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THE UNIQUENESS OF THE ADJOINT OPERATION. III

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1. INTRODUCTION

Let H be a complex, infinite-dimensional Hilbert space with inner product $(\cdot,\cdot)$. Let $B(H)$ denote the set of operators on H (i.e. bounded linear transformations of H into itself) and let $\mathrm{GL}(H)$ denote the set of invertible operators. If $T \in B(H)$ then T *is positive* or equivalently $T \geqslant 0$ means that $(Tx, x) \geqslant 0$ for all $x \in H$. The purpose of this paper is to prove the following result.

THEOREM 1. *Let $h\colon \mathrm{GL}(H) \to B(H)$ be a function such that $h(ST) = h(T)h(S)$ and $h(S)S \geqslant 0$ for all $S, T \in \mathrm{GL}(H)$. If $h(I) \neq 0$, then either $h(S) = S^*$ for all $S \in \mathrm{GL}(H)$ or $h(S) = S^{-1}$ for all $S \in \mathrm{GL}(H)$.*

In [6] it is shown that if the domain of h is all of $B(H)$, then h must be the adjoint, and in [7] the case in which H is finite dimensional is considered.

The proof of Theorem 1 contained in this paper combines results and techniques from [6] and [7] as well as some new ideas.

The last section of this paper contains some open questions.

2. THE PROOF AND SOMETHING EXTRA

LEMMA 1. [6, Corollary 3.2]. *Let $h\colon \mathrm{GL}(H) \to B(H)$ be a function such that $h(ST) = h(T)h(S)$ and $h(S)S \geqslant 0$ for all $S, T \in \mathrm{GL}(H)$. If $h(I) \neq 0$ then $h(U) = U^*$ whenever U is a unitary operator (*i.e. *$U^*U = I = UU^*$).*

REMARK 1. How do things stand in light of this lemma? Well, if $T \in \mathrm{GL}(H)$ then the Polar Decomposition Theorem shows that there exist a unique unitary operator U and a unique invertible operator $P \geqslant 0$ such that $T = UP$. If $h(I) \neq 0$ then $h(T) = h(P)U^*$. Thus, the problem is reduced to determining $h(P)$.

An operator D is said to be a *diagonal* operator if there exist an orthonormal basis $\{e_i\}$ for H and a bounded set of complex numbers $\{d_i\}$ such that $De_i = d_ie_i$ for all i.

Let $\mathscr{D} = \{D \in \mathrm{GL}(H) : D \geqslant 0$ and D is a diagonal operator$\}$.

LEMMA 2. *If P is an invertible positive operator, then there exist invertible positive diagonal operators D_1 and D_2 and a unitary operator W such that $P = WD_1D_2$.*

REMARK 2. In general, D_1D_2 will not be a diagonal operator. To see this, consider the case in which $D_1D_2 = D_3$, a diagonal operator. Then $P = WD_3$, and this is equivalent to $PD_3^{-1} = W$. Since the spectrum of PD_3^{-1} is contained in the set of nonnegative real numbers (see the proof of the Proposition in [8] for more details) and the spectrum of W is contained in the set of complex numbers of modulus 1, it follows from the Spectral Theorem that $W = I$.

Proof of Lemma 2. If H is separable, then if follows from Halmos' improvement [3, Lemma 1] of Weyl's Theorem [9] that $P = D_2 + K$, where D_2 is in $\mathscr{D}$ and K is a self-adjoint compact operator. This can be rewritten as $P = (I + KD_2^{-1})D_2$. Since $I + KD_2^{-1}$ is invertible, there exist a unitary operator W and an invertible positive operator D_1 such that $I + KD_2^{-1} = WD_1$. Since $D_1^2 = (I + KD_2^{-1})^*(I + KD_2^{-1}) = I +$ self-adjoint compact operator, it follows that D_1 is a diagonal operator.

