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GENERALIZED CONDITIONAL EXPECTATIONS AND MARTINGALES IN NONCOMMUTATIVE L^p-SPACES

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INTRODUCTION

Since Umegaki's earlier works [25, 26], the martingale convergence theory for conditional expectations in von Neumann algebras has been developed by several authors (see [7, 15, 24] for example). The conditional expectation of a von Neumann algebra M onto its von Neumann subalgebra N does not generally exist relative to a faithful normal state (or semifinite weight) φ on M. Indeed, according to the well-known theorem of Takesaki [21], the existence is equivalent to the global invariance of N under the modular automorphism group associated with φ. Nevertheless, the generalized conditional expectation introduced by Accardi and Cecchini [1] always exists relative to any N and φ (whenever $\varphi \upharpoonright N_+$ is semifinite), while it is not necessarily a projection onto N. The strong martingale convergence of generalized conditional expectations was obtained in [12, 17].

After Haagerup [10] introduced L^p-spaces over general von Neumann algebras, several other constructions of noncommutative L^p-spaces have been known (see [2, 4, 13, 14, 23]). Although those L^p-spaces constructed so far are mutually isometrically isomorphic, the interpolation L^p-spaces in [14, 23] have the advantage of enjoying the complex interpolation technique. With respect to a faithful normal state φ on a von Neumann algebra M, Kosaki's L^p-spaces $L^p(M; \varphi)_\eta$, $1 < p < \infty$, are defined with the parameter $0 \leqslant \eta \leqslant 1$ corresponding to the way of imbedding of M into M_*. On the other hand, Terp's L^p-spaces are defined in one way without the parameter but with respect to a faithful normal semifinite weight φ on M.

Concerning the martingale convergence in noncommutative L^p-spaces, some results have been obtained by Cecchini and Petz [5] and Goldstein [8, 9]. But these are not yet complete. The purpose of this paper is to study more thoroughly generalized conditional expectations and their martingale convergence in L^p-spaces.

In Section 1 of this paper, we give a brief survey on Kosaki's and Terp's interpolation L^p-spaces for later convenience. In Section 2, fixing a unital von Neumann subalgebra N of M, we introduce the generalized L^p-conditional expectations on Kosaki's L^p-spaces $L^p(M;\varphi)_\eta$, $1 < p < \infty$, $0 \leqslant \eta \leqslant 1$, relative to a faithful normal state φ. These are regarded as natural extensions of the generalized conditional expectation $\varepsilon : M \to N$ and becomes linear contractions between L^p-spaces. The main tool is the complex interpolation theorem of Riesz-Thorin type. Our class of (generalized) L^p-conditional expectations includes those given in [4, 5, 8, 9]. Furthermore we give some characterizations of $\varepsilon : M \to N$ being the conditional expectation (i.e. norm one projection onto N) in terms of generalized L^p-conditional expectations. In Section 3, we discuss the norm convergence of generalized martingales in L^p-spaces under an increasing or decreasing net of unital von Neumann subalgebras of M in the same situation as Section 2. Finally in Section 4, we consider generalized L^p-conditional expectations and martingales in Terp's L^p-spaces relative to a faithful normal semifinite weight.

1. PRELIMINARIES ON L^p-SPACES

In this section, we briefly summarize Kosaki's and Terp's interpolation L^p-spaces to give preliminaries and notations for later discussions. We fix a von Neumann algebra M on a Hilbert space $\mathscr{H}$ with a faithful normal semifinite weight φ. The following are the usual notations in the Tomita-Takesaki theory: $\mathfrak{n}_\varphi = \{x \in M : \varphi(x^*x) < \infty\}$, $\mathfrak{m}_\varphi = \operatorname{span} \mathfrak{n}_\varphi^* \mathfrak{n}_\varphi$, the GNS representation $(\mathscr{H}_\varphi, \pi)$ of M induced by φ, the canonical injection Λ of $\mathfrak{n}_\varphi$ into $\mathscr{H}_\varphi$, the modular operator Δ, the modular conjugation J, the modular automorphism group σ_t, $t \in \mathbb{R}$, associated with φ.

We begin with Haagerup's L^p-spaces. Let R denote the crossed product $M \times_\sigma \mathbf{R}$ which admits the canonical faithful normal semifinite trace τ and the dual action θ_s, $s \in \mathbf{R}$, satisfying $\tau \circ \theta_s = e^{-s}\tau$, $s \in \mathbb{R}$. The set of all τ-measurable operators affiliated with R is denoted by $\tilde{R}$ (cf. [16], [22, Chapter I]). For each $\psi \in M_*^+$, let $\tilde{\psi}$ be its dual weight on R and h_ψ the element of $\tilde{R}$ satisfying $\tilde{\psi} = \tau(h_\psi \cdot)$. The mapping $\psi \mapsto h_\psi$ is extended to a linear bijection (still denoted by $\psi \mapsto h_\psi$) of M_* onto $\{a \in \tilde{R} : \theta_s(a) = e^{-s}a,\ s \in \mathbb{R}\}$. For each $1 \leqslant p \leqslant \infty$, *Haagerup's L^p-space* $L^p(M)$ introduced in [10] is

$$L^p(M) = \{a \in \tilde{R} : \theta_s(a) = e^{-s/p}a,\ s \in \mathbb{R}\}.$$

When $1 \leqslant p < \infty$, $L^p(M)$ coincides with the set of $a \in \tilde{R}$ having the polar decomposition $a = u|a|$ such that $u \in M$ and $|a|^p \in L^1(M)$. The linear functional tr on $L^1(M)$ is defined by $\operatorname{tr}(h_\psi) = \psi(1)$, $\psi \in M_*$. Then $L^p(M)$ is a Banach space with the norm

$$\|a\|_p = \operatorname{tr}(|a|^p)^{1/p}, \quad a \in L^p(M),\ 1 \leqslant p < \infty,$$

$$\|a\|_\infty = \|a\|, \quad a \in L^\infty(M)\ (=M).$$

In particular, $M = L^\infty(M)$ and $M_* \cong L^1(M)$ by the isometry $\psi \mapsto h_\psi$. The detailed expositions on Haagerup's L^p-spaces are found in [22, Chapter II].

Now let φ be a faithful normal state on M (hence M is σ-finite). By taking the GNS representation of M induced by φ, we may assume that M has a cyclic and separating vector $\xi \in \mathscr{H}$ and $\varphi = (\cdot\xi|\xi)$. We denote $h_\varphi \in L^1(M)$ simply by h. For each $0 \leqslant \eta \leqslant 1$, M is imbedded into $L^1(M)$ by $x \mapsto h^\eta x h^{1-\eta}$, $x \in M$. Define the norm $\|h^\eta x h^{1-\eta}\|_{\infty,\eta} = \|x\|$ on $h^\eta M h^{1-\eta}(\subset L^1(M))$, i.e. $h^\eta M h^{1-\eta} \cong M$. Then $(h^\eta M h^{1-\eta}, L^1(M))$ becomes a pair of compatible Banach spaces. For each $1 < p < \infty$ and $0 \leqslant \eta \leqslant 1$, *Kosaki's L^p-space* $L^p(M;\varphi)_\eta$ with respect to φ is defined as the complex interpolation space $C_\theta(h^\eta M h^{1-\eta}, L^1(M))$, $\theta = 1/p$, equipped with the complex interpolation norm $\|\cdot\|_{p,\eta}$ $(= \|\cdot\|_{C_\theta})$. In particular, $L^p(M;\varphi)_0$, $L^p(M;\varphi)_1$ and $L^p(M;\varphi)_{1/2}$ are called the left, right and symmetric L^p-spaces, respectively. For the general theory of complex interpolation spaces, see [3] for example. According to [14, Theorem 9.1], $L^p(M;\varphi)_\eta$ is exactly $h^{\eta/q}L^p(M)h^{(1-\eta)/q}$ where $1/p + 1/q = 1$, and

$$\|h^{\eta/q} a h^{(1-\eta)/q}\|_{p,\eta} = \|a\|_p, \quad a \in L^p(M).$$

That is,

$$L^p(M;\varphi)_\eta = h^{\eta/q}L^p(M)h^{(1-\eta)/q} \cong L^p(M).$$

Furthermore, when $1 < p' < p < \infty$,

$$h^\eta M h^{1-\eta} \subset L^p(M;\varphi)_\eta \subset L^{p'}(M;\varphi)_\eta \subset L^1(M),$$

$$\|x\| = \|h^\eta x h^{1-\eta}\|_{\infty,\eta} \geqslant \|h^\eta x h^{1-\eta}\|_{p,\eta} \geqslant$$

$$\geqslant \|h^\eta x h^{1-\eta}\|_{p',\eta} \geqslant \|h^\eta x h^{1-\eta}\|_1, \quad x \in M.$$

Also $h^\eta M h^{1-\eta}$ is dense in $L^p(M;\varphi)_\eta$ for every $1 < p < \infty$. Let $1 < p, q < \infty$ with $1/p + 1/q = 1$. Then $L^p(M;\varphi)_\eta$ $(\cong L^p(M))$ becomes the dual Banach space of $L^q(M;\varphi)_{\eta'}$ $(\cong L^q(M))$ for any $0 \leqslant \eta, \eta' \leqslant 1$. Especially when $\eta' = 1-\eta$, the duality between $L^p(M;\varphi)_\eta$ and $L^q(M;\varphi)_{1-\eta}$ is given by

$$\langle h^{\eta/q} a h^{(1-\eta)/q}, h^{(1-\eta)/p} b h^{\eta/p}\rangle_{p,q} = \mathrm{tr}(ab), \quad a \in L^p(M),\ b \in L^q(M).$$

This duality is convenient in the sense that, for every $x, y \in M$,

$$\langle h^\eta x h^{1-\eta}, h^{1-\eta} y h^\eta\rangle_{p,q} = \mathrm{tr}((h^{\eta/p} x h^{(1-\eta)/p})(h^{(1-\eta)/q} y h^{\eta/q})) =$$

$$= \mathrm{tr}(h^\eta x h^{1-\eta} y)$$

is independent of the pair (p, q). Since

$$\|h^{\eta}xh^{1-\eta}\|_{2,\eta} = \|h^{\eta/2}xh^{(1-\eta)/2}\|_2 =$$

$$= \operatorname{tr}(h^{\eta}xh^{1-\eta}x^*)^{1/2} = \|\Delta^{\eta/2}x\xi\|, \quad x \in M,$$

we can define a surjective linear isometry $\Theta^{\eta} : \mathscr{H} \to L^2(M; \varphi)_{\eta}$, $0 \leqslant \eta \leqslant 1$, by

$$\Theta^{\eta}(\Delta^{\eta/2}x\xi) = h^{\eta}xh^{1-\eta}, \quad x \in M.$$

We have also

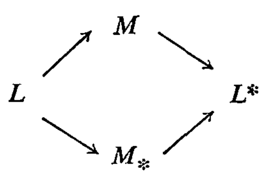
$$\operatorname{tr}(h^{\eta}xh^{1-\eta}y) = (\Delta^{\eta/2}x\xi \mid \Delta^{\eta/2}y^*\xi) = (\Delta^{\eta/2}x\xi \mid J\Delta^{(1-\eta)/2}y\xi)$$

for every $0 \leqslant \eta \leqslant 1$ and $x, y \in M$.

We next turn to Terp's L^p-spaces. Now let φ be a faithful normal semifinite weight on M. Let L denote the set of all $x \in M$ such that there exists a (unique) $\psi_x \in M_*$ satisfying

$$\psi_x(z^*y) = (J\pi(x)^*J\Lambda(y) \mid \Lambda(z)), \quad y, z \in \mathfrak{n}_{\varphi},$$

which is a Banach space with the norm $\|x\|_L = \max\{\|x\|, \|\psi_x\|\}$. We have $\mathfrak{m}_{\varphi} \subset L$ (see [23, Proposition 4]). Let L^* be the dual Banach space of L. Taking injective linear contractions $x \in L \mapsto x \in M$ and $x \in L \mapsto \psi_x \in M_*$ and their transposes, we obtain the commutative diagram of canonical imbeddings as follows:



Then (M, M_*) becomes a pair of compatible Banach spaces. For each $1 < p < \infty$, *Terp's L^p-space* $L^p(M; \varphi)$ (denoted in [23] by V_p) with respect to φ is defined as the complex interpolation space $C_{\theta}(M, M_*)$, $\theta = 1/p$, with the complex interpolation norm $\|\cdot\|_p (= \|\cdot\|_{C_\theta})$. When φ is a state (or $\varphi(1) < \infty$), the imbedding $x \mapsto \psi_x$ of M $(= L)$ into M_* corresponds with $x \mapsto h^{1/2}xh^{1/2}$ of M into $L^1(M)$ and Terp's L^p-spaces are exactly Kosaki's symmetric L^p-spaces $L^p(M; \varphi)_{1/2}$ (cf. [14, Remark 12.3]).

