of $K_X(T)$ for finite matrices which is independent of n. Currently the best general linear bound is that $M_X(T) \leqslant nK_X(T)$, which is obtained through the theory of completely bounded maps [11, Exercise 3.11]. Recently, Bourgain [3] has obtained the estimate $M_X(T) \leqslant K_X(T)^4 \log n$.

Holbrook [6] proved that if X denotes the closed unit disk, then $K_X(T) = M_X(T)$ when T is a 2 by 2 matrix. Misra [8] proved that if T is a 2 by 2 matrix with a single eigenvalue and $K_X(T) = 1$, then $M_X(T) = 1$, for any compact set X.

In this paper we prove that for a 2 by 2 matrix T and arbitrary compact set X, $K_X(T) = M_X(T)$, and obtain an explicit formula for this number in terms of the matrix T and an analytic constant associated with the set X. Thus, in particular a 2 by 2 matrix T with $K_X(T) = 1$ has a ∂X-normal dilation. Our method uses Holbrook's and Misra's techniques, as well as the two characterizations of $M_X(T)$ given above, i.e., as a cb-norm and as a minimum condition number. The proof of Misra's result obtained by specializing the above result is very different from the original proof.

Finally, we study $K_X(T)$ and $M_X(T)$ when T is an elementary Jordan block matrix and obtain some partial results. We believe that these elementary Jordan block matrices are quite central to any general study of the relationship between $K_X(T)$ and $M_X(T)$, since they are the models for the "localizations" of a general operator to the eigenspace above a particular eigenvalue.

2. UNEQUAL EIGENVALUES

Let X be a compact set in the complex plane and let λ, μ be in X with $\lambda \neq \mu$. Let $T_t = \begin{bmatrix} \lambda & t \\ 0 & \mu \end{bmatrix}$, where t is any non-negative real number. Since T_t is similar to T_0 and $K_X(T_0) = M_X(T_0) = 1$, it is easy to see that $K_X(T_t)$ and $M_X(T_t)$ are finite and in fact are both bounded by $c(S)$, where S is any similarity carrying T_t to T_0. Note that for any f in $R(X)$,

$$f(T) = \begin{bmatrix} f(\lambda) & t(f(\lambda) - f(\mu))/(\lambda - \mu) \\ 0 & f(\mu) \end{bmatrix}.$$

Let u and v be any pair of unit eigenvectors for T_t with eigenvalues λ and μ, respectively. We define the *eccentricity* of T_t, $e(T_t)$ to be the modulus of the inner product of u and v. Note that $e(T_t)$ is independent of the choice of u and v and that if f is in $R(X)$ and $f(\lambda) \neq f(\mu)$, then $e(f(T_t)) = e(T_t)$, since u and v are eigenvectors for $f(T_t)$. Set

$$h(T_t) = (1 + e(T_t))^{1/2}/(1 - e(T_t))^{1/2}.$$

Finally, we let

$$a_X(\lambda, \mu) = \sup\{|f(\lambda)| : f(\lambda) = -f(\mu),\ \|f\| \leqslant 1,\ f \text{ in } R(X)\}.$$

A straightforward calculation yields

$$e(T_t) = t/(|\lambda - \mu|^2 + t^2)^{1/2},$$

and

$$h(T_t) = \frac{(|\lambda - \mu|^2 + t^2)^{1/2} + t}{|\lambda - \mu|}.$$

LEMMA 2.1. (Holbrook). *Let* $B = \begin{bmatrix} b & c \\ 0 & -b \end{bmatrix}$ *with* $b \neq 0$, *then* $\|B\| = b \cdot h(B)$.

Proof. See the calculation in [6, p. 239].

THEOREM 2.2. *Let* T_t *be as above. Then* $K_X(T_t) = 1$ *if and only if* $a_X(\lambda, \mu)\, h(T_t) \leqslant 1$.

Proof. If $K_X(T_t) = 1$ and f is in $R(X)$ with $\|f\| \leqslant 1$, and $f(\lambda) = -f(\mu)$, $f(\lambda) \neq 0$, then by Lemma 2.1,

$$1 \geqslant \|f(T_t)\| = |f(\lambda)| \cdot h(T_t),$$

from which one implication follows.

Conversely, assume that the inequality in the statement of the theorem is met, let $\varepsilon > 0$ be given and choose f with $\|f\| \leqslant 1$ and $f(\lambda) \neq f(\mu)$ such that $K_X(T_t) \leqslant \|f(T_t)\| + \varepsilon$.

There exists a Mobius map ψ from the disk to the disk, such that $\psi(f(\lambda)) = - \psi(f(\mu))$. Hence, by Lemma 2.1 and our hypothesis, $\|\psi(f(T_t))\| = |\psi(f(\lambda))| h(T_t) \leqslant 1$. But, now by von Neumann's inequality, $\|f(T_t)\| = \|\psi^{-1}(\psi(f(T_t)))\| \leqslant \|\psi^{-1}\| \leqslant 1$. Thus, $K_X(T_t) = 1$, as desired.

Note that $e(T_t), h(T_t)$, and $K_X(T_t)$ are all non-decreasing functions of t. It is also the case that $M_X(T_t)$ is a non-decreasing function of t. This is most easily seen using the unitary equivalence introduced in Proposition 2.6. Let t_* denote the value of t, with T_* the corresponding matrix, for which $a_X(\lambda, \mu)\, h(T_*) = 1$. By Theorem 2.2, $t_* = \sup\{t : K_X(T_t) = 1\}$.