If H is nonseparable, then the representation due to Weyl's Theorem used above is no longer valid for all invertible positive operators [2, Exercise 1 of Problem 4]. However, there exist infinite-dimensional, separable subspaces H_α contained in H such that $P = \sum^\oplus P_\alpha$ with respect to $\sum^\oplus H_\alpha$. The reasoning used above shows that there exist invertible, positive diagonal operators D_α and R_α (mapping H_α into itself) and unitary operators W_α such that $P_\alpha = W_\alpha D_\alpha R_\alpha$. This shows that $P = (\sum^\oplus W_\alpha)(\sum^\oplus D_\alpha)(\sum^\oplus R_\alpha) = WD_1D_2$, where D_1 and D_2 are in $\mathscr{D}$ and W is a unitary operator. ∎

REMARK 1 (continued). It now follows from Lemma 2 that $h(P) = h(WD_1D_2) = h(D_2)h(D_1)W^*$. Thus, to prove Theorem 1, it suffices to show that either $h(D) = D$ for all $D \in \mathscr{D}$ or $h(D) = D^{-1}$ for all $D \in \mathscr{D}$. In the former case, it follows that $h(S) = S^*$ for all $S \in \mathrm{GL}(H)$, and in the latter case, it folows that $h(S) = S^{-1}$ for all $S \in \mathrm{GL}(H)$.

An immediate consequence of the next result is that $h: \mathscr{D} \to \mathscr{D}$. This is a good sign, but there is still a lot to be done.

LEMMA 3. [6, Corollary 3.1]. *If $h: \mathrm{GL}(H) \to B(H)$ is a function such that $h(ST) = h(T)h(S)$ and $h(S)S \geqslant 0$ for all $S, T \in \mathrm{GL}(H)$, then each orthogonal projection commuting with an element S of $\mathrm{GL}(H)$ commutes with $h(S)$.*

The proof of the following result will complete the proof of Theorem 1.

LEMMA 4. *Let $h: \mathrm{GL}(H) \to B(H)$ be a function such that $h(ST) = h(T)h(S)$ and $h(S)S \geqslant 0$ for all $S, T \in \mathrm{GL}(H)$. If $h(I) \neq 0$, then either $h(D) = D$ for all $D \in \mathscr{D}$ or $h(D) = D^{-1}$ for all $D \in \mathscr{D}$.*

Proof. (Some techniques come from the proof of Lemma 4.1 in [7].) If c is a nonzero complex number, then it follows from Lemma 1 that $h(cI)$ commutes with all symmetries (self-adjoint unitary operators). This implies that there exists a multiplicative homomorphism ν on the nonzero complex numbers such that $h(bI) = \nu(b)I$ for all nonzero complex numbers b.

Let K be a subspace of H. Let $T_c = cI \oplus I$ (c as above) with respect to to $K \oplus K^\perp$. It will be demonstrated that $h(T_c) = T_{\nu(c)}$.

It follows from Lemma 3 that $h(T_c) = A_c \oplus B_c$ with respect to $K \oplus K^\perp$. If $J \in B(K)$ is a symmetry, then T_c commutes with the symmetry $J \oplus I$. It then follows from Lemma 1 that A_c commutes with J. Since this is true for an arbitrary symmetry, it follows that A_c is a scalar multiple of I. Say $A_c = \gamma(c)I$. Similarly, $B_c = \zeta(c)I$. Since $I = h(T_c)h(T_c^{-1})$, it follows that $\gamma(c) \neq 0 \neq \zeta(c)$.

It is now necessary to consider a special case. Assume that K and $K^\perp$ are isomorphic. Let $K = H_1 \oplus H_2$ be a direct sum decomposition of K into two isomorphic orthogonal subspaces. Then $cI \oplus cI \oplus I$ ($= T_c$ with respect to $H_1 \oplus H_2 \oplus K^\perp$) is unitarily equivalent to both $cI \oplus I \oplus I$ and $I \oplus cI \oplus I$. If W and U are unitary operators which satisfy $W^*T_cW = cI \oplus I \oplus I$ and $U^*T_cU = I \oplus cI \oplus I$ (e.g. if $\{e_i\}$, $\{f_i\}$, and $\{g_i\}$ are orthonormal bases for H_1, H_2, and $K^\perp$, respectively, then W is determined by $Wf_i = g_{2i}$, $Wg_i = g_{2i+1}$, $We_{2i} = f_i$, $We_{2i+1} = e_i$), then $T_c = W^*T_cWU^*T_cU$. If we apply h to both sides of this equation and do a little manipulating, we get

$$\gamma(c)I \oplus \gamma(c)I \oplus \zeta(c)I = (\zeta(c)I \oplus \gamma(c)I \oplus \zeta(c)I)(\gamma(c)I \oplus \zeta(c)I \oplus \zeta(c)I).$$

This shows that $\zeta(c) = 1$.

