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REPRESENTATIONS OF OPERATOR BIMODULES AND THEIR APPLICATIONS

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1. INTRODUCTION

It is well-know that any normed vector space V is isometric to a *function space*, i.e., a linear subspace of $\ell^\infty(X)$, the bounded functions on a set X. One may, for example, let X be the closed unit ball of the dual space, and use the canonical injection of V into its second dual. Similarly, it was shown in [14] that any L^∞-matricially normed space $\mathscr{V}$ is completely isometric to an *operator space*, i.e., a linear subspace of $\mathscr{B}(H)$, the bounded operators on a Hilbert space H. In this paper we turn our attention to matricially normed and dual matricially normed bimodules for operator algebras. Such bimodules naturally arise when one considers mapping spaces for operator algebras. We show that these spaces may often be realized as *operator bimodules*, i.e., in many cases we can replace abstract bimodule multiplications by operator products. These representations enable one to extend various concrete bimodule results, including a surprising theorem of May ([9], §4.13), to mapping space bimodules (see (c) below). The latter was needed to complete an argument in [5].

Given C^*-algebras $\mathscr{A}_1, \mathscr{A}_2 \subseteq \mathscr{B}(H)$, with $I \in \mathscr{A}_1, \mathscr{A}_2$ we say that a linear space $\mathscr{V} \subseteq \mathscr{B}(H)$ is a (concrete) $\mathscr{A}_1, \mathscr{A}_2$ *operator bimodule* if $\mathscr{A}_1\mathscr{V} \subseteq \mathscr{V}$ and $\mathscr{V}\mathscr{A}_2 \subseteq \mathscr{V}$. If $\mathscr{A}_1 = \mathscr{A}_2 = \mathscr{A}$, we say that $\mathscr{V}$ is a (concrete) $\mathscr{A}$ *operator bimodule*. The $\mathscr{A}_1, \mathscr{A}_2$ operator bimodules were abstractly characterized in [2]. That approach is inadequate for the case $\mathscr{A}_1 = \mathscr{A}_2 = \mathscr{A}$ since it will generally provide different representations of $\mathscr{A}$ on H for left and right multiplication. The desired characterization is obtained in Section 2 by using the Christensen-Sinclair theory of completely positive trilinear maps [3].

A σ-weakly closed operator space $\mathscr{V} \subseteq \mathscr{B}(H)$ is a dual Banach space, and we say that $\mathscr{V}$ together with its σ-weak (or weak*) topology is a (concrete) *dual operator space*. These spaces are characterized in Theorem 3.3. If we are also given von Neumann algebras $\mathscr{R}_1, \mathscr{R}_2 \subseteq \mathscr{B}(H)$ for which $\mathscr{R}_1\mathscr{V} \subseteq \mathscr{V}$ and $\mathscr{V}\mathscr{R}_2 \subseteq \mathscr{V}$, we say that $\mathscr{V}$ is a (concrete) *dual* $\mathscr{R}_1, \mathscr{R}_2$ *operator bimodule*, and as above, if $\mathscr{R}_1 = \mathscr{R}_2$

we say that $\mathscr{V}$ is a (concrete) *dual $\mathscr{R}$ operator bimodule.* The normal dual $\mathscr{R}_1, \mathscr{R}_2$ operator bimodules are characterized in Theorem 3.4, and the dual $\mathscr{R}$ operator bimodules are determined in Theorem 4.2. From these representations we conclude that for any (abstract) normal dual $\mathscr{R}_1, \mathscr{R}_2$ operator bimodule $\mathscr{V}$:

a) given bounded nets $r_\nu \in \mathscr{R}_1, s_\nu \in \mathscr{R}_2$ and $v_\nu \in \mathscr{V}$ with $r_\nu^* \to r^*, s_\nu \to s$ strongly and $v_\nu \to v$ in the weak* topology, then $r_\nu v_\nu s_\nu \to rvs$ in the weak* topology.

b) $\mathbf{M}_\infty(\mathscr{V})$ is a normal dual $\mathbf{M}_\infty(\mathscr{R}_1)$, $\mathbf{M}_\infty(\mathscr{R}_2)$-bimodule,

and given two abstract normal dual $\mathscr{R}$ operator bimodules $\mathscr{V}_1$ and $\mathscr{V}_2$,

c) if $\Phi: \mathscr{V}_1 \to \mathscr{V}_2$ is a (*not necessarily weak*-continuous*) completely bounded $\mathscr{R}$-bimodule map, then

$$\Phi_\infty: \mathbf{M}_\infty(\mathscr{V}_1) \to \mathbf{M}_\infty(\mathscr{V}_2)$$

is an $\mathbf{M}_\infty(\mathscr{R})$-bimodule map.

It is interesting to note that the representations for matricially normed modules considered in this paper do not have analogues in the "classical" theory of normed modules.

We consider only unital C^*-algebras, and unital bimodules $\mathscr{V}$, i.e., we assume that $1 \cdot v = v \cdot 1 = v$ for $v \in \mathscr{V}$. To avoid confusion from the many uses of the star symbol, we reserve the symbol "*" for dual spaces, and the symbol "*" for the adjoints of operators, and "conjugate" complex vector spaces. This will naturally lead to such hybrids as the weak* topology on a dual Banach space, and the strong* topology on operators.

2. OPERATOR BIMODULES

Reviewing the definitions of [13] and [14], a *matricially normed space* is a complex vector space together with a norm on each matrix space $\mathbf{M}_n(\mathscr{V})$ satisfying

$$\|\alpha v \beta\| \leqslant \|\alpha\| \|v\| \|\beta\|$$

$$\|v \oplus 0_m\| = \|v\| \quad (\alpha, \beta \in \mathbf{M}_n(\mathbf{C}), v \in \mathbf{M}_n(\mathscr{V})),$$

and an L^∞-*matricially* (resp., L^1-*matricially*) normed space satisfies the additional condition $\|v \oplus w\| = \max\{\|v\|, \|w\|\}$ (resp., $\|v \oplus w\| = \|v\| + \|w\|$) for $v \in \mathbf{M}_n(\mathscr{V})$, $w \in \mathbf{M}_m(\mathscr{V})$. One defines the notions of complete boundedness, and complete isometries between such spaces in the usual manner. It is clear that any operator space $\mathscr{V} \subseteq \mathscr{B}(H)$ with the matricial norms determined by the inclusions $\mathbf{M}_n(\mathscr{V}) \subseteq \mathscr{B}(H^n)$ is an L^∞-matricially normed space, and conversely it was shown in [14] that any L^∞-matricially normed space is completely isometric to an operator space. We therefore will refer to these spaces as (abstract) *operator spaces.*

An earlier notion that was introduced in [1] was that of an *operator system*, i.e., a space of operators $\mathcal{L} \subseteq \mathcal{B}(H)$ for which $\mathcal{L} = \mathcal{L}^*$, and $I \in \mathcal{L}$. These have a matricial ordering determined by the cones

$$\mathbf{M}_n(\mathcal{L})^+ = \mathbf{M}_n(\mathcal{L}) \cap \mathcal{B}(H^n)^+.$$

Such spaces were abstractly characterized in [1]. In all of our representation theorems for operator spaces, we first embed the operator space in an operator system. This procedure was first used for concrete operator spaces in [10].

Given a matrix of operators

$$A = \begin{bmatrix} a & v \\ v^* & b \end{bmatrix},$$

a simple calculation shows that $A \geqslant 0$ if and only if $a, b \geqslant 0$ and for all $\varepsilon > 0$, $\|(a + \varepsilon)^{-1/2}v(b + \varepsilon)^{-1/2}\| \leqslant 1$. In particular, we have that $\|v\| \leqslant 1$ if and only if $T(v) \geqslant 0$, where

$$T(v) = \begin{bmatrix} I & v \\ v^* & I \end{bmatrix}. \tag{2.1}$$

This link between the order and norm plays a key rôle in what is to follow. In particular, it determines the canonical L^∞-matricial normed structure on an abstract operator system.

Let us suppose that $\mathcal{A}_1, \mathcal{A}_2$ (resp., $\mathcal{A}$) are unital C^*-algebras and that $\mathcal{V}$ is an operator space which is an $\mathcal{A}_1, \mathcal{A}_2$ (resp., $\mathcal{A}$) bimodule. We say that $\mathcal{V}$ is an $\mathcal{A}_1, \mathcal{A}_2$ ***operator bimodule*** (resp. $\mathcal{A}$ ***operator bimodule***) if the bimodule map

$$\Phi : \mathcal{A}_1 \times \mathcal{V} \times \mathcal{A}_2 \to \mathcal{V} : (a, v, b) \mapsto a \circ v \circ b \tag{2.2}$$

(resp., with $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{A}$) is completely contractive. It follows from (2.2) that if $v \in \mathbf{M}_n(\mathcal{V})$ and $a \in \mathbf{M}_n(\mathcal{A})^+$ satisfy $\|(a + \varepsilon)^{-1/2}v(a + \varepsilon)^{-1/2}\| \leqslant 1$ for all $\varepsilon > 0$, then for any $b \in \mathbf{M}_n(\mathcal{A})$ and $\varepsilon > 0$,

$$\|(b^*ab + \varepsilon)^{-1/2}b^* \circ v \circ b(b^*ab + \varepsilon)^{-1/2}\| \leqslant 1. \tag{2.3}$$