LEMMA 2.3. (Holbrook). *Let* $r \leqslant t$, *then* $\inf\{c(S) : S^{-1}T_tS = T_r\} = h(T_t)/h(T_r)$.

Proof. See [6, Lemma 3.2] and [6, p. 240]. •

THEOREM 2.4. *Let* $t \geqslant t_*$, *then* $K_X(T_t) = a_X(\lambda, \mu)h(T_t)$. *Furthermore, if* f *is in* $R(X)$ *with* $\|f\| \leqslant 1$, $f(\lambda) = -f(\mu)$, *and* $|f(\lambda)| = a_X(\lambda, \mu)$, *then* $K_X(T_t) = \|f(T_t)\|$.

Proof. The inequality $K_X(T_t) \geqslant a_X(\lambda, \mu)h(T_t)$ follows by applying Lemma 2.1 to $f(T_t)$ when $\|f\| \leqslant 1$, $f(\lambda) = -f(\mu)$. To obtain the other inequality note that if $S^{-1}T_*S = T_t$, then $K_X(T_t) \leqslant c(S)$, and so $K_X(T_t) \leqslant \inf\{c(S) : S^{-1}T_*S = T_t\} = h(T_t)/h(T_*) = a_X(\lambda, \mu)h(T_t)$.

In order to prove the last statement, note that for such an f, by Lemma 2.1, $\|f(T_t)\| = a_X(\lambda, \mu)h(T_t) = K_X(T_t)$.

If we knew that $M_X(T_*) = 1$, then Lemma 2.3 and the proof of Theorem 2.4 imply that $K_X(T_t) = M_X(T_t)$, for all t. This is the case, for example, when X is a disk or an annulus. However, determining that $M_X(T_*) = 1$ is precisely the problem that was left open in [8]. We show below that this is indeed the case, but our proof is rather roundabout. Essentially, we obtain estimates on $K_X(T_t)$ and $M_X(T_t)$, then by considering the behaviour of these estimates as $t \to \infty$, we find that $K_X(T_t) = M_X(T_t)$, for all t.

We now turn our attention to $M_X(T_t)$. Set $t_\# = \sup\{t : M_X(T_t) \leqslant 1\}$, and let $T_\#$ denote the respective matrix. Note that $t_\# \leqslant t_*$. We shall eventually show that $t_\# = t_*$.

LEMMA 2.5. *Let* $t \geqslant t_\#$, *then* $M_X(T_t) = h(T_t)/h(T_\#) = \inf\{c(S) : S^{-1}T_tS = T_\#\}$.

Proof. Recall $M_X(T_t) = \inf\{c(S) : M_X(S^{-1}T_tS) = 1\}$ and note that $S^{-1}T_tS$ is unitarily equivalent to T_s for some s with $s \leqslant t_\#$. Applying Lemma 2.3 yields,

$$M_X(T_t) = \inf\{h(T_t)/h(T_s) : s \leqslant t_\#\} = h(T_t)/h(T_\#).$$

For the next result we need to introduce another constant related to the analytic structure of X. Let

$$\gamma_X(\lambda, \mu) = \sup \left\{ \left| \frac{f(\lambda) - f(\mu)}{\lambda - \mu} \right| : \|f\| \leqslant 1, f \text{ in } R(X) \right\}.$$

PROPOSITION 2.6. $\gamma_X(\lambda, \mu) = \lim_{t\to\infty} K_X(T_t)/t = \lim_{t\to\infty} M_X(T_t)/t.$

Proof. Recalling the form of $f(T_t)$, we see that

$$t\left| \frac{f(\lambda) - f(\mu)}{\lambda - \mu} \right| \leqslant \|f(T_t)\| \leqslant |f(\lambda)| + |f(\mu)| + t\left| \frac{f(\lambda) - f(\mu)}{\lambda - \mu} \right|. \tag{1}$$

From this is follows that,

$$t\gamma_X(\lambda, \mu) \leqslant K_X(T_t) \leqslant 2 + t\gamma_X(\lambda, \mu), \tag{2}$$

and hence, $\lim_{t\to\infty} K_X(T_t)/t = \gamma_X(\lambda, \mu)$.

Similarly, if $(f_{i,j})$ is an n by n matrix of functions, then $(f_{i,j}(T_t))$ is an n by n matrix of 2 by 2 matrices, and is unitarily equivalent to,

$$\begin{bmatrix} F(\lambda) & t(F(\lambda) - F(\mu))/(\lambda - \mu) \\ 0 & F(\mu) \end{bmatrix},$$

where $F(z)$ denotes the matrix-valued function $(f_{i,j}(z))$. Thus, inequality (1) still holds with f replaced by F, and so (2) holds with $M_X(T_t)$ replacing $K_X(T_t)$, and $\tilde{\gamma}_X(\lambda, \mu)$ replacing $\gamma_X(\lambda, \mu)$, where $\tilde{\gamma}_X(\lambda, \mu) = \sup \left\{ \left\| \frac{F(\lambda) - F(\mu)}{\lambda - \mu} \right\| : \|F\| \leqslant 1, F \text{ in } M_n(R(X)) \right\}$. If we fix vectors x and y in $\mathbf{C}^n$, with $\|x\| = \|y\| = 1$, then $f(z) = \langle F(z)x, y\rangle$ is in $R(X)$ and $\|f\| \leqslant 1$. Since x and y were arbitrary, we have that $\tilde{\gamma}_X(\lambda, \mu) = \gamma_X(\lambda, \mu)$. The proof that $\lim_{t\to\infty} M_X(T_t)/t = \gamma_X(\lambda, \mu)$, now follows as for $K_X(T_t)$.