We here recall spatial L^p-spaces of Connes and Hilsum. For details, see [13] and [22, Chapter IV]. Besides φ on M, let φ' be a faithful normal semifinite weight on the commutant M' of M and $\mathrm{d} = \dfrac{\mathrm{d}\varphi}{\mathrm{d}\varphi'}$ the spatial derivative of φ with respect to φ' (cf. [6]). For $\psi \in M_*^+$ with the polar decomposition $\psi = u|\psi|$, define $\frac{\mathrm{d}\psi}{\mathrm{d}\varphi'} = u\frac{\mathrm{d}|\psi|}{\mathrm{d}\varphi'}$ and $\int \frac{\mathrm{d}\psi}{\mathrm{d}\varphi'}\mathrm{d}\varphi' = \psi(1)$. For each $1 \leqslant p < \infty$, the *spatial L^p-space* $L^p(\varphi')$ with respect to φ' is the space of all closed densely-defined operators a on $\mathscr{H}$ having the polar decomposition $a = u|a|$ such that $u \in M$ and $|a|^p = \frac{\mathrm{d}\psi}{\mathrm{d}\varphi'}$ for some $\psi \in M_*^+$, equipped with the norm $\|a\|_p = \left(\int |a|^p \mathrm{d}\varphi'\right)^{1/p}$. For $p = \infty$, $L^\infty(\varphi') = M$ with $\|a\|_\infty = \|a\|$. In particular, $M_* \cong L^1(M)$ by the isometry $\psi \mapsto \frac{\mathrm{d}\psi}{\mathrm{d}\varphi'}$.

According to [23, Theorem 23], there exists a surjective linear isometry $\mathscr{P}: \mathscr{H}_\varphi \to L^2(\varphi')$ such that $\mathscr{P}(\Lambda(x)) = [x\mathrm{d}^{1/2}]$, $x \in \mathfrak{n}_\varphi$, where $[x\mathrm{d}^{1/2}]$ is the closure of $x\mathrm{d}^{1/2}$. For $1 < p < \infty$, Terp constructed injective linear contractions $\mu_p: L \to L^p(\varphi')$ and $\nu_p : L^p(\varphi') \to M + M_* (\subset L^*)$ where $\nu_p \circ \mu_p$ coincides with the canonical imbedding $L \to L^*$, and proved (see [23, Theorem 36]) that ν_p maps $L^p(\varphi')$ onto $L^p(M; \varphi)$ with $\|\nu_p(a)\|_p = \|a\|_p$, $a \in L^p(\varphi')$. So we can take a surjective isometry $\Theta = \nu_2 \circ \mathscr{P} : \mathscr{H}_\varphi \to L^2(M; \varphi)$. When $1 < p, q < \infty$ and $1/p + 1/q = 1$, $L^p(M; \varphi)$ is the dual Banach space of $L^q(M; \varphi)$ with the duality given by

$$\langle \nu_p(a), \nu_q(b)\rangle_{p,q} = \int a\cdot b\,\mathrm{d}\varphi', \quad a \in L^p(\varphi'),\ b \in L^q(\varphi'),$$

where $a\cdot b$ is the closure $[ab]$ of ab. For every $x, y \in L$, by [23, (56)] we have

$$\langle x, y\rangle_{p,q} = \int \mu_p(x)\cdot\mu_q(y)\mathrm{d}\varphi' = \psi_x(y)$$

independently of (p, q).

2. GENERALIZED CONDITIONAL EXPECTATIONS

Throughout this and next sections, let M be a von Neumann algebra on a Hilbert space $\mathscr{H}$ with a cyclic and separating vector ξ. The faithful normal state φ on M is given by $\varphi = (\cdot\,\xi|\xi)$. In this section, let N be a fixed unital von Neumann subalgebra of M which acts on $\mathscr{H}_N = \overline{N\xi}$ with a cyclic and separating vector ξ. Let $\varphi_N = \varphi \restriction N$ and P_N be the orthogonal projection of $\mathscr{H}$ onto $\mathscr{H}_N$. We take Δ_N, J_N, σ_t^N, Haagerup's L^p-spaces $L^p(N)$, $1 \leqslant p \leqslant \infty$, and Kosaki's L^p-spaces $L^p(N; \varphi_N)_\eta$, $1 < p < \infty$, $0 \leqslant \eta \leqslant 1$, associated with (N, φ_N) as well as Δ, J, σ_t, $L^p(M)$ and $L^p(M; \varphi)_\eta$ associated with (M, φ). Then $L^p(N; \varphi_N)_\eta = h_N^{\eta/q} L^p(N) h_N^{(1-\eta)/q}$ $(\subset L^1(N))$ where $h_N = h_{\varphi_N}$ and $1/p + 1/q = 1$.

Let $\varepsilon : M \to N$ be the *generalized conditional expectation* relative to φ introduced by Accardi and Cecchini [1], that is, ε is given by

$$\varepsilon(x) = J_N P_N J x J J_N, \quad x \in M,$$

and is a faithful normal unital completely positive map of M into N with $\varphi = \varphi \circ \varepsilon$. This ε coincides with the conditional expectation (as a norm one projection onto N) relative to φ whenever the latter exists. The aim of this section is to extend $\varepsilon : M \to N$ to linear contractions between Kosaki's L^p-spaces in natural way.

LEMMA 2.1. *For each* $0 \leqslant \eta \leqslant 1$, *the following inequalities hold:*

(1) $$\|h_N^{\eta}\varepsilon(x)h_N^{1-\eta}\|_{2,\eta} \leqslant \|h^{\eta}xh^{1-\eta}\|_{2,\eta}, \quad x \in M.$$

(2) $$\|h^{\eta}yh^{1-\eta}\|_{2,\eta} \leqslant \|h_N^{\eta}yh_N^{1-\eta}\|_{2,\eta}, \quad y \in N.$$

Proof. (1) Since $\|h_N^{\eta}\varepsilon(x)h_N^{1-\eta}\|_{2,\eta} = \|\Delta_N^{\eta/2}\varepsilon(x)\xi\|$ and $\|h^{\eta}xh^{1-\eta}\|_{2,\eta} = \|\Delta^{\eta/2}x\xi\|$, it suffices to show that

$$\|\Delta_N^{\eta/2}\varepsilon(x)\xi\| \leqslant \|\Delta^{\eta/2}x\xi\|, \quad x \in M.$$

Suppose that $x \in M$ is σ-analytic and $y \in N$ is σ^N-analytic. Define an entire function $F(z)$ by

$$F(z) = (\varepsilon(\sigma_{iz/2}(x))\xi \mid \sigma^N_{-i\bar{z}/2}(y)\xi) = (J_N P_N J \Delta^{-z/2}x\xi \mid \Delta_N^{\bar{z}/2}y\xi).$$

For every $t \in \mathbf{R}$, we have $|F(it)| \leqslant \|x\xi\| \|y\xi\|$ and

$$|F(1 + it)| = |(\Delta_N^{1/2}\varepsilon(\sigma_{(i-t)/2}(x))\xi \mid \Delta_N^{-it/2}y\xi)| =$$

$$= |(J_N\varepsilon(\sigma_{-(i+t)/2}(x^*))\xi \mid \Delta_N^{-it/2}y\xi)| =$$

$$= |(P_N J\Delta^{1/2}\sigma_{-t/2}(x^*)\xi \mid \Delta_N^{-it/2}y\xi)| = |(\Delta^{-it/2}x\xi \mid \Delta_N^{-it/2}y\xi)| \leqslant \|x\xi\| \|y\xi\|.$$

Hence the three lines theorem implies

$$|(\Delta_N^{\eta/2}\varepsilon(\sigma_{i\eta/2}(x))\xi \mid y\xi)| = |F(\eta)| \leqslant \|x\xi\| \|y\xi\|.$$

Replacing x by $\sigma_{-i\eta/2}(x)$, we get

$$\|\Delta_N^{\eta/2}\varepsilon(x)\xi\| \leqslant \|\sigma_{-i\eta/2}(x)\xi\| = \|\Delta^{\eta/2}x\xi\|.$$

For each $x \in M$, there is a sequence $\{x_n\}$ of σ-analytic elements of M such that $\|x_n\xi - x\xi\| \to 0$ and $\|\Delta^{\eta/2}x_n\xi - \Delta^{\eta/2}x\xi\| \to 0$. Since

$$\|\varepsilon(x_n)\xi - \varepsilon(x)\xi\| \leqslant \|(x_n - x)\xi\| \to 0$$

and

$$\|\Delta_N^{\eta/2}\varepsilon(x_m)\xi - \Delta_N^{\eta/2}\varepsilon(x_n)\xi\| \leqslant \|\Delta^{\eta/2}(x_m - x_n)\xi\| \to 0$$

as $m, n \to \infty$, it follows that

$$\|\Delta_N^{\eta/2}\varepsilon(x)\xi\| = \lim_{n\to\infty} \|\Delta_N^{\eta/2}\varepsilon(x_n)\xi\| \leqslant \lim_{n\to\infty} \|\Delta^{\eta/2}x_n\xi\| = \|\Delta^{\eta/2}x\xi\|.$$

(2) Although (2) can be proved by the three lines theorem as in (1), we prefer the following proof for later reference. As seen from the proof of (1), the closure $[\Delta_N^{\eta/2}J_NP_NJ\Delta^{-\eta/2}]$ of $\Delta_N^{\eta/2}J_NP_NJ\Delta^{-\eta/2}$ is a contraction $\mathscr{H}$ into $\mathscr{H}_N$. If $x \in M$ and $y \in N$, then

$$(\Delta_N^{\eta/2}J_NP_NJ\Delta^{-\eta/2}(\Delta^{\eta/2}x\xi) \mid J_N\Delta_N^{\eta/2}y\xi) =$$

$$= (J_NP_NJx\xi \mid J_Ny\xi) = (x\xi \mid Jy\xi) = (\Delta^{\eta/2}x\xi \mid J\Delta^{\eta/2}y\xi) =$$

$$= (\Delta^{\eta/2}x\xi \mid J\Delta^{\eta/2}\Delta_N^{-\eta/2}J_N(J_N\Delta_N^{\eta/2}y\xi)).$$

This shows that $[J\Delta^{\eta/2}\Delta_N^{-\eta/2}J_N]$ is the adjoint of $[\Delta_N^{\eta/2}J_NP_NJ\Delta^{-\eta/2}]$ and is a contraction of $\mathscr{H}_N$ into $\mathscr{H}$. For every $y \in N$, we hence have

$$\|h^\eta yh^{1-\eta}\|_{2,\eta} = \|\Delta^{\eta/2}y\xi\| =$$

$$= \|J\Delta^{\eta/2}\Delta_N^{-\eta/2}J_N(J_N\Delta_N^{\eta/2}y\xi)\| \leqslant \|\Delta_N^{\eta/2}y\xi\| = \|h_N^\eta yh_N^{1-\eta}\|_{2,\eta}. \quad ▨$$

Because of Lemma 2.1 and the density of $h^\eta Mh^{1-\eta}$ (resp. $h_N^\eta Nh_N^{1-\eta}$) in $L^2(M;\varphi)_\eta$ (resp. $L^2(N;\varphi_N)_\eta$), the linear contractions $\varepsilon^\eta : L^2(M;\varphi)_\eta \to L^2(N;\varphi_N)_\eta$ and $\varkappa^\eta : L^2(N;\varphi_N)_\eta \to L^2(M;\varphi)_\eta$, $0 \leqslant \eta \leqslant 1$, are determined by

$$\varepsilon^\eta(h^\eta xh^{1-\eta}) = h_N^\eta\varepsilon(x)h_N^{1-\eta}, \quad x \in M,$$

$$\varkappa^\eta(h_N^\eta yh_N^{1-\eta}) = h^\eta yh^{1-\eta}, \quad y \in N.$$

Then $\varepsilon^\eta(h^\eta Mh^{1-\eta}) \subset h_N^\eta Nh_N^{1-\eta}$, $\varkappa^\eta(h_N^\eta Nh_N^{1-\eta}) \subset h^\eta Mh^{1-\eta}$, and

$$\|\varepsilon^\eta(h^\eta xh^{1-\eta})\|_{\infty,\eta} = \|\varepsilon(x)\| \leqslant \|x\| = \|h^\eta xh^{1-\eta}\|_{\infty,\eta}, \quad x \in M,$$

$$\|\varkappa^\eta(h_N^\eta yh_N^{1-\eta})\|_{\infty,\eta} = \|y\| = \|h_N^\eta yh_N^{1-\eta}\|_{\infty,\eta}, \quad y \in N.$$

Furthermore, by the reiteration theorem for complex interpolation spaces (cf. [3, Theorem 4.6.1]),

$$L^p(M;\varphi)_\eta = C_{2/p}(h^\eta Mh^{1-\eta}, L^2(M;\varphi)_\eta),$$

$$L^p(N;\varphi_N)_\eta = C_{2/p}(h_N^\eta Nh_N^{1-\eta}, L^2(N;\varphi_N)_\eta), \quad 2 < p < \infty.$$

Thus the abstract version of the Riesz-Thorin theorem (cf. [3, Theorem 4.1.2]) implies

THEOREM 2.2. *For each* $2 \leqslant p < \infty$ *and* $0 \leqslant \eta \leqslant 1$, ε^η *maps* $L^p(M; \varphi)_\eta$ *into* $L^p(N; \varphi_N)_\eta$ *with* $\|\varepsilon^\eta(x)\|_{p,\eta} \leqslant \|x\|_{p,\eta}$, $x \in L^p(M; \varphi)_\eta$, *and* $\varkappa^\eta$ *maps* $L^p(N; \varphi_N)_\eta$ *into* $L^p(M; \varphi)_\eta$ *with* $\|\varkappa^\eta(y)\|_{p,\eta} \leqslant \|y\|_{p,\eta}$, $y \in L^p(N; \varphi_N)_\eta$.