To see this let us fix $0 \neq b \in \mathbf{M}_n(\mathcal{A})$ (the case $b = 0$ is trivial) and $\varepsilon > 0$, and let $\varepsilon_1 = \varepsilon/\|b^*b\|$. By assumption $\|(a + \varepsilon_1)^{-1/2} \circ v \circ (a + \varepsilon_1)^{-1/2}\| \leqslant 1$, and thus

$$\|(b^*ab + \varepsilon)^{-1/2}b^* \circ v \circ b(b^*ab + \varepsilon)^{-1/2}\| =$$

$$= \|(b^*ab + \varepsilon)^{-1/2}b^*(a + \varepsilon_1)^{1/2} \circ (a + \varepsilon_1)^{-1/2} \circ v \circ (a + \varepsilon_1)^{-1/2} \circ (a + \varepsilon_1)^{1/2}b(b^*ab +$$

$$+ \varepsilon)^{-1/2}\| \leqslant \|(b^*ab + \varepsilon)^{-1/2}b^*(a + \varepsilon_1)^{1/2}\| \, \|(a + \varepsilon_1)^{1/2}b(b^*ab + \varepsilon)^{-1/2}\| =$$

$$= \|(b^*ab + \varepsilon)^{-1/2}b^*(a + \varepsilon/\|b^*b\|)b(b^*ab + \varepsilon)^{-1/2}\| \leqslant$$

$$\leqslant \|(b^*ab + \varepsilon)^{-1/2}(b^*ab + \varepsilon)(b^*ab + \varepsilon)^{-1/2}\| = \|1\| = 1.$$

Turning to some examples, let us suppose that W is a normed vector space, and that $\mathscr{L} = \mathscr{L}(W, \mathscr{A})$ is the space of bounded linear maps $\varphi: W \to \mathscr{A}$. Identifying $\mathbf{M}_n(\mathscr{L})$ with $\mathscr{L}(W, \mathbf{M}_n(\mathscr{A}))$, it is easy to verify that $\mathscr{L}$ is an L^∞-matricially normed space. Similarly if $\mathscr{W}$ is matricially normed and $\mathscr{M} = \mathscr{M}(\mathscr{W}, \mathscr{A})$ consists of the completely bounded linear maps $\varphi: \mathscr{W} \to \mathscr{A}$, then indentifying $\mathbf{M}_n(\mathscr{M})$ with $\mathscr{M}(\mathscr{W}, \mathbf{M}_n(\mathscr{A}))$, $\mathscr{M}$ is again an L^∞-matricially normed space. It is evident that letting $\mathscr{L}$ and $\mathscr{M}$ have the "range bimodule operations" $(a \circ \varphi \circ b)(w) = a\varphi(w)b$, we obtain $\mathscr{A}$ operator bimodules.

THEOREM 2.1. *Suppose that $\mathscr{V}$ is an operator $\mathscr{A}$ bimodule. Then there exists a Hilbert space H, a complete isometry $\theta: \mathscr{V} \to \mathscr{B}(H)$, and a faithful unital representation $\pi: \mathscr{A} \to \mathscr{B}(H)$ for which*

$$\theta(a \circ v \circ b) = \pi(a)\theta(v)\pi(b).$$

Proof. We let $\mathscr{V}^* = \{v^*: v \in \mathscr{V}\}$ denote the "conjugate" abstract operator space of $\mathscr{V}$. This is most easily defined by assuming that $\mathscr{V}$ is represented on some Hilbert space H_0, and then letting v^* denote the usual adjoint of v. Given $v = [v_{ij}] \in \mathbf{M}_n(\mathscr{V})$, we define $v^* \in \mathbf{M}_n(\mathscr{V}^*)$ by $(v^*)_{ij} = [v_{ji}^*]$. The matricial norms on $\mathscr{V}^*$ are then given by $\|v^*\| = \|v\|$. We let $\mathscr{V}^*$ have the $\mathscr{A}$ operator bimodule structure determined by

$$a \circ w^* \circ b = (b^* \circ w \circ a^*)^*.$$

The bimodule multiplication map

$$\Phi: \mathscr{A} \times \mathscr{V}^* \times \mathscr{A} \to \mathscr{V}^*: (a, w^*, b) \to a \circ w^* \circ b$$

is again completely contractive since given $a, b \in \mathbf{M}_n(\mathscr{A})$ and $w^* = [w_{ji}^*] \in \mathbf{M}_n(\mathscr{V}^*)$, we have that

$$\|\Phi_n(a, w^*, b)\| = \|[\sum_{p,q} \Phi(a_{ip}, w_{qp}^*, b_{qj})]\| =$$

$$= \|[\sum_{p,q} a_{ip} \circ w_{qp}^* \circ b_{qj}]\| = \|[(\sum_{p,q} b_{qj}^* \circ w_{qp} \circ a_{ip}^*)^*]\| =$$

$$= \|[\sum_{p,q} b_{qi}^* \circ w_{qp} \circ a_{jp}^*]\| = \|b^* \circ w \circ a^*\| \leqslant$$

$$\leqslant \|b\| \|w\| \|a\| = \|b\| \|w^*\| \|a\|.$$

We let $\mathscr{L}_{\mathscr{A}} = \mathscr{A} \oplus \mathscr{V} \oplus \mathscr{V}^*$ be the linear space of expressions of the form

$$a \oplus v \oplus w^* = \begin{bmatrix} a & v \\ w^* & a \end{bmatrix}$$

with $a \in \mathscr{A}$, $v \in \mathscr{V}$, and $w^* \in \mathscr{V}^*$. We let

$$\mathbf{M}_n(\mathscr{L}_{\mathscr{A}}) = \mathbf{M}_n(\mathscr{A}) \oplus \mathbf{M}_n(\mathscr{V}) \oplus \mathbf{M}_n(\mathscr{V}^*).$$

Following [14], we see that $\mathscr{L}_{\mathscr{A}}$ is an operator system with unit $1 = 1 \oplus 0 \oplus 0$, $*$-operation

$$(a \oplus v \oplus w^*)^* = (a^* \oplus w \oplus v^*),$$

and the matricial ordering on $\mathbf{M}_n(\mathscr{L}_{\mathscr{A}})$ determined by $a \oplus v \oplus v^* \geqslant 0$ if and only if $a \geqslant 0$, and for all $\varepsilon > 0$, $\|(a+\varepsilon)^{-1/2} \circ v \circ (a+\varepsilon)^{-1/2}\| \leqslant 1$ ($a \in \mathbf{M}_n(\mathscr{A})$, $v \in \mathbf{M}_n(\mathscr{V})$). We may fix a Hilbert space H_1 and a unital representation $\mathscr{L}_{\mathscr{A}} \subseteq \mathscr{B}(H_1)$. The map $\mathscr{A} \to \mathscr{L}_{\mathscr{A}}$ (resp., $\mathscr{V} \to \mathscr{L}_{\mathscr{A}}$) determined by $a \mapsto a \oplus 0 \oplus 0$ (resp., $v \mapsto 0 \oplus v \oplus 0$) is a complete order isomorphism (resp., a complete isometry). It is important to note that the resulting embedding $\mathscr{A} \to \mathscr{B}(H_1)$ is not a $*$-homomorphism. This is remedied below.

We define a unital $\mathscr{A}$ operator bimodule structure on $\mathscr{L}_{\mathscr{A}}$ by

$$a \circ (a_0 \oplus v \oplus w^*) \circ b = aa_0b \oplus a \circ v \circ b \oplus a \circ (w^*) \circ b.$$

The corresponding map

$$\Phi : \mathscr{A} \times \mathscr{L}_{\mathscr{A}} \times \mathscr{A} \to \mathscr{L}_{\mathscr{A}} : (a, x, b) \mapsto a \circ x \circ b \tag{2.4}$$

is symmetric in the sense of [3] since

$$(a \circ x \circ b)^* = b^* \circ x^* \circ a^*.$$

To see that it is completely positive according to the definition of [3], let us suppose that we are given $x = a \oplus v \oplus v^* \in \mathbf{M}_n(\mathscr{L}_{\mathscr{A}})^+$ and $b \in \mathbf{M}_n(\mathscr{A})$. Then

$$b^* \circ x \circ b = b^*ab \oplus b^* \circ v \circ b \oplus (b^* \circ v \circ b)^* \geqslant 0$$

since from (2.3), $\|(a+\varepsilon)^{-1/2} \circ v \circ (a+\varepsilon)^{-1/2}\| \leqslant 1$ for all $\varepsilon > 0$ implies that

$$\|(b^*ab+\varepsilon)^{-1/2} b^* \circ v \circ b (b^*ab+\varepsilon)^{-1/2}\| \leqslant 1$$

for all $\varepsilon > 0$. The operation is completely contractive since given $x \in \mathbf{M}_n(\mathscr{L}_{\mathscr{A}})$ with $\|x\| \leqslant 1$, the positivity of Φ_{2n} implies that

$$0 \leqslant \begin{bmatrix} a & 0 \\ 0 & b^* \end{bmatrix} \circ \begin{bmatrix} 1 & x \\ x^* & 1 \end{bmatrix} \circ \begin{bmatrix} a & 0 \\ 0 & b^* \end{bmatrix}^* = \begin{bmatrix} aa^* & a \circ x \circ b \\ b^* \circ x^* \circ a^* & b^*b \end{bmatrix} \leqslant \begin{bmatrix} \|a\|^2 1 & a \circ x \circ b \\ b^* \circ x^* \circ a^* & \|b\|^2 1 \end{bmatrix}.$$

Since $\mathscr{L}_{\mathscr{A}}$ is an operator system, it follows that $\|a \circ x \circ b\| \leqslant \|a\|\,\|b\|$, and thus for general x, $\|a \circ x \circ b\| \leqslant \|a\|\,\|x\|\,\|b\|$.