Combining Proposition 2.6 with Theorem 2.4 yields,

$$\gamma_X(\lambda, \mu) = \lim_{t\to\infty} a_X(\lambda, \mu)h(T_t)/t = \frac{2a_X(\lambda, \mu)}{|\lambda - \mu|}.$$

This relationship is also easily obtained by elementary complex analysis. In particular, it implies that

$$K_X(T_t) = \gamma_X(\lambda, \mu)\left[\frac{[(|\lambda - \mu|^2 + t^2)^{1/2} + t]}{2}\right], \quad \text{for } t \geqslant t_*.$$

THEOREM 2.7. *$K_X(T_t) = M_X(T_t)$, and, in particular, $t_{\#} = t_{*}$.*

Proof. By Lemma 2.5 and Theorem 2.4, $M_X(T_t)/K_X(T_t) = h(T_*)/h(T_{\#})$. But by Proposition 2.6, this constant ratio approaches 1 as t approaches infinity, and so consequently is always 1. Furthermore, if this ratio equals 1, then $e_{\#} = e_*$ and hence $t_{\#} = t_*$.

REMARK 2.8. Misra [8], calculates t_* in terms of the constant $\delta = \sup\{|f(\lambda)| : \|f\| \leqslant 1, f(\mu) = 0\}$ and obtains $t_*^2 = |\lambda - \mu|^2(1 - \delta^2)/\delta^2$. Using the above results, one has $h(T_*) = a_X(\lambda, \mu)^{-1}$, which allows one to solve for t_* in terms of $a_X(\lambda, \mu)$. In this manner, one obtains, $t_* = |\lambda - \mu|(1 - a^2)/2a$, where $a = a_X(\lambda, \mu)$. These two formulas imply a relationship between the constants a and δ, which is also readily derived by standard complex analytic techniques.

REMARK 2.9. For the case $R_t = \begin{bmatrix} \lambda & t \\ 0 & \lambda \end{bmatrix}$, one can argue heuristically that $K_X(R_t) = M_X(R_t)$ by applying Theorem 2.7 to T_t, with $\lambda \neq \mu$, and arguing that as μ approaches λ, $K_X(T_t)$ approaches $K_X(R_t)$, and $M_X(T_t)$ approaches $M_X(R_t)$. By evaluating the limiting value of $K_X(T_t)$ as μ approaches λ, one obtains a formula for $K_X(R_t)$. In fact, if we let

$$\gamma_X(\lambda) = \sup\{|f'(\lambda)| : \|f\| \leqslant 1, f \in R(X)\},$$

then

$$\lim_{\mu \to \lambda} \gamma_X(\lambda, \mu) = \gamma_X(\lambda).$$

Using the formula preceding Theorem 2.7, we obtain,

$$K_X(R_t) = \gamma_X(\lambda)t, \quad \text{for } t \geqslant t_*.$$

With care, these arguments can be made precise.

In Section 3, we derive this result by quite different arguments. Part of the arguments of Section 3 extend to Jordan blocks of arbitrary size.

REMARK 2.10. Stampfli [12] studies the problem of determining if each of two sets is a spectral set for an operator, then is their intersection a K-spectral set for the operator. In the case of two disks he proves that this is the case provided that the spectrum of the operator is bounded away from the "corners" of the intersection. Some insight into this problem can be gained by considering the matrices T_t. Let X_1, X_2 be two compact sets and X their intersection. If $h(T_t) = \min\{a_{X_1}(\lambda, \mu)^{-1}, a_{X_2}(\lambda, \mu)^{-1}\}$, then X_1 and X_2, will each be spectral sets for T_t, but $K_X(T_t) = \min\{a_X(\lambda, \mu)/a_{X_1}(\lambda, \mu), a_X(\lambda, \mu)/a_{X_2}(\lambda, \mu)\}$. By considering a direct sum of matrices of the above form, we can see that for X to always be a K-spectral set, for some number K, for every operator for which X_1 and X_2 are both spectral sets, necessarily, the above ratios would need to be bounded, independent of λ and μ, by K.

3. JORDAN BLOCKS

In this section we study $K_X(T)$ and $M_X(T)$ for T the elementary Jordan block corresponding to a single eigenvalue λ. When T is 2 by 2, we prove that $K_X(T) = M_X(T)$ and obtain a formula for this quantity. We recover Misra's result [8] that if a set X is a spectral set for such a matrix, then the matrix has a ∂X-normal dilation. Our proof is quite different from Misra's in that for T 2 by 2, we show that $K_X(T) = 1$ implies $M_X(T) = 1$ directly, rather than deducing it by constructing a dilation.

When T is a Jordan block of size other than 2, our results are very incomplete and largely negative. We prove that a likely conjectured formula for $K_X(T)$ and $M_X(T)$ can only hold in very special circumstances.