Let $E^\eta = \varkappa^\eta \circ \varepsilon^\eta$. Then $E^\eta \restriction L^p(M; \varphi)_\eta$ is a linear contraction of $L^p(M; \varphi)_\eta$ into itself for $2 \leqslant p < \infty$ and $0 \leqslant \eta \leqslant 1$.

Corresponding to the contractions $\psi \in M_* \mapsto \psi \restriction N \in N_*$ and $\psi \in N_* \mapsto$ $\mapsto \psi \circ \varepsilon \in M_*$, we define the linear contractions $\tilde{\varepsilon} : L^1(M) \to L^1(N)$ and $\tilde{\varkappa} : L^1(N) \to$ $\to L^1(M)$ by

$$\mathrm{tr}_N(\tilde{\varepsilon}(a)y) = \mathrm{tr}(ay), \quad a \in L^1(M),\ y \in N,$$

$$\mathrm{tr}(\tilde{\varkappa}(b)x) = \mathrm{tr}_N(b\varepsilon(x)), \quad b \in L^1(N),\ x \in M,$$

where tr_N is the linear functional tr on $L^1(N)$. Also define $\tilde{E} : L^1(M) \to L^1(M)$ by $\tilde{E} = \tilde{\varkappa} \circ \tilde{\varepsilon}$.

THEOREM 2.3. *Let* $1/p + 1/q = 1$, $1 < q \leqslant 2$, *and* $0 \leqslant \eta \leqslant 1$. *Then* $\tilde{\varepsilon}$ *maps* $L^q(M; \varphi)_\eta$ *into* $L^q(N; \varphi_N)_\eta$ *with* $\|\tilde{\varepsilon}(x)\|_{q,\eta} \leqslant \|x\|_{q,\eta}$, $x \in L^q(M; \varphi)_\eta$, *and* $\tilde{\varkappa}$ *maps* $L^q(N; \varphi_N)_\eta$ *into* $L^q(M; \varphi)_\eta$ *with* $\|\tilde{\varkappa}(y)\|_{q,\eta} \leqslant \|y\|_{q,\eta}$, $y \in L^q(N; \varphi_N)_\eta$. *Moreover, under the duality* $\langle \cdot, \cdot \rangle_{p,q}$ *between* $L^p(M; \varphi)_\eta$ *and* $L^q(M; \varphi)_{1-\eta}$ *and that between* $L^p(N; \varphi_N)_\eta$ *and* $L^q(N; \varphi_N)_{1-\eta}$ *(see Section 1), the transpose of* $\varepsilon^\eta \restriction L^p(M; \varphi)_\eta$ *is* $\tilde{\varkappa} \restriction L^q(N; \varphi_N)_{1-\eta}$ *and the transpose of* $\varkappa^\eta \restriction L^p(N; \varphi_N)_\eta$ *is* $\tilde{\varepsilon} \restriction L^q(M; \varphi)_{1-\eta}$.

Proof. The first assertion follows from the second and Theorem 2.2. To show the second, let $(\varepsilon_p^\eta)^t$ and $(\varkappa_p^\eta)^t$ be the transposes of $\varepsilon_p^\eta = \varepsilon^\eta \restriction L^p(M; \varphi)_\eta$ and $\varkappa_p^\eta = \varkappa^\eta \restriction L^p(N; \varphi_N)_\eta$. If $x \in M$ and $y \in N$, then

$$\mathrm{tr}(\tilde{\varkappa}(h_N^{1-\eta} y h_N^\eta)x) = \mathrm{tr}_N(h_N^{1-\eta} y h_N^\eta \varepsilon(x)) =$$

$$= \langle h_N^\eta \varepsilon(x) h_N^{1-\eta},\ h_N^{1-\eta} y h_N^\eta \rangle_{p,q} =$$

$$= \langle h^\eta x h^{1-\eta},\ (\varepsilon_p^\eta)^t (h_N^{1-\eta} y h_N^\eta) \rangle_{p,q} = \mathrm{tr}((\varepsilon_p^\eta)^t (h_N^{1-\eta} y h_N^\eta) x),$$

so that $\tilde{\varkappa}(h_N^{1-\eta} y h_N^\eta) = (\varepsilon_p^\eta)^t (h_N^{1-\eta} y h_N^\eta)$. Since $h_N^{1-\eta} N h_N^\eta$ is dense in $L^q(N; \varphi_N)_{1-\eta}$, we obtain $(\varepsilon_p^\eta)^t = \tilde{\varkappa} \restriction L^q(N; \varphi_N)_{1-\eta}$ from $\|\cdot\|_1 \leqslant \|\cdot\|_{q,1-\eta}$ and Theorem 2.2. The proof of $(\varkappa_p^\eta)^t = \tilde{\varepsilon} \restriction L^q(M; \varphi)_{1-\eta}$ is similar. ▨

By Theorem 2.3, for p, q and η as above, $\tilde{E}$ maps $L^q(M; \varphi)_\eta$ into it self and the transpose of $E^\eta \restriction L^p(M; \varphi)_\eta$ is $\tilde{E} \restriction L^q(M; \varphi)_{1-\eta}$.

LEMMA 2.4. *The map* $\tilde{\varepsilon}$ *extends* $\varepsilon^{1/2}$ *and* $\tilde{\varkappa}$ *does* $\varkappa^{1/2}$.

Proof. It suffices to show the following equalities:

$$\tilde{\varepsilon}(h^{1/2}xh^{1/2}) = \varepsilon^{1/2}(h^{1/2}xh^{1/2}), \quad x \in M,$$

$$\tilde{\varkappa}(h_N^{1/2}yh_N^{1/2}) = \varkappa^{1/2}(h_N^{1/2}yh_N^{1/2}), \quad y \in N.$$

For every $x \in M$ and $y \in N$, we have

$$\mathrm{tr}_N(\tilde{\varepsilon}(h^{1/2}xh^{1/2})y) = \mathrm{tr}(h^{1/2}xh^{1/2}y) =$$

$$= (x\xi \mid Jy\xi) = (J_N P_N Jx\xi \mid J_N y\xi) =$$

$$= (\Delta_N^{1/4}\varepsilon(x)\xi \mid J_N \Delta_N^{1/4} y\xi) = \mathrm{tr}_N(\varepsilon^{1/2}(h^{1/2}xh^{1/2})y),$$

implying the first equality. The second is analogously proved. ▨

When $\tilde{\varepsilon}$ and $\tilde{E}$ (resp. $\tilde{\varkappa}$) are restricted on $L^q(M;\varphi)_\eta$ (resp. $L^q(N;\varphi_N)_\eta$), we denote them by $\tilde{\varepsilon}^\eta$ and $\tilde{E}^\eta$ (resp. $\tilde{\varkappa}^\eta$). Lemma 2.4 shows that $\varepsilon^{1/2} = \tilde{\varepsilon}^{1/2}$, $E^{1/2} = \tilde{E}^{1/2}$ on $L^2(M;\varphi)_{1/2}$ and $\varkappa^{1/2} = \tilde{\varkappa}^{1/2}$ on $L^2(N;\varphi_N)_{1/2}$.

The next lemma is useful in Section 3.

LEMMA 2.5. *For each* $0 \leqslant \eta \leqslant 1$,

$$(\Theta^\eta)^{-1} \circ E^\eta \circ \Theta^\eta = (\Theta^{1-\eta})^{-1} \circ \tilde{E}^{1-\eta} \circ \Theta^{1-\eta} = [\Delta^{\eta/2} J_N P_N J \Delta^{-\eta/2}]$$

where $\Theta^\eta : \mathscr{H} \to L^2(M;\varphi)_\eta$ *is the isometry given in Section* 1.

Proof. We have

$$\langle \Theta^\eta(\Delta^{\eta/2}x\xi),\ \Theta^{1-\eta}(\Delta^{(1-\eta)/2}y\xi)\rangle_{2,2} = \mathrm{tr}(h^\eta x h^{1-\eta} y) =$$

$$= (\Delta^{\eta/2}x\xi \mid J\Delta^{(1-\eta)/2}y\xi)$$

for all $x, y \in M$. Hence

$$\langle \Theta^\eta \zeta_1,\ \Theta^{1-\eta}\zeta_2\rangle_{2,2} = (\zeta_1 \mid J\zeta_2), \quad \zeta_1,\ \zeta_2 \in \mathscr{H}.$$

Since the transpose of $E^\eta \restriction L^2(M;\varphi)_\eta$ is $\tilde{E}^{1-\eta} \restriction L^2(M;\varphi)_{1-\eta}$, this shows that

$$(\Theta^{1-\eta})^{-1} \circ \tilde{E}^{1-\eta} \circ \Theta^{1-\eta} = J((\Theta^\eta)^{-1} \circ E^\eta \circ \Theta^\eta)^* J.$$

It is immediate from definitions of Θ^η and E^η that

$$(\Theta^\eta)^{-1} \circ E^\eta \circ \Theta^\eta = [\Delta^{\eta/2} J_N P_N J \Delta^{-\eta/2}].$$

Furthermore, since

$$(\Delta^{\eta/2}J_N P_N J\Delta^{-\eta/2}(\Delta^{\eta/2}x\xi) \mid J\Delta^{\eta/2}y\xi) =$$

$$= (x\xi \mid JJ_N P_N Jy\xi) = (\Delta^{\eta/2}x\xi \mid J\Delta^{\eta/2}J_N P_N J\Delta^{-\eta/2}J(J\Delta^{\eta/2}y\xi)),$$

we get

$$[\Delta^{\eta/2}J_N P_N J\Delta^{-\eta/2}]^* = J[\Delta^{\eta/2}J_N P_N J\Delta^{-\eta/2}]J.$$

Thus the lemma is proved. ▨

In the rest of this section, we discuss when ε becomes a norm one projection onto N.

THEOREM 2.6. *Let* $1 < q \leqslant 2 \leqslant p < \infty$ *and* $0 \leqslant \eta \leqslant 1$. *Then the following conditions are equivalent:*

(i) ε *is the conditional expectation;*

(ii) $\varepsilon \circ \varepsilon = \varepsilon$ *on* M *(*$\varepsilon(M) = N$ *is not required);*

(iii) $E^\eta \circ E^\eta = E^\eta$ *on* $L^p(M; \varphi)_\eta$*;*

(iv) $\tilde{E}^\eta \circ \tilde{E}^\eta = \tilde{E}^\eta$ *on* $L^q(M; \varphi)_\eta$.

Proof. It is obvious that (i) implies (ii)—(iv). If either (iii) or (iv) holds, then we obtain (ii). Finally (ii) implies

$$(J_N P_N J)^2 x\xi = J_N P_N Jx\xi, \quad x \in M,$$

so that $(J_N P_N J)^2 = J_N P_N J$. Because an idempotent contraction on a Hilbert space is an orthogonal projection, we get $J_N P_N J = P_N$, showing (i). ▨

THEOREM 2.7. *Let* $0 \leqslant \eta \leqslant 1$ *with* $\eta \neq 1/2$. *Then the following conditions are equivalent:*

(i) ε *is the conditional expectation;*

(ii) $\varepsilon^\eta = \tilde{\varepsilon}^\eta$ *on* $L^2(M; \varphi)_\eta$*;*

(iii) $\varkappa^\eta = \tilde{\varkappa}^\eta$ *on* $L^2(N; \varphi_N)_\eta$*;*

(iv) $E^\eta = \tilde{E}^\eta$ *on* $L^2(M; \varphi)_\eta$.