We now apply the Christensen-Sinclair Representation Theorem for completely positive multilinear maps to Φ (see [3], Theorem 2.8 and Lemma 3.1). It should be noted that in that proof *it is nowhere necessary to assume that the middle space is a C^*-algebra rather than just an operator system.* The argument shows that there exists a Hilbert space K, a contraction $S: H_1 \to K$, a completely positive map $\varphi: \mathscr{L}_{\mathscr{A}} \to \mathscr{B}(K)$, and a representation π of $\mathscr{A}$ on K such that

$$a \circ x \circ b = S^*\pi(a)\varphi(x)\pi(b)S. \tag{2.5}$$

Consider the projection E' onto $[\pi(\mathscr{A})SH_1]$. This commutes with π, and since

$$S^*\pi(a_1)\varphi(a \circ x \circ b)\pi(b_1)S = S^*\pi(a_1)\pi(a)\varphi(x)\pi(b)\pi(b_1)S,$$

we have that

$$E'\pi(a)\varphi(x)\pi(b)E' = E'\varphi(a \circ x \circ b)E'.$$

Replacing K, S, π, and φ by $E'K$, $E'S$, $E'\pi E'$, and $E'\varphi E'$, respectively, we conclude that (2.5) is still valid, π is unital, and in addition,

$$\pi(a)\varphi(x)\pi(b) = \varphi(a \circ x \circ b). \tag{2.6}$$

Letting $T = \varphi(1)$, we have that $T \geqslant 0$ and

$$\pi(a)T = \varphi(a) = T\pi(a).$$

It follows that $T^{1/2}$ also commutes with π. From [1], Lemma 2.2, $\varphi = T^{1/2}\psi T^{1/2}$ where $\psi: \mathscr{L}_{\mathscr{A}} \to \mathscr{B}(K)$ is completely positive and unital. Thus letting $\tilde{S} = T^{1/2}S$, we have that

$$a \circ x \circ b = S^*\pi(a)T^{1/2}\psi(x)T^{1/2}\pi(b)S = \tilde{S}^*\pi(a)\psi(x)\pi(b)\tilde{S}.$$

Since

$$\tilde{S}^*\pi(a)\psi(x)\pi(b)\tilde{S} = \tilde{S}^*\psi(a \circ x \circ b)\tilde{S},$$

we have as above that

$$F'\pi(a)\psi(x)\pi(b)F' = F'\psi(a \circ x \circ b)F',$$

where F' is the projection onto $[\pi(\mathscr{A})\tilde{S}H]$, which again commutes with π. Replacing K by $H = F'K$, φ by $F'\psi F'$, π by $F'\pi F'$, and S by $\tilde{S}$, we may assume that we have

(2.5), (2.6), and $\varphi(1) = 1$. π is unital and faithful, since if $a \neq 0$, then

$$0 \neq a \oplus 0 \oplus 0 = a \circ 1 = S^* \pi(a) \varphi(1) S,$$

and thus $\pi(a) \neq 0$.

The map $\varphi: \mathscr{L}_{\mathscr{A}} \to \mathscr{B}(H)$ is a unital complete order isomorphism since $S^* \varphi S$ is the identity map $\mathscr{L}_{\mathscr{A}} \to \mathscr{L}_{\mathscr{A}}$, and thus it is completely isometric. Composing with the completely isometric bimodule injection $\mathscr{V} \to \mathscr{L}_{\mathscr{A}}$, we obtain the desired map θ.

3. NORMAL DUAL OPERATOR BIMODULES

Dual topologies must be handled with care. A Banach space may be isometric to the dual of more than one Banach space, and the corresponding weak* topologies may be badly behaved with respect to auxiliarly structures on the Banach space. Thus the Banach space ℓ^1 is the dual of $C(X)$ for any countable compact metric space X, and it is also the dual of a Banach space in which the positive cone of ℓ^1 is not weak* closed. We will use the following well-known facts regarding the dual V of a Banach space V_* (see [4], §V.5.5). Letting K be the norm closed unit ball of V,

W_1: A convex subset $D \subseteq V$ is weak* closed if and only if $D \cap \alpha K$ is weak* closed for each $\alpha > 0$. In particular, if D is a convex cone, one need only verify the case $\alpha = 1$.

W_2: A linear functional f on V is weak* continuous if and only if its restriction $f|_K$ is weak* continuous. In particular if a net of weak* continuous functionals f_ν converges uniformly on K to a functional f, then f must also be weak* continuous. More generally, a linear map θ of V into a locally convex space is weak* continuous if and only if it is weak* continuous on K. Since K is compact, any weak* continuous isometry of dual Banach spaces must be a weak* homeomorphism.

The following result is useful for constructing preduals.

LEMMA 3.1. *Suppose that $(X_*, \| \|)$ is a normed vector space, and that $(X, \| \|)$ is its dual Banach space. Given an equivalent norm $|||\ |||$ on X, one has an equivalent norm $|||\ |||$ on X_* with $(X_*, |||\ |||)^* = (X, |||\ |||)$ if and only if the set*

$$D = \{x \in X : |||x||| \leqslant 1\}$$

is weak closed.*

Proof. The necessity of the condition is obvious. Conversely, we may identify the dual Banach spaces $(X, \| \|)^*$ and $(X, |||\ |||)^*$ as the same vector space X^* with equivalent norms $\| \|$ and $|||\ |||$. The evaluation map $\varepsilon: X_* \to X^*$ is isometric in the $\| \|$ norms, and we will use it to identify X_* with a subspace of X^*. We thus have a relative $|||\ |||$ norm on X_*. Letting D^* be the norm closed unit ball in $(X^*, |||\ |||)$, $D_* = D^* \cap X_*$ is the corresponding relative unit ball. We claim that

D_* is weak* dense in D^*. To prove this, let us recall that the *circled polar* S_0 of a set $S \subseteq X^*$ (resp., S^0 of a set $S \subseteq X$) is the $x \in X$ for which $|x^*(x)| \leqslant 1$ for all $x^* \in S$ (resp., the $x^* \in X^*$ such that $|x^*(x)| \leqslant 1$ for all $x \in S$). It is obvious that $(D_*)_0 \supseteq D$. On the other hand, if $x \notin D$, then since D is weak* closed and convex, we may choose an element $x_* \in X_*$ with $|x_*(D)| \leqslant 1$ and $|x_*(x)| > 1$, i.e., we have that $x \notin (D_*)_0$. It follows that

$$((D_*)_0)^0 = D^0 = D^*,$$

and the density assertion follows from the Bipolar Theorem. As a result, the map $X \to (X_*)^*$ is $|||\ |||$ isometric. On the other hand, the norms $\|\ \|$ and $|||\ |||$ on X_* are equivalent since they are equivalent on X^*. It follows that if F is a bounded linear functional on $(X_*, |||\ |||)$ it is determined by an element of $X = (X_*, \|\ \|)^*$, and thus the map $X \to (X_*)^*$ is surjective.

Given a Hilbert space H, we recall that $\mathscr{B}(H)$ has a natural predual, which we will denote by $\mathscr{B}(H)_*$. The weak* topology is also called the σ-weak operator topology, and we shall use these terms interchangeably. This topology coincides on bounded sets with the weak operator topology. The predual was shown by Sakai be unique, but we shall not need this fact. Since any Banach space can be embedded in $\mathscr{B}(H)$ for some H, the uniqueness statement is definitely not true for subspaces of $\mathscr{B}(H)$. Given von Neumann algebras $\mathscr{R}_k \subseteq \mathscr{B}(H_k)$, $k = 1, 2$, any weak* continuous homomorphism $\varphi: \mathscr{R}_1 \to \mathscr{R}_2$ is strong* continuous on bounded sets (and in particular is σ-strongly continuous), and the same is true for states on $\mathscr{R}_1$. States and representations of a von Neumann algebra are also said to be *normal* if they are weak* continuous.

Operator multiplication is singly but not jointly continuous in the weak* topology. For joint continuity we will use the following simple result:

LEMMA 3.2. *Suppose that r_ν, v_ν, and s_ν are bounded nets in $\mathscr{B}(H)$, and that $r_\nu^* \to r^*$, $s_\nu \to s$ in the strong topology, and $v_\nu \to v$ in the weak* topology. Then $r_\nu v_\nu s_\nu \to rvs$ in the weak* topology.*

Proof. Assuming that r_ν, v_ν, and s_ν are contractions, we have for unit vectors η and ζ,

$$|(r_\nu v_\nu s_\nu - rvs)\eta \cdot \zeta| \leqslant |v_\nu s_\nu \eta \cdot (r_\nu - r)^* \zeta| + |(s_\nu - s)\eta \cdot v_\nu^* r^* \zeta| + |(v_\nu - v)s\eta \cdot r^* \zeta| \leqslant$$

$$\leqslant \|(r_\nu - r)^* \zeta\| + \|(s_\nu - s)\eta\| + |(v_\nu - v)s\eta \cdot r^* \zeta|.$$

If $\mathscr{V}$ is a weak* closed subspace of $\mathscr{B}(H)$, then letting $\mathscr{V}_\perp$ denote the pre-annihilator, the restriction map $\mathscr{B}(H)_* \to \mathscr{V}_*$ determines a natural isometry $\mathscr{V} \cong (\mathscr{V}_*)^*$, where $\mathscr{V}_* = \mathscr{B}(H)_*/\mathscr{V}_\perp$.