Let X be a compact subset of the complex plane, λ a point in X, t a non-negative number, and let T_t be the $n+1$ by $n+1$ matrix which has λ for its diagonal entries, t for its super-diagonal entries, and 0 for its remaining entries. Let J denote the $n+1$ by $n+1$ matrix of the above form which corresponds to $\lambda = 0$ and $t = 1$, so that $T_t = \lambda I + tJ$ and J^k is the matrix which has a 1 for each entry of its k-th super-diagonal and 0 for its remaining entries. In particular, $J^{n+1} = 0$. It is not difficult to see that for f in $R(X)$,

$$f(T) = \sum_{j=0}^{n} f^{(j)}(\lambda) t^j J^j / j!\,.$$

In general, if λ lies in the boundary of X, one expects to find f in $R(X)$ with $\|f\| \leqslant 1$, but $f'(\lambda)$ arbitrarily large. In this case, one has $K_X(T_t) = M_X(T_t) = +\infty$, except when $t = 0$. To avoid this case, we shall assume in the remainder of this section that λ lies in the interior of X, and thus $K_X(T_t)$ and $M_X(T_t)$ will be finite as the following proposition shows.

PROPOSITION 3.1. *Let d denote the distance from λ to ∂X. Then $M_X(T_t) = 1$ for $t \leqslant d$, and $M_X(T_t) \leqslant (t/d)^n$ for $t \geqslant d$.*

Proof. Let B denote the backwards bilateral shift on ℓ_2, so that B is unitary, and let $t \leqslant d$. Since the spectrum of $\lambda I + tB$ is contained in X, $M_X(\lambda I + tB) = 1$. The subspace spanned by any $n+1$ consecutive vectors from the canonical basis for ℓ_2 is semi-invariant for $\lambda I + tB$ and its compression to this space is $\lambda I + tJ = T_t$. Thus, $M_X(T_t) = 1$.

Finally, for $t \geqslant d$, if S denotes the diagonal matrix with entries $1, t/d, \ldots$ $\ldots, (t/d)^n$, then $S(T_d)S^{-1} = T_t$. Hence, $M_X(T_t) \leqslant c(S)M_X(T_d) = (t/d)^n$.

Let $\gamma_n(\lambda) = \sup\{|f^{(n)}(\lambda)| : f \in R(X),\ \|f\| \leqslant 1\}$.

PROPOSITION 3.2. *Let T_t be as above, then*

$$\gamma_n(\lambda)/n! = \lim_{t\to\infty} K_X(T_t)/t^n = \lim_{t\to\infty} M_X(T_t)/t^n.$$

Proof. The proof is identical to the proof of Proposition 2.6. One first observes that $\gamma_n(\lambda)t^n/n! \leqslant K_x(T_t) \leqslant M_x(T_t) \leqslant \sum_{j=0}^{n} \gamma_j(\lambda)t^j/j!$.

LEMMA 3.3. *Let $s \leqslant t$, then* $\inf\{c(S) : ST_tS^{-1} = T_s\} = (t/s)^n$ *and is attained by letting S be the diagonal matrix whose diagonal entries are $1, s/t, \ldots, (s/t)^n$.*

Proof. Let D denote the above diagonal matrix, then $DT_tD^{-1} = T_s$ and $c(D) = (t/s)^n$, so it will be enough to show that if $ST_tS^{-1} = T_s$, then $c(S) \geqslant c(D)$. Let S be such a matrix and set $A = D^{-1}S$, then $AT_tA^{-1} = T_t$ and so A commutes with J. Thus, A necessarily has the form, $A = a_0I + a_1J + \ldots + a_nJ^n$.

Define $A(z) = a_0I + a_1zJ + \ldots + a_nz^nJ^n$, for z any complex number, and set $S(z) = DA(z)$, so that $A(z)$ and $S(z)$ are analytic matrix-valued functions. Note that if $z_1 = \alpha z_2$ with $|\alpha| = 1$, then $A(z_1) = U^*A(z_2)U$, where U is the diagonal unitary with entries $1, \alpha, \ldots, \alpha^n$, and $S(z_1) = U^*S(z_2)U$. Thus we have that $\|S(z_1)\| = \|S(z_2)\|$ for $|z_1| = |z_2|$. By the maximum modulus theorem $\|S(z)\|$ is an increasing function of $|z|$.

Also, if $A^{-1} = b_0I + b_1J + \ldots + b_nJ^n$, and we set $B(z) = b_0I + b_1zJ + \ldots$ $\ldots + b_nz^nJ^n$, then $A(z)^{-1} = B(z)$, since both are analytic and are clearly equal when $|z| = 1$. So $S(z)^{-1} = B(z)D^{-1}$. By repeating the above argument, we see that $\|S(z)^{-1}\|$ is also an increasing function of $|z|$.

Finally, note that $S(z)T_tS(z)^{-1} = T_s$ and that $S(1) = S$. Hence, $c(S) =$ $= c(S(1)) \geqslant c(S(0)) = c(D)$ as desired.

LEMMA 3.4. (Misra). *Let $n = 1$, then $K_X(T_t) = 1$ if and only if $t \leqslant \gamma_1^{-1}(\lambda)$.*

Proof. This is [8, Corollary 1.1] combined with the observation that, $\gamma_1(\lambda) =$ $= \sup\{|f'(\lambda)| : f \in R(X),\ \|f\| \leqslant 1$, and $f(\lambda) = 0\}$.

