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NORMAL OPERATORS ARE DIAGONAL PLUS HILBERT SCHMIDT

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In [8], Voiculescu proves that every normal operator can be expressed as the sum of a diagonalizable operator and a small Hilbert-Schmidt operator. The purpose of this note is to simplify a few points in his argument. Voiculescu proves much more. He provides a good technique for obtaining ideal perturbations of sets of operators. In particular, for $n \geqslant 2$ he shows that n-tuples of commuting Hermitian operators are $\mathscr{C}_n$ perturbations of an n-tuple of commuting diagonal operators. Moreover, he identifies an ideal $\mathscr{C}_n^-$ containing $\mathscr{C}_p$ for all $p < n$. The absolutely continuous part of a commuting n-tuple is shown to be the obstruction to a $\mathscr{C}_n^-$ perturbation to a diagonal n-tuple. These results strengthen and generalize results of Voigt [9] and Kato and Rosenblum [5, 6].

The key observation is that Voiculescu's generalized Weyl — von Neumann Theorem [7] (henceforth refereed to as Voiculescu's Theorem) will apply to norm ideals such as $\mathscr{C}_p$ provided that a quasicentral approximate unit modulo the ideal is available. This notion was introduced by Arveson [2] in his paper which clarifies and amplifies many aspects of Voiculescu's theorem.

In this note, we will consider only $\mathscr{C}_2$ perturbations of single operators. This is done for the sake of clarity and simplicity. (After all, that is the whole point of writing this.) We were led to this because of our study of the distance between unitary orbits [4]. We thank I. David Berg for some helpful conversations.

1. BACKGROUND

All operators in this paper are continuous linear operators acting on a separable Hilbert space $\mathscr{H}$. The set of all operators is denoted by $\mathscr{B}(\mathscr{H})$, and $\mathscr{K}$ denotes the ideal of compact operators. The Hilbert-Schmidt class is denoted $\mathscr{C}_2$ with the usual norm

$$\|K\|_2 = [\mathrm{trace}(K^*K)]^{1/2}.$$

Two operators are *approximately unitarily equivalent* modulo an ideal $\mathscr{J}$, denoted

$A \underset{\mathscr{I}}{\sim} B$, provided there is a sequence U_n of unitary operators such that $A - U_nBU_n^*$ belongs to $\mathscr{I}$ and

$$\lim_{n\to\infty} \|A - U_nBU_n^*\|_{\mathscr{I}} = 0.$$

When the ideal is $\mathscr{K}$, we write $A \underset{a}{\sim} B$ and suppress the mention of $\mathscr{K}$. We also write $A \overset{\varepsilon}{\underset{\mathscr{I}}{\sim}} B$ to mean there is a unitary U such that $\|A - UBU^*\|_{\mathscr{I}} < \varepsilon$.

The fundamental theorem about approximate unitary equivalence is:

THEOREM (Voiculescu). *Let A be an operator on $\mathscr{H}$, and let ρ be a representation of $C^*(A)$ on a separable space which annihilates $C^*(A) \cap \mathscr{K}$. Then $A \underset{a}{\sim} A \oplus \rho(A)$.*

The *unitary orbit* of an operator A is

$$\mathscr{U}(A) = \{UAU^* : U \in \mathscr{U}(\mathscr{H})\}$$

where $\mathscr{U}(\mathscr{H})$ denote the set of all unitary operators. An important corollary of Voiculescu's Theorem is the relation

$$\mathscr{U}(A)^- = \{T : T \underset{a}{\sim} A\}.$$

Arveson [2] and Akemann--Pedersen [1] introduced the notion of *quasicentral approximate unit*. If $\mathscr{S}$ is a set of operators and $\mathscr{I}$ is an ideal, then a (countable) quasicentral approximate unit modulo $\mathscr{I}$ for $\mathscr{S}$ is an increasing sequence E_n of positive finite rank contractions with $\sup E_n = I$ such that

$$\lim_{n\to\infty} \|[S, E_n]\|_{\mathscr{I}} = 0 \quad \text{for all } S \text{ in } \mathscr{S}.$$

By clarifying the role of quasicentral approximate units in approximation, Arverson was able to provide a straightforward proof of Voiculescu's theorem that was conducive to generalization to other ideals. Voiculescu does this in [8]. We state a special version of this, but all the important ideas are present.