Given a matricially normed space $\mathcal{V}_*$, we say that a matricially normed space $\mathcal{V}$ is the *matricial normed dual* of $\mathcal{V}_*$, if $\mathcal{V}$ is the dual Banach space of $\mathcal{V}_*$, and the pairing

$$\text{(3.1)} \qquad \mathbf{M}_n(\mathcal{V}_*) \times \mathbf{M}_n(\mathcal{V}) \to \mathbf{C} : (f, v) \mapsto \sum \langle f_{ij}, v_{ji} \rangle,$$

determines $\mathbf{M}_n(\mathcal{V})$ as the Banach dual of $\mathbf{M}_n(\mathcal{V}_*)$. If $\mathcal{V}$ is an operator space, it follows that $\mathcal{V}_*$ is an L^1-matricially normed space, i.e., $\|f \oplus g\| = \|f\| + \|g\|$ (see [13]). We shall on occasion refer to $\mathcal{V}_*$ as the *predual* of $\mathcal{V}$ — by this we mean the *given* space of which $\mathcal{V}$ is the dual. Since we wish to use (W_1) and (W_2) above, *we shall only consider complete, i.e., Banach preduals* $\mathcal{V}_*$. (3.1) differs slightly from the usual pairing $(f, v) \mapsto \sum f_{ij}(v_{ij})$ employed in earlier papers (see e.g. [6]). Its advantages will become clear in the discussion of dual bimodules below.

We are now ready to prove a weak* version of the representation theorem for operator spaces [14].

THEOREM 3.3. *Suppose that $\mathcal{V}_*$ is an L^1-matricially normed Banach space, and that $\mathcal{V} = (\mathcal{V}_*)^*$ is the corresponding dual operator space. Then there is a completely isometric, weak* homeomorphism of $\mathcal{V}$ onto a weak* closed linear space of operators $\mathcal{V}_1 \subseteq \mathcal{B}(K)$ for some Hilbert space K.*

Proof. We define

$$\mathcal{L} = \mathbf{C} \oplus \mathcal{V} \oplus \mathcal{V}^*,$$

where $\mathcal{V}^* = \{v^* : v \in \mathcal{V}\}$ is the conjugate matricially normed space (see the proof of Theorem 2.1), and we have a corresponding identification

$$\mathbf{M}_n(\mathcal{L}) = \mathbf{M}_n(\mathbf{C}) \oplus \mathbf{M}_n(\mathcal{V}) \oplus \mathbf{M}_n(\mathcal{V}^*).$$

We let $\| \ \|_\infty$ be the norm of $\mathbf{M}_n(\mathcal{L})$ defined by

$$\|\alpha \oplus v \oplus w^*\|_\infty = \max\{\|\alpha\|, \|v\|, \|w\|\}.$$

Given f in the L^1-matricially normed Banach space $\mathcal{V}_*$, we define a linear functional f^* on $\mathcal{V}^*$ by $f^*(v^*) = f(v)^-$. It is evident that these functions form an L^1-matricially normed predual space $\mathcal{V}_*^*$, which we may identify with the conjugate space $(\mathcal{V}_*)^*$ of $\mathcal{V}_*$. Defining

$$\mathcal{L}_* = \mathbf{C} \oplus \mathcal{V}_* \oplus \mathcal{V}_*^*$$

and letting

$$\mathbf{M}_n(\mathcal{L}_*) = \mathbf{M}_n(\mathbf{C})_* \oplus \mathbf{M}_n(\mathcal{V}_*) \oplus \mathbf{M}_n(\mathcal{V}_*^*)$$

have the norm

$$\|\alpha \oplus f \oplus g\|_1 = \|\alpha\| + \|f\| + \|g\|,$$

we may in the usual manner regard $\mathbf{M}_n(\mathscr{L})$ as the Banach dual of $\mathbf{M}_n(\mathscr{L}_*)$. In fact we see that with the norms $\|\ \|_\infty$ and $\|\ \|_1$, the L^∞-matricially normed Banach space $\mathscr{L}$ is the dual of the L^1-matricially normed Banach space $\mathscr{L}_*$. However, we wish to change to the operator system norm on $\mathscr{L}$, and use Lemma 3.1 to make a corresponding modification in $\mathscr{L}_*$.

As in the proof of Theorem 2.1, $\mathscr{L}$ is an operator system, and we have a corresponding matricial norm structure determined by $\|x\| \leqslant 1$ if and only if $T(x) \geqslant 0$ (see 2.1)). We claim that the corresponding unit ball

$$D = \{x \in \mathbf{M}_n(\mathscr{L}) : \|x\| \leqslant 1\}$$

is weak* closed. To prove this, we use the fact that the operator orders and norms are intrinsically determined by the matricial normed structure of $\mathscr{V}$.

We have that $\alpha \oplus v \oplus w^* \geqslant 0$ if and only if $\alpha \geqslant 0$, $w = v^*$ and for all $\varepsilon > 0$, $\|(\alpha + \varepsilon)^{-1/2} v (\alpha + \varepsilon)^{-1/2}\| \leqslant 1$. It follows that $\mathbf{M}_n(\mathscr{L})^+$ is weak* closed. To see this let us suppose that $\alpha_\nu \oplus v_\nu \oplus v_\nu^* \to \alpha \oplus v \oplus v^*$ where $\alpha_\nu \oplus v_\nu \oplus v_\nu^* \geqslant 0$. Then $\alpha_\nu \to \alpha$ in matrix norm. Fixing $\varepsilon > 0$, $(\alpha_\nu + \varepsilon)^{-1/2} \to (\alpha + \varepsilon)^{-1/2}$ in norm, and thus

$$(\alpha_\nu + \varepsilon)^{-1/2} v_\nu (\alpha_\nu + \varepsilon)^{-1/2} \to (\alpha + \varepsilon)^{-1/2} v (\alpha + \varepsilon)^{-1/2}$$

in the weak* topology (after applying these expressions to matrices $f = [f_{ij}] \in \mathbf{M}_n(\mathscr{L}_*)$ by using (3.1), this is a consequence of the fact that the scalar multiplication map $\mathbf{C} \times \mathscr{V} \times \mathbf{C} \to \mathscr{V}$ is jointly weak* continuous). Since the terms on the left have norm $\leqslant 1$ and the norm is semicontinuous in the weak* topology, the term on the right also has norm $\leqslant 1$. On the other hand, we have that $\|x\| \leqslant 1$ if and only if $T(x) \geqslant 0$ (see (2.1)). It follows that the norm closed unit ball D of $\mathbf{M}_n(\mathscr{L})$ is also weak* closed since if $\|x_\nu\| \leqslant 1$ and $x_\nu \to x$ in the weak* topology, then $T(x_\nu) \to T(x)$ weakly*, and $T(x_\nu) \geqslant 0$ imply that $T(x) \geqslant 0$. From Lemma 3.1, we have that $\mathbf{M}_n(\mathscr{L}_*)$ has a predual norm $\|\ \|$ such that $(\mathbf{M}_n(\mathscr{L}), \|\ \|) = (\mathbf{M}_n(\mathscr{L}_*), \|\ \|)^*$.

Modifying the proof of [1], Theorem 4.4, we let Φ_m be all of the weak* continuous completely positive unital maps $\varphi : \mathscr{L} \to \mathbf{M}_m$. Letting $M_m^\varphi = M_m$, and

$$\mathscr{B}_m = \bigoplus_{\varphi \in \Phi_m} \mathbf{M}_m^\varphi$$

be the L^∞-direct sum, we define

$$J^m : \mathscr{L} \to \mathscr{B}_m$$

by letting $J^m(v) = (\varphi(v))_{\varphi \in \Phi_m}$. The argument in [1] shows that this is a completely positive map. It is obviously weak* continuous (providing the von Neumann algebra $\oplus \mathbf{M}_m^\varphi$ with the usual weak* $= \sigma$-weak topology). We claim that

$$(J^m)_m : \mathbf{M}_n(\mathscr{L}) \to \mathbf{M}_m(\mathscr{B}_m) = \bigoplus_{\varphi \in \Phi_m} \mathbf{M}_m(\mathbf{M}_m^\varphi)$$

is an order isomorphism onto its image, i.e., if $(J^m)_m(x) \geqslant 0$, then $x \geqslant 0$. Since $\mathbf{M}_m(\mathscr{L})^+$ is weak* closed, it suffices to show that if $f\colon \mathbf{M}_m(\mathscr{L}) \to \mathbf{C}$ is positive and weak* continuous, then $f(x) \geqslant 0$ for such x. Using the notation of [1], we have that

$$f(x) = \psi(x)\cdot\varepsilon$$

where $\psi = \Lambda f(x)\colon \mathscr{L} \to \mathbf{M}_m$ is weak* continuous and completely positive, and $\varepsilon = [\varepsilon_{ij}]$ is the matrix of matrix units in $\mathbf{M}_m$ (this is a positive matrix). We let $b = \psi(1)$. Replacing f by a positive scalar multiple, we may assume that $\|b\| = 1$. From [1], Lemma 2.2, we have that $\psi = b^{1/2}\varphi b^{1/2}$, where $\varphi \in \Phi_m$. Thus $\psi_m = b_m^{1/2}\varphi_m b_m^{1/2}$, where $b_m = \underbrace{b \oplus \ldots \oplus b}_{m \text{ times}}$. It should be noted that in the proof of that result, the matrices converge in norm, and thus the limit functional is weak* continuous. By assumption we have that $\varphi(x) \geqslant 0$, and thus $f(x) \geqslant 0$.

It follows that

$$J = \oplus\, J^m\colon \mathscr{L} \to \mathscr{B} = \bigoplus_{m \in \mathbf{N}} \mathscr{B}_m$$

is a complete unital order isomorphism. Since

$$J_n\colon \mathbf{M}_n(\mathscr{L}) \to \mathbf{M}_n(\mathscr{B})$$

is weak* continuous and one-to-one on the norm-closed unit ball D of $\mathbf{M}_n(\mathscr{L})$, and D is weak* compact, J_n is a homeomorphism on the unit ball of the image $J_n(\mathbf{M}_n(\mathscr{L}))$. It follows that the image subspace is weak* closed, and that J_n is a weak* homeomorphism onto the image.