THEOREM 3.5. *Let $n = 1$, then for $t \leqslant \gamma_1(\lambda)^{-1}$, $K_X(T_t) = M_X(T_t) = 1$, while for $t \geqslant \gamma_1(\lambda)^{-1}$, $K_X(T_t) = M_X(T_t) = t\gamma_1(\lambda)$.*

Proof. Let $t_* = \gamma_1^{-1}(\lambda)$, $T_* = T_{t_*}$, set $t_\# = \sup\{t : M_X(T_t) = 1\}$ and let $T_\# = T_{t_\#}$. Recall that $M_X(T_t) = \inf\{c(S) : M_X(S^{-1}T_tS) = 1\}$. But up to a unitary equivalence $S^{-1}T_tS = T_s$, for some s, and $M_X(T_s) = 1$ if and only if $s \leqslant t_\#$. Hence, by Lemma 3.3, $M_X(T_t) = \inf\{c(S) : S^{-1}T_tS = T_s,\ s \leqslant t_\#\} = \inf\{t/s : s \leqslant t_\#\} =$ $= t/t_\#$, when $t \geqslant t_\#$ and 1 when $t \leqslant t_\#$.

On the other hand, if $t \geqslant t_*$, then since $\inf\{c(S) : S^{-1}T_tS = T_*\} = t/t_*$, we have that $K_X(T_t) \leqslant t/t_*$. But if $\|f\| \leqslant 1$, then $K_X(T_t) \geqslant \|f(T_t)\| \geqslant t|f'(\lambda)|$. Thus, $K_X(T_t) \geqslant t\gamma_1(\lambda) = t/t_*$ and so, $K_X(T_t) = t/t_*$ for $t \geqslant t_*$. Clearly, for $t \leqslant t_*$, $K_X(T_t) = 1$.

The proof will be completed by showing that $t_* = t_\#$. To see this, note that $t_\# \leqslant t_*$ and that $1 \leqslant t_*/t_\# = \dfrac{t/t_\#}{t/t_*} = \dfrac{M_X(T_t)}{K_X(T_t)}$, but this last expression approaches 1 as t approaches infinity, by Proposition 3.2. Hence, the constant ratio $t_*/t_\#$ must be equal to one.

COROLLARY 3.6. *Let $n = 1$ and set $t_* = \gamma_1(\lambda)^{-1}, T_* = T_{t_*}$. Then for $t \geqslant t_*$, $K_X(T_t) = M_X(T_t) = t/t_* = \inf\{c(S) : S^{-1}T_tS = T_*\}$. Furthermore, if $f \in R(X)^-$ with $\|f\| = 1$ and $f'(\lambda) = \gamma_1(\lambda)$, then for $t \geqslant t_*$, $K_X(T_t) = \|f(T_t)\|$.*

REMARK 3.7. Theorem 3.5 makes it possible to do some explicit calculations. We return to the problem of Stampfli discussed in Remark 2.10. Let D_+ and D_- denote the closed disks of radius 1 with centers at $+\sin\theta$ and $-\sin\theta$, respectively, for fixed θ, $0 < \theta < \pi/2$. If each of these disks is a spectral set for an operator T, then by Stampfli's result [12], their intersection will be a K-spectral set for T, provided that the spectrum of T contains neither of the points $\pm i\cos\theta$.

For the above matrices T_t, we wish to calculate this value K explicitly and then examine the behaviour as we allow the eigenvalue of T_t to approach the points $\pm i\cos\theta$.

Let X denote the intersection of D_+ and D_-, and let $\gamma_+(\lambda)$, $\gamma_-(\lambda)$ and $\gamma_X(\lambda)$ denote the values of $\gamma_1(\lambda)$ for the sets D_+, D_- and X, respectively. If $\lambda \in X$ and we set $t = \min\{\gamma_+(\lambda)^{-1}, \gamma_-(\lambda)^{-1}\}$, then by Theorem 3.5, D_+ and D_- will both be spectral sets for T_t, i.e., $K_{D_+}(T_t) = K_{D_-}(T_t) = 1$. However, again by Theorem 3.5, $K_X(T_t) = t\gamma_X(\lambda) = \min\{\gamma_X(\lambda)/\gamma_+(\lambda), \gamma_X(\lambda)/\gamma_-(\lambda)\}$.

This quantity is fairly computable. Recall that if f is the Riemann maping of a region onto the unit disk with $f(\lambda) = 0$, then $|f'(\lambda)| = \gamma_1(\lambda)$ for the region.

If we set $\lambda = \alpha i$, $-\cos\theta < \alpha < \cos\theta$, then by computing f, we find,

$$\gamma_+(\alpha i) = \gamma_-(\alpha i) = 2^{-1}(\cos^2\theta - \alpha^2)^{-1},$$

$$\gamma_X(\alpha i) = \pi\cos(\theta/(\pi - 2\theta))(\cos^2\theta - \alpha^2).$$

Thus, $K_X(T_t) = 2\pi\cos\theta/(\pi - 2\theta)$, independent of α.

It is interesting to see how many of the above arguments can be extended to the $n \geqslant 2$ case. Surprisingly, if a result like Corollary 3.6 were true for $n \geqslant 2$, then the whole theory carries over. However, there is a function theoretic obstruction that prevents a result like Corollary 3.6 from being true for $n \geqslant 2$, at least for most sets X. We make this precise in what follows.

Let T be an arbitrary operator with $M_X(T)$ finite, for some set X. If B is similar to T, $M_X(B) = 1$ and $M_X(T) = \inf\{c(S) : S^{-1}TS = B\}$, then we call B a *nearst conjugate* of T. Note that when $M_X(T) = 1$, B must be unitarily equivalent to T.

Let T_t be the $(n+1) \times (n+1)$ matrices considered above and set

$$t_n = \sup\{t : K_X(T_t) = 1\},$$

$$r_n = \sup\{t : M_X(T_t) = 1\}.$$

Thus, in our above notation $t_1 = t_*$, $r_1 = t_\#$ and $t_1 = r_1$.