For the proof, we need

LEMMA 2.8. (1) $[\Delta_N^{1/2}J_N P_N J\Delta^{-1/2}] = P_N$.

(2) $[\Delta^{1/2}J_N P_N J\Delta^{-1/2}] = JJ_N P_N$.

Proof. (1) For every $x \in M$,

$$\Delta_N^{1/2}J_N P_N J\Delta^{-1/2}(\Delta^{1/2}x\xi) = \Delta_N^{1/2}\varepsilon(x)\xi = J_N\varepsilon(x^*)\xi =$$

$$= P_N Jx^*\xi = P_N(\Delta^{1/2}x\xi).$$

(2) For every $x \in M$,

$$\Delta^{1/2}J_N P_N J\Delta^{-1/2}(\Delta^{1/2}x\xi) = \Delta^{1/2}\varepsilon(x)\xi = J\varepsilon(x^*)\xi =$$

$$= JJ_N P_N Jx^*\xi = JJ_N P_N(\Delta^{1/2}x\xi).$$ ▨

Proof of Theorem 2.7. First suppose (i). We thus have $\Delta_N = \Delta \upharpoonright \mathscr{H}_N$ and $J_N = J \upharpoonright \mathscr{H}_N$ (cf. [21]). If $x \in M$ and $y \in N$, then

$$\mathrm{tr}_N(\varepsilon^\eta(h^\eta x h^{1-\eta})y) = (\Delta_N^{\eta/2}\varepsilon(x)\xi \mid \Delta_N^{\eta/2}y^*\xi) =$$

$$= (\Delta^{\eta/2}x\xi \mid \Delta^{\eta/2}y^*\xi) = \mathrm{tr}(h^\eta x h^{1-\eta}y) = \mathrm{tr}_N(\tilde{\varepsilon}(h^\eta x h^{1-\eta})y),$$

so that $\varepsilon^\eta(h^\eta x h^{1-\eta}) = \tilde{\varepsilon}(h^\eta x h^{1-\eta})$, showing (ii). Similarly we obtain (iii) and hence (iv).

By Theorem 2.3, $\varepsilon^\eta = \tilde{\varepsilon}^\eta$ on $L^2(M;\varphi)_\eta$ if and only if $\varkappa^{1-\eta} = \tilde{\varkappa}^{1-\eta}$ on $L^2(N;\varphi_N)_{1-\eta}$, and $\tilde{E}^\eta = E^\eta$ on $L^2(M;\varphi)_\eta$ if and only if $E^{1-\eta} = \tilde{E}^{1-\eta}$ on $L^2(M;\varphi)_{1-\eta}$. So we may assume $1/2 < \eta \leqslant 1$ to prove that each of (ii)—(iv) implies (i).

(ii) $\Rightarrow$ (i). Suppose (ii). For every $x \in M$ and $y \in N$, by Lemma 2.8(1) we have

$$(\Delta^{1/2}x\xi \mid \Delta_N^{\eta-1/2}y\xi) = (\Delta_N^{1/2}J_N P_N Jx\xi \mid \Delta_N^{\eta-1/2}y\xi) =$$

$$= (\Delta_N^{\eta/2}\varepsilon(x)\xi \mid \Delta_N^{\eta/2}y\xi) = \mathrm{tr}_N(\varepsilon^\eta(h^\eta x h^{1-\eta})y^*) =$$

$$= \mathrm{tr}(h^\eta x h^{1-\eta}y^*) = (\Delta^{1/2}x\xi \mid \Delta^{\eta-1/2}y\xi).$$

Hence $\Delta_N^{\eta-1/2}y\xi = \Delta^{\eta-1/2}y\xi$ for all $y \in N$. This shows $\Delta_N = \Delta \upharpoonright \mathscr{H}_N$, so that

$$JJ_N P_N = [\Delta^{1/2}J_N P_N J\Delta^{-1/2}] = [\Delta_N^{1/2}J_N P_N J\Delta^{-1/2}] = P_N$$

by Lemma 2.8. Therefore $J_N P_N J = (JJ_N P_N)^* = P_N$, implying (i).

(iii) $\Rightarrow$ (i) is analogously proved.

(iv) $\Rightarrow$ (i). For every $x, y \in M$, we have

$$\mathrm{tr}(E^\eta(h^\eta x h^{1-\eta})y^*) = \mathrm{tr}(h^\eta \varepsilon(x)h^{1-\eta}y^*) =$$

$$= (\Delta^{\eta-1/2}J_N P_N Jx\xi \mid \Delta^{1/2}y\xi)$$

and

$$\mathrm{tr}(\tilde{E}^\eta(h^\eta x h^{1-\eta})y^*) = \mathrm{tr}(h^\eta x h^{1-\eta}\varepsilon(y)^*) =$$

$$= (\Delta^{\eta-1/2}x\xi \mid \Delta^{1/2}J_N P_N Jy\xi) = (J_N P_N J\Delta^{\eta-1/2}x\xi \mid \Delta^{1/2}y\xi)$$

by Lemma 2.8(2). Hence (iv) implies that $J_N P_N J \Delta^{\eta-1/2} \subset \Delta^{\eta-1/2} J_N P_N J$. This shows $J_N P_N J \Delta^{1/2} \subset \Delta^{1/2} J_N P_N J$, so that $J_N P_N J = J J_N P_N$ by Lemma 2.8(2). Therefore, since

$$(J_N P_N J)^2 = J_N P_N J J J_N P_N = P_N,$$

we get $\varepsilon(\varepsilon(x)) = x$ for all $x \in N$. So $\varepsilon \circ \varepsilon$ is a norm one projection onto N with $\varphi \circ \varepsilon \circ \varepsilon = \varphi$. Thus $\varepsilon = \varepsilon \circ \varepsilon$ and we obtain (i). ▨

If ε is the conditional expectation, then $\tilde{\varkappa}^\eta \,(= \varkappa^\eta)$ is an isometry of $L^p(N; \varphi_N)_\eta$ into $L^p(M; \varphi)_\eta$ for each $1 < p < \infty$ and $0 \leqslant \eta \leqslant 1$. In this case, we can regard $L^p(N; \varphi_N)_\eta$ as the subspace of $L^p(M; \varphi)_\eta$ and $\tilde{E}^\eta \,(= E^\eta)$ as the projection of $L^p(M; \varphi)_\eta$ onto $L^p(N; \varphi_N)_\eta$. In fact, this case was stated in [14, Proposition 4.1] and [9].

When ε is not the conditional expectation and $\eta \neq 1/2$, it is a problem to decide whether ε^η (resp. $\varkappa^\eta$) is extended to a linear contraction on $L^p(M; \varphi)_\eta$ (resp. $L^p(N; \varphi_N)_\eta$) for $p < 2$, or equivalently whether $\tilde{\varkappa}^{1-\eta}$ (resp. $\tilde{\varepsilon}^{1-\eta}$) is contractive in the norm $\|\cdot\|_{q,1-\eta}$ for $q > 2$.

We regard the linear contractions ε^η, E^η on $L^p(M; \varphi)_\eta$ and $\tilde{\varepsilon}^\eta$, $\tilde{E}^\eta$ on $L^q(M; \varphi)_\eta$, $1 < q \leqslant 2 \leqslant p < \infty$, as the generalized L^p-conditional expectations relative to φ. Notice that the L^p-conditional expectations given in [4, 5] coincide with ours in the symmetric case $\eta = 1/2$. Also $\tilde{\varepsilon}^\eta$ on $L^q(M; \varphi)_\eta$, $1 < q \leqslant 2$, $0 \leqslant \eta \leqslant 1$, are already given in [8] by a similar method of complex interpolation.

3. MARTINGALE CONVERGENCE

In this section, we discuss the norm convergence of increasing or decreasing generalized martingales in L^p-spaces. First let $\{N_\alpha\}$ be an increasing net of unital von Neumann subalgebras of M with $N_\infty = \bigvee_\alpha N_\alpha$. Let $\varphi_\alpha = \varphi \restriction N_\alpha$, $\mathscr{H}_\alpha = \overline{N_\alpha \xi}$ and P_α be the orthogonal projection of $\mathscr{H}$ onto $\mathscr{H}_\alpha$. We take Δ_α, J_α and $L^p(N_\alpha, \varphi_\alpha)_\eta$, $1 < p < \infty$, $0 \leqslant \eta \leqslant 1$, associated with $(N_\alpha, \varphi_\alpha)$. Let $\varepsilon_\alpha : M \to N_\alpha$ be the generalized conditional expectation relative to φ. The corresponding linear contractions on Kosaki's L^p-spaces are given as follows: for $2 \leqslant p < \infty$ and $0 \leqslant \eta \leqslant 1$,

$$L^p(M; \varphi)_\eta \xrightarrow{\varepsilon_\alpha^\eta} L^p(N_\alpha; \varphi_\alpha)_\eta \xrightarrow{\varkappa_\alpha^\eta} L^p(M; \varphi)_\eta, \quad E_\alpha^\eta = \varkappa_\alpha^\eta \circ \varepsilon_\alpha^\eta,$$

and for $1 < q \leqslant 2$ and $0 \leqslant \eta \leqslant 1$,

$$L^q(M; \varphi)_\eta \xrightarrow{\tilde{\varepsilon}_\alpha^\eta} L^q(N_\alpha; \varphi_\alpha)_\eta \xrightarrow{\tilde{\varkappa}_\alpha^\eta} L^q(M; \varphi)_\eta, \quad \tilde{E}_\alpha^\eta = \tilde{\varkappa}_\alpha^\eta \circ \tilde{\varepsilon}_\alpha^\eta.$$

Also $\varepsilon_\infty : M \to N_\infty$, E^η_∞ and $\tilde{E}^\eta_\infty$ are given. Moreover, when $\alpha \geqslant \beta$ in the directed set of indices, we take the generalized conditional expectation $\varepsilon_{\alpha\beta} : N_\alpha \to N_\beta$ relative to φ_α and the corresponding linear contractions $\varepsilon^\eta_{\alpha\beta} : L^p(N_\alpha; \varphi_\alpha)_\eta \to$ $\to L^p(N_\beta; \varphi_\beta)_\eta$, $\tilde{\varepsilon}^\eta_{\alpha\beta} : L^q(N_\alpha; \varphi_\alpha)_\eta \to L^q(N_\beta; \varphi_\beta)_\eta$ for $1 < q \leqslant 2 \leqslant p < \infty$ and $0 \leqslant \eta \leqslant 1$.

As generalized martingales in L^p-spaces, we consider nets $\{x_\alpha\}$ of $x_\alpha \in L^p(N_\alpha; \varphi_\alpha)_\eta$ where $2 \leqslant p < \infty$ (resp. $1 < p \leqslant 2$) such that $\varepsilon^\eta_{\alpha\beta}(x_\alpha) = x_\beta$ (resp. $\tilde{\varepsilon}^\eta_{\alpha\beta}(x_\alpha) = x_\beta$) for any $\alpha \geqslant \beta$. We discuss the norm convergence of $\{\varkappa^\eta_\alpha(x_\alpha)\}$ (resp. $\{\tilde{\varkappa}^\eta_\alpha(x_\alpha)\}$) in $L^p(M; \varphi)_\eta$ for such martingales $\{x_\alpha\}$. Since $\varepsilon^\eta_{\alpha\beta} \circ \varepsilon^\eta_\alpha = \varepsilon^\eta_\beta$ (see [1, (3.30)]) and $\tilde{\varepsilon}^\eta_{\alpha\beta} \circ \tilde{\varepsilon}^\eta_\alpha = \tilde{\varepsilon}^\eta_\beta$ for $\alpha \geqslant \beta$, it follows that if $x_\alpha = \varepsilon^\eta_\alpha(x)$ (or $x_\alpha = \tilde{\varepsilon}^\eta_\alpha(x)$) for some $x \in L^p(M; \varphi)_\eta$, then $\{x_\alpha\}$ is a martingale in the above sense. In this case, $\{x_\alpha\}$ is called to be simple. We point out here that the definition of martingales in [5] (in the symmetric L^p-spaces) seems somewhat inadequate because $E_\beta \circ E_\alpha \neq E_\beta$, $\alpha \geqslant \beta$, for the case of generalized conditional expectations (see Theorem 2.6) and so simple martingales are not necessarily martingales in the sense in [5].