REMARK. It follows from Theorem 3.3 that if $\mathscr{X}$ is a complete L^1-matricially normed space, then it is a quotient matricially normed space of $\mathscr{B}(H)_*$ for some Hilbert space H. To see this we let $\mathscr{V} = \mathscr{X}^*$. From Theorem 3.3 there is a completely isometric weak* continuous homeomorphism θ of $\mathscr{V}$ into $\mathscr{B}(H)$ for some Hilbert space H. It follows that $\theta = \eta^*$, where $\eta\colon \mathscr{B}(H)_* \to \mathscr{X}$ is a quotient map. This result was proved in [6], Corollary 2.3 under the additional hypothesis that $\mathscr{X}$ is a dual matricially normed space.

Having characterized the dual operator spaces, it is natural to ask whether or not the σ-strong and σ-strong* topologies are intrinsically determined on such a space. That this is not the case is seen as follows: Fixing a Hilbert space H with basis $\xi_1, \xi_2, \ldots$ we let E_n be the projection onto $[\xi_1, \ldots, \xi_n]$. Then it is readily verified that the map

$$\Phi\colon \mathscr{B}(H) \to \oplus\mathbf{M}_n\colon T \to \oplus\, E_n T E_n$$

is a completely isometric weak* continuous map onto a subspace $\mathscr{W}$ of $\oplus \mathbf{M}_n$. It follows from W_1 and W_2 (see above) that Φ is a weak* homeomorphic complete isometry of $\mathscr{B}(H)$ onto $\mathscr{W}$. Letting $v_n \in \mathscr{B}(H)$ be the rank one partial isometry carrying ξ_n onto ξ_1, we have that v_n converges σ-strongly to 0, but v_n^* does not converge σ-strongly to 0. It is easy to directly verify that $\Phi(v_n)$ converges to 0 in the σ-strong and σ-strong* topologies, or one can use the fact that Φ must be σ-strongly continuous (see below) and these topologies must agree on the finite von Neumann algebra $\oplus \mathbf{M}_n$. Thus $\Phi(v_n^*) = \Phi(v_n)^*$ converges σ-strongly to 0. It follows that although $\Phi^{-1}: \mathscr{W} \to \mathscr{B}(H)$ is a weak* homeomorphic complete isometry, it is not σ-strongly continuous, since v_n^* does not converge to 0 σ-strongly. Similarly it is not σ-strongly* continuous since v_n does not converge to 0 σ-strongly*.

On the other hand, if $\mathscr{R} \subseteq \mathscr{B}(H)$ is a von Neumann algebra, a σ-weakly continuous completely bounded map $\theta: \mathscr{R} \to \mathscr{B}(K)$ must be continuous in the σ-strong and σ-strong* topologies, since using the Wittstock-Stinespring decomposition, one has that $\theta(r) = V\pi(r)W$, where π is a normal representation (see, e.g., [3], §5.7). Since π is then continuous in these topologies, the same is true for θ. It follows that a weak* homeomorphic complete isometry between von Neumann algebras must be homeomoprhic in the σ-strong and σ-strong* topologies.

Finally we note that *the Arveson-Wittstock extension theorem cannot be adapted to weak* continuous completely bounded maps on dual operator spaces.* To see this, consider the mapping $\Phi: \mathscr{B}(H) \to \mathscr{W} \subseteq \oplus \mathbf{M}_n$ described previously. If we can extend $\Phi^{-1}: \mathscr{W} \to \mathscr{B}(H)$ to a completely bounded weak* continuous map $\Psi: \oplus \mathbf{M}_n \to \mathscr{B}(H)$, it will follow from above that Ψ and thus Φ^{-1} are σ-strongly and σ-strongly* continuous, a contradiction.

Let us suppose that $\mathscr{R}_1$ and $\mathscr{R}_2$ are von Neumann algebras, and that $\mathscr{V}_*$ is an L^1-matricially normed space which is an $\mathscr{R}_2, \mathscr{R}_1$ bimodule for which the map

$$\Phi: \mathscr{R}_2 \times \mathscr{V}_* \times \mathscr{R}_1 \to \mathscr{V}_*: (s, f, r) \mapsto s \circ f \circ r$$

is completely contractive. We define $\mathscr{R}_1, \mathscr{R}_2$ bimodule operations on $\mathscr{V} = (\mathscr{V}_*)^*$ by

$$\langle f, r \circ v \circ s\rangle = \langle s \circ f \circ r, v\rangle \quad (r \in \mathscr{R}_1, s \in \mathscr{R}_2).$$

Owing to our convention (3.1), we have for matrices $r \in M_n(R_1)$, $s \in \mathbf{M}_n(\mathscr{R}_2)$ and $f \in \mathbf{M}_n(\mathscr{V})_*$, $v \in \mathbf{M}_n(\mathscr{V})$ that

$$\begin{aligned}\langle f, r \circ v \circ s\rangle &= \sum_{i,j} \langle f_{ij}, (r \circ v \circ s)_{ji}\rangle = \sum_{i,j} \langle f_{ij}, \sum_{p,q} r_{jp} v_{pq} s_{qi}\rangle = \\ &= \sum_{i,j} \langle \sum_{p,q} s_{qi} f_{ij} r_{jp}, v_{pq}\rangle = \langle s \circ f \circ r, v\rangle.\end{aligned}$$

It follows that $\mathscr{V} = (\mathscr{V}_*)^*$ is an $\mathscr{R}_1$, $\mathscr{R}_2$ operator bimodule, since if we choose f with $\|f\| \leqslant 1$,

$$|\langle f, r \circ v \circ s\rangle| \leqslant \|s \circ f \circ r\| \, \|v\| \leqslant \|s\| \, \|r\| \, \|v\|.$$

If one did not use the modified pairing (3.1), the bimodule operations would instead have been completely contractive for the opposite algebras $\mathscr{R}_k^{op}$.

If the bimodule $\mathscr{V}$ arises in this manner, we say that it is a *dual* $\mathscr{R}_1$, $\mathscr{R}_2$ operator bimodule. The mappings $v \to r \circ v \circ s$ are automatically weak* continuous, and we say that $\mathscr{V}$ is *normal* if in addition the maps $r \mapsto r \circ v$ (resp., $s \mapsto v \circ s$) of $\mathscr{R}_1$ (resp., $\mathscr{R}_2$) into $\mathscr{V}$ are continuous in the weak* topologies for each $v \in \mathscr{V}$. Again if $\mathscr{R} = \mathscr{R}_1 = \mathscr{R}_2$, we say that $\mathscr{V}$ is a dual or normal dual $\mathscr{R}$ operator bimodule.

The following is the normal dual version of [2], Corollary 3.3.

THEOREM 3.4. *Suppose that $\mathscr{V}$ is a normal dual $\mathscr{R}_1$, $\mathscr{R}_2$ operator bimodule. Then there exists a Hilbert space K, a completely isometric and weak* homeomorphic map θ of $\mathscr{V}$ onto a weak* closed subspace of $\mathscr{B}(K)$, and faithful normal representations π_k of $\mathscr{R}_k$ on K $(k = 1, 2)$, such that*

$$\theta(r \circ v \circ s) = \pi_1(r)\theta(v)\pi_2(s).$$

Proof. From Theorem 3.3 we may assume that $\mathscr{V}$ is a σ-weakly closed space of operators on some Hilbert space H, and that the initial weak* topology coincides with the σ-weak topology. From [3] and [12] there exists a representation

$$r \circ v \circ s = R\pi_1(r)\theta(v)\pi_2(s)T,$$

where π_1 and π_2 are representations of $\mathscr{R}_1$ and $\mathscr{R}_2$, θ is a representation of the C^*-algebra $\mathscr{A}$ generated by $\mathscr{V}$ on a common Hilbert space K_0, and

$$H \xrightarrow{T} K_0 \xrightarrow{R} H$$

is a diagram of contractions. As in the proof of Theorem 2.1 we may assume that π_1 and π_2 are unital.

Since the bimodule is normal, we may replace π_1 by its "normal part" $\tilde{\pi}_1$. Recalling the standard argument for this, we have that the identity map $\mathscr{R}_1 \to \mathscr{R}_1$ has a normal homomorphic extension $\rho : \mathscr{R}_1^{**} \to \mathscr{R}_1$. It follows that there is a central projection $z \in \mathscr{R}_1^{**}$ for which the restriction of ρ, $\rho_1 : \mathscr{R}_1^{**}z \to \mathscr{R}_1$ is a normal isomorphism, and if $r \in \mathscr{R}_1$, $\rho_1(rz) = \rho(r) = r$. We may choose a net r_ν in $\mathscr{R}_1$ which converges σ-weakly to z in $\mathscr{R}_1^{**}$. It follows that $r_\nu z$ converges σ-weakly to z in $\mathscr{R}_1^{**}$, and since ρ_1 is a σ-weakly continuous, $r_\nu = \rho_1(r_\nu z)$ converges σ-weakly to $I = \rho_1(z)$ in $\mathscr{R}_1$.