THEOREM 1.([8]). *Suppose A and B are operators such that $A \underset{a}{\sim} A \oplus B$. If B has a quasicentral approximate unit modulo $\mathscr{C}_2$, then $A \underset{\mathscr{C}_2}{\sim} A \oplus B$.*

2. IDEAL PERTURBATIONS

In this section we prove Theorem 1. The first step is to modify the quasicentral approximate unit for B. This proof works perfectly well for other ideals.

LEMMA 2. *Suppose that B has a $\mathscr{C}_2$ quasicentral approximate unit. Then there is such an approximate unit F_n such that*

$$\lim_{m\to\infty} \sup_{n\geqslant m} \|[(F_n - F_m)^{1/2}, B]\|_2 = 0 = \lim_{n\to\infty} \|[F_n^{1/2}, B]\|_2.$$

Proof. Let E_n be a given $\mathscr{C}_2$ approximate unit. It is routine to obtain a new $\mathscr{C}_2$ quasicentral approximate unit with the additional property that $E_nE_{n+1} = E_n$ for all $n \geqslant 1$. (Indeed this is achieved by a sequence which is $\mathscr{C}_2$ asymptotic to a subsequence of the original one.) Assuming this additional feature, we define

$$F_n = \sin^2(\pi/2E_n).$$

Then $F_n^{1/2} = \sin(\pi/2E_n)$ and $(F_n - F_m)^{1/2} = \sin(\pi/2E_n)\cos(\pi/2E_m)$. A well known estimate is needed:

$$\|[E^k, B]\|_2 \leqslant \sum_{j=0}^{k-1} \|E^j[E, B]E^{k-1-j}\|_2 \leqslant k\|[E, B]\|_2.$$

Hence a routine power series estimate yields

$$\|[\sin(\pi/2E), B]\|_2 \leqslant \sum_{n\geqslant 0} \frac{(\pi/2)^{2n+1}}{(2n+1)!} 2n+1\|[E, B]\|_2 =$$

$$= \pi/2 \cosh(\pi/2)\|[E, B]\|_2 < 4\|[E, B]\|_2.$$

Similarly, $\|[\cos(\pi/2E), B]\|_2 < 4\|[E, B]\|_2$. Thus

$$\|[(F_n - F_m)^{1/2}, B]\|_2 \leqslant \|[\sin(\pi/2E_n), B]\cos(\pi/2E_m)\| +$$

$$+ \|\sin(\pi/2E_n)[\cos(\pi/2E_m), B]\| \leqslant$$

$$\leqslant 4\|[E_n, B]\|_2 + 4\|[E_m, B]\|_2.$$

Similarly, $\|[F_n^{1/2}, B]\|_2 \leqslant 4\|[E_n, B]\|_2$ and $\|[F_n, B]\|_2 \leqslant 8\|[E_n, B]\|_2$. It is now immediate that F_n has the desired properties. ▨

Proof of Theorem 1. Fix $\varepsilon > 0$. Let F_n be a $\mathscr{C}_2$ quasicentral approximate unit for B with the additional properties provided by Lemma 2. Drop to a subsequence if necessary so that

$$\sum_{n\geqslant 1} \|[(F_n - F_{n-1})^{1/2}, B]\|_2 < \varepsilon.$$

Let P_n be the (finite rank) projection onto the range of F_n.