For simple martingales in L^p-spaces, we have the next theorem extending [5, Theorem 8] and [8, Theorem 8].

THEOREM 3.1. (1) $\|E^\eta_\alpha(x) - E^\eta_\infty(x)\|_{p,\eta} \to 0$ *for every* $x \in L^p(M; \varphi)_\eta$ *where* $2 \leqslant p < \infty$ *and* $0 \leqslant \eta \leqslant 1$.

(2) $\|\tilde{E}^\eta_\alpha(x) - \tilde{E}^\eta_\infty(x)\|_{q,\eta} \to 0$ *for every* $x \in L^q(M; \varphi)_\eta$ *where* $1 < q \leqslant 2$ *and* $0 \leqslant \eta \leqslant 1$.

Before the proof, we state two useful lemmas.

LEMMA 3.2. *If* $1 \leqslant p, p_1, p_2 \leqslant \infty$ *and* $1/p = (1-\theta)/p_1 + \theta/p_2$ *with* $0 < \theta < 1$, *then*

$$\|h^\eta x h^{1-\eta}\|_{p,\eta} \leqslant \|h^\eta x h^{1-\eta}\|^{1-\theta}_{p_1,\eta} \|h^\eta x h^{1-\eta}\|^{\theta}_{p_2,\eta}$$

or every $x \in M$ *and* $0 \leqslant \eta \leqslant 1$, *where* $\|\cdot\|_{1,\eta} = \|\cdot\|_1$.

Because the reiteration theorem (cf. [3, Theorem 4.6.1]) gives

$$L^p(M; \varphi)_\eta = C_\theta(L^{p_1}(M; \varphi)_\eta,\ L^{p_2}(M; \varphi)_\eta),$$

the lemma is an easy consequence of the abstract Riesz-Thorin theorem (cf. [3, Theorem 4.1.2]). A special case of Lemma 3.2 is [8, Theorem 1].

LEMMA 3.3. *If* $0 \leqslant \eta,\ \eta_1, \eta_2 \leqslant 1$ *and* $\eta = (1-\theta)\eta_1 + \theta\eta_2$ *with* $0 < \theta < 1$, *then*

$$\|h^\eta x h^{1-\eta}\|_{2,\eta} \leqslant \|h^{\eta_1} x h^{1-\eta_1}\|^{1-\theta}_{2,\eta_1} \|h^{\eta_2} x h^{1-\eta_2}\|^{\theta}_{2,\eta_2}$$

for all $x \in M$. *In particular,*

$$\|h^\eta x h^{1-\eta}\|_{2,\eta} \leqslant \|x\xi\|^{1-\eta} \|x^*\xi\|^\eta$$

for all $x \in M$ *and* $0 \leqslant \eta \leqslant 1$.

Since $\|h^{\eta}xh^{1-\eta}\|_{2,\eta} = \|\Delta^{\eta/2}x\xi\|$, we can show the lemma by applying the three lines theorem to $\Delta^{z/2}x\xi$ on the strip $0 \leqslant \operatorname{Re} z \leqslant 1$. See also [2, (C.7)] and [18, Lemma 1].

Proof of Theorem 3.1. (1) We may show the assertion for $x = h^{\eta}ah^{1-\eta}$, $a \in M$. When $p = 2$, we have

$$\|E_{\alpha}^{\eta}(x) - E_{\infty}^{\eta}(x)\|_{2,\eta} = \|h^{\eta}(\varepsilon_{\alpha}(a) - \varepsilon_{\infty}(a))\, h^{1-\eta}\|_{2,\eta} \leqslant$$

$$\leqslant \|(\varepsilon_{\alpha}(a) - \varepsilon_{\infty}(a))\xi\|^{1-\eta} \|(\varepsilon_{\alpha}(a^{*}) - \varepsilon_{\infty}(a^{*}))\xi\|^{\eta}$$

by Lemma 3.3. Since $\varepsilon_{\alpha}(a) \to \varepsilon_{\infty}(a)$ strongly for every $a \in M$ (see [12, Theorem 3] and [17, Theorem 11]), it follows that $\|E_{\alpha}^{\eta}(x) - E_{\infty}^{\eta}(x)\|_{2,\eta} \to 0$. The assertion for the case $2 < p < \infty$ is obtained from the case $p = 2$ and Lemma 3.2.

(2) Since $\|\cdot\|_{q,\eta} \leqslant \|\cdot\|_{2,\eta}$ for $1 < q \leqslant 2$, it suffices to show the case $q = 2$. But this follows from (1) and Lemma 2.5. ▨

The next theorem is our main result concerning the increasing martingale convergence. We note that, even in the symmetric case $\eta = 1/2$, this is different from [5, Theorem 9] in view of the formulation of generalized martingales.

THEOREM 3.4. *Assume* $N_{\infty} = M$ *(i.e.* $N_{\alpha} \nearrow M$*).*

(1) *Let* $2 \leqslant p < \infty$ *and* $0 \leqslant \eta \leqslant 1$. *If* $\{x_{\alpha}\}$ *is a net of* $x_{\alpha} \in L^{p}(N_{\alpha}, \varphi_{\alpha})_{\eta}$ *satisfying* $\varepsilon_{\alpha\beta}^{\eta}(x_{\alpha}) = x_{\beta}$ *for* $\alpha \geqslant \beta$ *and* $\sup_{\alpha}\|x_{\alpha}\|_{p,\eta} < \infty$, *then there exists an* $x \in L^{p}(M; \varphi)_{\eta}$ *such that* $x_{\alpha} = \varepsilon_{\alpha}^{\eta}(x)$ *for all* α *and* $\|\varkappa_{\alpha}^{\eta}(x_{\alpha}) - x\|_{p,\eta} \to 0$.

(2) *Let* $1 < q \leqslant 2$ *and* $0 \leqslant \eta \leqslant 1$. *If* $\{x_{\alpha}\}$ *is a net of* $x_{\alpha} \in L^{q}(N_{\alpha}, \varphi_{\alpha})_{\eta}$ *satisfying* $\tilde{\varepsilon}_{\alpha\beta}^{\eta}(x_{\alpha}) = x_{\beta}$ *for* $\alpha \geqslant \beta$ *and* $\sup_{\alpha}\|x_{\alpha}\|_{q,\eta} < \infty$, *then there exists an* $x \in L^{q}(M; \varphi)_{\eta}$ *such that* $x_{\alpha} = \tilde{\varepsilon}_{\alpha}^{\eta}(x)$ *for all* α *and* $\|\tilde{\varkappa}_{\alpha}^{\eta}(x_{\alpha}) - x\|_{q,\eta} \to 0$.

Now let T_{α} and T be positive selfadjoint operators on $\mathscr{H}$ such that $T_{\alpha} \to T$ strongly in the generalized sense, equivalently $(1 + T_{\alpha})^{-1} \to (1 + T)^{-1}$ strongly (cf. [19, VIII.7, Problem VIII. 27]). We then have

LEMMA 3.5. *If* $\zeta_{\alpha} \in \mathscr{D}(T_{\alpha})$, $\zeta \in \mathscr{D}(T)$, $\|\zeta_{\alpha} - \zeta\| \to 0$ *and* $\|T_{\alpha}\zeta_{\alpha} - T\zeta\| \to 0$, *then* $\|T_{\alpha}^{\eta}\zeta_{\alpha} - T^{\eta}\zeta\| \to 0$ *for all* $0 < \eta \leqslant 1$.

Proof. Let $T_{\alpha} = \int_{0}^{\infty} \lambda \, de_{\alpha}(\lambda)$ and $T = \int_{0}^{\infty} \lambda \, de(\lambda)$ be the spectral decompositions. For any $\varepsilon > 0$, take an $s \geqslant 1$ with $\int_{s-1}^{\infty} \lambda^{2} d\|e(\lambda)\zeta\|^{2} \leqslant \varepsilon^{2}$ and define two bounded continuous functions f, g on $[0, \infty)$ by

$$f(\lambda) = \begin{cases} \lambda, & 0 \leqslant \lambda \leqslant s-1, \\ (s-1)(s-\lambda), & s-1 \leqslant \lambda \leqslant s, \\ 0, & \lambda \geqslant s, \end{cases}$$

$$g(\lambda) = \begin{cases} \lambda^\eta, & 0 \leqslant \lambda \leqslant s, \\ s^\eta(s+1-\lambda), & s \leqslant \lambda \leqslant s+1, \\ 0, & \lambda \geqslant s, \end{cases}$$

where $0 < \eta \leqslant 1$. Since $f(T_\alpha) \to f(T)$ strongly (cf. [19, Theorem VIII.20]), we get $\|f(T_\alpha)\zeta_\alpha - f(T)\zeta\| \to 0$. Similarly $\|g(T_\alpha)\zeta_\alpha - g(T)\zeta\| \to 0$. Choose an α_0 such that $\|T_\alpha\zeta_\alpha\|^2 \leqslant \|T\zeta\|^2 + \varepsilon^2$ and $\|f(T_\alpha)\zeta_\alpha\|^2 \geqslant \|f(T)\zeta\|^2 - \varepsilon^2$ for all $\alpha \geqslant \alpha_0$. If $\alpha \geqslant \alpha_0$, then

$$\int_s^\infty \lambda^{2\eta}\,\mathrm{d}\|e_\alpha(\lambda)\zeta_\alpha\|^2 \leqslant \int_s^\infty \lambda^2\,\mathrm{d}\|e_\alpha(\lambda)\zeta_\alpha\|^2 \leqslant$$

$$\leqslant \|T_\alpha\zeta_\alpha\|^2 - \|f(T_\alpha)\zeta_\alpha\|^2 \leqslant \|T\zeta\|^2 - \|f(T)\zeta\|^2 + 2\varepsilon^2 \leqslant$$

$$\leqslant \int_{s-1}^\infty \lambda^2\,\mathrm{d}\|e(\lambda)\zeta\|^2 + 2\varepsilon^2 \leqslant 3\varepsilon^2,$$

so that

$$\|T_\alpha^\eta\zeta_\alpha - T^\eta\zeta\| \leqslant \|g(T_\alpha)\zeta_\alpha - g(T)\zeta\| +$$

$$+ \left\{\int_s^\infty \lambda^{2\eta}\,\mathrm{d}\|e_\alpha(\lambda)\zeta_\alpha\|^2\right\}^{1/2} + \left\{\int_s^\infty \lambda^{2\eta}\,\mathrm{d}\|e(\lambda)\zeta\|^2\right\}^{1/2} \leqslant$$

$$\leqslant \|g(T_\alpha)\zeta_\alpha - g(T)\zeta\| + (\sqrt{3} + 1)\varepsilon.$$

Hence there exists an $\alpha_1 (\geqslant \alpha_0)$ such that $\|T_\alpha^\eta\zeta_\alpha - T^\eta\zeta\| \leqslant 3\varepsilon$ for all $\alpha \geqslant \alpha_1$. ▨

Proof of Theorem 3.4. We show that $\{x_\alpha\}$ is simple. When this is shown, the convergence of $\{\varkappa_\alpha^\eta(x_\alpha)\}$ or $\{\tilde{\varkappa}_\alpha^\eta(x_\alpha)\}$ follows from Theorem 3.1.