Letting $\bar{\pi}_1 : \mathscr{R}_1^{**} \to \mathscr{B}(K_0)$ be the normal extension of π_1 to the second dual, we define $\tilde{\pi}_1 : \mathscr{R}_1 \to \mathscr{B}(K_0)$ by $\tilde{\pi}_1(r) = \bar{\pi}_1(rz)$. This is a normal representation of $\mathscr{R}_1$ since $r \mapsto rz$ $(r \in \mathscr{R}_1)$ is the inverse of the normal isomorphism ρ_1. In addition, we have that for $r \in \mathscr{R}_1$,

$$\tilde{\pi}_1(r) = \bar{\pi}_1(rz) = \pi_1(r)\bar{\pi}_1(z) = \lim \pi_1(r)\bar{\pi}_1(r_\nu) = \lim \pi_1(r\, r_\nu).$$

Since r_ν converges σ-weakly to I in $\mathscr{R}_1$ and $\mathscr{V}$ is a normal $\mathscr{R}_1$, $\mathscr{R}_2$ operator bimodule,

$$R\tilde{\pi}_1(r)\theta(v)T = \lim R\pi_1(rr_\nu)\theta(v)T = \lim rr_\nu \circ v = r \circ v.$$

Similarly we may replace π_2 by its normal part $\tilde{\pi}_2$. Deleting the tildes, we may initially assume that π_1 and π_2 are normal. Letting π_k' ($k = 1, 2$) be faithful normal representations of $\mathscr{R}_k$ on a single Hilbert space H', and replacing K_0 by $K = K_0 \oplus H'$, π_k by $\pi_k \oplus \pi_k'$, θ by $\theta \oplus 0_{H'}$, and T by $T \oplus 0_{H'}$, we may assume that the π_k are also faithful.

On the other hand, letting $E' = [\pi_1(\mathscr{R}_1)R^*H]$, $F' = [\pi_2(\mathscr{R}_2)TH]$, and $\tilde{\theta} = E'\theta F'$, we have that E' and F' commute with π_1 and π_2, respectively, and

$$r \circ v \circ s = R\pi_1(r)\tilde{\theta}(v)\pi_2(s)T.$$

We claim that $\tilde{\theta}$ is weak* continuous. To see this let us suppose that we have a net $v_\nu \in \mathscr{V}$ with $\|v_\nu\| \leq 1$ and $v_\nu \to v$ weak*. Then given $\eta, \xi \in H$ and $r \in \mathscr{R}_1$, $s \in \mathscr{R}_2$ we have that

$$\theta(v_\nu)\pi_2(s)T\eta \cdot \pi_1(r)R^*\xi = r^* \circ v_\nu \circ s\eta \cdot \xi \to$$

$$\to r^* \circ v \circ s\eta \cdot \xi = \theta(v)\pi_2(s)T\eta \cdot \pi(r)R^*\xi,$$

and it follows that $E'\theta(v_\nu)F'$ converges weakly* to $E'\theta(v)F'$, i.e., $\tilde{\theta}$ is weak* continuous on the unit ball of $\mathscr{V}$. From W_2 (see above), $\tilde{\theta}: \mathscr{V} \to \mathscr{B}(K)$ is weak* continuous.

Again deleting the tilde, we have that

$$r \circ v \circ s = R\pi_1(r)\theta(v)\pi_2(s)T, \tag{3.2}$$

where π_1, θ and π_2 are weak* continuous. Since $\mathscr{V}$ is unital, and $v = R\theta(v)S$, it is evident that θ is the desired weak* homeomorphism.

COROLLARY 3.5. *Suppose that $\mathscr{V}$ is a normal dual $\mathscr{R}_1$, $\mathscr{R}_2$ operator bimodule. Then the bimodule operations are continuous in the sense that given bounded convergent nets $r_\nu^* \to r^*$, $s_\nu \to s$ in the strong topology, and $v_\nu \to v$ in the weak* topology, then $r_\nu \circ v_\nu \circ s_\nu \to r \circ v \circ s$ in the weak* topology.*

Proof. From Theorem 3.4, we may identify $\mathscr{V}$ with a weak* closed linear space of operators on a Hilbert space H, and the bimodule operations with left and right multiplication by $\pi_1(r)$ and $\pi_2(s)$, respectively, where π_1 and π_2 are weak* continuous representations of $\mathscr{R}_1$ and $\mathscr{R}_2$, respectively. But a weak* continuous representation of a von Neumann algebra must be σ-strong* continuous and our assertion follows from Lemma 3.2.

Given an abstract operator space $\mathscr{V}$, we let $\mathbf{M}_\infty(\mathscr{V})$ denote the linear space of matrices $v = [v_{ij}]_{i,j \in \mathbf{N}}$ which are *bounded* in the sense that there is a constant K

such that for all $n \in \mathbf{N}$, $\|[v_{ij}]_{i,j\leqslant n}\| \leqslant K$. We obtain a norm on $\mathbf{M}_\infty(\mathscr{V})$ by letting

$$\|v\| = \sup\{\|[v_{ij}]_{i,j\geqslant n}\| : n \in \mathbf{N}\}.$$

Identifying $\mathbf{M}_n(\mathbf{M}_\infty(\mathscr{V}))$ with $\mathbf{M}_\infty(\mathbf{M}_n(\mathscr{V}))$, it is a simple matter to verify that $\mathbf{M}_\infty(\mathscr{V})$ is again an operator space. Letting $\mathbf{M}_\infty^{\mathrm{f}}$ be the matrices with only finitely many non-zero scalar entries, $\mathbf{M}_\infty(\mathscr{V})$ is an $\mathbf{M}_\infty^{\mathrm{f}}$-bimodule under matrix multiplication. In particular, letting $E_n = I_n \oplus 0$, we identify $[v_{ij}]_{i,j\leqslant n}$ with $E_n v E_n$. Any completely bounded map of operator spaces $\varphi : \mathscr{V} \to \mathscr{W}$ determines a completely bounded map $\varphi_\infty : \mathbf{M}_\infty(\mathscr{V}) \to \mathbf{M}_\infty(\mathscr{W})$ via $\varphi_\infty([v_{ij}]) = [\varphi(v_{ij})]$. It is evident that φ_∞ is an $\mathbf{M}_\infty^{\mathrm{f}}$-bimodule map.

If $\mathscr{V}_*$ is an L^1-matricially normed space, then we have completely isometric inclusions $\mathbf{M}_n(\mathscr{V}_*) \hookrightarrow \mathbf{M}_{n+1}(\mathscr{V}_*)$. We let $\mathbf{M}_\infty(\mathscr{V}_*)$ be the completion of $\bigcup \mathbf{M}_n(\mathscr{V}_*)$. It is easy to check that it is a predual of $\mathbf{M}_\infty(\mathscr{V})$.

If $\mathscr{V}$ is a subspace of $\mathscr{B}(H)$, we may identify $\mathbf{M}_\infty(\mathscr{V})$ with the bounded operators $v = [v_{ij}]$ on H^∞ with $v_{ij} \in \mathscr{V}$. In this representation, the operators $[v_{ij}]_{i,j\leqslant n} = E_n v E_n$ converge strongly and thus weakly* to v. If $\mathscr{V}$ is weak* closed in $\mathscr{B}(H)$, then $\mathbf{M}_\infty(\mathscr{V})$ is weak* closed in $\mathscr{B}(H^\infty)$. To see this let us suppose that $v^\nu \in \mathbf{M}_\infty(\mathscr{V})$ converges weakly* to v. Then letting $F_k = E_k - E_{k-1}$, it follows that $v_{ij}^\nu = F_i v^\nu F_j$ converges weakly* to v_{ij}, and thus $v \in \mathbf{M}_\infty(\mathscr{V})$. If $\mathscr{V} = \mathscr{R} \subseteq \mathscr{B}(H)$ is a von Neumann algebra, then $\mathbf{M}_\infty(\mathscr{R}) \subseteq \mathscr{B}(H^\infty)$ is again a von Neumann algebra. Letting H_0 be a countably infinite dimensional Hilbert space, $\mathbf{M}_\infty(\mathscr{R})$ is isomorphic to the usual von Neumann tensor product $\mathscr{R} \overline{\otimes} \mathscr{B}(H_0)$. For $r \in \mathbf{M}_\infty(\mathscr{R})$ we denote $r_n = E_n r E_n \in \mathbf{M}_n(\mathscr{R})$.

COROLLARY 3.6. *Suppose that $\mathscr{V}$ is a normal dual $\mathscr{R}_1$, $\mathscr{R}_2$ operator bimodule. Then $\mathbf{M}_\infty(\mathscr{V})$ is a normal dual $\mathbf{M}_\infty(\mathscr{R}_1)$, $\mathbf{M}_\infty(\mathscr{R}_2)$ operator bimodule under the operations*

$$r \circ v \circ s = \lim r_n \circ v_n \circ s_n \ \ (\textit{weak* topology}).$$

Proof. We choose faithful normal representations π_k and a weak* homeomorphic complete isometry $\theta : \mathscr{V} \to \mathscr{V}_1 \subseteq \mathscr{B}(K)$ as in Theorem 3.4. Given $v \in \mathbf{M}_\infty(\mathscr{V})$, we have that the operator $\theta_\infty(v) = [\theta(v_{ij})]$ on K^∞ is just the weak* limit of the operators $\theta_n(v_n)$. Since θ is weak* continuous, the same applies to θ_∞. To see this we note that given $\eta, \xi \in K^\infty$, the function $f(v) = \theta_\infty(v)\eta \cdot \xi$ is the uniform limit on the closed unit ball D of $\mathbf{M}_\infty(\mathscr{V})$ of the functions

$$f_n(v) = \theta_n(v_n)\eta_n \cdot \xi_n,$$

where $\eta_n = E_n \eta$, and $\xi_n = E_n \xi$. Since the functions f_n are weak* continuous on D, the same is true for f. It follows that θ_∞ is a weak* homeomorphism on D and thus a weak* homeomorphism of $\mathbf{M}_\infty(\mathscr{V})$ onto $\mathbf{M}_\infty(\mathscr{V}_1)$. We note that the same argument shows that $(\pi_k)_\infty$ are weak* and thus strong* continuous representations of $\mathbf{M}_\infty(\mathscr{R}_k)$ $(k = 1, 2)$. We let $\mathscr{R}_{k\infty} = \pi_{k\infty}(\mathbf{M}_\infty(\mathscr{R}_k))$.