Using Voiculescu's Theorem, it is easy to deduce that $A \underset{a}{\sim} A \oplus B^{(\infty)}$ where $B^{(\infty)}$ represents the direct sum of countably many copies of B acting on $H^{(\infty)}$, the Hilbert space direct sum of countably many copies of $\mathscr{H}$. Thus we obtain isometries V_n with pairwise orthogonal ranges such that

$$\lim_{n\to\infty} \|AV_n - V_nB\| = 0 = \lim_{n\to\infty} \|A^*V_n - V_nB^*\|.$$

Since P_k is finite rank,

$$\lim_{n\to\infty} \|(AV_n - V_nB)P_k\|_1 + \|(A^*V_n - V_nB^*)P_k\|_1 = 0.$$

(Here $\|\cdot\|_1$ is the trace norm. All we need is $\|X\|_1 \leqslant \|X\|_2$ for all operators X.) By dropping to a subsequence of $\{V_n\}$, also denoted V_n, we may suppose that

$$\sum_{k\geqslant 1} \|(AV_k - V_kB)P_k\|_2 + \|(A^*V_k - V_kB^*)P_k\|_2 < \varepsilon.$$

Now, we define an isometry $W = \sum_{k\geqslant 1} V_k(F_k - F_{k-1})^{1/2}$ where $F_0 = 0$ by convention. Consider the following computation:

$$\|AW - WB\|_2 = \|\sum_{k\geqslant 1}(AV_k - V_kB)P_k(F_k - F_{k-1})^{1/2} + V_k[B, (F_k - F_{k-1})^{1/2}]\|_2 \leqslant$$

$$\leqslant \sum_{k\geqslant 1} \|(AV_k - V_kB)P_k\| + \sum_{k\geqslant 1} \|[B, (F_k - F_{k-1})^{1/2}]\|_2 < 2\varepsilon.$$

Similarly, $\|A^*W - WB^*\|_2 < 2\varepsilon$.

Let $P = WW^*$ be the projection onto the range of W. Set $C = P^\perp A|P^\perp\mathscr{H}$ and $B' = WBW^*|P\mathscr{H}$. Then B' is unitarily equivalent to B, and $\|A - B' \oplus \oplus C\|_2 < 4\varepsilon$. That is,

$$A \underset{\mathscr{C}_2}{\overset{4\varepsilon}{\sim}} B \oplus C.$$

Since B has a $\mathscr{C}_2$ quasicentral approximate unit, so does $B^{(\infty)}$. So by using $B^{(\infty)}$ in place of B above, one obtains an operator C such that

$$A \underset{\mathscr{C}_2}{\overset{4\varepsilon}{\sim}} B^{(\infty)} \oplus C \simeq B^{(\infty)} \oplus C \oplus B \underset{\mathscr{C}_2}{\overset{4\varepsilon}{\sim}} A \oplus B.$$

As ε is arbitrary, this concludes the proof. ∎

Corollary 3. *If $A \underset{a}{\sim} B$ and both A and B have a $\mathscr{C}_2$ approximate identity, then $A \underset{\mathscr{C}_2}{\sim} B$.*

Proof. By Theorem 3.2 of [4], it follows that there are operators A_0, B_0, C, and D such that $A \simeq A_0 \oplus C$, $B \simeq B_0 \oplus C$, and $A \underset{a}{\sim} B \underset{a}{\sim} C \oplus A_0^{(\infty)} \oplus B_0^{(\infty)}$. It is easy to obtain a $\mathscr{C}_2$ quasicentral approximate unit for A_0 or B_0 by compressing the approximate unit for A or B to the domain of A_0 or B_0. Thus by Theorem 1,

$$A \underset{\mathscr{C}_2}{\sim} A \oplus A_0^{(\infty)} \oplus B_0^{(\infty)} \cong C \oplus A_0^{(\infty)} \oplus B_0^{(\infty)} \cong$$

$$\cong B \oplus A_0^{(\infty)} \oplus B_0^{(\infty)} \underset{\mathscr{C}_2}{\sim} B.$$ ▧

3. NORMAL OPERATORS

In this section, we give a modified proof of Voiculescu's result which diagonalizes all normal operators modulo the Hilbert—Schmidt class.