Note that for $t \neq 0$ the matrices T_t are all similar, and that for $t > r_n$, $M_X(T_t) > 1$. Thus it makes sense to ask whether or not there exists a single matrix B, which is a nearest conjugate for all T_t, $t > r_n$. When this occurs we say that B is a *nearest conjugate for the family* $\{T_t\}$, $t > r_n$. Corollary 3.6 says that there is a nearest conjugate for the family $\{T_t\}$, $t \geqslant r_n$ when $n = 1$. We shall see below in Theorem 3.9 and Example 3.11 that this generally fails for $n \geqslant 2$, i.e., when T_t is 3×3 or larger, and that the obstruction is function theoretic. In Section 2, for the unequal eigenvalue 2×2 case, we saw that the family T_t, $t \geqslant t_*$ also had a nearest conjugate.

We begin with some elementary inequalities.

PROPOSITION 3.8. *The sequences $\{t_n\}$ and $\{r_n\}$ are both non-increasing, and if d denotes the distance from λ to ∂X, then, $d \leqslant r_n \leqslant t_n \leqslant (n!/\gamma_n(\lambda))^{1/n}$.*

Proof. The $n+1$ by $n+1$ version of T_t is the compression to a semi-invariant subspace of the $k+1$ by $k+1$ version when $n \leqslant k$. From this it easily follows that t_n and r_n are non-increasing.

Note that the $(1, n+1)$-entry of $f(T_t)$ is $t^n f^{(n)}(\lambda)/n!$. Thus, if $\|f\| \leqslant 1$ and $K_X(T_t) = 1$, then $t^n f^{(n)}(\lambda)/n! \leqslant 1$, so that $t^n \leqslant n!/\gamma_n(\lambda)$, from which the inequality $t_n \leqslant (n!/\gamma_n(\lambda))^{1/n}$ follows. The fact that $d \leqslant r_n$ follows from Proposition 3.1.

Note that $M_X(T_t) \leqslant \inf\{c(S) : S^{-1}T_tS = T_{r_n}\} = (t/r_n)^n$ and that $K_X(T_t) \leqslant (t/t_n)^n$. When $n = 1$, both these inequalities are equalities. The following results show how different the $n \geqslant 2$ case is.

THEOREM 3.9. *Let $n \geqslant 1$. The following are equivalent:*

i) *there exists a nearest conjugate for the family $\{T_t\}$, $t > r_n$,*

ii) *T_{r_n} is a nearest conjugate for the family $\{T_t\}$, $t \geqslant r_n$,*

iii) $r_n = t_n = (n!/\gamma_n(\lambda))^{1/n}$,

iv) *for $t \geqslant r_n$, $K_X(T_t) = M_X(T_t) = (t/r_n)^n$.*

Moreover, if any of these equivalent conditions is met and if $\{f_j\}$ is a sequence in $R(X)$ with $\|f_j\| \leqslant 1$ for all j, such that $\lim_{j\to\infty} f_j^{(n)}(\lambda) = \gamma_n(\lambda)$, then $\lim_{j\to\infty} f_j^{(k)}(\lambda) = 0$ for $k = 0, 1, \ldots, n-1$.

Proof. If B is a nearest conjugate for the family $\{T_t\}$, $t > r_n$, then a limiting argument shows that B is also a nearest conjugate for T_{r_n}. Since $M_X(T_{r_n}) = 1$ we must have that B and T_{r_n} are unitarily equivalent. Thus, i) implies ii).

Assuming ii), by Lemma 3.3, we have that $M_X(T_t) = (t/r_n)^n$, for $t \geqslant r_n$. Applying Proposition 3.2, we have that $\gamma_n(\lambda)/n! = (1/r_n)^n$. Hence, by Proposition 3.8, $r_n = t_n = (n!/\gamma_n(\lambda))^{1/n}$. Thus, ii) implies iii).

By considering the $(1, n)$-entry of $f(T_t)$ we see that $t^n|f^{(n)}(\lambda)|/n! \leqslant \|f(T_t)\| \leqslant \leqslant K_X(T_t)$, for $\|f\| \leqslant 1$. Taking the supremum of the left-hand side yields,

$$t^n\gamma_n(\lambda)/n! \leqslant K_X(T_t) \leqslant M_X(T_t) \leqslant (t/r_n)^n.$$

Now iii) implies that the first and last terms of the above inequality are equal, and hence iv) follows.

If iv) is true, then $M_X(T_t) = (t/r_n)^n = \inf\{c(S) : S^{-1}T_tS = T_{r_n}\}$ by Lemma 3.3, for $t \geqslant r_n$. Thus, T_{r_n} is a nearest conjugate for the family $\{T_t\}$, $t > r_n$, and so i) follows.

Finally, assuming i) -- iv) and with $\{f_j\}$ as above, we have that the 2-norm of the first row of $f_j(T_t)$ satisfies,

$$\gamma_n(\lambda)t^n/n! = K_X(T_t) \geqslant \|f_j(T_t)\| \geqslant (|f_j(\lambda)|^2 + \ldots$$

$$\ldots + t^{2n}|f_j^{(n)}(\lambda)|^2/(n!)^2)^{1/2} \geqslant t^n|f_j^{(n)}(\lambda)|/n!\ .$$

Since the last term in this inequality is approaching the first term, $f_j(\lambda), \ldots, f_j^{(n-1)}(\lambda)$, must all be approaching 0.