Using Lemma 3.2, it is a simple exercise to verify that $\mathbf{M}_\infty(\mathscr{V}_1)$ is an $\mathscr{R}_{1\infty}$, $\mathscr{R}_{2\infty}$ operator bimodule under operator multiplication in $\mathscr{B}(K^\infty)$. We use θ_∞ to define the bimodule operations on $\mathbf{M}_\infty(\mathscr{V})$, i.e., we let

$$r \circ v \circ s = R_\infty \pi_{1\infty}(r)\theta_\infty(v)\pi_{2\infty}(s)S_\infty,$$

where we let $R_n = R \oplus R \oplus \ldots$ (n copies, $1 \leqslant n \leqslant \infty$). It follows from (3.2) that for any $n \in \mathbf{N}$, $r \in \mathbf{M}_n(\mathscr{R}_1)$, $s \in \mathbf{M}_n(\mathscr{R}_2)$, and $v \in \mathbf{M}_n(\mathscr{V})$,

$$R_n(\pi_1)_n(r)\theta_n(v)(\pi_2)_n(s)T_n = r \circ v \circ s. \tag{3.3}$$

Thus given $r \in \mathbf{M}_\infty(\mathscr{R}_1)$, $s \in \mathbf{M}_\infty(\mathscr{R}_2)$, and $v \in \mathbf{M}_\infty(\mathscr{V})$,

$$r \circ v \circ s = \lim R_n(\pi_1)_n(r_n)\theta_n(v)(\pi_2)_n(s_n)T_n = \lim r_n \circ v_n \circ s_n.$$

Since it is evident that $\mathbf{M}_\infty(\mathscr{V}_1)$ is a normal dual $\mathscr{R}_{1\infty}$, $\mathscr{R}_{2\infty}$ operator bimodule, the same is true for $\mathbf{M}_\infty(\mathscr{V})$.

4. NORMAL OPERATOR DUAL BIMODULES

Our next task is to adapt the construction in Theorem 2.1 to normal dual operator bimodules. Throughout this section $\mathscr{R}$ will denote a fixed von Neumann algebra.

THEOREM 4.1. *Suppose that $\mathscr{V}$ is normal dual $\mathscr{R}$ operator bimodule. Then there exist a Hilbert space K, a weak* homeomorphic, completely isometric mapping $\theta: \mathscr{V} \to \mathscr{B}(K)$, and a faithful unital weak* continuous representation $\pi: \mathscr{R} \to \mathscr{B}(K)$ for which*

$$\theta(r \circ v \circ s) = \pi(r)\theta(v)\pi(s).$$

Proof. As in the proof of Theorem 2.1, we let $\mathscr{L}_\mathscr{R}$ be all expressions

$$r \oplus v \oplus w^* = \begin{bmatrix} r & v \\ w^* & r \end{bmatrix}$$

with $r \in \mathscr{R}$, $v \in \mathscr{V}$, and $w^* \in \mathscr{V}^*$, and we use the same definitions for the matricial ordering, the $*$-operation, and the $\mathscr{R}$ bimodule operations. Again we have that $\mathscr{L}_\mathscr{R}$ is an operator system with unit $I \oplus 0 \oplus 0$ (which we shall denote by $I \in \mathscr{L}_\mathscr{R}$).

On the other hand we may use the technique of Theorem 3.3, to regard $\mathscr{L}_\mathscr{R}$ as a dual operator system. We let

$$\mathbf{M}_n(\mathscr{L}_\mathscr{R}) = \mathbf{M}_n(\mathscr{R}) \oplus \mathbf{M}_n(\mathscr{V}) \oplus \mathbf{M}_n(\mathscr{V}^*)$$

have the auxiliary norm

$$\|r \oplus v \oplus w^*\|_\infty = \max\{\|r\|, \|v\|, \|w\|\}.$$

Defining

$$\mathscr{L}_{\mathscr{R}_*} = \mathscr{R}_* \oplus \mathscr{V}_* \oplus \mathscr{V}_*^*$$

and letting

$$\mathbf{M}_n(\mathscr{L}_{\mathscr{R}_*}) = \mathbf{M}_n(\mathscr{R}_*) \oplus \mathbf{M}_n(\mathscr{V}) \oplus \mathbf{M}_n(\mathscr{V}_*^*)$$

have the norm

$$\|\omega \oplus f \oplus g\|_1 = \|\omega\| + \|f\| + \|g\|,$$

we have that $\mathbf{M}_n(\mathscr{L}_{\mathscr{R}})$ is the Banach dual of $\mathbf{M}_n(\mathscr{L}_{\mathscr{R}_*})$. Since $\mathscr{V}$ is a normal dual $\mathscr{R}$ operator module, it is clear that the same is true for $\mathscr{L}_{\mathscr{R}}$. The map $\mathscr{V} \to \mathscr{L}_{\mathscr{R}}$ is a weak* homeomorphic completely isometric injection.

We claim that the cone $M_n(\mathscr{L}_{\mathscr{R}})^+$ is weak* closed. Given a net $r_\nu \oplus v_\nu \oplus v_\nu^*$ in $\mathbf{M}_n(\mathscr{L}_{\mathscr{R}})^+$ converging to $x = r \oplus v \oplus v^*$ with

$$\|r_\nu \oplus v_\nu \oplus v_\nu^*\|_\infty \leqslant 1,$$

we have that in particular, r_ν converges weakly* to r. A net of convex combinations of the form

$$\sum \alpha_k r_{\nu_k} \quad (\alpha_k \geqslant 0,\ \sum \alpha_k = 1)$$

must converge strongly to r, and replacing the $r_\nu \oplus v_\nu \oplus v_\nu^*$ by corresponding convex combinations

$$\sum \alpha_k (r_{\nu_k} \oplus v_{\nu_k} \oplus v_{\nu_k}^*) = (\sum \alpha_k r_{\nu_k}) \oplus (\sum \alpha_k v_{\nu_k}) \oplus (\sum \alpha_k v_{\nu_k}^*),$$

we may initially assume that r_ν converges strongly to r. Since $r_\nu \geqslant 0$ and $\|r_\nu\| \leqslant 1$, it follows that for any $\varepsilon > 0$, $(r_\nu + \varepsilon)^{-1/2}$ converges strongly* to $(r + \varepsilon)^{-1/2}$ (see [8], Lemma 2). Since v_ν converges weakly* to v, we have from Corollary 3.5 that $(r_\nu + \varepsilon)^{-1/2} \circ v_\nu \circ (r_\nu + \varepsilon)^{-1/2}$ converges weakly* to $(r + \varepsilon)^{-1/2} \circ v \circ (r + \varepsilon)^{-1/2}$. Since the terms in the net have norm $\leqslant 1$, the same applies to the limit, and thus $r \oplus v \oplus v^* \geqslant 0$.

As in the proof of Theorem 3.3, it follows that the closed unit ball D in the operator space norm is weak* closed, and from Lemma 3.1, $\mathscr{L}_{\mathscr{R}}$ is a dual operator system. Again we see that there is a weak* homeomorphic complete isometry of $\mathscr{L}_{\mathscr{R}}$ into $\mathscr{B}(H)$ for some Hilbert space H, and we shall identify $\mathscr{L}_{\mathscr{R}}$ with its image.

Turning to the proof of Theorem 2.1, we again have that there exists a faithful unital representation π of $\mathscr{R}$ and a unital complete order isomorphism $\varphi: \mathscr{L}_{\mathscr{R}} \to \mathscr{B}(K)$ such that

$$r \circ x \circ s = S^*\pi(r)\varphi(x)\pi(s)S$$

and

$$\pi(r)\varphi(x)\pi(s) = \varphi(rxs).$$

The argument used in the proof of Theorem 3.4 shows that we may also assume that π is weak* continuous, and that φ is weak* continuous on $\mathscr{V}$. Letting θ be the composition of φ with the map $\mathscr{V} \to \mathscr{L}_{\mathscr{R}}$, we obtain the desired embedding.

The following generalizes a result of May (see [9], § 4.13).

THEOREM 4.2. *Suppose that $\mathscr{V}_1$ and $\mathscr{V}_2$ are operator spaces which are normal dual $\mathscr{R}$ operator bimodules. If $\Phi: \mathscr{V}_1 \to \mathscr{V}_2$ is a completely bounded (not necessarily weak* continuous) $\mathscr{R}$ bimodule map, then*

$$\Phi_\infty: \mathbf{M}_\infty(\mathscr{V}_1) \to \mathbf{M}_\infty(\mathscr{V}_2)$$

is an $\mathbf{M}_\infty(\mathscr{R})$ bimodule map.

Proof. From the proof of Theorem 4.1, we may assume that $\mathscr{L}_{\mathscr{R}}(\mathscr{V}_k)$ is a weak* closed operator system on a Hilbert space H_k ($k = 1, 2$), that we have unital weak* continuous representations $\pi_k: \mathscr{R} \to \mathscr{B}(H_k)$ for which the bimodule operations are given by $r \circ x \circ s = \pi_k(r)x\pi_k(s)$, and that the injections $\mathscr{V}_k \to \mathscr{L}_{\mathscr{R}}(\mathscr{V}_k)$ are weak* homeomorphisms. With these assumptions, we note that $\pi_k(\mathscr{R}) \subseteq \mathscr{L}_{\mathscr{R}}(\mathscr{V}_k)$.