Let V be a unitary operator that satisfies $VT_cV^* = cT_{1/c}$ (i.e. $V(cI \oplus I)V^* = I \oplus cI$). Then

$$\nu(c)I = h(cI) = h(T_c \cdot cT_{1/c}) =$$

$$= h(cT_{1/c})h(T_c) = Vh(T_c)V^*h(T_c) = VT_{\gamma(c)}V^*T_{\gamma(c)} = \gamma(c)I.$$

Thus, $h(T_c) = T_{\nu(c)}$ when $\dim K = \dim K^\perp$.

Suppose $\dim K > \dim K^\perp$ (as cardinal numbers). As above, $T_c = cI \oplus cI \oplus I$ with respect to $H_1 \oplus H_2 \oplus K^\perp$. Then

$$h(T_c) = h(cI \oplus I \oplus I)h(I \oplus cI \oplus I) = (\nu(c)I \oplus I \oplus I)(I \oplus \nu(c)I \oplus I)$$

(by the above). Thus, $h(T_c) = T_{\nu(c)}$.

Finally, suppose $\dim K < \dim K^\perp$. Then

$$\nu(c)I = h(cI) = h(I \oplus cI)\, h(cI \oplus I) = (I \oplus \nu(c)I)h(T_c)$$

(by the previous paragraph). Thus, $h(T_c) = T_{\nu(c)}$.

Let M be a subspace of H of codimension 1. In the following, all matrices are written with respect to $M \oplus M^{\perp}$. Let $W: M \to M^{\perp}$, let $S_W = \begin{pmatrix} I & 0 \\ W & -1 \end{pmatrix}$ and let $h(S_W) = \begin{pmatrix} A & B \\ C & a \end{pmatrix}$. If r is a positive real number, let $Q_r = rI \oplus r^{-1}$. Then

$$h(Q_r) = h((rI \oplus 1)(I \oplus r^{-1})) = \nu(r)I \oplus \nu(r)^{-1}.$$

Since $h(Q_r)Q_r \geqslant 0$, it follows that $\nu(r) \geqslant 0$.

Let $0 < r < 1$ and let m be a nonnegative integer. Since $S_{r^{2m}W} = Q_{r^{-m}} S_W Q_{r^m} = (Q_{r^{-1}})^m S_W (Q_r)^m$, it follows that

$$h(S_{r^{2m}W}) = \begin{pmatrix} A & \nu(r)^{2m}B \\ \nu(r)^{-2m}C & a \end{pmatrix}.$$

Since $h(S_{r^{2m}W})S_{r^{2m}W} \geqslant 0$, it follows that

$$\begin{pmatrix} A + (r\nu(r))^{2m}BW & -\nu(r)^{2m}B \\ \nu(r)^{-2m}C + ar^{2m}W & -a \end{pmatrix} \geqslant 0.$$

This shows that

(†) $a \leqslant 0$, $A + (r\nu(r))^{2m}BW \geqslant 0$ and $-\nu(r)^{2m}B^* = \nu(r)^{-2m}C + ar^{2m}W$.

Can $\nu(r) = 1$? If $\nu(r) = 1$ then $-B^* = C + ar^{2m}W$. This shows that $a = 0$ and $-B^* = C$. Then

$$I = h(S_W^2) = (h(S_W))^2 = \begin{pmatrix} * & * \\ * & -B^*B \end{pmatrix}$$

shows that $-B^*B = 1$. This is impossible. Thus, $\nu(r) \neq 1$.

It now follows that $a \neq 0$. That is, if $a = 0$ then $-\nu(r)^{4m}B^* = C$. Since $\nu(r) \neq 1$, it follows that $B = 0$ and $C = 0$. This is impossible as $h(S_W)$ is invertible.