(1) We first prove the case $p = 2$. Let $\Theta^\eta : \mathscr{H} \to L^2(M; \varphi)_\eta$ and $\Theta_\alpha^\eta : \mathscr{H}_\alpha \to L^2(N_\alpha; \varphi_\alpha)_\eta$ be the surjective isometries as in Section 1. Define $\mathscr{E}_\alpha^\eta = (\Theta_\alpha^\eta)^{-1} \circ \varepsilon_\alpha^\eta \circ \Theta^\eta$ and $\mathscr{E}_{\alpha\beta}^\eta = (\Theta_\beta^\eta)^{-1} \circ \varepsilon_{\alpha\beta}^\eta \circ \Theta_\alpha^\eta$ for $\alpha \geqslant \beta$. Then $\mathscr{E}_\alpha^\eta = [\Delta_\alpha^{\eta/2} J_\alpha P_\alpha J \Delta^{-\eta/2}]$ and the adjoint $(\mathscr{E}_{\alpha\beta}^\eta)^*$ of $\mathscr{E}_{\alpha\beta}^\eta$ is given by $(\mathscr{E}_{\alpha\beta}^\eta)^* = [J_\alpha \Delta_\alpha^{\eta/2} \Delta_\beta^{-\eta/2} J_\beta]$ as seen from the proof of Lemma

2.1(2). It was shown in [11] that $\Delta_\alpha P_\alpha + (1 - P_\alpha) \to \Delta$ strongly in the generalized sense. Since $P_\alpha \nearrow 1$ and

$$(1 + \Delta_\alpha P_\alpha)^{-1} = \{1 + \Delta_\alpha P_\alpha + (1 - P_\alpha)\}^{-1} + \frac{1}{2}(1 - P_\alpha),$$

we get $\Delta_\alpha P_\alpha \to \Delta$ strongly in the generalized sense. Moreover it was shown in [12, 17] that $J_\alpha P_\alpha \to J$ strongly. For each $a \in M$, since $\|\varepsilon_\alpha(a)\xi - a\xi\| \to 0$ (see [12, 17]) and

$$\|(\Delta_\alpha P_\alpha)^{1/2}\varepsilon_\alpha(a)\xi - \Delta^{1/2}a\xi\| = \|J_\alpha \varepsilon_\alpha(a^*)\xi - Ja^*\xi\| \leqslant$$

$$\leqslant \|\varepsilon_\alpha(a^*)\xi - a^*\xi\| + \|(J_\alpha P_\alpha - J)a^*\xi\| \to 0,$$

Lemma 3.5 implies

$$\|(\Delta_\alpha P_\alpha)^{\eta/2}\varepsilon_\alpha(a)\xi - \Delta^{\eta/2}a\xi\| \to 0, \quad 0 < \eta \leqslant 1.$$

But

$$\mathscr{E}_\alpha^\eta(\Delta^{\eta/2}a\xi) = \Delta_\alpha^{\eta/2}\varepsilon_\alpha(a)\xi = (\Delta_\alpha P_\alpha)^{\eta/2}\varepsilon_\alpha(a)\xi.$$

Therefore $\mathscr{E}_\alpha^\eta \to 1$ strongly for $0 < \eta \leqslant 1$. Also $\mathscr{E}_\alpha^0 = J_\alpha P_\alpha J \to 1$ strongly. Now define a net $\{\zeta_\alpha\}$ of $\zeta_\alpha \in \mathscr{H}_\alpha$ $(\subset \mathscr{H})$ by $\zeta_\alpha = (\Theta_\alpha^\eta)^{-1}x_\alpha$. Then $\mathscr{E}_{\alpha\beta}^\eta \zeta_\alpha = \zeta_\beta$ for $\alpha \geqslant \beta$ and $\sup_\alpha \|\zeta_\alpha\| < \infty$. Hence there exists a subnet $\{\zeta_{\alpha'}\}$ of $\{\zeta_\alpha\}$ which converges weakly to some $\zeta \in \mathscr{H}$. For each α, if $\alpha' \geqslant \alpha$ and $b \in N_\alpha$, then

$$(\zeta_\alpha - \mathscr{E}_\alpha^\eta\zeta \mid J_\alpha\Delta_\alpha^{\eta/2}b\xi) = (\mathscr{E}_{\alpha'\alpha}^\eta(\zeta_{\alpha'} - \mathscr{E}_{\alpha'}^\eta\zeta) \mid J_\alpha\Delta_\alpha^{\eta/2}b\xi) =$$

$$= (\zeta_{\alpha'} - \mathscr{E}_{\alpha'}^\eta\zeta \mid (\mathscr{E}_{\alpha'\alpha}^\eta)^* J_\alpha\Delta_\alpha^{\eta/2}b\xi) = (\zeta_{\alpha'} - \mathscr{E}_{\alpha'}^\eta\zeta \mid J_{\alpha'}\Delta_{\alpha'}^{\eta/2}b\xi).$$

We have $\sup_{\alpha'}\|\zeta_{\alpha'} - \mathscr{E}_{\alpha'}^\eta\zeta\| < \infty$ and $\zeta_{\alpha'} - \mathscr{E}_{\alpha'}^\eta\zeta \to 0$ weakly since $\mathscr{E}_{\alpha'}^\eta \to 1$ strongly. On the other hand, since

$$\|(\Delta_{\alpha'}P_{\alpha'})^{1/2}b\xi - \Delta^{1/2}b\xi\| = \|(J_{\alpha'}P_{\alpha'} - J)b\xi\| \to 0,$$

using Lemma 3.5 we have

$$\|J_{\alpha'}\Delta_{\alpha'}^{\eta/2}b\xi - J\Delta^{\eta/2}b\xi\| \leqslant$$

$$\leqslant \|(\Delta_{\alpha'}P_{\alpha'})^{\eta/2}b\xi - \Delta^{\eta/2}b\xi\| + \|(J_{\alpha'}P_{\alpha'} - J)\Delta^{\eta/2}b\xi\| \to 0$$

for $0 < \eta \leqslant 1$. This holds for $\eta = 0$ as well. Therefore $(\zeta_{\alpha'} - \mathscr{E}_{\alpha'}^\eta\zeta \mid J_{\alpha'}\Delta_{\alpha'}^{\eta/2}b\xi) \to 0$, so that $(\zeta_\alpha - \mathscr{E}_\alpha^\eta\zeta \mid J_\alpha\Delta_\alpha^{\eta/2}b\xi) = 0$ for every $b \in N_\alpha$, showing $\zeta_\alpha = \mathscr{E}_\alpha^\eta\zeta$. Letting $x \in \Theta^\eta\zeta$, we obtain $x \in L^2(M;\varphi)_\eta$ and $x_\alpha = \varepsilon_\alpha^\eta(x)$ for all α.

We next prove the case $2 < p < \infty$. Since $L^p(N_\alpha;\varphi_\alpha)_\eta \subset L^2(N_\alpha;\varphi_\alpha)_\eta$ and $\|x_\alpha\|_{2,\eta} \leqslant \|x_\alpha\|_{p,\eta}$, it follows from the case $p = 2$ that there exists an $x \in L^2(M;\varphi)_\eta$

satisfying $x_\alpha = \varepsilon_\alpha^\eta(x)$ for all α. But there exists a subnet $\{\varkappa_{\alpha'}^\eta(x_{\alpha'})\}$ of $\{\varkappa_\alpha^\eta(x_\alpha)\}$ which converges in the weak topology of $L^p(M;\varphi)_\eta$ to some $y \in L^p(M;\varphi)_\eta$. For each $a \in M$, since

$$\|\varkappa_{\alpha'}^\eta(x_{\alpha'}) - x\|_1 \leqslant \|E_{\alpha'}^\eta(x) - x\|_{2,\eta} \to 0$$

by Theorem 3.1, we get

$$\operatorname{tr}(xa) = \lim_{\alpha'} \operatorname{tr}(\varkappa_{\alpha'}^\eta(x_{\alpha'})a) =$$

$$= \lim_{\alpha'} \langle \varkappa_{\alpha'}^\eta(x_{\alpha'}), h^{1-\eta}ah^\eta\rangle_{p,q} = \operatorname{tr}(ya).$$

Thus $x = y \in L^p(M;\varphi)_\eta$.

(2) Let $h_\alpha = h_{N_\alpha}$ and $\operatorname{tr}_\alpha = \operatorname{tr}_{N_\alpha}$ on $L^1(N_\alpha)$. Since $\|\tilde\varkappa_\alpha^\eta(x_\alpha)\|_{q,\eta} \leqslant \|x_\alpha\|_{q,\eta}$, there exists a subnet $\{\tilde\varkappa_{\alpha'}^\eta(x_{\alpha'})\}$ of $\{\tilde\varkappa_\alpha^\eta(x_\alpha)\}$ which converges in the weak topology of $L^q(M;\varphi)_\eta$ to some $x \in L^q(M;\varphi)_\eta$. For each α, if $\alpha' \geqslant \alpha$ and $b \in N_\alpha$, then

$$\operatorname{tr}(\tilde\varkappa_{\alpha'}^\eta(x_{\alpha'})b) = \operatorname{tr}_{\alpha'}(x_{\alpha'}\varepsilon_{\alpha'}(b)) =$$

$$= \operatorname{tr}_\alpha(x_\alpha b) + \operatorname{tr}_{\alpha'}(x_{\alpha'}(\varepsilon_{\alpha'}(b) - b))$$

since $\tilde\varepsilon_{\alpha'\alpha}^\eta(x_{\alpha'}) = x_\alpha$. With $1/p + 1/q = 1$, using Lemmas 3.2 and 3.3 we have

$$|\operatorname{tr}_{\alpha'}(x_{\alpha'}(\varepsilon_{\alpha'}(b) - b))| = |\langle x_{\alpha'}, h_{\alpha'}^{1-\eta}(\varepsilon_{\alpha'}(b) - b)h_{\alpha'}^\eta\rangle_{q,p}| \leqslant$$

$$\leqslant \|x_{\alpha'}\|_{q,\eta}\|h_{\alpha'}^{1-\eta}(\varepsilon_{\alpha'}(b) - b)h_{\alpha'}^\eta\|_{p,1-\eta} \leqslant$$

$$\leqslant \|x_{\alpha'}\|_{q,\eta}\|h_{\alpha'}^{1-\eta}(\varepsilon_{\alpha'}(b) - b)h_{\alpha'}^\eta\|_{2,1-\eta}^{2/p}\,\|h_{\alpha'}^{1-\eta}(\varepsilon_{\alpha'}(b) - b)h_{\alpha'}^\eta\|_{\infty,1-\eta}^{1-2/p} \leqslant$$

$$\leqslant \|x_{\alpha'}\|_{q,\eta}(2\|b\|)^{1-2/p}\{\|(\varepsilon_{\alpha'}(b) - b)\xi\|^\eta\|(\varepsilon_{\alpha'}(b^*) - b^*)\xi\|^{1-\eta}\}^{2/p} \to 0,$$

so that

$$\operatorname{tr}_\alpha(x_\alpha b) = \lim_{\alpha'} \operatorname{tr}(\tilde\varkappa_{\alpha'}^\eta(x_{\alpha'})b) =$$

$$= \lim_{\alpha'} \langle \tilde\varkappa_{\alpha'}^\eta(x_{\alpha'}), h^{1-\eta}bh^\eta\rangle_{q,p} = \operatorname{tr}(xb)$$

for every $b \in N_\alpha$. Hence $x_\alpha = \tilde\varepsilon_\alpha^\eta(x)$ for all α. ▨

The assumption $N_\alpha \nearrow M$ is essential in Theorem 3.4. For instance, let N be a von Neumann subalgebra for which the generalized conditional expectation $\varepsilon : M \to$ $\to N$ is not surjective. If we take $N_\alpha = N$ and $x_\alpha = h_N^\eta a h_N^{1-\eta}$ for all α where $a \in N \setminus \varepsilon(M)$, then the conclusion of Theorem 3.4(1) fails to hold.

From the last argument in the proof of Theorem 3.4(1), we have the following result as well: if $N_\alpha \nearrow M$ and if $\{x_\alpha\}$ is a net of $x_\alpha \in N_\alpha$ satisfying $\varepsilon_{\alpha\beta}(x_\alpha) = x_\beta$ for $\alpha \geqslant \beta$ and $\sup_\alpha \|x_\alpha\| < \infty$, then $x_\alpha = \varepsilon_\alpha(x)$ and $x_\alpha \to x$ strongly for some $x \in M$.

In the rest of this section, we discuss the convergence of decreasing generalized martingales in L^p-spaces. Let $\{N_\alpha\}$ be a decreasing net of unital von Neumann subalgebras of M with $N_\infty = \bigcap_\alpha N_\alpha$. As in the increasing case, we use the notations P_α, J_α, ε_α, E_α^η, $\tilde{E}_\alpha^\eta$ (or $\tilde{E}_\alpha$) associated with $(N_\alpha, \varphi_\alpha = \varphi \restriction N_\alpha)$, and P_∞, J_∞, ε_∞, E_∞^η, $\tilde{E}_\infty^\eta$ (or $\tilde{E}_\infty$) associated with $(N_\infty, \varphi_\infty = \varphi \restriction N_\infty)$.

Let $1 < q \leqslant 2 \leqslant p < \infty$ and $0 \leqslant \eta \leqslant 1$. We consider the following conditions:

$(\mathrm{C})_\infty$ $\varepsilon_\alpha(x) \to \varepsilon_\infty(x)$ strongly for every $x \in M$;

$(\mathrm{C})_1$ $\|\psi \circ \varepsilon_\alpha - \psi \circ \varepsilon_\infty\| \to 0$ for every $\psi \in M_*$ (i.e. $\|\tilde{E}_\alpha(a) - \tilde{E}_\infty(a)\|_1 \to 0$ for every $a \in L^1(M)$);

$(\mathrm{C})_{p,\eta}$ $\|E_\alpha^\eta(x) - E_\infty^\eta(x)\|_{p,\eta} \to 0$ for every $x \in L^p(M; \varphi)_\eta$;

$(\tilde{\mathrm{C}})_{q,\eta}$ $\|\tilde{E}_\alpha^\eta(x) - \tilde{E}_\infty^\eta(x)\|_{q,\eta} \to 0$ for every $x \in L^q(M; \varphi)_\eta$.

