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INFINITE COXETER GROUPS DO NOT HAVE KAZHDAN'S PROPERTY

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1. INTRODUCTION

Recall that a *kernel of negative type* on a set Y is a function $f: Y \times Y \to \mathbf{R}$ such that for all n-tuples $(y_1, \dots, y_n)$ of elements of Y and any n-tuple $(\lambda_1, \dots, \lambda_n)$ of complex numbers such that $\sum_{i=1}^{n} \lambda_i = 0$ we have

$$\sum_{i,j=1}^{n} \lambda_i \overline{\lambda}_j f(y_i, y_j) \leqslant 0.$$

A *function of negative type* on a group G is a function $\varphi : G \to \mathbf{R}$ such that $f(g, h) = \varphi(g^{-1}h)$ defines a kernel of negative type on G.

It was proved by [3] that any function of negative type on a Kazhdan group is bounded (cf. [4] for the definition and basic properties of Kazhdan groups).

Let (W, S) be a Coxeter group. Then W acts on two complexes canonically associated with W, the *Cayley complex* $C(W)$ defined below in Section 3 and the *Coxeter complex* $\Delta(W)$ [6, Chapter II]. We define "embeddings" α_1 and α_2 of $C(W)$ and $\Delta(W)$ respectively into a Hilbert space $\mathscr{H}$ in such a way that W acts by isometries with respect to the induced metrics. Actually while for $C(W)$ we get an embedding of the whole complex, for $\Delta(W)$ we just embed the set of chambers. If W is infinite both embeddings are unbounded. Then

$$f_i(g, h) = \|\alpha_i(g \cdot \sigma_i) - \alpha_i(h \cdot \sigma_i)\|^2$$

is an unbounded kernel of negative type (see [1]) where σ_i are fixed base points in the complexes. Note that any infinite subgroup of a Coxeter group inherits a kernel of negative type. Thus we obtain the

THEOREM. *No infinite subgroup of a Coxeter group has Kazhdan's property.*

This answers a question of de la Harpe [5].

It turns out that $f_2(x, 1)$ equals the length function $l(x)$ on W for the set of generators S, i.e. the minimal length of a word in S representing x. By a result of Schoenberg [1, Theorem 7.8], $e^{-\lambda l(x)}$ is a positive definite function for all $\lambda \geq 0$.

Let us emphasize that our method for embedding the Cayley complex is quite general. For example, suppose a cubical complex Δ has an enumeration $(C_n)_{n \in N}$ of the cubes such that for all n, C_n intersects all C_i, all $i < n$, in the star of single vertex. Then Δ has an embedding into a Hilbert space such that the combinatorial distance between vertices is the square of the distance in Hilbert space. In particular, no group acting on such a complex with an infinite orbit can have Kazhdan's property. This includes the case of trees.

Also note that this technique gives nonembeddability results. For example, the Euclidean building of $SL(3, \mathbf{Q}_p)$ with the combinatorial distance does not embed isometrically into a Hilbert space, since it carries an action of a Kazhdan group.

2. THE COXETER COMPLEX AND ITS EMBEDDING

We refer to [6] for the definitions and basic properties of Coxeter complexes. Let (W, S) be a Coxeter group, $\Delta = \Delta(W)$ its Coxeter complex, $\mathscr{R}$ the set of roots and ρ the distance function on the chambers of Δ. Tits proves in [6, 2.22] that

$$\rho(x, y) = \#\{R \in \mathscr{R} : x \in R \text{ and } y \notin R\}.$$

Let $\mathscr{H} = \ell^2(\mathscr{R})$. Fix a chamber x_0, define

$$\alpha : \{x : x \text{ is a chamber of } \Delta\} \to \mathscr{H}$$

by

$$\alpha(x)(R) = \chi_R(x) - \chi_R(x_0)$$

for all $R \in \mathscr{R}$. It follows from Tits' formula that $\alpha(x)$ has finite support and that

$$2\rho(x, y) = \sum_{R \in \mathscr{R}} |\chi_R(x) - \chi_R(y)|^2 = \|\alpha(x) - \alpha(y)\|^2.$$

Note that this also defines an embedding of the chambers of the Coxeter complex into all $\ell^p(\mathscr{R})$ for $1 \leqslant p < \infty$ such that $2\rho(x, y) = \|\alpha(x) - \alpha(y)\|_p^p$.

3. THE CAYLEY COMPLEX AND ITS EMBEDDING

The complex we use is a refinement of the Cayley graph of the group, hence the name: the *Cayley complex* $C(W)$. It is defined as follows: the set of k-cells is indexed by $\coprod_P W/P$, where P runs through all finite k-parabolics, i.e. subgroups spanned by k generators from S. Two vertices x, y belong to the same cell $A \in W/P$ for some P if and only if $xy^{-1} \in P$.

These conditions completely determine the combinatorial structure of $C(W)$. Clearly, the 1-dimensional skeleton of $C(W)$ is just the Cayley graph of W.

In the case of finite groups there is an alternative description of the combinatorial structure of $C(W)$:

Take the canonical representation of W on $\mathbf{R}^n$ by reflections and take a point $p \in \mathbf{R}^n$ with trivial stabilizer. Let $P(W)$ be the convex closure of the orbit of p.

PROPOSITION. *The polyhedra $P(W)$ and $C(W)$ are combinatorially isomorphic. Moreover, this isomorphism is equivariant with respect to the obvious actions of W on these polyhedra.*

Proof. The isomorphism is defined by taking a k-face $g \cdot P$ in $C(W)$ to the convex closure of $g \cdot P(p)$. ▨

It is clear that the cell decomposition of the boundary of $P(W)$ is dual to the triangulation of the unit sphere in $\mathbf{R}^n$ given by Weyl chambers. Moreover, since the barycentric subdivision of a simplex is combinatorially equivalent with the cube of the same dimension, the quotient of $P(W)$ by W is combinatorially the cube.