Can $\nu(r) > 1$? If $\nu(r) > 1$, then

$$\|-B^*\| \leqslant \nu(r)^{-4m}\|C\| + (r\nu(r)^{-1})^{2m}\|aW\|$$

(by (†)) shows that $B^* = 0$ as the right side of the inequality can be made arbitrarily small. Thus, $-C = a(r\nu(r))^{2m}W$. This implies that $\nu(r) = r^{-1}$ and $C = -aW$. Since $h(S_W) = \begin{pmatrix} A & 0 \\ -aW & a \end{pmatrix}$, $h(S_W)S_W \geqslant 0$ and $(h(S_W))^2 = I$, it follows that $h(S_W) = S_W$ $(= S_W^{-1})$.

Can $v(r) < 1$? If $v(r) < 1$ then

$$\|-C\| \leqslant v(r)^{4m}\|B^*\| + (rv(r))^{2m}\|aW\|$$

(by (†)) shows that $C = 0$ as the right side of the inequality can be made arbitrarily small. Thus, $-B^* = a(rv(r)^{-1})^{2m}W$. This implies that $v(r) = r$ and $B = -aW^*$. Since $a \leqslant 0$, $A - ar^{4m}W^*W \geqslant 0$ (by (†)), $0 < r < 1$ and m can be any nonnegatve integer, it follows that $A \geqslant 0$. This time, since $A \geqslant 0$, $a \leqslant 0$ and $(h(S_W))^2 = I$, it follows that $h(S_W) = S_W^*$.

If $W \neq 0$ then it is impossible to have $v(r_1) = r_1$ for some $0 < r_1 < 1$ and $v(r_2) = r_2^{-1}$ for another $0 < r_2 < 1$ as this gives $S_W^* = S_W$.

Case 1. $v(r) = r$ for all $0 < r < 1$. Since v is multiplicative and $1 = v(r \cdot r^{-1})$, it follows that $v(r) = r$ for all $r > 0$. Choose $T \in \mathscr{D}$ such that $M^\perp$ is an eigenspace for T. Then there exists a real number t and an invertible, positive diagonal operator T_1 such that $T = T_1 \oplus t$ with respect to $M \oplus M^\perp$. By Lemma 3, $h(T) = B_1 \oplus b$ with respect to $M \oplus M^\perp$. Choose a unit vector $e \in M^\perp$ and an orthonormal basis $\{e_i\}$ for M such that $T_1 e_i = t_i e_i$. Let $W_i x = (x, e_i)e$ for all $x \in M$. Since

$$\begin{pmatrix} I & 0 \\ t^{-1}W_iT_1 & -1 \end{pmatrix} = \begin{pmatrix} T_1^{-1} & 0 \\ 0 & t^{-1} \end{pmatrix}\begin{pmatrix} I & 0 \\ W_i & -1 \end{pmatrix}\begin{pmatrix} T_1 & 0 \\ 0 & t \end{pmatrix},$$

it follows that

$$\begin{pmatrix} I & t^{-1}T_1W_i^* \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} B_1 & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} I & W_i^* \\ 0 & -1 \end{pmatrix}\begin{pmatrix} B_1^{-1} & 0 \\ 0 & b^{-1} \end{pmatrix}.$$

Thus, $t^{-1}T_1W_i^* = b^{-1}B_1W_i^*$. Evaluating each side of this equation at e shows that $bt^{-1}t_ie_i = B_1e_i$. This shows that there exists a positive real number $\lambda(T)$ such that $h(T) = \lambda(T)T$ (in fact, $\lambda(T) = bt^{-1}$). If U is a unitary operator then $h(UTU^*) = Uh(T)U^* = \lambda(T)UTU^*$ shows that $\lambda(T)$ is independent of the orthonormal basis used for H. Thus, for every $D \in \mathscr{D}$, there exists a positive real number $\lambda(D)$ such that $h(D) = \lambda(D)D$.

Choose any operator $D_0 \in \mathscr{D}$ that has 1 as an eigenvalue and all of its eigenvalues have multiplicity equal to the dimension of H. "Split" D_0 in half. That is, choose a subspace K_0 of H such that K_0 and $K_0^\perp$ have the same dimension, $D_0 = D_1 \oplus D_2$ with respect to $K_0 \oplus K_0^\perp$ where D_1 and D_2 are diagonal and unitarily equivalent to each other (so they both "look like" D_0). The conditions on D_0, D_1 and D_2 show that $D_1 \oplus D_2$ is unitarily equivalent to both $D_1 \oplus I$ and $I \oplus D_2$. It then follows that $\lambda(D_1 \oplus I) = \lambda(D_1 \oplus D_2) = \lambda(I \oplus D_2)$ and that $\lambda(D_1 \oplus D_2) = \lambda(D_1 \oplus I)\lambda(I \oplus D_2)$. Thus, $\lambda(D_0) = 1$.