THEOREM. ([8]). *Let N be a normal operator and let $\varepsilon > 0$ be given. Then there is a diagonal operator D and a Hilbert—Schmidt operator K such that $\|K\|_2 < \varepsilon$ and $N = D + K$.*

REMARK. Before giving the proof, we wish to point out that, for normal operators, one does not need Voiculescu's Theorem to obtain Theorem 1 and Corollary 3. A formulation of the Weyl—von Neumann—Berg Theorem [3] states: every normal operator N is approximately unitarily equivalent to any diagonal operator D such that $\sigma(N) = \sigma(D)$, and N and D have the same multiplicity at isolated eigenvalues. This suffices to show that $N \underset{a}{\sim} N \oplus M$ implies $N \underset{a}{\sim} N \oplus M^{(\infty)}$ for normal operators, for which we needed Voiculescu's Theorem in general. And it implies that if $N \underset{a}{\sim} M$, then these operators decompose as $N \simeq D \oplus N'$ and $M \simeq D \oplus M'$ where D is a diagonal operator corresponding to the isolated eigenvalues, and $\sigma(N') \cup \sigma(M')$ is contained in $\sigma_e(N) = \sigma_e(M)$. So $N \underset{a}{\sim} D \oplus (N' \oplus M')^{(\infty)} \underset{a}{\sim} M$. Thus Corollary 3 also follows without recourse to Voiculescu's Theorem.

Proof. By the Weyl—von Neumann—Berg Theorem [3], there is a diagonal operator D such that $N \underset{a}{\sim} D$. There is an increasing sequence E_n of finite rank projections in the commutant of D increasing strongly to I; so D has a central approximate unit. By Corollary 3, it suffices to produce a $\mathscr{C}_2$ quasicentral approximate unit for N. Decompose N into a direct sum $N \cong \sum^{\oplus} N_k$ where each N_k has a cyclic vector and either

(i) the spectral measure of N_k is singular to planar measure, or

(ii) the spectral measure is absolutely continuous.

Case (i) can further be refined in case

(i') $\sigma(N_k)$ has planar measure 0.

This is because spectral measures are regular, so one can find countably many disjoint compact sets of Lebesgue measure 0 whose union has full spectral measure. The restrictions to the corresponding spectral subspaces have the desired property.

It suffices to find a $\mathscr{C}_2$-quasicentral approximate unit for each summand. For Case (i′), let M be normal with cyclic vector x and planar measure of $\sigma(M)$ equal to 0. For any $\varepsilon > 0$, cover $\sigma(M)$ with finitely many discs D_j of radius r_j and center λ_j so that $\pi\sum r_j^2 < \varepsilon^2$. Let B_j be pairwise disjoint Borel subsets of D_j with $\cup B_j = \cup D_j$. Let $x_j = t_j E_m(B_j)x$ where t_j is chosen to make $\|x_j\| = 1$. Set P_ε to be the projection onto $\operatorname{span}\{x_j\}$. Then

$$\|[M, P_\varepsilon]\|_2^2 = \sum_j \|E_M(B_j)[M, P]\|_2^2 = \|\sum_j [M, x_j \otimes x_j^*]\|_2^2 =$$

$$= \sum_j \|[M, x_j \otimes x_j^*]\|_2^2 \leqslant 2 \sum_j \|(M - \lambda_j)x_j\|^2 \leqslant 2 \sum_j r_j^2 < \varepsilon^2.$$

Moreover, $P_\varepsilon x = x$ for all $\varepsilon > 0$. Thus if $p(x, y)$ is any polynomial,

$$\lim_{\varepsilon\to 0}\|P_\varepsilon p(M, M^*)x - p(M, M^*)x\| \leqslant \lim_{\varepsilon\to 0}\|[P_\varepsilon, p(M, M^*)]\|\,\|x\| = 0.$$

So P_ε increases strongly to I as ε decreases to 0. Hence M is $\mathscr{C}_2$-quasicentral.