It was shown in [12, Theorem 4] that $P_\alpha \searrow P_\infty \Leftrightarrow (\mathrm{C})_\infty \Rightarrow (\mathrm{C})_1$. Also $(\mathrm{C})_{2,\eta} \Leftrightarrow$ $\Leftrightarrow (\tilde{\mathrm{C}})_{2,1-\eta}$ is seen from Lemma 2.5. The next theorem establishes the relations among the above conditions.

THEOREM 3.8. (1) *For each* $2 \leqslant p < \infty$, *conditions* $(\mathrm{C})_\infty$, $(\mathrm{C})_{p,0}$ *and* $(\mathrm{C})_{p,1}$ *are equivalent.*

(2) *For each* $2 \leqslant p < \infty$ *and* $0 < \eta < 1$, *condition* $(\mathrm{C})_{p,\eta}$ *is equivalent to* $(\mathrm{C})_1$. *For each* $1 < q < 2$ *and* $0 \leqslant \eta \leqslant 1$, *condition* $(\tilde{\mathrm{C}})_{q,\eta}$ *is equivalent to* $(\mathrm{C})_1$.

Proof. (1) It is immediately seen from Lemmas 2.5 and 2.8(2) that each of $(\mathrm{C})_\infty$, $(\mathrm{C})_{2,0}$ and $(\mathrm{C})_{2,1}$ is equivalent to $J_\alpha P_\alpha \to J_\infty P_\infty$ strongly. For each $2 < p < \infty$ and $0 \leqslant \eta \leqslant 1$, we have $(\mathrm{C})_{2,\eta} \Leftrightarrow (\mathrm{C})_{p,\eta}$ from $\|\cdot\|_{2,\eta} \leqslant \|\cdot\|_{p,\eta}$ and Lemma 3.2.

(2) Suppose that $(\mathrm{C})_1$ holds. By Lemma 3.2, we then obtain $(\tilde{\mathrm{C}})_{q,\eta}$ for every $1 < q < 2$ and $0 \leqslant \eta \leqslant 1$. Furthermore $(\mathrm{C})_{2,1/2}$ $(= (\tilde{\mathrm{C}})_{2,1/2})$ is satisfied in view of Lemma 2.4. Hence Lemma 3.3 gives $(\mathrm{C})_{2,\eta}$ for every $0 < \eta < 1$. So, by Lemma 3.2 again, we obtain $(\mathrm{C})_{p,\eta}$ for every $2 \leqslant p < \infty$ and $0 < \eta < 1$.

Conversely if $(\mathrm{C})_{p,\eta}$ holds for some $2 \leqslant p < \infty$ and $0 \leqslant \eta \leqslant 1$, then $(\mathrm{C})_{2,\eta}$ follows from $\|\cdot\|_{2,\eta} \leqslant \|\cdot\|_{p,\eta}$, so that we get $(\tilde{\mathrm{C}})_{2,1-\eta}$. On the other hand, if $(\tilde{\mathrm{C}})_{q,\eta}$ holds for some $1 < q \leqslant 2$ and $0 \leqslant \eta \leqslant 1$, then $(\mathrm{C})_1$ follows from $\|\cdot\|_1 \leqslant \|\cdot\|_{q,\eta}$. Thus (2) is proved. ▨

In contrast with the increasing case, $(\mathrm{C})_\infty$ is not satisfied in general. Indeed it happens that ξ is cyclic for each N_α while $N_\infty = \mathbf{C}1$ (cf. [1, 12]). But the question is whether it is possible that $(\mathrm{C})_1$ holds while $(\mathrm{C})_\infty$ does not.

4. GENERALIZED CONDITIONAL EXPECTATIONS RELATIVE TO WEIGHTS

Throughout this section, let M be a von Neumann algebra with a fixed faithful normal semifinite weight φ. Let N be a unital von Neumann subalgebra of M such that $\varphi_N = \varphi \restriction N_+$ is semifinite. We take $\mathfrak{n}_N = \mathfrak{n}_\varphi \cap N$, $\mathfrak{m}_N = \mathfrak{m}_\varphi \cap N$ and the GNS representation $(\mathscr{H}_N, \pi_N)$ of N induced by φ_N where $\mathscr{H}_N$ is identified with the closure of $\Lambda(\mathfrak{n}_N)$ in $\mathscr{H}_\varphi$. Let P_N be the orthogonal projection of $\mathscr{H}_\varphi$ onto $\mathscr{H}_N$, then $P_N \in \pi(N)'$ and $\pi_N(x) = \pi(x)P_N$, $x \in N$. Let J_N be the modular conjugation associated with φ_N. Moreover, for $1 < p < \infty$, we take Terp's L^p-space $L^p(N; \varphi_N)$ as well as $L^p(M; \varphi)$. That is, $L^p(N; \varphi_N)$ is the complex interpolation space $C_{1/p}(N, N_*)$ where N and N_* are imbedded in the dual L_N^* of the Banach space L_N consisting of all $x \in N$ such that there exists a $\psi_x^N \in N_*$ with

$$\psi_x^N(z^*y) = (J_N\pi_N(x)^*J_N\Lambda(y) \mid \Lambda(z)), \quad y, z \in \mathfrak{n}_N.$$

The generalized conditional expectation $\varepsilon : M \to N$ relative to φ (see [1, Theorem 7.5]) is given by

$$\pi_N(\varepsilon(x)) = J_N P_N J\pi(x)JJ_N, \quad x \in M,$$

which has the same properties as that relative to a state (see Section 2). To extend $\varepsilon : M \to N$ to linear contractions between Terp's L^p-spaces, we first give

LEMMA 4.1. (1) *If* $x \in L$, *then* $\varepsilon(x) \in L_N$ *and* $\psi^N_{\varepsilon(x)} = \psi_x \restriction N$.
(2) *If* $x \in L_N$, *then* $x \in L$ *and* $\psi_x = \psi_x^N \circ \varepsilon$.

Proof. (1) If $x \in L$ and $y, z \in \mathfrak{n}_N$, then

$$\psi_x(z^*y) = (J\pi(x)^*J\Lambda(y) \mid \Lambda(z)) =$$

$$= (J_N(J_NP_NJ\pi(x)^*JJ_N)J_N\Lambda(y) \mid \Lambda(z)) = (J_N\pi_N(\varepsilon(x))^*J_N\Lambda(y) \mid \Lambda(z)).$$

(2) Let $x \in \mathfrak{m}_N$. For every $y, z \in \mathfrak{n}_\varphi$, $\varepsilon(z^*y) \in \mathfrak{m}_N$ and $\Lambda(\varepsilon(z^*y)) = J_NP_NJ\Lambda(z^*y)$ by [12, Lemma 1]. Hence, using [23, Proposition 7, Lemma 3], we have

$$\psi_x^N(\varepsilon(z^*y)) = \psi^N_{\varepsilon(z^*y)}(x) = (\Lambda(x) \mid J_N\Lambda(\varepsilon(z^*y)) =$$

$$= (\Lambda(x) \mid P_NJ\Lambda(z^*y)) = (\Lambda(y) \mid \pi(z)J\Lambda(x)) =$$

$$= (\Lambda(y) \mid J\pi(x)J\Lambda(z)) = (J\pi(x)^*J\Lambda(y) \mid \Lambda(z)),$$

so that $x \in L$ and $\psi_x = \psi_x^N \circ \varepsilon$. Now let $x \in L_N$. According to [23, Theorem 8], there exists a net $\{x_j\}$ in $\mathfrak{m}_N$ such that $x_j \to x$ σ-weakly and $\|\psi^N_{x_j} - \psi^N_x\| \to 0$. Since $\{x_j\} \subset L$ with $\psi_{x_j} = \psi^N_{x_j} \circ \varepsilon$ and $\|\psi_{x_j} - \psi^N_x \circ \varepsilon\| \leqslant \|\psi^N_{x_j} - \psi^N_x\| \to 0$, we obtain $x \in L$ and $\psi_x = \psi_x^N \circ \varepsilon$. ▨

We notice (see [23, Proposition 7]) that $L = M \cap M_*$ when M and M_* are imbedded in L^* with identification $x = \psi_x$. Lemma 4.1(1) asserts that if $x \in L$ then $\varepsilon(x) = \psi_x \restriction N$ as elements in L_N^*. So we can extend $\varepsilon : M \to N$ to a linear map (denoted by the same ε) of $M + M_*$ $(\subset L^*)$ into $N + N_*$ $(\subset L_N^*)$ by

$$\varepsilon(x + \psi) = \varepsilon(x) + \psi \restriction N, \quad x \in M, \ \psi \in M_*.$$

Similarly Lemma 4.1(2) enables us to define a linear map $\varkappa : N + N_* \to M + M_*$ by

$$\varkappa(x + \psi) = x + \psi \circ \varepsilon \quad x \in N, \ \psi \in N_*.$$

THEOREM 4.2. *Let $1 < p < \infty$ and $1/p + 1/q = 1$. Then ε maps $L^p(M;\varphi)$ into $L^p(N;\varphi_N)$ with $\|\varepsilon(x)\|_p \leq \|x\|_p$, $x \in L^p(M;\varphi)$, and $\varkappa$ maps $L^p(N;\varphi_N)$ into $L^p(M;\varphi)$ with $\|\varkappa(x)\|_p \leq \|x\|_p$, $x \in L^p(N;\varphi_N)$. Moreover the transpose of $\varepsilon \restriction L^p(M;\varphi)$ is $\varkappa \restriction L^q(N;\varphi_N)$ under the duality $\langle\cdot,\cdot\rangle_{p,q}$ between $L^p(M;\varphi)$ and $L^q(M;\varphi)$ and that between $L^p(N;\varphi_N)$ and $L^q(N;\varphi_N)$ (see Section 1).*

Proof. Since ε (resp. $\varkappa$) is contractive on both M (resp. N) and M_* (resp. N_*), the first assertion follows from the abstract Riesz-Thorin theorem. By Lemma 4.1(1), we get

$$\langle\varepsilon(x), y\rangle_{p,q} = \psi^N_{\varepsilon(x)}(y) = \psi_x(y) = \langle x, \varkappa(y)\rangle_{p,q}, \quad x \in L, \ y \in L_N.$$

This shows the second assertion, since L and L_N are dense in $L^p(M;\varphi)$ and $L^q(N;\varphi_N)$ respectively (see [23, Theorem 27]). ▨

When φ is a state, Theorem 4.2 is the same as the case $\eta = 1/2$ of Theorems 2.2 and 2.3 because $\varepsilon = \tilde{\varepsilon}$ on $M_* = L^1(M)$ and $\varkappa = \tilde{\varkappa}$ on $N_* = L^1(N)$.

Let $E = \varkappa \circ \varepsilon$. Then $E \restriction L^p(M;\varphi)$ is a linear contraction of $L^p(M;\varphi)$ into itself for $1 < p < \infty$. The contraction $\Theta^{-1} \circ E \circ \Theta$ on $\mathscr{H}_\varphi$ is naturally connected with ε, where $\Theta : \mathscr{H}_\varphi \to L^2(M;\varphi)$ is the isometry given in Section 1. Because $\Lambda(\varepsilon(x)) = J_N P_N J \Lambda(x)$ for all $x \in \mathfrak{n}_\varphi$ (see [12, Lemma 1]), another related contraction on $\mathscr{H}_\varphi$ is $J_N P_N J$.