Let W be infinite. Then $X = C(W)/W$ embeds into $C(W)$ as a fundamental domain for W and consists of finitely many cubes with common vertex 1. In fact, each finite parabolic subgroup F of W determines a cube $C(F)/F$, and these are all of them. This determines a cubical subdivision of $C(W)$. We will work with this cubical Cayley complex in the following. Define a panel structure (cf. [2]) on X by setting $X_\alpha = \{x \in X \mid \alpha \cdot x = x\}$ for $\alpha \in S$. Then $C(W)$ can be identified with the Γ-space associated to (W, X). Recall that this is the space $W \times X/\sim$ where $(\gamma, x) \sim (\delta, y)$ if and only if $x = y$ and $\delta^{-1}\gamma \in V_x = \langle \alpha \in S | x_\alpha \in X_\alpha \rangle$ (cf. [2, §13]). Observe that the general results on reflection systems on manifolds in [2] apply to W-spaces associated to panel structures (see also the proof of [2, 13.5]. Note also that $C(W)$ is the cubical analogue of Davis' universal W-complex [2, §14].

Now we embed the cubical Cayley comples into a Hilbert space $\mathscr{H}$. The idea is to embed cubes orthogonally.

First consider the case of a finite Coxeter graph W. Recall that $P(W)$ is dual to the triangulation of the unit sphere in $\mathbf{R}^n$ by Weyl chambers. Pick an orthonormal basis $\{e_v\}$ of some $\mathbf{R}^n$ indexed by the vertices v of the Weyl chamber triangulation. Each cube σ in $P(W)$ is spanned by the edges from 0 to the vertices $v_1, \ldots, v_k$ of a Weyl chamber. Map σ to the set $\{t_1 e_{v_1} + \ldots + t_k e_{v_k} \,|\, t_i \in [0,1]\}$ which we call the *cube on* $e_{v_1}, \ldots, e_{v_k}$. Now let W be arbitrary. Let $C(W) = \prod_{g \in W} g \cdot X$. Embed $g \cdot X$ into some $\mathbf{R}_g^N$ as follows. Let the generating set S enumerate an orthonormal basis $\{e_s^g\}$ of $\mathbf{R}_g^N$. If $T \subset S$, fill in the cube on $\{e_s^g \mid s \in T\}$ whenever T generates a finite parabolic. All the edges of the cubes are translates of the basis $\{e_s^g\}_{s \in S}$.

Enumerate the elements of $W: g_1, g_2, \dots$ such that $l(g_i) \leqslant l(g_{i+1})$ where l is the length function for S. Recall Lemma 8.2 from [2]:

LEMMA. *The set $g_{k+1} \cdot X$ intersects $\coprod_{i \leqslant k} g_i X$ along a set F_{k+1} of faces which are contained in the orbit of a finite parabolic.*

We define an orthogonal basis $\{e_i\}$ for the Hilbert space $\mathscr{H}$ by induction on k:

Step 1: Pick the vectors $\{e_i^1\}$ from $\mathbf{R}_1$.

Step $k+1$: Identify the $e_i^{g_{k+1}}$ parallel to an edge in F_{k+1} with the previously embedded e_i's. Add the remaining $e_i^{g_{k+1}}$ as new basis vectors.

This also determines a map $\alpha: \{e_i^g\} \to \{e_i\}$. We define an embedding α of $C(W)$ into $\mathscr{H}$ by induction on k:

Step 1: Embed X into $\mathscr{H}$ by sending $\sum t_i e_i^1$ to $\sum t_i \alpha(e_i^1)$.

Step $k+1$: Since we enumerate W by nondecreasing word length there is a vertex x of $g_{k+1}X$ in $\coprod_{i \leqslant k} g_i X$. Let $x = \sum c_i e_i^{g_{k+1}}$ in $\mathbf{R}^N_{g_{k+1}}$. Send 0 in $g_{k+1}X$ to $0_{k+1} = \alpha(x) - \sum c_i \alpha(e_i^{g_{k+1}})$. Now map $\mathbf{R}^N_{g_{k+1}}$ affinely to the affine space at 0_{k+1} by sending $e_i^{g_{k+1}}$ to $\alpha(e_i^{g_{k+1}})$. The consistency of this construction follows from the lemma and the consistency for finite Coxeter groups.

Note that if two faces meet they are embedded orthogonally. Therefore α is injective provided α is injective on vertices. This is clear since at each induction step we insert new edges orthogonally. Finally note that W acts isometrically with respect to the induced metric since the embedding and thus the distance are determined combinatorially.

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REFERENCES

1. BERG, C.; FORST, G., *Potential theory on locally compact abelian groups*, Ergebnisse der Mathematik, **87**, Springer-Verlag, New York, 1975.
2. DAVIS, M., Groups generated by reflections and aspherical manifolds not covered by Euclidean space, *Ann. of Math.*, **117**(1983), 293–324.
3. DELORME, P., 1-cohomologie des représentations unitaires des groupes de Lie semisimples et resolubles, *Bull. Soc. Math. France*, **105**(1977), 281–336.
4. DELAROCHE, C.; KIRILLOV, A., Sur les relations entre l'espace dual d'un groupe et la structure de ses sous-groupes fermés, (d'apres Kazhdan), *Sem. Bourbaki*, Exposé **343**, 1967–68.

5. DE LA HARPE, P., Coxeter groups and amenability, preprint, Université de Genève, 1985.
6. TITS, J., *Buildings of spherical type and finite* BN-*pairs*, Lecture Notes in Mathematics, no. **386**, Springer-Verlag, New York, 1974.

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