Now consider two bilateral shifts of infinite multiplicity acting on $\mathscr{H} = \operatorname{span}\{e_{nm} : n, m \in \mathbf{Z}\}$ by $U_1 e_{nm} = e_{n+1,m}$ and $U_2 e_{nm} = e_{n,m+1}$. Let P_n be the orthogonal projection onto $\operatorname{span}\{e_{jk} : |j| \leqslant n, |k| \leqslant n\}$. Note that

$$[U_i, P_n] = (P_{n+1} - P_n)U(P_n - P_{n-1}) - (P_n - P_{n-1})U(P_{n+1} - P_n)$$

is the difference of two partial isometries of rank $2n - 1$. These two terms have orthogonal domains and ranges, which are orthogonal to the domains and ranges of $[U_i, P_m]$, $m \neq n$. For each N, let $S_N = \sum_{k=N}^{N^2} \frac{1}{k} \approx \log N$. Set $E_N = S_N^{-1} \sum_{k=N}^{N^2} \frac{1}{k} P_k$. Then

$$\|[U_i, E_N]\|_2^2 = S_N^{-2} \sum_{k=N}^{N^2} \frac{1}{k^2} \|[U_i, P_k]\|_2^2 =$$

$$= S_N^{-2} \sum_{k=N}^{N^2} \frac{1}{k^2} \|[U_i, P_k]\|_2^2 = S_N^{-2} \sum_{k=N}^{N^2} \frac{4k-2}{k^2} < 4S_N^{-1}.$$

Thus E_N is a $\mathscr{C}_2$-quasicentral unit for U_1 and U_2 simultaneously.

The Fourier transform carries $\mathscr{H}$ onto L^2 of the torus $\mathbf{T}^2$ with Lebesgue measure, and carries U_1 and U_2 to the multiplication operators M_{z_1} and M_{z_2} by the coordinate functions. Let M be any cyclic normal operator with $\|M\| \leqslant 1$ and absolutely continuous spectral measure. Now $T = \operatorname{Re} U_1 + \mathrm{i} \operatorname{Im} U_2$ has spectrum $\{x + \mathrm{i}y : |x| \leqslant 1, |y| \leqslant 1\}$. So there is a projection Q in $\{U_1, U_2\}'$ such that $T_0 = T|Q\mathscr{H}$ is unitarily equivalent to M. Let $F_N = QE_N|Q\mathscr{H}$. Clearly, F_N tends strongly to I. Moreover,

$$\|[F_N, T_0]\|_2 = \|Q[E_N, T]Q\|_2 \leqslant \|[E_N, T]\|_2$$

tends to 0. Thus, F_N is a $\mathscr{C}_2$-quasicentral unit for T_0 and so M has one also.

The proof now follows from Corollary 3. ▨

4. AN EXAMPLE

The purpose of this short section is to demonstrate that without a $\mathscr{C}_2$ quasicentral approximate unit around, $A \underset{a}{\sim} B$ need not imply that $A \underset{\mathscr{C}_2}{\sim} B$. The example is based on a result by Voigt [9]. (See also [8].) For $n \geqslant 2$, let $(A_1, \ldots, A_n)$ be an n-tuple of commuting Hermitian operators such that the joint spectral measure on $(A_1, \ldots, A_n)$ is mutually absolutely continuous with respect to Lebesgue measure. Then $(A_1, \ldots, A_n)$ are not simultaneously $\mathscr{C}_{n-1}$ approximately unitarily equivalent to an n-tuple of diagonal operators.