THEOREM 4.3. *The following conditions are equivalent:*
(i) *$\varepsilon : M \to N$ is the conditional expectation;*
(ii) *$\varepsilon \circ \varepsilon = \varepsilon$ on M ($\varepsilon(M) = N$ is not required);*
(iii) *$E \circ E = E$ on $L^p(M;\varphi)$, where $1 < p < \infty$;*
(iv) *$\Theta^{-1} \circ E \circ \Theta = J_N P_N J$.*

LEMMA 4.4. *If $x \in \mathfrak{m}_\varphi$, then $\Theta(\Delta^{1/4}\Lambda(x)) = x$.*

Proof. Let φ' be a faithful normal semifinite weight on M' and $d = \dfrac{d\varphi}{d\varphi'}$. Since $\Theta = \nu_2 \circ \mathscr{P}$ and $\nu_2(\mu_2(x)) = x$ for $x \in L$ $(\supset \mathfrak{m}_\varphi)$ (see Section 1), it suffices to show that

$$\mathscr{P}(\Delta^{1/4}\Lambda(x)) = \mu_2(x), \quad x \in \mathfrak{m}_\varphi .$$

Since $\mathfrak{m}_\varphi = \mathrm{span}(\mathfrak{m}_\varphi)_+$, we may assume $x \in (\mathfrak{m}_\varphi)_+$. Taking $a = x^{1/2}$, we define

$$a_n = \sqrt{n/\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-nt^2}\sigma_t(a)\,\mathrm{d}t$$

and $x_n = a_n^2$ for $n \geqslant 1$. Then $\|a_n\| \leqslant \|a\|$, $a_n \to a$ strongly and $\|\Lambda(a_n) - \Lambda(a)\| \to 0$ (cf. [20, p. 29]). Note (cf. [23, Lemma 22]) that $vd^s \subset d^s\sigma_{is}(v)$, $s \geqslant 0$, for any σ-analytic $v \in M$. Since $[a_n d^{1/4}]$ and $d^{1/4}\sigma_{i/4}(a_n) = [\sigma_{-i/4}(a_n)d^{1/4}]^*$ are in $L^4(\varphi')$ by [23, Theorem 26], we get $[a_n d^{1/4}] = d^{1/4}\sigma_{i/4}(a_n)$ and $d^{1/4}a_n = [\sigma_{-i/4}(a_n)d^{1/4}]$. From definition of μ_2 in the proof of [23, Theorem 27], it follows that

$$\mu_2(x_n) = d^{1/4}a_n\cdot[a_n d^{1/4}] \supset \sigma_{-i/4}(a_n)d^{1/2}\sigma_{i/2}(\sigma_{-i/4}(a_n)) \supset$$

$$\supset \sigma_{-i/4}(a_n)\sigma_{-i/4}(a_n)d^{1/2} = \sigma_{-i/4}(x_n)d^{1/2},$$

and hence $\mu_2(x_n) = [\sigma_{-i/4}(x_n)d^{1/2}]$ since both sides are in $L^2(\varphi')$. Therefore

$$\mathscr{P}(\Delta^{1/4}\Lambda(x_n)) = \mathscr{P}(\Lambda(\sigma_{-i/4}(x_n))) = \mu_2(x_n), \quad n \geqslant 1,$$

by definition of $\mathscr{P}$. Since $\|\psi_{x_n}\| \leqslant \|\Lambda(a_n)\|^2$ by [23, Proposition 4] and $\|\mu_2(x_n)\|_2 \leqslant \|\psi_{x_n}\|^{1/2}\|x_n\|^{1/2}$ by [23, Theorem 27], we have $\sup_n\|\mu_2(x_n)\|_2 < \infty$. Since $\mu_2(L)$ is dense in $L^2(\varphi')$ and

$$\int \mu_2(y)\mu_2(x_n)\,\mathrm{d}\varphi' = \psi_y(x_n) \to \mu_y(x) = \int \mu_2(y)\mu_2(x)\,\mathrm{d}\varphi', \quad y \in L,$$

we have $\mu_2(x_n) \to \mu_2(x)$ weakly, so that $\Delta^{1/4}\Lambda(x_n) = \mathscr{P}^{-1}(\mu_2(x_n)) \to \mathscr{P}^{-1}(\mu_2(x))$ weakly. On the other hand,

$$\|\Lambda(x_n) - \Lambda(x)\| = \|\pi(a_n)\Lambda(a_n) - \pi(a)\Lambda(a)\| \to 0.$$

Thus $\Delta^{1/4}\Lambda(x) = \mathscr{P}^{-1}(\mu_2(x))$ as desired. ▨

Proof of Theorem 4.3. Clearly (i) implies (ii) and (iii). Since $L^p(M;\varphi) \cap M\,(\supset L)$ is σ-weakly dense in M (see [23, Corollary 5]), we have (iii) $\Rightarrow$ (ii). Furtherm ore (ii) $\Rightarrow$ (i) is seen as in the proof of Theorem 2.6.

We now show (i) $\Leftrightarrow$ (iv). If $x \in \mathfrak{m}_\varphi$, then $\varepsilon(x) \in \mathfrak{m}_\varphi$ and Lemma 4.4 gives

$$\Theta^{-1}\circ E\circ\Theta(\Delta^{1/4}\Lambda(x)) = \Theta^{-1}(\varepsilon(x)) = \Delta^{1/4}\Lambda(\varepsilon(x)) = \Delta^{1/4}J_N P_N J\Lambda(x).$$

So condition (iv) is equivalent to $J_N P_N J \Delta^{1/4} \subset \Delta^{1/4} J_N P_N J$. Hence (i) $\Leftrightarrow$ (iv) is shown as in the proof of Theorem 2.7. ▨

We hereafter consider the martingale convergence of generalized conditional expectations relative to the weight φ. Let $\{N_\alpha\}$ be an increasing net of unital von Neumann subalgebras of M with $N_\infty = \bigvee_\alpha N_\alpha$. Assume that $\varphi_\alpha = \varphi \restriction (N_\alpha)_+$ is semifinite for each α and hence also $\varphi_\infty = \varphi \restriction (N_\infty)_+$ is semifinite. We take $n_\alpha = n_\varphi \cap N_\alpha$, the orthogonal projection P_α of $\mathscr{H}_\varphi$ onto $\overline{\Lambda(n_\alpha)}$, the modular conjugation J_α associated with φ_α, and analogously P_∞, J_∞. Let $\varepsilon_\alpha : M \to N_\alpha$ and $\varepsilon_\infty : M \to N_\infty$ be the generalized conditional expectations relative to φ. For each $1 < p < \infty$, ε_α and ε_∞ are extended to linear contractions E_α and E_∞ of $L^p(M; \varphi)$ into itself. We then have

THEOREM 4.5. *The following conditions are equivalent:*
(i) $\varepsilon_\alpha(x) \to \varepsilon_\infty(x)$ *strongly for every* $x \in M$;
(ii) $\|\psi \circ \varepsilon_\alpha - \psi \circ \varepsilon_\infty\| \to 0$ *for every* $\psi \in M_*$;
(iii) $\|E_\alpha(x) - E_\infty(x)\|_p \to 0$ *for every* $x \in L^p(M; \varphi)$, *where* $1 < p < \infty$.

Proof. We proved in [12, Theorem 3] that (i), (ii) and $J_\alpha P_\alpha \to J_\infty P_\infty$ strongly are equivalent.

(ii) $\Rightarrow$ (iii). Let $x \in L$. By Lemma 4.1, we get $\varepsilon_\alpha(x), \varepsilon_\infty(x) \in L$ with $\psi_{\varepsilon_\alpha(x)} = \psi_x \circ \varepsilon_\alpha$, $\psi_{\varepsilon_\infty(x)} = \psi_x \circ \varepsilon_\infty$. When $1/p + 1/q = 1$, it follows from [23, Theorem 27] that

$$\|E_\alpha(x) - E_\infty(x)\|_p = \|\mu_p(\varepsilon_\alpha(x) - \varepsilon_\infty(x))\|_p \leqslant$$
$$\leqslant \|\psi_{\varepsilon_\alpha(x) - \varepsilon_\infty(x)}\|^{1/p} \|\varepsilon_\alpha(x) - \varepsilon_\infty(x)\|^{1/q} \leqslant$$
$$\leqslant \|\psi_x \circ \varepsilon_\alpha - \psi_x \circ \varepsilon_\infty\|^{1/p} (2\|x\|)^{1/q} \to 0.$$

Since L is dense in $L^p(M; \varphi)$, we obtain (iii).

(iii) $\Rightarrow$ (i). Let $x, y \in L$. Using Hölder's inequality on spatial L^p-spaces (cf. [22, Chapter IV]), we have

$$\left| \int \mu_2(\varepsilon_\alpha(x) - \varepsilon_\infty(x)) \mu_2(y) \, d\varphi' \right| = \left| \int \mu_p(\varepsilon_\alpha(x) - \varepsilon_\infty(x)) \mu_q(y) \, d\varphi' \right| \leqslant$$
$$\leqslant \|\mu_p(\varepsilon_\alpha(x) - \varepsilon_\infty(x))\|_p \|\mu_q(y)\|_q = \|E_\alpha(x) - E_\infty(x)\|_p \|y\|_q \to 0.$$

This shows $\mu_2(\varepsilon_\alpha(x)) \to \mu_2(\varepsilon_\infty(x))$ weakly, because $\mu_2(L)$ is dense in $L^2(\varphi')$ and

$$\|\mu_2(\varepsilon_\alpha(x) - \varepsilon_\infty(x))\|_2 \leqslant \|\psi_x \circ (\varepsilon_\alpha - \varepsilon_\infty)\|^{1/2} \|\varepsilon_\alpha(x) - \varepsilon_\infty(x)\|^{1/2} \leqslant$$
$$\leqslant 2\|\psi_x\|^{1/2} \|x\|^{1/2}.$$

In particular let $x \in \mathfrak{m}_\varphi$. Then $\varepsilon_\alpha(x)$, $\varepsilon_\infty(x) \in \mathfrak{m}_\varphi$ and, by Lemma 4.4, we have

$$\Delta^{1/4}\Lambda(\varepsilon_\alpha(x)) = \mathscr{P}^{-1}(\mu_2(\varepsilon_\alpha(x))) \to \mathscr{P}^{-1}(\mu_2(\varepsilon_\infty(x))) = \Delta^{1/4}\Lambda(\varepsilon_\infty(x))$$

weakly, so that

$$(\Lambda(\varepsilon_\alpha(x)) - \Lambda(\varepsilon_\infty(x)) \mid \Delta^{1/4}\zeta) =$$

$$= (\Delta^{1/4}(\Lambda(\varepsilon_\alpha(x)) - \Lambda(\varepsilon_\infty(x))) \mid \zeta) \to 0, \quad \zeta \in \mathscr{D}(\Delta^{1/4}).$$

Since $\|\Lambda(\varepsilon_\alpha(x))\| \leqslant \|\Lambda(\varepsilon_\infty(x))\|$, this gives $\Lambda(\varepsilon_\alpha(x)) \to \Lambda(\varepsilon_\infty(x))$ weakly and hence $\|\Lambda(\varepsilon_\alpha(x)) - \Lambda(\varepsilon_\infty(x))\| \to 0$. Thus $J_\alpha P_\alpha \to J_\infty P_\infty$ strongly. ▨

It is known (see [12, Theorem 3]) that the conditions in Theorem 4.5 hold if and only if $\bigcup_\alpha \Lambda(\mathfrak{n}_\alpha \cap \mathfrak{n}_\alpha^*)$ is a core of $\Delta_\infty^{1/2}$, where $\Lambda(\mathfrak{n}_\alpha \cap \mathfrak{n}_\alpha^*)$ is the left Hibert algebra associated with φ_α and Δ_∞ is the modular operator associated with φ_∞. When $\varphi(1) < \infty$, this condition is satisfied and Theorem 4.5 is reduced to Theorem 3.1 with $\eta = 1/2$. But, for the weight case, this seems to be a rather strong condition (cf. [11, Example 1.6]).

Next let $\{N_\alpha\}$ be a decreasing net of unital von Neumann subalgebras of M with $N_\infty = \bigcap_\alpha N_\alpha$. Assume that $\varphi_\infty = \varphi \restriction (N_\infty)_+$ is semifinite and hence each $\varphi_\alpha = \varphi \restriction (N_\alpha)_+$ is semifinite. Let P_α, ε_α, E_α and P_∞, ε_∞, E_∞ be as above.

THEOREM 4.6. *If* $P_\alpha \searrow P_\infty$, *then the following conditions hold:*

(i) $\varepsilon_\alpha(x) \to \varepsilon_\infty(x)$ *strongly for every* $x \in M$;

(ii) $\|\psi \circ \varepsilon_\alpha - \psi \circ \varepsilon_\infty\| \to 0$ *for every* $\psi \in M_*$;

(iii) $\|E_\alpha(x) - E_\infty(x)\|_p \to 0$ *for every* $x \in L^p(M; \varphi)$, $1 < p < \infty$.

Proof. It was proved in [12, Theorem 4] that if $P_\alpha \searrow P_\infty$ then (i) and (ii) hold. (ii) $\Rightarrow$ (iii) is seen as in the proof of Theorem 4.5. ▨

When each ε_α is the conditional expectation, all the conditions in Theorems 4.5 and 4.6 are satisfied (cf. [12, 24]).

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