Now choose any operator $D \in \mathscr{D}$. Choose a subspace K of H such that K and $K^\perp$ have the same dimension and $D = D_3 \oplus D_4$ with respect to $K \oplus K^\perp$ where

D_3 and D_4 are diagonal. Now replace D_4 by an operator D_5 which makes $D_3 \oplus D_5$ "look like" D_0'. Specifically, 1 is an eigenvalue of D_5, every eigenvalue of D_3 is an eigenvalue of D_5 and all eigenvalues of D_5 have multiplicity equal to the dimension of H. The previous paragraph shows that $\lambda(D_3 \oplus D_5) = 1 = \lambda(I \oplus D_5)$. It then follows that $\lambda(D_3 \oplus I) = 1$. Similar reasoning shows that $\lambda(I \oplus D_4) = 1$. This shows that $\lambda(D) = 1$ and thus, $h(D) = D$ for all $D \in \mathcal{D}$.

Case 2. $\gamma(r) = r^{-1}$ for all $0 < r < 1$. Thus, $\gamma(r) = r^{-1}$ for all $r > 0$. Let T, T_1, t, B_1, b, e, $\{e_i\}$ and W_i be as described in Case 1. Now since

$$\begin{pmatrix} I & 0 \\ tW_iT_1^{-1} & -1 \end{pmatrix} = \begin{pmatrix} T_1 & 0 \\ 0 & t \end{pmatrix} \begin{pmatrix} I & 0 \\ W_i & -1 \end{pmatrix} \begin{pmatrix} T_1^{-1} & 0 \\ 0 & t^{-1} \end{pmatrix},$$

it follows that

$$\begin{pmatrix} I & 0 \\ tW_iT_1^{-1} & -1 \end{pmatrix} = \begin{pmatrix} B_1^{-1} & 0 \\ 0 & b^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ W_i & -1 \end{pmatrix} \begin{pmatrix} B_1 & 0 \\ 0 & b \end{pmatrix}.$$

Thus, $tW_iT_1^{-1} = b^{-1}W_iB_1$. Take the adjoint of both sides of this equation and then evaluate both sides of the resulting equation at e. It then follows that there exists a positive real number $\lambda(T)$ such that $h(T) = \lambda(T)T^{-1}$ (in fact, $\lambda(T) = bt$). This time it follows that there exists a positive real number $\lambda(D)$ such that $h(D) = \lambda(D)D^{-1}$ for all $D \in \mathcal{D}$.

The rest of the reasoning done in Case 1 also applies here and thus, $h(D) = D^{-1}$ for all $D \in \mathcal{D}$. □

Let $L(H)$ denote the set of left-invertible operators on H. (I.e. S belongs to $L(H)$ if there exists an operator T in $B(H)$ such that $TS = I$.) If $S \in L(H)$, then it follows, from the Polar Decomposition Theorem, that there is a unique invertible positive operator P and a unique isometry U ($(Ux, Ux) = (x, x)$ for all $x \in H$) such that $S = UP$. This factorization leads to a natural left-inverse of S, $P^{-1}U^*$, which will be denoted by $S^\dagger$. (Actually, $S^\dagger$ denotes the generalized inverse of S. Theorem 4.4. in [1] shows that $S^\dagger = P^{-1}U^*$.) Let Is(H) denote the set of isometries on H.

LEMMA 5. [6, Theorem 4.1]. *Let h : Is$(H) \to B(H)$ be a function such that $h(ST) = h(T)h(S)$ and $h(S)S \geq 0$ for all S, $T \in$ Is(H). If $h(I) \neq 0$ then there exists a homomorphism φ from the additive semigroup $\{0, 1, 2, \ldots, \dim H\}$ to the multiplicative semigroup $\mathbf{R}^+$ (positive real numbers) such that $h(S) = \varphi(\text{nullity } S^*)S^*$ for all $S \in$ Is(H).*

(Note: If $h(I) \neq 0$ is replaced by $h(U) \neq 0$ for some shift U of infinite multiplicity then $\varphi(\text{nullity } S^*) = 1$ for all $S \in$ Is(H). See [6, Theorem 3.1] for the details.)