We may assume that the given map Φ is a complete contraction. The map $\Psi: \mathscr{L}_{\mathscr{R}}(\mathscr{V}_1) \to \mathscr{L}_{\mathscr{R}}(\mathscr{V}_2)$ determined by

$$\Psi(r \oplus v \oplus w^*) = r \oplus \Phi(v) \oplus \Phi(w)^*$$

is then completely positive (see [11], Lemma 7.1), and an $\mathscr{R}$ bimodule map. The corresponding map

$$\Psi_\infty: \mathbf{M}_\infty(\mathscr{L}_{\mathscr{R}}(\mathscr{V}_1)) \to \mathbf{M}_\infty(\mathscr{L}_{\mathscr{R}}(\mathscr{V}_2))$$

is given by

$$\Psi_\infty(r \oplus v \oplus w^*) = r \oplus \Phi_\infty(v) \oplus \Phi_\infty(w)^*.$$

It clearly suffices to show that Ψ_∞ is an $\mathscr{R}_\infty = \mathbf{M}_\infty(\mathscr{R})$ bimodule map, i.e., that for $r, s \in \mathscr{R}_\infty$,

$$\Psi_\infty(\pi_{1\infty}(r)x\pi_{1\infty}(s)) = \pi_{2\infty}(r)\Psi_\infty(x)\pi_{2\infty}(s).$$

We let I_n denote the identity in $\mathbf{M}_n(\mathscr{L}_{\mathscr{R}}(\mathscr{V}_k))$. Since Ψ is a unital $\mathscr{R}$ bimodule map, a simple algebraic calculation shows that Ψ_n is a unital $\mathscr{R}_n$ bimodule map for $n \in \mathbf{N}$. Given $r \in \mathbf{M}_\infty(\mathscr{R})$, we have the strong limits

$$\Psi_\infty(\pi_{1\infty}(r)) = \lim_n \Psi_n(\pi_{1n}(r_n)) = \lim_n \pi_{2n}(r_n)\Psi_n(I_n) = \pi_{2\infty}(r).$$

On the other hand we have that

$$[\pi_{1\infty}(r(I_\infty - E_n))x]\,[\pi_{1\infty}(r(I_\infty - E_n))x]^* = \pi_{1\infty}(r(I_\infty - E_n))xx^*\pi_{1\infty}((I_\infty - E_n)r^*) \leqslant$$
$$\leqslant \|x\|^2\pi_{1\infty}(r(I - E_n)r^*).$$

Ψ_∞ is completely positive (strong limits of positive operators are again positive) and thus it has a completely positive extension

$$\tilde{\Psi}_\infty : \mathscr{B}(H_1^\infty) \to \mathscr{B}(H_2^\infty).$$

Applying the Kadison-Schwarz inequality (see [8], or simply use the Stinespring representation), we conclude that

$$\Psi_\infty(\pi_{1\infty}[r(I_\infty - E_n)]x)\,\Psi_\infty(\pi_{1\infty}[r(I_\infty - E_n)]x)^* \leqslant$$
$$\leqslant \tilde{\Psi}_\infty[\{\pi_{1\infty}[r(I_\infty - E_n)]x\}\,\{\pi_{1\infty}[r(I_\infty - E_n)]x\}^*] \leqslant$$
$$\leqslant \|x\|^2\Psi_\infty[\pi_{1\infty}[r(I - E_n)r^*]] = \|x\|^2\pi_{2\infty}[r(I - E_n)r^*].$$

Since the latter converges weakly to zero, it follows that $\Psi_\infty(\pi_{1\infty}[r(I_\infty - E_n)]x)$ converges strongly to zero. We have that for $m \geqslant n$, $E_n = E_nE_m$, and thus

$$\Psi_\infty(\pi_{1\infty}(rE_n)x) = \lim_{m \geqslant n} \Psi_m(\pi_{1m}(E_mrE_n)E_mxE_m) =$$

$$= \lim_{m \geqslant n} \pi_{2m}(E_mrE_n)\Psi_m(E_mxE_m) = \pi_{2\infty}(rE_n)\Psi_\infty(x).$$

We conclude that

$$\Psi_\infty(\pi_{1\infty}(r)x) = \lim_n \Psi_\infty(\pi_{1\infty}(rE_n)x) =$$

$$= \lim_n \pi_{2\infty}(rE_n)\Psi_\infty(x) = \pi_{2\infty}(r)\Psi_\infty(x).$$

A similar calculation shows that Ψ_∞ is also a right $\mathbf{M}_\infty(\mathscr{R})$-module map.

Theorem 4.2 may be generalized to suitable submodules of a normal dual $\mathscr{R}$ operator bimodule $\mathscr{V}_1$. We say that a subspace $\mathscr{W}_1$ of $\mathscr{V}_1$ is an $\mathbf{M}_\infty$-*submodule* of

$\mathscr{V}_1$ if given $r, s \in \mathbf{M}_\infty(\mathscr{R})$, and $w \in \mathbf{M}_\infty(\mathscr{W}_1)$, it follows that $r \circ v \circ s \in \mathbf{M}_\infty(\mathscr{W}_1)$. Using the representation constructed in Theorem 4.1, it is evident that the proof of Theorem 4.2 applies to bimodule maps $\Phi: \mathscr{W}_1 \to \mathscr{V}_2$.

To illustrate theabove theory, let us suppose that $\mathscr{R}$ is a von Neumann algebra, and that $\mathscr{V}$ is an operator space. We saw above that the space $\mathscr{M} = \mathscr{M}(\mathscr{V}, \mathscr{R})$ of completely bounded maps $\varphi: \mathscr{V} \to \mathscr{R}$ is a normal dual $\mathscr{R}$ operator bimodule. Furthermore, we may identify $\mathbf{M}_\infty(\mathscr{M}(\mathscr{V}, \mathscr{R}))$ with $\mathscr{M}(\mathscr{V}, \mathbf{M}_\infty(\mathscr{R}))$, letting the first space have the weak* topology determined by $\mathbf{M}_\infty(\mathscr{M}_*)$ (see above), and $\mathscr{M}(\mathscr{V}, \mathbf{M}_\infty(\mathscr{R}))$ have the topology of point-weak* convergence. In particular, the obvious "range" $\mathbf{M}_\infty(\mathscr{R})$-bimodule operations on the latter space correspond to the operations described above on $\mathbf{M}_\infty(\mathscr{M}(\mathscr{V}, \mathscr{R}))$. To see this we observe that given $S, T \in \mathbf{M}_\infty(\mathscr{R})$, then letting $S_n = E_n S E_n$, $T_n = E_n T E_n$ and $\varphi_n = E_n \varphi E_n$, we have that $S_n \varphi_n T_n \to S\varphi T$ in the point-strong, and thus the point-weak* topology.

Now let us suppose that $\mathscr{V}$ is a dual operator space, and let $\mathscr{M}^\sigma = \mathscr{M}^\sigma(\mathscr{V}, \mathscr{R})$ be the subspace of weak* continuous maps $\varphi \in \mathscr{M} = \mathscr{M}(\mathscr{V}, \mathscr{R})$. We claim that *$\mathscr{M}^\sigma$ is an $\mathbf{M}_\infty$-submodule of $\mathscr{M}$*. It is a simple matter to verify that

$$\mathscr{M}^\sigma(\mathscr{V}, \mathbf{M}_\infty(\mathscr{R})) = \mathbf{M}_\infty(\mathscr{M}^\sigma(\mathscr{V}, \mathscr{R})),$$

i.e., a completely bounded map $\varphi: \mathscr{V} \to \mathbf{M}_\infty(\mathscr{R})$ is weak* continuous if and only if $\varphi = [\varphi_{ij}]$ with $\varphi_{ij}: \mathscr{V} \to \mathscr{R}$ weak* continuous for all i, j, and that any bounded (see Section 3) matrix $[\varphi_{ij}]$, $\varphi_{ij} \in \mathscr{M}^\sigma(\mathscr{V}, \mathscr{R})$ determines a corresponding $\varphi \in \mathscr{M}^\sigma(\mathscr{V}, \mathbf{M}_\infty(\mathscr{R}))$. Given $S, T \in \mathscr{B}(H^\infty)$, the map $b \mapsto SbT$ is weak* continuous and completely bounded on $\mathscr{B}(H)$. It follows that if $\varphi \in \mathscr{M}^\sigma(\mathscr{V}, \mathbf{M}_\infty(\mathscr{R}))$, then

$$S \circ \varphi \circ T(v) = S\varphi(v)T$$

determines an element of $\mathscr{M}^\sigma(\mathscr{V}, \mathbf{M}_\infty(\mathscr{R}))$.

Turning to the case considered in [5], we fix a von Neumann algebra $\mathscr{R} \subseteq \mathscr{B}(H)$ and we regard $\mathscr{B}(H)$ and thus $\mathscr{M}^\sigma(\mathscr{R}', \mathscr{B}(H))$ as $\mathscr{R}$ operator bimodules. Then from above we obtain a result used in [5], Lemma 4.1.

COROLLARY 4.3. *Suppose that $\Phi: \mathscr{M}^\sigma(\mathscr{R}', \mathscr{B}(H)) \to \mathscr{B}(H)$ is a completely bounded $\mathscr{R}$ operator bimodule map. Then*

$$\Phi_\infty: \mathbf{M}_\infty(\mathscr{M}^\sigma(\mathscr{R}', \mathscr{B}(H))) \to \mathbf{M}_\infty(\mathscr{B}(H))$$

is an $\mathbf{M}_\infty(\mathscr{R})$-bimodule map.

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