For $n \geqslant 1$, let $I_n = \left[\frac{1}{2n+1}, \frac{1}{2n}\right]$; and let $X = \prod_{n\geqslant 1} I_n$. Let μ be the product measure on X of Lebesgue measure on each I_n. Let A_n be multiplication by the n-th coordinate function on $L^2(\mu)$. Then $\{A_n, n \geqslant 1\}$ is a commuting family of self adjoint operators with spectral measure equivalent to μ. Moreover, $\sigma(A_n) = I_n$ are pairwise disjoint. For $N \geqslant 2$, define an operator on $\mathscr{H}^{(N)}$ by

$$T_N = \begin{bmatrix} A_1 & \frac{1}{2}I & 0 & & 0 & 0 \\ 0 & A_2 & \frac{1}{3}I & & 0 & 0 \\ 0 & 0 & A_3 & & 0 & 0 \\ & & & \ddots & & \\ 0 & 0 & 0 & & A_{N-1} & \frac{1}{N}I \\ 0 & 0 & 0 & & 0 & A_N \end{bmatrix}.$$

Similarly, we can define an infinite version which we denote T_ω.

Let $\{D_n, n \geqslant 1\}$ be a family of diagonal operators with joint spectrum equal to X. By Voiculescu's Theorem applied to $C^*\{A_n, n \geqslant 1\}$ and $C^*\{D_n, n \geqslant 1\}$, we find that $A_n \underset{a}{\sim} D_n$ simultaneously. That is, there are unitary operators U_k such that $A_nU_k - U_kD_n$ are compact for all n and k, and

$$\lim_{k\to\infty} \|A_nU_k - U_kD_n\| = 0 \quad \text{for all } n \geqslant 1.$$

Now, let S_N be the operator on $\mathscr{H}^{(N)}$ defined like T_N with A_n replaced by D_n in all places. Then $S_N \underset{a}{\sim} T_N$ for all $N \geqslant 2$, and $S_\omega \underset{a}{\sim} T_\omega$.

We claim that S_N is not $\mathscr{C}_{N-1}$ approximately unitarily equivalent to T_N; and S_ω is not $\mathscr{C}_p$ approximately unitarily equivalent to T for any $p < \infty$. The point is that $\sigma(T_N) = \bigcup_{j=1}^{N} I_j$, and the Riesz projection $E_T\left\{\bigcup_{j=1}^{n} I_j\right\}$ equals $\mathscr{H}^{(n)} \oplus 0^{(N-n)}$ for each $1 \leqslant n \leqslant N$. The same holds for S_N. If $S_N \underset{\mathscr{C}_{N-1}}{\sim} T_N$, the approximate unitary equivalence must identify the Riesz projections modulo $\mathscr{C}_{N-1}$. It is not hard to then modify the sequence of unitaries so that $U_j\mathscr{H}^{(n)} \oplus 0^{(N-n)} = \mathscr{H}^{(n)} \oplus 0^{(N-n)}$ for all $1 \leqslant n \leqslant N$. Therefore it follows that U_j has the diagonal form $U_j = \sum_{n=1}^{N}{}^{\oplus} U_{j,n}$.

Now compute the $(n, n+1)$ entry of $T_NU_j - U_jS_N$. This is $(n+1)^{-1}(U_{j,n} - U_{j,n+1})$. Thus, U_j is $\mathscr{C}_{N-1}$ asymptotic to $U_{j,1}^{(N)}$. Consequently, $U_{j,1}^{(N)}$ it also yield a $\mathscr{C}_{N-1}$ approximate unitary equivalence of T_N and S_N. But this means that $U_{j,}$ provides joint $\mathscr{C}_{N-1}$ approximate unitary equivalence between $\{A_1, \ldots, A_n\}$ and $\{D_1, \ldots, D_N\}$. This is impossible. The case of S_ω and T_ω is handled in the same way.

Finally, notice that S_N and S_ω have a central approximate unit. So T_N cannot have a $\mathscr{C}_{N-1}$ quasicentral approximate unit, nor can T_ω have a $\mathscr{C}_p$ quasicentral approximate unit for any $p < \infty$.

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