THEOREM 2. *Let h: $L(H) \to B(H)$ be a function such that $h(ST) = h(T)h(S)$ and $h(S)S \geq 0$ for all S, $T \in L(H)$. If $h(I) \neq 0$ then there exists a homomorphism φ*

from the additive semigroup $\{0, 1, 2, \ldots, \dim H\}$ *to the multiplicative semigroup* $\mathbf{R}^+$ *such that either* $h(S) = \varphi(\text{nullity } S^*)S^*$ *for all* $S \in L(H)$ *or* $h(S) = \varphi(\text{nullity } S^\dagger)S^\dagger$ *for all* $S \in L(H)$.

Proof. The proof of Theorem 1 shows that either $h(P) = P$ for all invertible positive operators P or $h(P) = P^{-1}$ for all invertible positive operators P. The rest of the proof is an immediate consequence of Lemma 5 and the remarks immediately preceding it. ▨

3. QUESTIONS AND COMMENTS

If Lemma 2, the Polar Decomposition Theorem, and the fact that every unitary operator is the product of four symmetries [5] are combined, then it follows that every invertible operator is the product of six diagonal operators. Is six best possible? What if the invertibility condition is dropped? Is every positive operator the product of a finite number of diagonal operators? Is every operator the product of a finite number of diagonal operators? The answer to the last question is no because the unilateral shift is not the product of finitely many normal operators [4, Problem 144]. This answer shows that the wrong question was asked. A better question: Is every operator that is the product of finitely many normal operators also the product of finitely many diagonal operators? P. Y. Wu has a result that might help answer this question.

THEOREM 3. [10, Theorem 1.1]. *If* $T \in B(H)$, *then the following are equivalent:*

1. *T is the product of finitely many normal operators;*
2. *T is the product of finitely many self-adjoint operators;*
3. *T is the product of finitely many positive operators;*
4. *$T = SP$ or PS depending on whether* $\dim\ker T \geqslant \dim\ker T^*$ *or* $\dim\ker T \leqslant \dim\ker T^*$ *for some operator S which is one-to-one with dense range and some orthogonal projection P;*
5. $\dim\ker T = \dim\ker T^*$ *or* $\operatorname{ran} T$ *is not closed.*

Moreover, in this case, T can be expressed as the product of 3 *normal operators,* 6 *self-adjoint operators, or* 18 *positive operators.*

What happens if $h(S)S \geqslant 0$ is replaced by $Sh(S) \geqslant 0$? We expect nothing to happen and this is the case. When the domain of h is closed under taking adjoints (e.g. $B(H)$, $\mathrm{GL}(H)$), consider the function $g(S) = (h(S^*))^*$. Since $g(S)S = (S^*h(S^*))^*$ and $g(ST) = g(T)g(S)$, we get the same results. When the domain of h is $\mathrm{Is}(H)$ then $Sh(S) \geqslant 0$ for all $S \in \mathrm{Is}(H)$ implies that $S^*(Sh(S))S \geqslant 0$ for all $S \in \mathrm{Is}(H)$ and thus, $h(S)S \geqslant 0$ for all $S \in \mathrm{Is}(H)$. Once again we get the same results. These observations show that the case in which the domain of h is $L(H)$ also remains unchanged.

What happens when other subsets of $B(H)$ become the domain of h? The only restriction on these subsets is that they be closed under multiplication. One natural candidate is $R(H)$, the set of right-invertible operators. This case is easily derived from Theorem 2. After all, $S \in L(H)$ if and only if $S^* \in R(H)$. Then by considering $g(S) = (h(S^*))^*$, the desired results are obtained. Other natural domains on which h has not been characterized are the compact operators and any von Neumann algebra (the latter set was suggested by W. B. Arveson). If H is finite dimensional, then the singular operators are a domain on which h has not been characterized.

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Added in proofs. A revised version of [10] contains the following result, which affirmatively answers some of the questions raised at the beginning of Section 3.

Theorem. *T is the product of positive diagonal operators if and only if* $\dim\ker T = \dim\ker T^*$ *or* $\operatorname{ran} T$ *is not closed.*