REMARK 3.10. If X is the closure of an open set G, and if every element of $H^\infty(G)$ is the pointwise limit of a bounded sequence of elements of $R(X)$, then by a normal families argument there exists an analytic function f from G to the unit disk, such that $\gamma_n(\lambda) = f^{(n)}(\lambda)$. (Such a G is called, "approachable" in [8].) In this case, the last statement in Theorem 3.8 simplifies. Namely, for such an f, $f(\lambda) = \ldots = f^{n-1}(\lambda) = 0$.

EXAMPLE 3.11. We show that for $n \geqslant 2$ and X the closed unit disk, the equivalent conditions of Theorem 3.9 are not met.

It is well-known that the disk is a spectral set for an operator if and only if the operator is a contraction, and that, in this case the disk is a complete spectral set. Thus $r_n = t_n$, in this case, and their common value is the unique value of t for which $\|T_t\| = 1$.

Now consider the problem of finding,

$$(*) \qquad \sup\{|f^{(n)}(\lambda)| : f(\lambda) = \ldots = f^{(n-1)}(\lambda) = 0\},$$

where f is an analytic function from the disk to the disk. Let $\psi_\lambda(z)$ denote the elementary Möbius map which is 0 at λ, i.e., $\psi_\lambda(z) = (z - \lambda)/(1 - \bar{\lambda}z)$. For f as in (*), there will exist a function g, analytic on the disk, such that $f(z) = \psi_\lambda(z)^n g(z)$. Since $|\psi_\lambda(z)| = 1$ for $|z| = 1$, by the maximum modulus principle, $|g(z)| \leqslant 1$. But, $f^{(n)}(\lambda) = (\psi_\lambda^n)^{(n)}(\lambda)g(\lambda)$, and so we have that (*) is attained by the function ψ_λ^n. We calculate,

$$(**)\qquad (\psi_\lambda^n)^{(n)}(\lambda) = n!/(1 - |\lambda|^2)^n.$$

If the hypotheses of Theorem 3.9 were met in this case, then we would have that this last value agrees with $\gamma_n(\lambda)$, by Remark 3.10. This can be shown to be false, except when $n = 1$ or $\lambda = 0$, by a direct calculation. In fact, for odd $n = 2m + 1$, $\gamma_n(\lambda)$ is attained by the function $f(Z) = Z^m \psi_\lambda(Z)^{m+1}$ and $\gamma_n(\lambda) = |f^{(n)}(\lambda)| = = n!/(1 - |\lambda|^n) \left\{ \sum_{k=0}^{m} \binom{m}{k}^2 |\lambda|^{2k} \right\}$.

Also, for even n, $\gamma_n(\lambda) \neq n!/(1 - |\lambda|^2)^n$ although the expression is more complicated. See [4, Chapter 8, Exercise 7], and [13].

Another less direct way to see that both these hypotheses are false, is to note that, if not, then by iii) we would have that, $r_n = t_n = (n!/\gamma_n(\lambda))^{1/n} = 1 - |\lambda|^2$, a value independent of $n!$. This would say that $T_{(1-|\lambda|^2)}$ has norm 1, for our prescribed n, which can again be shown to be false by a direct calculation, except when $n = 1$ or $\lambda = 0$.

Thus, we see that for $n > 1$ and $\lambda \neq 0$, the family $\{T_t\}$, $t \geqslant r_n$ does not have a nearest conjugate. The obstruction is simply the fact that the function where $\gamma_n(\lambda)$ is attained does not have the property that its first $(n - 1)$ derivatives vanish at λ, except when $\lambda = 0$ or $n = 1$. The $n = 1$ case, i.e., the 2 by 2 case we have already treated.

When $\lambda = 0$, a Cauchy estimate shows that $\gamma_n(0) \leqslant n!$, but the function $f(Z) = Z^n$ has $f^{(n)}(0) = n!$, so $n! = \gamma_n(0)$. On the other hand, for $\lambda = 0$, $\|T_t\| = 1$ when $t = 1$. Thus $t_n = r_n = 1$, and so Theorem 3.9 iii) is met.

Using the disk, it is possible to construct other simply connected regions with a single point where the hypotheses of Theorem 3.9 are satisfied, for some n. We do not know any examples of connected sets with more than one such point.

The results of this section show that the problem of determining $M_X(T_t)$ and $K_X(T_t)$ is, for $n \geqslant 2$, considerably more subtle than in the 2 by 2 case. Indeed, we do not even know if $M_X(T_t) = K_X(T_t)$ when X is the closed unit disk. Another important question, to which we do not know the answer, is whether or not the ratio $M_X(T_t)/K_X(T_t)$ is bounded, independent of λ, t, and n. Again, even when X is the closed unit disk this is not clear. If this ratio is not bounded, then it is likely that one could find an operator A, which is the direct sum of these elementary Jordan matrices, with $K_X(A)$ finite, but $M_X(A)$ infinite.

Theorem 3.9 also leaves open the possibility that for some $r > r_n$, the family $\{T_t\}$, $t \geqslant r$ has a nearest conjugate. In fact, we conjecture that in some limiting sense this is true, and that is the topic of the next section.

4. ASYMPTOTIC NEAREST CONJUGATES

Let X be a compact set in the complex plane and let T be an operator with $M_X(T)$ finite. If B is similar to T, then we set

$$m_B(T) = \inf\{c(S) : S^{-1}TS = B\}.$$

If $M_X(B) = 1$, then $M_X(T) \leqslant m_B(T)$ and we call B a *nearest conjugate* when $M_X(T) = m_B(T)$.

Let $\{T_t\}$ denote the family of $(n + 1)$ by $(n + 1)$ matrices considered in the previous section. In that section we considered the problem of finding a matrix B with $M_X(T_t) = m_B(T_t)$. We call B an *asymptotic nearest conjugate for the family* $\{T_t\}$ if $M_X(B) = 1$ and

$$\limsup_{t\to\infty}(m_B(T_t) - M_X(T_t))/t^n = 0. \tag{1}$$

By Proposition 3.2 this is the same as requiring that $M_X(B) = 1$ and

$$\lim_{t\to\infty} m_B(T_t)/t^n = \gamma_n(\lambda)/n!. \tag{2}$$

We conjecture that the family $\{T_t\}$ always has an asymptotic nearest conjugate. In this section we verify this conjecture in one simple case.

Note that for any matrix B similar to the family $\{T_t\}$ with $M_X(B) = 1$ the quantity

$$\liminf_{t\to\infty} m_B(T_t)/t^n$$

gives an upper bound on $\gamma_n(\lambda)$. Thus, our conjecture is equivalent to requiring that

$$\inf_B \limsup_{t\to\infty} m_B(T_t)/t^n = \gamma_n(\lambda)/n!,$$

where the infimum is taken over all matrices B, similar to $\{T_t\}$ with $M_X(B) = 1$.

Let X be the closed unit disk, and let J be the $(n + 1)$ Jordan block with 0 eigenvalues, i.e., the entries of J are all 0 except for the superdiagonal which consists of 1's. Let λ be an arbitrary point in the open unit disk, let $T_t = \lambda I + tJ$, and let $\psi(Z) = (Z + \lambda)/(1 + \bar{\lambda}Z)$ be the elementary Möbius map which carries 0 to λ. We conjecture, that for all n, $\psi(J)$ is an asymptotic nearest conjugate for $\{T_t\}$. We shall verify this only in the case $n = 2$.

When $n = 2$,

$$\psi(J) = \begin{bmatrix} \lambda & 1-|\lambda|^2 & -\bar{\lambda}(1-|\lambda|^2) \\ 0 & \lambda & 1-|\lambda|^2 \\ 0 & 0 & \lambda \end{bmatrix}.$$

A direct calculation shows that if $S^{-1}T_tS = \psi(J)$ and S is normalized such that its $(1, 1)$ entry is 1, then

$$S = \begin{bmatrix} 1 & a & b \\ 0 & (1-|\lambda|^2)/t & (1-|\lambda|^2)(a-\bar{\lambda})/t \\ 0 & 0 & (1-|\lambda|^2)^2/t^2 \end{bmatrix},$$

and

$$S^{-1} = \begin{bmatrix} 1 & -at/(1-|\lambda|^2) & (a^2-a\bar{\lambda}-b)t^2/(1-|\lambda|^2)^2 \\ 0 & t/(1-|\lambda|^2) & -(a-\bar{\lambda})t^2/(1-|\lambda|^2)^2 \\ 0 & 0 & t^2/(1-|\lambda|^2)^2 \end{bmatrix},$$

where a, b are arbitrary complex numbers.

Letting $S(t, a, b)$ denote this family of matrices

$$m_{\psi(J)}(T_t) = \inf_{a,b} c(S(t, a, b)),$$

thus

$$\gamma_2(\lambda)/2 \leqslant \limsup_{t\to\infty} m_{\psi(J)}(T_t)/t^2 \leqslant \inf_{a,b} \limsup_{t\to\infty} c(S(t, a, b))/t^2$$

clearly,

$$\lim_{t\to\infty} c(S(t, a, b))/t^2 = (1 + |a|^2 + |b|^2)^{1/2}(1 + |a - \bar{\lambda}|^2 + |a^2 - a\bar{\lambda} - b|^2)^{1/2}/(1 - |\lambda|^2)^2.$$

Setting $a = +\bar{\lambda}/2$, $b = -\bar{\lambda}^2/8$ yields,

$$\gamma_2(\lambda) \leqslant 2 \cdot \limsup_{t\to\infty} m_{\psi(J)}(T_t)/t^2 \leqslant 2(1 + |\lambda|^2/8)^2/(1 - |\lambda|^2)^2.$$

In [13], the value of $\gamma_2(\lambda)$ is calculated and shown to be equal to the right-hand side of the above inequality. Thus, for $n = 2$, $\psi(J)$ is an asymptotic nearest conjugate as claimed.

In fact, we do not need the full strength of [13] in order to complete the above argument. Szasz produces the function,

$$f(Z) = \frac{(Z-\lambda)^2 + \frac{1}{2}\lambda(Z-\lambda)(1-\bar{\lambda}Z) - \frac{1}{8}\lambda^2(1-\bar{\lambda}Z)^2}{(1-\bar{\lambda}Z)^2 + \frac{1}{2}\bar{\lambda}(Z-\lambda)(1-\bar{\lambda}Z) - \frac{1}{8}\bar{\lambda}^2(Z-\lambda)^2}$$

and then argues that this is the extremal function, i.e., that $\gamma_2(\lambda) = |f''(\lambda)|$. However, since we already have an upper bound on $\gamma_2(\lambda)$, we need only calculate that $|f''(\lambda)|$ is equal to the right-hand side of the above inequality, which is tedious but straightforward.

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